A nonlinear stochastic heat equation: Hölder continuity and smoothness of the density of the solution

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Abstract

In this paper, we establish a version of the Feynman-Kac formula for multidimensional stochastic heat equation driven by a general semimartingale. This Feynman-Kac formula is then applied to study some nonlinear stochastic heat equations driven by nonhomogenous Gaussian noise: First, it is obtained an explicit expression for the Malliavin derivatives of the solutions. Based on the representation we obtain the smooth property of the density of the law of the solution. On the other hand, we also obtain the Hölder continuity of the solutions.

1 Introduction

In this paper we consider the following nonlinear stochastic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\triangle u + b(u) + \sigma(u)\dot{W}(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^d \\ u(0,x) = u_0(x)\,, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, b and σ are globally Lipschitz continuous functions, and W is a zero mean Gaussian random field, which is a Brownian motion in the time variable and it has a nonhomogeneous spatial covariance with density $q(x,y)$ (see (2.1) for the precise definition). Here $\dot{W}(t,x)$ denotes the generalized random field $\dfrac{\partial^{d+1}W}{\partial t\partial x_1\cdots\partial x_d}$.

The case of an homogeneous covariance kernel $q(x,y) = q(x-y)$ has been studied in the seminal paper by Dalang [3]. In this case, the existence, uniqueness and Hölder continuity of $u(t,x)$ with respect to both parameters t and x is obtained in [15] under integrability conditions on the spectral measure μ of the noise. We extend these results to the nonhomogeneous case in Section 4.

On the other hand, using the techniques of Malliavin calculus, and assuming suitable nondegeneracy conditions, one can show that for a fixed (t,x), $t > 0$, the random variable $u(t,x)$, solution to (1.1), has an absolutely continuous probability law and the density is smooth. The results that have been obtained so far along this direction can be summarized as follows.

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(i) In [13] Pardoux and Zhang considered Equation (1.1) when x is in the interval $(0,1)$ with Dirichlet boundary conditions, assuming that W is a space-time white noise. In this case, if the coefficients are Lipschitz, then $u(t,x)$ has an absolutely continuous distribution for any $t>0$ provided $\sigma(u_0(x_0)) \neq 0$ for some $x_0 \in (0,1)$. The smoothness of the density in this framework was proved by Mueller and Nualart in [10], assuming that the coefficients are infinitely differentiable with bounded derivatives. On the other hand, under the stronger nondegeneracy condition $|\sigma(x)| \geq c > 0$, and smooth coefficients, Bally and Pardoux [1] proved that the law of any vector of the form $(u(t,x_1),\dots,u(t,x_n))$, $0 \leq x_1 < \cdots < x_n \leq 1$, $t>0$, has an infinitely differentiable density, assuming Neumann boundary conditions on $(0,1)$.

(ii) For the d-dimensional heat equation with an homogeneous spatial covariance, Nualart and Quer-Sardanyons have provided sufficient conditions for the existence and smoothness of the density of $u(t,x)$ for $t>0$ and $x \in \mathbb{R}^d$, assuming $|\sigma(x)| \geq c > 0$, in the paper [12] (see also [4]).

An open problem for the stochastic heat equation with colored spatial covariance is to derive the existence and smoothness of the density under a nondegeneracy condition of the form $\sigma(u_0(x_0)) \neq 0$ for some $x_0 \in \mathbb{R}^d$. The main purpose of this paper is obtain new results in this direction. To prove such results we need to show that the norm of the Malliavin derivative of the solution $\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds$ is either strictly positive almost surely (for the absolute continuity) or it has negative moments of all orders (for the smoothness of the density), where $\mathcal{H}$ is the Hilbert space associated with the spatial covariance.

We develop a new approach to prove these results based on the Feynman-Kac representation for the solution to the heat equation with multiplicative noise driven by a general continuous semimartingale. The main idea is to express $\|D_s u(t,x)\|_{\mathcal{H}}^2$ as the norm in $L^2(\mathbb{R}^d)$ of a function $V_{s,\xi}(t,x)$ given by $V_{s,\xi}(t,x) = \int_{\mathbb{R}^d} c(\xi,y) D_{s,y} u(t,x) dy$, where c is the square root of the kernel q as an operator. Then for any fixed (s,ξ), $V_{s,\xi}(t,x)$ satisfies the linear stochastic heat equation with random coefficients

$$\frac{\partial V_{s,\xi}}{\partial t} = \frac{1}{2}\triangle V_{s,\xi} + b'(u)V_{s,\xi} + \sigma'(u)V_{s,\xi}\dot{W}(t,x), \quad t \geq s, x \in \mathbb{R}^d, \tag{1.2}$$

with initial condition $V_{s,\xi}(s,x) = c(\xi,x)\sigma(u(s,x))$.

In order to establish a Feynman-Kac representation for the solution to Equation (1.2) we need to assume that the covariance kernel $q(x,y)$ is non-singular and this implies the existence of a random field $W_1(t,x)$ such that $\dot{W}(t,x) = \frac{\partial W_1}{\partial t}(t,x)$. Then, Equation (1.2) is a particular case of a more general stochastic heat equation of the form

$$\frac{\partial V}{\partial t}(t,x) = \frac{1}{2}\triangle V(t,x) + V\frac{\partial F}{\partial t}(t,x), \tag{1.3}$$

where $\{F(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a continuous semimartingale in the sense of Kunita [9], with local characteristic $b(t,x) = b'(u(t,x))$ and $a(t,x,y) = \sigma'(u(t,x))\sigma'(u(t,y))q(x,y)$. In Section 3 (see Theorem 3.1) we derive a Feynman-Kac formula for the solution of (1.3) assuming that the functions b and a are bounded by $C(1+|x|^\beta)$, and $C(1+|x|^\beta+|y|^\beta)$ for some $0 \leq \beta < 2$. This result has its own interest. The proof is based on a generalized Itô formula proved in [9].

There have been other papers on the Feynman-Kac formula for the stochastic heat equation. We can mention the recent works [6] and [7] on the stochastic heat equation driven by fractional white noise. We refer to the references in these papers for related works.

In Section 4 we show the existence and uniqueness of a solution for the general stochastic heat equation (1.1) with a nonhomogeneous spatial covariance and we deduce the Hölder continuity of the solution. This result is an extension of the results proved in [15]. Finally, in Section 5, assuming that the covariance kernel is continuous and under a nondegeneracy condition of the form $q(x_0, x_0) > 0$ and $\sigma(u_0(x_0)) \neq 0$ for some $x_0 \in \mathbb{R}$, we establish the absolute continuity of the law of the solution and the smoothness of the density if the coefficients are smooth.

To simplify the presentation we have assumed that the functions b and σ depend only on the variable u. All the results of this paper could be extended without difficulty to the case of coefficients $b(t, x, u)$ and $\sigma(t, x, u)$ such that they are Lipschitz and with linear growth in u, uniformly in $(t, x) \in [0, T] \times \mathbb{R}^d$ for any $T > 0$. In this case, the nondegeneracy condition would be $\sigma(0, x_0, u_0(x_0)) \neq 0$, for some $x_0 \in \mathbb{R}^d$.

The results of this paper can be extended to the stochastic heat equation on an open and bounded set $A \subset \mathbb{R}^d$, with Dirichlet boundary conditions. In this case, the Feynman-Kac formula involves a d-dimensional Brownian motion starting form a point $x \in A$, and killed when it leaves the set A. On the other hand, the existence and smoothness of the density have been deduced, applying techniques of Malliavin calculus, for stochastic differential equations of the form $Lu = b(u) + \sigma(u)\dot{W}$, where L is a differential operator more general than $\partial_t - \frac{1}{2}\Delta$ (see, for instance, [8, 16] where L is a pseudodifferential operator and [12] where L is a general parabolic or hyperbolic operator). In all these examples, one assumes that σ is bounded away from the origin. Our approach to handle a nondegeneracy of the form $\sigma(0, x_0, u_0(x_0))$ only works if a Feynman-Kac representation is available for the corresponding stochastic linear equation satisfied by the Malliavin derivative. This happens, for instance, for parabolic operators of the form $L = \partial_t - \sum_i b_i \partial_{x_i} - \frac{1}{2}\sum_{i,j} a_{i,j}\partial^2_{x_i,x_j}$. The methodology developed in this paper could be extended to these operators, replacing the Brownian motion by the diffusion process with generator $\sum_i b_i \partial_{x_i} - \frac{1}{2}\sum_{i,j} a_{i,j}\partial^2_{x_i,x_j}$.

2 Preliminaries

2.1 Malliavin calculus

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Consider a family of zero mean Gaussian random variables $W = \{W_t(\varphi), \varphi \in C_0^\infty(\mathbb{R}^d), t \geq 0\}$, where $C_0^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support, with covariance

$$E\left[W_t(\varphi)W_s(\psi)\right] = (t \wedge s)\int_{\mathbb{R}^{2d}} \varphi(x)\psi(y)q(x, y)dxdy, \tag{2.1}$$

where q is a nonnegative definite and locally integrable function.

Let $\mathcal{H}$ be the Hilbert space defined as the completion of $C_0^\infty(\mathbb{R}^d)$ by the inner product

$$\langle \varphi, \psi\rangle_{\mathcal{H}} := \int_{\mathbb{R}^{2d}} \varphi(x)\psi(y)q(x, y)dxdy.$$

The mapping $\mathbf{1}_{[0,t]}\varphi \mapsto W_t(\varphi)$ can be extended to a linear isometry between $\mathcal{H}_\infty := L^2([0,\infty);\mathcal{H})$ and the L^2 space spanned by W. Then $\{W(h), h \in \mathcal{H}_\infty\}$ is an isonormal Gaussian process associated with the Hilbert space $\mathcal{H}_\infty$.

We will denote by D the derivative operator in the sense of Malliavin calculus. That is, if F is a smooth and cylindrical random variable of the form

$$F = f(W(h_1), \dots, W(h_n)),$$

$h_i \in \mathcal{H}_\infty$, $f \in C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth), then DF is the $\mathcal{H}_\infty$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(h_1), \dots, W(h_n))h_j.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega;\mathcal{H}_\infty)$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|^2_{\mathcal{H}_\infty})}.$$

We denote by δ the adjoint of the derivative operator, given by duality formula

$$E(\delta(u)F) = E\left(\langle DF, u\rangle_{\mathcal{H}_\infty}\right), \tag{2.2}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega;\mathcal{H}_\infty)$ in the domain of δ. The operator δ is also called the Skorohod integral. The higher Malliavin derivatives can be defined in similar way and we can define $\mathbb{D}^{k,p}$ for any integer $k \geq 1$ and real number $p \geq 1$. Set $\mathbb{D}^\infty = \bigcap_{k\geq 1, p\geq 2} \mathbb{D}^{k,p}$.
To obtain the existence and smoothness of the density, we make use of the following criteria.

Theorem 2.1 *Let $F : \Omega \to \mathbb{R}$ be a random variable. If $F \in \mathbb{D}^{1,2}$ and $\|DF\|_{\mathcal{H}_\infty} > 0$ almost surely, then the probability law of F is absolutely continuous with respect to the Lebesgue measure. Moreover, if $F \in \mathbb{D}^\infty$ and $E\left[\|DF\|^{-p}_{\mathcal{H}_\infty}\right] < \infty$ for all $p \geq 1$, then the density of F is infinitely differentiable.*

For the proof of this result and a detailed presentation of the Malliavin calculus we refer to [11] and the references therein.

2.2 Generalized Itô formula

In this section we introduce some preliminaries on continuous semimartingales depending on a parameter and the corresponding generalized Itô formula. We refer to [9] for more details.

Fix a time interval $[0,T]$, a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$ satisfying the usual conditions (increasing, right-continuous, and $\mathcal{F}_0$ contains all the null sets). Let $\{F(t,x), 0 \leq t \leq T, x \in O\}$ be a family of real valued processes with parameter $x \in O$, where O is a domain in $\mathbb{R}^d$. We can regard it as random field with double parameters x and t. If $F(t,x)$ is m-times continuously differentiable with respect to x a.s. for any t, it can be regarded as stochastic process with values in C^m or a C^m-process.

Here we denote by $C^m = C^m(O, \mathbb{R})$ the set of all real valued functions on O which are m times continuously differentiable. If furthermore, for each multi-index $\alpha \in \{1, \dots, d\}^k$ with $|\alpha| = k \le m$, $\{D_x^\alpha F(t,x), x \in O\}$ is a family of continuous semimartingales, then $F(t,x)$ is called a C^m-semimartingale. Here we have used the notation $D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_k}}$.

We denote by $C^{1,1}$ the set of all functions on $a : [0,T] \times O \times O \to \mathbb{R}$ such that the partial derivatives $\frac{\partial a}{\partial x_i}(t,x,y)$, $\frac{\partial a}{\partial y_j}(t,x,y)$ and $\frac{\partial^2 a}{\partial x_i \partial y_j}(t,x,y)$ exist for any $1 \le i, j \le d$ and are continuous in (x,y), and for any compact set $K \subset O$ and $1 \le i,j \le d$

$$\int_0^T \sup_{x,y\in K} \left(|a(t,x,y)| + |\frac{\partial a}{\partial x_i}(t,x,y)| + |\frac{\partial a}{\partial y_j}(t,x,y)| + |\frac{\partial^2 a}{\partial x_i \partial y_j}(t,x,y)| \right) dt < \infty.$$

We also denote C^1 the set of all functions on $b : [0,T] \times O \to \mathbb{R}$ which are continuously differentiable in x, and for any compact set $K \subset O$ and any $1 \le i \le d$

$$\int_0^T \sup_{x\in K} \left(|b(t,x)| + |\frac{\partial b}{\partial x_i}(t,x)| \right) dt < \infty.$$

Let $\{F(t,x), x \in O\}$ be a family of continuous semimartingales decomposed as $F(t,x) = M(t,x) + B(t,x)$, where $M(t,x)$ is a continuous local martingale and $B(t,x)$ is a continuous process of bounded variation. Let $A(t,x,y)$ be the joint quadratic variation of $M(t,x)$ and $M(t,y)$ and assume that $A(t,x,y) = \int_0^t a(s,x,y)ds$ and $B(t,x) = \int_0^t b(s,x)ds$, where $a(t,x,y)$ and $b(t,x)$ are predictable processes. Then $(a(t,x,y), b(t,x))$ is called the local characteristic of the family of semimartingales $\{F(t,x), x \in O\}$. Following Section 3.2 of [9], we say that the local characteristic (a,b) belongs to the class $B^{1,0}$ if $a(t,x,y)$ and $b(t,x)$ are predictable processes with values in $C^{1,1}$ and C^1, respectively.

Now let $\{F(t,x), x \in O\}$ be a continuous semimartingale with local characteristic (a,b). Let $\{f_t, 0 \le t \le T\}$ be a predictable process with values in O satisfying

$$\int_0^T a(s, f_s, f_s)ds < \infty, \quad \int_0^T |b(s,f_s)|ds < \infty \quad a.s. \tag{2.3}$$

Then, the Itô stochastic integral of f_t based on the kernel $F(dt,x)$ is defined as the following limit in probability if it exists

$$\int_0^t F(ds, f_s) = \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} \{F(t_{k+1} \wedge t, f_{t_k \wedge t}) - F(t_k \wedge t, f_{t_k \wedge t})\},$$

where $\Delta = \{0 = t_0 < \cdots < t_n = T\}$, and $|\Delta| = \max_{1 \le i \le n}(t_i - t_{i-1})$.

The joint quadratic variation of the Itô integrals $\int_0^t F(ds, f_s)$ and $\int_0^t F(ds, g_s)$ satisfies

$$\langle \int_0^\cdot F(ds, f_s), \int_0^\cdot F(ds, g_s) \rangle_t = \int_0^t a(s, f_s, g_s)ds.$$

The following is the generalized Itô formula (see Theorem 3.3.1 in [9]).

Theorem 2.2 (Generalized Itô formula) *Let $\{F(t,x), x \in O\}$ be a continuous C^2-process and a continuous C^1-semimartingale with local characteristic belonging to the class $B^{1,0}$ and* *let* $\{X_t, 0 \le t \le T\}$ *be a continuous semimartingale with values in* O*. Then* $\{F(t, X_t), 0 \le t \le T\}$ *is a continuous semimartingale and satisfies*

$$\begin{aligned} F(t,X_t) =& F(0,X_0) + \int_0^t F(dr, X_r) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(r, X_r) dX_r^i \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(r, X_r) d\langle X^i, X^j\rangle_r + \sum_{i=1}^d \langle \int_0^\cdot \frac{\partial F}{\partial x_i}(dr, X_r), X^i\rangle_t\,, \end{aligned} \tag{2.4}$$

for any $t \in [0, T]$.

3 Feynman-Kac formula

In this section we establish a general Feynman-Kac formula for the d-dimensional heat equation driven by a continuous semimartingale. Suppose that $F = \{F(t, x), 0 \le t \le T, x \in \mathbb{R}^d\}$ is a continuous semimartingale with local characteristic (a, b). We are going to impose the following condition.

(H1) Assume that $a(t, x, y)$ and $b(t, x)$ are continuous and satisfy

$$|a(t,x,y)| \le C(1 + |x|^\beta + |y|^\beta), \tag{3.1}$$

$$|b(t,x)| \le C(1 + |x|^\beta), \tag{3.2}$$

for $t \in [0, T]$, with $0 \le \beta < 2$.

Consider the stochastic heat equation

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) = \dfrac{1}{2}\triangle V(t,x) + V\dfrac{\partial F}{\partial t}(t,x) \\ V(x,0) = h(x). \end{cases} \tag{3.3}$$

An adapted random field $\{V(t, x), 0 \le t \le T, x \in \mathbb{R}^d\}$ is called a mild solution to the above equation if $V(t, x)$ satisfies the following integral equation

$$V(t,x) = \int_{\mathbb{R}^d} p_t(x-z)h(z)dz + \int_{\mathbb{R}^d}\left(\int_0^t p_{t-r}(x-z)V(r,z)F(dr,z)\right)dz, \tag{3.4}$$

where $p_t(x) = (2\pi t)^{-\frac{d}{2}} \exp(-|x|^2/2t)$.

Theorem 3.1 (Feynman-Kac Formula) *Let* $h(x)$ *be continuous and with polynomial growth. Then the process*

$$V(t,x) = E^B\left(h(x+B_t)\exp\left(\int_0^t F(dr, x+B_t-B_r) - \frac{1}{2}\int_0^t \bar{a}(r, x+B_t-B_r)dr\right)\right), \tag{3.5}$$

where B *is a* d*-dimensional standard Brownian motion independent of* F*,* E^B *denotes the mathematical expectation with respect to* B*, and* $\bar{a}(t, x) = a(t, x, x)$*, is a mild solution to Equation (3.3).*

Proof.

We divide the proof into three steps.

Step 1. First we show that the process (3.5) is well defined. In the sequel we denote by E the mathematical expectation in the probability space where F is defined, and E^B denotes the expectation with respect to the independent Brownian motion B. Set

$$Y_t = \int_0^t F(dr, x + B_t - B_r) - \frac{1}{2} \int_0^t \bar{a}(r, x + B_t - B_r) dr.$$

Notice that $\int_0^t F(dr, x + B_t - B_r)$ is a well defined Itô stochastic integral, because the process $\{B_t - B_r, 0 \le r \le t\}$ is independent of the semimartingale F, and conditions (2.3) are satisfied. Then $V(t,x) = E^B\left(h(x + B_t)\exp(Y_t)\right)$. We claim that this expectation exists and $V(t,x)$ satisfies the following condition for any $x \in \mathbb{R}^d$ and $p \ge 1$,

$$\sup_{0 \le t \le T} E|V(t,x)|^p \le K_1 \exp\left(K_2|x|^\beta\right), \tag{3.6}$$

where the constants K_1 and K_2 depend on p and T. In particular, this implies that the stochastic integral in (3.4) is well defined. We can write

$$E|V(t,x)|^p \le \left(E^B|h(x + B_t|^{2p} \; EE^B \exp(2pY_t)\right)^{\frac{1}{2}}.$$

Let us denote by $M(t,x)$ the martingale part of $F(t,x)$. Then we make the decomposition

$$Y_t = Y_t^{(1)} + Y_t^{(2)},$$

where

$$Y_t^{(1)} = \int_0^t M(dr, x + B_t - B_r) - p \int_0^t \bar{a}(r, x + B_t - B_r) dr,$$

and

$$Y_t^{(2)} = \int_0^t \left[b(r, x + B_t - B_r) + \left(p - \frac{1}{2}\right) \bar{a}(r, x + B_t - B_r) \right] dr.$$

Using conditions (3.1) and (3.2) and taking into account that $\beta < 2$, we obtain for all $t \in [0,T]$

$$\begin{aligned} E^B \exp(2pY_t^{(2)}) &\le E^B \left(\exp\left(C \int_0^t (1 + |x + B_t - B_r|^\beta) dr \right) \right) \\ &\le K_1 \exp(K_2|x|^\beta). \end{aligned}$$

On the other hand, taking into account that $\exp(2pY_t^{(1)})$ is a martingale, we can write

$$EE^B \exp(2pY_t^{(1)}) = E^B E \exp(2pY_t^{(1)}) \le 1,$$

which completes the proof of (3.6).

Step 2. We now show that the process (3.5) is a solution to Equation (3.3) under some additional regularity assumptions on the semimartingale $F(t,x)$. Suppose that $F(t,x)$ is a C^3-semimartingale, such that the local characteristic (a,b) satisfies (3.1) and (3.2). We also assume that the functions $D_x^\alpha D_y^\alpha a(t,x,y)$ satisfy the estimate (3.1) for all multi-index

α with $1 \le |\alpha| \le 2$, and the functions $D_x^\alpha b(t,x)$ and $D_x^\alpha \bar{a}(t,x)$ satisfy the estimate (3.2) for all multi-index α with $1 \le |\alpha| \le 2$. Clearly this implies that the local characteristic belongs to the class $B^{1,0}$. Suppose also that the functions $D_x^\alpha h(t,x)$ have polynomial growth for all multi-index α with $1 \le |\alpha| \le 2$.

For fixed x, let

$$\Phi(t,y) = \int_0^t F(dr, x+y-B_r) - \frac{1}{2}\int_0^t \bar{a}(r, x+y-B_r)dr.$$

According to Theorem 3.3.3 in [9], $\Phi(t,y)$ is a C^2-semimartingale with local characteristic belonging to the class $B^{1,0}$. We can apply the generalized Itô formula (2.4) to the process $Y_t = \Phi(t, B_t)$, and we obtain

$$\begin{aligned} dY_t = F(dt,x) - \frac{1}{2}\bar{a}(t,x)dt + \sum_{i=1}^d \left[\int_0^t \frac{\partial F}{\partial x_i}(dr, x+B_t-B_r)\right] dB_t^i \\ - \frac{1}{2}\sum_{i=1}^d \left[\int_0^t \frac{\partial \bar{a}}{\partial x_i}(r, x+B_t-B_r)dr\right] dB_t^i + \frac{1}{2}\sum_{i=1}^d \left[\int_0^t \frac{\partial^2 F}{\partial x_i^2}(dr, x+B_t-B_r)\right] dt \\ - \frac{1}{4}\sum_{i=1}^d \left[\int_0^t \frac{\partial^2 \bar{a}}{\partial x_i^2}(r, x+B_t-B_r)dr\right] dt. \end{aligned}$$

The terms $\langle \int_0^\cdot \frac{\partial F}{\partial x_i}(dr,x), B_\cdot^i\rangle_t = \langle \int_0^\cdot \frac{\partial M}{\partial x_i}(dr,x), B_\cdot^i\rangle_t$ vanish since M and B are independent. The quadratic variation of the semimartingale Y is given by

$$d\langle Y\rangle_t = \bar{a}(t,x)dt + \sum_{i=1}^d \big[\int_0^t \frac{\partial F}{\partial x_i}(dr, x+B_t-B_r) - \frac{1}{2}\int_0^t \frac{\partial \bar{a}}{\partial x_i}(r, x+B_t-B_r)dr\big]^2 dt\,. \quad (3.7)$$

Consider the process $Z(t,x) = h(x+B_t)e^{Y_t}$. Applying Itô's formula to $h(x+B_t)e^{Y_t}$ yields

$$\begin{aligned} Z(t,x) =& h(x) + \int_0^t Z(s,x)dY_s + \sum_{i=1}^d \int_0^t \frac{\partial h}{\partial x_i}(x+B_s)e^{Y_s}dB_s^i \\ &+ \frac{1}{2}\sum_{i=1}^d \int_0^t \frac{\partial^2 h}{\partial x_i^2}(x+B_s)e^{Y_s}ds + \frac{1}{2}\int_0^t \hat{V}(s,x)d\langle Y\rangle_s \\ &+ \sum_{i=1}^d \int_0^t \frac{\partial h}{\partial x_i}(x+B_s)e^{Y_s}d\langle B^i, Y\rangle_s\,. \end{aligned} \quad (3.8)$$

We claim that the stochastic integrals with respect to B^i in the above expression have zero expectation with respect to B. This is a consequence of the following properties

$$\int_0^T E^B Z(t,x)^2 E^B \left|\sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(dr, x+B_t-B_r)\right|^2 dt < \infty, \quad (3.9)$$

$$\int_0^T E^B Z(t,x)^2 E^B \left|\sum_{i=1}^d \int_0^t \frac{\partial \bar{a}}{\partial x_i}(r, x+B_t-B_r)dr\right|^2 dt < \infty, \quad (3.10)$$

and

$$\int_0^T E^B \left| \sum_{i=1}^d \int_0^t \frac{\partial h}{\partial x_i}(x+B_s) \right|^2 e^{2Y_s} ds < \infty. \tag{3.11}$$

These properties follow from our additional assumptions. For instance, to show (3.9) for the martingale component of F, we take the expectation in the probability space where F is defined and we use the fact that for any $p \geq 2$

$$\begin{aligned} & E \left| \sum_{i=1}^d \int_0^t \frac{\partial M}{\partial x_i}(dr, x+B_t-B_r) \right|^p \\ & \quad \leq c_p E \left| \sum_{i,j=1}^d \int_0^t \frac{\partial^2 a}{\partial x_i \partial y_j}(r, x+B_t-B_r, x+B_t-B_r) dr \right|^{\frac{p}{2}} \\ & \quad \leq CE \int_0^t (1+|B_t-B_r|^\beta) ds < \infty. \end{aligned}$$

Then, taking the expectation with respect to B in (3.8) yields

$$\begin{aligned} & V(t,x) = h(x) + \int_0^t V(s,x) F(ds,x) \\ & + \frac{1}{2} \sum_{i=1}^d E^B \Bigg(\int_0^t V(s,x) \Big\{ \int_0^s \frac{\partial^2 F}{\partial x_i^2}(dr, x+B_s-B_r) - \frac{1}{2} \int_0^s \frac{\partial^2 \bar{a}}{\partial x_i^2}(r, x+B_s-B_r) dr \\ & + \Big[\int_0^s \frac{\partial F}{\partial x_i}(dr, x+B_s-B_r) - \frac{1}{2} \int_0^s \frac{\partial \bar{a}}{\partial x_i}(r, x+B_s-B_r) dr \Big]^2 \Big\} ds \\ & + \int_0^t \frac{\partial^2 h}{\partial x_i^2}(x+B_s) e^{Y_s} ds \\ & + 2 \int_0^t \frac{\partial h}{\partial x_i}(x+B_s) e^{Y_s} [\int_0^s \frac{\partial F}{\partial x_i}(dr, x+B_s-B_r) - \frac{1}{2} \int_0^s \frac{\partial \bar{a}}{\partial x_i}(r, x+B_s-B_r) dr] ds \Bigg) \end{aligned}$$

Using that

$$\frac{\partial Y_s}{\partial x_i} = \int_0^s \frac{\partial F}{\partial x_i}(dr, x+B_s-B_r) - \frac{1}{2} \int_0^s \frac{\partial \bar{a}}{\partial x_i}(r, x+B_s-B_r) dr\,,$$

we obtain easily

$$V(t,x) = h(x) + \int_0^t V(s,x) F(ds,x) + \frac{1}{2} \sum_{i=1}^d \int_0^t \frac{\partial^2 V}{\partial x_i^2}(s,x) ds\,.$$

This shows that under some the additional regularity conditions on F and h the process u defined by (3.5) is a strong solution to Equation (3.3), and also a mild solution.

Step 3. Consider now the case of a general semimartingale F. For any $\epsilon > 0$ we define

$$\begin{aligned} M^\epsilon(t,x) &= \int_{\mathbb{R}^d} M(t,y) p_\epsilon(x-y) dy, \\ B^\epsilon(t,x) &= \int_{\mathbb{R}^d} B(t,y) p_\epsilon(x-y) dy, \end{aligned}$$

and $h^\epsilon(x) = \int_{\mathbb{R}^d} h(y)p_\epsilon(x-y)dy$. It is easy to check that h^ϵ is infinitely differentiable and it has polynomial growth together with all its partial derivatives. Also $F^\epsilon(t,x) = M^\epsilon(t,x) + B^\epsilon(t,x)$ is a C^3-semimartingale with local characteristic given by

$$b^\epsilon(t,x) = \int_{\mathbb{R}^d} b(t,y)p_\epsilon(x-y)dy,$$

and

$$a^\epsilon(t,x,y) = \int_0^t \int_{\mathbb{R}^{2d}} a(s, x-z_1, y-z_2)p_\epsilon(z_1)p_\epsilon(z_2)dz_1dz_2ds.$$

It easy to check that a^ϵ and b^ϵ satisfy the estimates (3.1) and (3.2) respectively, the partial derivatives $D_x^\alpha D_y^\alpha a^\epsilon(t,x,y)$ satisfy the estimate (3.1) for all multi-index α with $1 \le |\alpha| \le 2$, and the functions $D_x^\alpha b^\epsilon(t,x)$ satisfy the estimate (3.2) for all multi-index α with $1 \le |\alpha| \le 2$. From Step 2 it follows that

$$V^\epsilon(t,x) = E^B\big\{h^\epsilon(x+B_t)\exp\big(\int_0^t F^\epsilon(dr, x+B_t-B_r) - \frac{1}{2}\int_0^t \bar{a}^\epsilon(dr, x+B_t-B_r)\big)\big\}$$

is the strong solution to

$$\begin{cases} \dfrac{\partial V^\epsilon}{\partial t}(t,x) = \dfrac{1}{2}\triangle V^\epsilon(t,x) + V^\epsilon \dfrac{\partial F^\epsilon}{\partial t}(t,x) \\ V^\epsilon(x,0) = h^\epsilon(x) \end{cases} \tag{3.12}$$

As a consequence, it is also a mild solution to (3.12), namely,

$$V^\epsilon(t,x) = \int_{\mathbb{R}^d} p_t(x-z)h^\epsilon(z)dz + \int_{\mathbb{R}^d}\left(\int_0^t p_{t-r}(x-z)V^\epsilon(r,z)F^\epsilon(dr,z)\right)dz.$$

Finally, we are going to take the limit as ϵ tends to zero in each term of the above expression in order to deduce the Feynman-Kac formula of V. The estimate (3.1) implies

$$\sup_{\epsilon>0} E\exp\left(p\int_0^t |\bar{a}^\epsilon(r, x+B_t-B_r)|dr\right) < \infty, .$$

for all $p \ge 1$ and, as a consequence, $V^\epsilon(t,x)$ converges to $V(t,x)$ in L^p for all $p \ge 1$. Clearly,

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^d} p_t(x-z)h^\epsilon(z)dz = \int_{\mathbb{R}^d} p_t(x-z)h(z)dz,$$

and

$$\lim_{\epsilon\downarrow 0}\int_0^t\int_{\mathbb{R}^d} p_{t-r}(x-z)V^\epsilon(r,z)b^\epsilon(r,z)dzdr = \int_0^t\int_{\mathbb{R}^d} p_{t-r}(x-z)V(r,z)b(r,z)dzdr,$$

also in L^p for all $p \ge 1$. The following limits in L^p are also easy to check:

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^d}\left(\int_0^t p_{t-r}(x-z)V^\epsilon(r,z)[M^\epsilon(dr,z) - M(dr,z)]\right)dz = 0$$

and

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^d}\left(\int_0^t p_{t-r}(x-z)[V^\epsilon(r,z) - V(r,z)]M(dr,z)\right)dz = 0.$$

This completes the proof of the theorem. ∎

4 Stochastic heat equation: Hölder continuity of the solution

Consider the following nonlinear stochastic partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\triangle u + b(u) + \sigma(u)\dot{W}(t,x)\,, \quad t \ge 0\,, \quad x \in \mathbb{R}^d \\ u(0,x) = u_0(x)\,. \end{cases} \tag{4.1}$$

where W is the Gaussian family introduced in Section 2.1 with covariance function given by (2.1). Let us recall that an adapted random field $\{u(t,x)\,, t \ge 0\,, x \in \mathbb{R}^d\}$ is called a mild solution to Equation (4.1) if u satisfies the following integral equation.

$$\begin{aligned} u(t,x) = & \int_{\mathbb{R}^d} p_t(x-z)u_0(z)dz + \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x-z)b(u(r,z))dzdr \\ & + \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x-z)\sigma(u(r,z))W(dr,dz)\,, \end{aligned} \tag{4.2}$$

where the stochastic integral is defined as the integral of an $\mathcal{H}$-valued predictable process.

We are going to impose the following condition on the covariance function.

(**H1**) For each $t \ge 0$,

$$\sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^{2d}} p_{t-s}(x-z_1)p_{t-s}(x-z_2)|q(z_1,z_2)|dz_1dz_2ds < \infty\,.$$

Theorem 4.1 *Suppose that b and σ are globally Lipschitz continuous functions and suppose that the covariance function q satisfies (**H1**). Let $u_0(x)$ be a bounded function in $\mathbb{R}^d$. Then there exists a unique adapted process $u = \{u(t,x), t \in [0,T], x \in \mathbb{R}^d\}$ satisfying (4.2). Moreover,*

$$\sup_{t\in[0,T],x\in\mathbb{R}^d} E|u(t,x)|^p < \infty, \quad \forall\ p \ge 2\,. \tag{4.3}$$

Proof. Fix $p \ge 2$. Let $\mathbb{B}_p$ be the Banach space of all adapted random fields u such that $\|u\|_p < \infty$, where $\|u\|_p^p = \sup_{t\in[0,T],x\in\mathbb{R}^d} E|u(t,x)|^p$. On $\mathbb{B}_p$, define the following mapping

$$\begin{aligned} \Psi(u)(t,x) := & \int_{\mathbb{R}^d} p_t(x-z)u_0(z)dz + \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x-z)b(u(r,z))dzdr \\ & + \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x-z)\sigma(u(r,z))W(dr,dz)\,. \end{aligned}$$

It is straightforward to obtain

$$\begin{aligned} E\left|\Psi(u)-\Psi(v)\right|^p(t,x) \le C\Bigg[& E\left(\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-z)|u(s,z)-v(s,z)|dzds\right)^p \\ + E\Bigg\{ & \int_0^t \int_{\mathbb{R}^{2d}} p_{t-s}(x-z_1)|u(s,z_1)-v(s,z_1)|p_{t-s}(x-z_2)|u(s,z_2)-v(s,z_2)| \end{aligned}$$

$$\times |q(z_1,z_2)|dz_1dz_2ds\Bigg\}^{p/2}\Bigg].$$

Taking the supremum with respect to t and x, we have

$$\|\Psi(u)-\Psi(v)\|_p^p \le C\int_0^T \|u-v\|_p^p ds \le CT\|u-v\|_p^p .$$

Consequently, Ψ is a contraction mapping on $\mathbb{B}_p$ when T sufficiently small. This proves the existence and uniqueness of the solution for some small T. From the above argument it is clear that the T such that Ψ is a contraction is independent of the initial value of the solution. This can be used to to show the existence and uniqueness of the solution for any T. The inequality (4.3) follows in a similar way. ■

Now we apply the factorization method to obtain the Hölder continuity of u. Fix an arbitrary $\alpha \in (0,1)$ and denote

$$Y_\alpha(r,z) = \int_0^r \int_{\mathbb{R}^d} p_{r-s}(z-y)\sigma(u(s,y))(r-s)^{-\alpha} W(ds,dy). \tag{4.4}$$

The semigroup property of the heat kernel and the stochastic Fubini's theorem yield

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)\\ =&\frac{\sin(\pi\alpha)}{\pi}\int_0^t \int_{\mathbb{R}^d} p_{t-r}(x-z)(t-r)^{\alpha-1}Y_\alpha(r,z)dzdr.\end{aligned} \tag{4.5}$$

Consider the following stronger condition on the covariance function.

(**H1a**) There exists $\gamma > -1$ such that for each $t \ge 0$,

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} p_t(x-z_1)p_t(x-z_2)|q(z_1,z_2)|dz_1dz_2 < Ct^\gamma.$$

Lemma 4.2 *Let the assumptions of Theorem 4.1 be satisfied. Assume the covariance function q satisfies (**H1a**). Then for any fixed $T>0, p\ge 1, \alpha \in (0, \frac{1+\gamma}{2})$, we have*

$$\sup_{r\in[0,T],z\in\mathbb{R}^d} E(|Y_\alpha(r,z)|^p) < \infty.$$

Proof. Since $\sup_{r\in[0,T],z\in\mathbb{R}^d} E(|u(r,z)|^p) < \infty$ from Theorem 4.1, and σ is Lipschitz continuous, we have

$$\sup_{r\in[0,T],z\in\mathbb{R}^d} E(|\sigma(u(r,z))|^p) < \infty.$$

Then we can write

$$\begin{aligned}E|Y_\alpha(r,z)|^p \le& C\Bigg(E\int_0^r \int_{\mathbb{R}^{2d}} p_{r-s}(z-y_1)p_{r-s}(z-y_2)\\ &\times \sigma(u(s,y_1))\sigma(u(s,y_2))(r-s)^{-2\alpha}q(y_1,y_2)dy_1dy_2ds\Bigg)^{\frac{p}{2}}\end{aligned}$$

$$\le C \sup_{r\in[0,T],z\in\mathbb{R}^d} E(|\sigma(u(r,z))|^p)$$
$$\times \left(\int_0^r \int_{\mathbb{R}^{2d}} p_{r-s}(z-y_1)p_{r-s}(z-y_2)(r-s)^{-2\alpha}|q(y_1,y_2)|dy_1dy_2ds\right)^{\frac{p}{2}}$$
$$\le C\left(\int_0^r (r-s)^{\gamma-2\alpha}ds\right)^{\frac{p}{2}} < \infty.$$

■

Equation (4.5) and Lemma 4.2 constitute the main ingredients to prove the following theorem concerning the Hölder continuity of the solution u.

Theorem 4.3 *Suppose that b and σ are globally Lipschitz continuous. Assume (**H1a**) and suppose that $u_0(x)$ is bounded and ρ-Hölder continuous. Then the solution u to the equation (4.1) is a.s. β_1-Hölder continuous in the time variable t and β_2-Hölder continuous in the space variable x for any $\beta_1 \in (0, \frac{1}{2}[\rho\wedge(1+\gamma)])$ and $\beta_2 \in (0, \rho\wedge(1+\gamma))$, respectively.*

Proof. It suffices to follow the idea of the proof of Theorem 2.1 in [15], and we provide a sketch of the proof for the reader's convenience. The proof contains two parts for time and space variables, respectively.

Part I

Fix $T, h > 0$ and $p \in [2,\infty)$. First we show that

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d} E(|u(t+h,x)-u(t,x)|^p) \le C(p,T)h^{\eta p}, \tag{4.6}$$

for any $\eta \in (0, \frac{1}{2}[\rho\wedge(1+\gamma)])$. Let Y_α be as defined in (4.4) with $\alpha \in (0, \frac{1+\gamma}{2})$ and denote $P_t f(x) = \int_{\mathbb{R}^d} p_t(x-z)f(z)dz$. We have

$$E(u(t+h,x)-u(t,x)|^p) \le C(p,\alpha)\sum_{i=1}^4 I_i(t,h,x),$$

where

$$I_1(t,h,x) = |P_{t+h}u_0(x) - P_t u_0(x)|^p,$$
$$I_2(t,h,x) = E\left(\left|\int_0^t\int_{\mathbb{R}^d}[p_{t+h-r}(x-z)(t+h-r)^{\alpha-1} - p_{t-r}(x-z)(t-r)^{\alpha-1}]Y_\alpha(r,z)\right|^p dzdr\right),$$
$$I_3(t,h,x) = E\left(\left|\int_t^{t+h}\int_{\mathbb{R}^d} p_{t+h-r}(x-z)(t+h-r)^{\alpha-1}Y_\alpha(r,z)\right|^p dzdr\right),$$
$$I_4(t,h,x) = E\left(\left|\int_0^{t+h}\int_{\mathbb{R}^d} p_{t+h-r}(x-z)b(u(r,z))dzdr - \int_0^t\int_{\mathbb{R}^d} p_{t-r}(x-z)b(u(r,z))\right|^p dzdr\right).$$

For the term $I_1(t,h,x)$, using the fact that u_0 is ρ-Hölder continuous we have $I_1(t,h,x) \le Ch^{\frac{\rho p}{2}}$. For any $\alpha \in (0, \frac{1+\gamma}{2})$ set $\psi^\alpha(t,x) = p_t(x)t^{\alpha-1}$. By Hölder's inequality and Lemma 4.2, we have

$$I_2(t,h,x) \le C\left(\int_0^t\int_{\mathbb{R}^d} |\psi^\alpha(t+h-r,x-z) - \psi^\alpha(t-r,x-z)|dzdr\right)^p.$$

Set

$$I_{2,1}(t,h,x) = \int_0^t \int_{\mathbb{R}^d} \exp\left(-\frac{|x-z|^2}{2(t-r)}\right) |(t+h-r)^{\alpha-1-\frac{d}{2}} - (t-r)^{\alpha-1-\frac{d}{2}}| dz dr,$$

and

$$I_{2,2}(t,h,x) = \int_0^t \int_{\mathbb{R}^d} (t+h-r)^{\alpha-1-\frac{d}{2}} \left| \exp\left(-\frac{|x-z|^2}{2(t+h-r)}\right) - \exp\left(-\frac{|x-z|^2}{2(t-r)}\right) \right| dz dr.$$

Then $I_2(t,h,x) \le C(I_{2,1}(t,h,x)^p + I_{2,2}(t,h,x)^p)$, and using the same arguments as in the proof of Theorem 2.1 in [15] to estimate the terms $I_{2,1}(t,h,x)^p$ and $I_{2,3}(t,h,x)^p$, we obtain for $\eta \in (0,\alpha)$ that $I_2(t,h,x) \le Ch^{\eta p}$. By Hölder's inequality and Lemma 4.2, we have

$$I_3(t,h,x) \le C\left(\int_t^{t+h} \int_{\mathbb{R}^d} p_{t+h-r}(x-z)(t+h-r)^{\alpha-1} dz dr\right)^p \le Ch^{\alpha p}.$$

A change of variable yields

$$I_4(t,h,x) \le C(I_{4,1}(t,h,x) + I_{4,2}(t,h,x)),$$

with

$$I_{4,1}(t,h,x) = E\left(\left|\int_0^h \int_{\mathbb{R}^d} P_{t+h-r}(x-z) b(u(r,z)) dz dr\right|^p\right),$$

$$I_{4,2}(t,h,x) = E\left(\left|\int_0^h \int_{\mathbb{R}^d} P_{t-r}(x-z)[b(u(r+h,z)) - b(u(r,z))] dz dr\right|^p\right),$$

Since b is Lipschitz, using Hölder's inequality and Equation (4.3), we have

$$I_{4,1}(t,h,x) \le Ch^p.$$

The Lipschitz property of b also implies

$$I_{4,2}(t,h,x) \le \int_0^t \sup_{z\in\mathbb{R}^d} E(|u(r+h,z) - u(r,z)|^p) dr.$$

Putting together all estimations for $I_i, i = 1,\dots,4$, we obtain

$$\sup_{x\in\mathbb{R}^d} E(|u(t+h,x) - u(t,x)|^p) \le C\left(h^{p\min(\frac{\rho}{2},\eta,\alpha)} + \int_0^t \sup_{x\in\mathbb{R}^d} E(|u(r+h,x) - u(r,x)|^p) dr\right).$$

Since $0 < \eta < \alpha < \frac{1+\gamma}{2}$, the estimate (4.6) follows by Gronwall's Lemma.

Part II

Now consider the increments in the space variable. We want to show that for any $T > 0, p \in [2,\infty), x, a \in \mathbb{R}^d$ and $\eta \in (0, \rho \wedge (1+\gamma))$,

$$\sup_{0\le t\le T} \sup_{x\in\mathbb{R}^d} E(|u(t,x+a) - u(t,x)|^p) \le C|a|^{\eta p}. \tag{4.7}$$

Fix $\alpha \in (0, \frac{1+\gamma}{2})$. Ee have

$$E(|u(t,x+a)-u(t,x)|^p) \le C \sum_{i=1}^{3} J_i(t,x,a),$$

with

$$\begin{aligned} J_1(t,x,a) &= |P_t u_0(x+a) - P_t u_0(x)|^p \\ J_2(t,x,a) &= E\left(\left| \int_0^t \int_{\mathbb{R}^d} [\psi^\alpha(t-r,x+a-z) - \psi^\alpha(t-r,x-z)] Y_\alpha(r,z) dz dr \right|^p \right), \\ J_3(t,x,a) &= E\left(\left| \int_0^t \int_{\mathbb{R}^d} [p_{t-r}(x+a-z) - p_{t-r}(x-z)] b(u(r,y)) dz dr \right|^p \right). \end{aligned}$$

It is easy to show that $J_1(t,x,a) \le C|a|^{\rho p}$. For the term $J_2(t,x,a)$, first we have using the mean value theorem,

$$\int_{\mathbb{R}^d} |\psi^\alpha(t-r,x+a-z) - \psi^\alpha(t-r,x-z)| dz \le C(t-r)^{\alpha-1-\frac{\eta}{2}} |a|^\eta,$$

where $\eta \in (0,1)$. Again by Hölder's inequality and Lemma 4.2, for $\alpha \in (0, \frac{1+\gamma}{2}), \eta \in (0, 2\alpha \wedge 1)$, we deduce

$$J_2(t,x,a) \le C \left(\int_0^t \int_{\mathbb{R}^d} |\psi^\alpha(t-r,x+a-z) - \psi^\alpha(t-r,x-z)| dz dr \right)^p \le C|a|^{\eta p}.$$

Finally, by a change of variable, the Lipschitz property of b, and Hölder's inequality,

$$\begin{aligned} J_3(t,x,a) &\le E\left(\left| \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x-y)[b(u(r,z+a) - b(u(r,z)] dz dr \right|^p \right) \\ &\le C \int_0^t \sup_{z \in \mathbb{R}^d} E(|u(r,a+z) - u(r,z)|^p) dr. \end{aligned}$$

Then (4.7) follows from the Gronwall's lemma and the estimates of $J_i, i = 1,2,3$. ■

Here are two examples.

Example 4.4 *A similar Hölder continuity result was obtained in [15] in the case of an homogeneous covariance function $q(x,y) = q(x-y) \ge 0$, where q is a nonnegative continuous function on $\mathbb{R}^d \backslash \{0\}$ such that it is the Fourier transform of a non-negative definite tempered measure μ on $\mathbb{R}^d$, and for some $\eta \in (0,1)$ we have*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty.$$

*This condition implies (**H1a**). In fact, we can write*

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} p_t(x-z_1) p_t(x-z_2) q(z_1-z_2) dz_1 dz_2 = \int_{\mathbb{R}^{2d}} p_t(x-z-z_2) p_t(x-z_2) q(z) dz_2 dz \\ &= \int_{\mathbb{R}^d} p_{2t}(z) q(z) dz = \int_{\mathbb{R}^d} e^{-2t\xi^2} \mu(d\xi) \\ &\le \int_{\mathbb{R}^d} e^{-2t(\xi^2+1)} \mu(d\xi) \le Ct^{-\eta} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}. \end{aligned}$$

Theorem 4.3 can be applied to noises which do not have an homogeneous spatial covariance like the following example.

Example 4.5 *Consider the case where* $d = 1$ *and the covariance structure in space is that of a bifractional Brownian motion with parameters* $H \in (0,1), K \in (0,1]$*, that is,*

$$q(x,y) = 2^{-K} \frac{\partial^2}{\partial x \partial y}((|x|^{2H} + |y|^{2H})^K - |x-y|^{2HK}),$$

where $2KH > 1$*. Then,* $B^{H,K}(t,x) = W(\mathbf{1}_{[0,t]\times[0,x]})$*, is a bifractional Brownian motion in* $x \geq 0$ *for each fixed* t*, and formally,* $W(t,x) = \frac{\partial}{\partial x} B^{H,K}(t,x)$*. Then*

$$|q(x,y)| \leq C[|x|^{2HK-2} + |y|^{2HK-2} + |x-y|^{2HK-2}]$$

and $\gamma = HK - 1 \in (-1, 0)$*. Thus, Theorem 4.3 can be applied to this case.*

5 Stochastic heat equation: Regularity of the density of the solution

In this section we consider again the solution $u(t,x)$ to (4.1), and we will impose the following condition on the covariance $q(x,y)$.

(**H2**) q is γ_0-Hölder continuous for some $\gamma_0 > 0$, and for some $\beta \in [0,2)$

$$|q(x_1,x_2)| \leq C(1 + |x_1|^\beta + |x_2|^\beta).$$

In this case we can assume that the random field $\{W_t(\varphi)\}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$. That means, we suppose that there exists a zero mean Gaussian random field $\{W_1(t,x), t \geq 0, x \in \mathbb{R}^d\}$ with covariance

$$E(W_1(t,x)W_1(s,y)) = (s \wedge t) q(x,y),$$

such that $W_t(\varphi) = \int_{\mathbb{R}^d} \varphi(x) W_1(t,x) dx$, for any $\varphi \in C_0(\mathbb{R}^d)$, where $q(x,y)$ is positive definite, namely, $\int_{\mathbb{R}^{2d}} q(x,y) f(x) f(y) dx dy \geq 0$ for all $f \in L^2(\mathbb{R}^d, dx)$. The additional regularity conditions imposed on q have allowed us to introduce the density process $W_1(t,x)$, which is a Brownian motion in the time variable and it has the spacial covariance q.

From a Theorem of Mercer's type (section 98 on page 245 in [14]) we know that if $\int_{\mathbb{R}^{2d}} |q(x,y)|^2 dx dy < \infty$, then

$$q(x,y) = \sum_{n=1}^{\infty} \lambda_n e_n(x) e_n(y)\,,$$

where $e_n, n = 1, 2, \cdots$ is an orthonormal basis of $L^2(\mathbb{R}^d, dx)$ and $\sum_{n=1}^\infty \lambda_n^2 < \infty$. The positive definite property of $q(x,y)$ implies $\lambda_n \geq 0$. If we take $C(x,y) = \sum_{n=1}^\infty \sqrt{\lambda_n} e_n(x) e_n(y)$, then $q(x,y) = \int_{\mathbb{R}^d} c(\xi,x) c(\xi,y) d\xi$. Thus it is without loss of generality for us to assume that $q(y_1,y_2) = \int_{\mathbb{R}^d} c(\xi,y_1) c(\xi,y_2) d\xi$ for some $c(\xi,y)$. Furthermore, we assume $c(x,y)$ has polynomial growth.

The following is the main result of this section.

Theorem 5.1 *Assume that q is γ_0-Hölder continuous for some $\gamma_0 > 0$ and satisfies (**H1a**). Suppose*

$$q(x_1, x_2) \le C(1 + |x_1|^\beta + |x_2|^\beta) \quad \text{for some } \beta \in [0, 2)\,. \tag{5.1}$$

Let u_0 be bounded and ρ-Hölder continuous for some $\rho > 0$. Suppose that there is a $x_0 \in \mathbb{R}^d$ such that $q(x_0, x_0) > 0$ and $\sigma(u_0(x_0)) \neq 0$. Then,

(1) If b and σ are continuous differentiable functions with bounded first order derivatives, for any $t > 0$ and $x \in \mathbb{R}^d$, the probability law of $u(t, x)$ is absolutely continuous with respect to the Lebesgue measure.

(2) If b and σ be infinitely differentiable with bounded derivatives of all orders, then for any $t > 0$ and $x \in \mathbb{R}^d$, the probability law of $u(t, x)$ has a smooth density with respect to Lebesgue measure

Proof. First we claim that for all (t, x) the random variable $u(t, x)$ belongs to the Sobolev space $\mathbb{D}^{1,2}$ under condition (1), and to the space $\mathbb{D}^\infty$ under condition (2). This follows from standard arguments and we omit the proof (see, for instance [11], Proposition 2.4.4 in the case of the stochastic heat equation). On the other hand, the Malliavin derivative $D_{s,y}u(t, x)$ satisfies the linear stochastic evolution equation

$$\begin{aligned} D_{s,y}u(t, x) =& p_{t-s}(x - y)\sigma(u(s, y)) + \int_{\mathbb{R}^d} \int_0^t p_{t-r}(x - z)b'(u(r, z))D_{s,y}u(r, z)drdz \\ &+ \int_{\mathbb{R}^d} \int_0^t p_{t-r}(x - z)\sigma'(u(r, z))D_{s,y}u(r, z)W_1(dr, z)dz\,. \end{aligned}$$

Denote

$$F := \|Du(t, x)\|^2_{\mathcal{H}_\infty} = \int_0^t \int_{\mathbb{R}^{2d}} D_{s,y_1}u(t, x)D_{s,y_2}u(t, x)q(y_1, y_2)dy_1dy_2ds\,,$$

where $\|\cdot\|_{\mathcal{H}_\infty}$ is the Hilbert norm introduced in Section 2. We are going to show only the statement (2), and the first one follows from similar arguments. It suffices to show that $E[F^{-p}] < \infty$ for any $p \ge 1$. We divide the proof into two steps.

Step 1. Introduce $V_{s,\xi}(t, x) = \int_{\mathbb{R}^d} c(\xi, y)D_{s,y}u(t, x)dy$. Then we can write

$$F = \int_0^t \int_{\mathbb{R}^d} V_{s,\xi}(t, x)^2 d\xi ds\,.$$

For any fixed (s, ξ), the random field $\{V_{s,\xi}(t, x), t \ge s, x \in \mathbb{R}^d\}$ satisfies the following linear stochastic heat equation for $t \ge s$, and $x \in \mathbb{R}^d$,

$$\begin{cases} \dfrac{\partial V_{s,\xi}}{\partial t} = \dfrac{1}{2}\triangle V_{s,\xi} + b'(u)V_{s,\xi} + \sigma'(u)V_{s,\xi}\dfrac{\partial W_1}{\partial t}(t, x), \\ V_{s,\xi}(s, x) = c(\xi, x)\sigma(u(s, x))\,. \end{cases}$$

Consider the continuous semimartingale $\{F(t, x), t \ge 0, x \in \mathbb{R}^d\}$ given by

$$F(t, x) = \int_0^t b'(u(r, x))dr + \int_0^t \sigma'(u(r, x))W_1(dr, x).$$

The local characteristic of this semimartingale are $b(t,x)=b'(u(t,x))$ and

$$a(t,x,y)=\int_0^t \sigma'(u(r,x))\sigma'(u(r,y))q(x,y)dr\,.$$

Notice that conditions (3.1) and (3.2) hold because b' and σ' are bounded and q satisfies (5.1). Then, Theorem 3.1 gives an explicit Feynman-Kac formula for the above equation. This means that we have

$$\begin{aligned}V_{s,\xi}(t,x) =&E^B\Bigg[c(\xi,x+B_{t-s})\sigma(u(s,x+B_{t-s}))\\ &\times\exp\left\{\int_s^t F(dr,x+B_{t-s}-B_{r-s})-\frac12\int_s^t \bar a(r+x,B_{t-s}-B_{r-s})dr\right\}\Bigg].\end{aligned}$$

Step 2. Let

$$Y(s,t;B)=\int_s^t F(dr,x+B_{t-s}-B_{r-s})-\frac12\int_s^t \bar a(r+x,B_{t-s}-B_{r-s})dr.$$

Then

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^d}|V_{s,\xi}(t,x)|^2d\xi ds &= \int_0^t\int_{\mathbb{R}^d}E^{B,\tilde B}\Bigg[c(\xi,x+B_{t-s})c(\xi,x+\tilde B_{t-s})\\ &\quad\times\sigma(u(s,x+B_{t-s}))\sigma(u(s,x+\tilde B_{t-s}))\exp\{Y(s,t;B)+Y(s,t;\tilde B)\}\Bigg]d\xi ds\\ &=\int_0^t E^{B,\tilde B}\Bigg[q(x+B_{t-s},x+\tilde B_{t-s})\\ &\quad\times\sigma(u(s,x+B_{t-s}))\sigma(u(s,x+\tilde B_{t-s}))\exp\{Y(s,t;B)+Y(s,t;\tilde B)\}\Bigg]ds\\ &=\int_0^t H(s)ds,\end{aligned}$$

where $\tilde B$ is a standard Brownian motion independent of B. If we can show that $E(H(0)^{-p})<\infty$ for all $p>1$, and $H(s)$ is Hölder continuous, then by Lemma 5.2 below we deduce

$$E\left(\int_0^t\int_{\mathbb{R}^d}|V_{s,\xi}(t,x)|^2d\xi ds\right)^{-p}=E\left(\int_0^t H(s)ds\right)^{-p}<\infty$$

for all $p\ge 1$. The Hölder continuity of $H(s)$ can be verified from the following inequality:

$$E|H(s_1)-H(s_2)|^p\le C|s_2-s_1|^{\frac p2\min\{\rho,\gamma_0,1+\gamma\}},$$

where C is determined by

$$\sup_{s\in[0,t]}\left\{E|q(x+B_{t-s},x+\tilde B_{t-s})|^{8p},E|\sigma(u(s,x+B_{t-s}))|^{8p},E\exp\{8pY(s,t;B)\}\right\}.$$

It remains to show that $E(H(0)^{-p})<\infty$. Notice that

$$H(0)=E^{B,\tilde B}\left(G_x\exp\{Y(0,t;B)+Y(0,t;\tilde B)\}\right),$$

where

$$G_x = q(x + B_t, x + \tilde{B}_t)\sigma(u_0(x + B_t))\sigma(u_0(x + \tilde{B}_t)).$$

We can write, by Jensen's inequality,

$$\begin{aligned}
&E\left(E^{B,\tilde{B}}\left[G_x \exp\{Y(0,t;B) + Y(0,t;\tilde{B})\}\right]\right)^{-p} \\
=&E\left|E^{B,\tilde{B}}\left[|G_x| \operatorname{sign}(G_x) \exp\{Y(0,t;B) + Y(0,t;\tilde{B})\}\right]\right|^{-p} \\
\leq&\left[E^{B,\tilde{B}}|G_x|\right]^{-p-1} E\left[|G_x| \exp\{-p(Y(0,t;B) + Y(0,t;\tilde{B}))\}\right].
\end{aligned}$$

Our nondegeneracy hypotheses imply that $E^{B,\tilde{B}}G > 0$, and this allows us to conclude the proof. ∎

Lemma 5.2 *Let $\{S_t, 0 \leq t \leq 1\}$ be a non-negative stochastic process. If $ES_0^{-a} < \infty$ for some $a > 0$, and $\sup_{0\leq s\leq t} |S_s - S_0| \leq Gt^\gamma$ where G is a positive random variable with $EG^b < \infty$ for some $b > 0$, then we have*

$$E\left|\int_0^1 S_t dt\right|^{-p} < \infty, \quad \text{for} \quad 0 < p < ab\gamma/(a + b + b\gamma).$$

In particular, if a and b can be arbitrarily large, then p can also be chosen arbitrarily large.

Proof. Let $\alpha, \beta > 0$, where $\alpha + \beta < 1$ and $b\beta\gamma - b\alpha \geq a\alpha$, and $0 < \epsilon < 2^{\alpha+\beta-1}$. We have

$$\begin{aligned}
&P\left[\int_0^1 S_t dt < \epsilon\right] \leq P\left[\int_0^{\epsilon^\beta} S_t dt < \epsilon, \quad S_0 > \epsilon^\alpha\right] + P\left[S_0 < \epsilon^\alpha\right] \\
\leq&P\left[\sup_{0\leq t\leq\epsilon^\beta} |S_t - S_0| > \frac{1}{2}\epsilon^\alpha\right] + P\left[S_0^{-a} > \epsilon^{-a\alpha}\right] \\
\leq&2^b\epsilon^{-b\alpha}E\left(\sup_{0\leq t\leq\epsilon^\beta} |S_t - S_0|^b\right) + \epsilon^{a\alpha}ES_0^{-a} \leq C\left(\epsilon^{b\beta\gamma-b\alpha} + \epsilon^{a\alpha}\right) \leq C\epsilon^{a\alpha}.
\end{aligned}$$

Then $E\left|\int_0^1 S_t dt\right|^{-p} < \infty$, for $0 < p < a\alpha$. The lemma follows with the choice of α and β such that $\alpha < b\gamma/(a + b + b\gamma)$ and $\beta = (a + b)/(a + b + b\gamma)$. ∎

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