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CHARACTERIZATION OF SMOOTH SCHUBERT VARIETIES IN RATIONAL HOMOGENEOUS MANIFOLDS OF PICARD NUMBER 1

JAEHYUN HONG AND NGAIMING MOK

Abstract

In a series of works one of the authors has developed with J.-M. Hwang a geometric theory of uniruled projective manifolds basing on the study of varieties of minimal rational tangents, and the geometric theory has especially been applied to rational homogeneous manifolds of Picard number 1. In Mok [Astérisque 322, pp. 151–205] and Hong-Mok [J. Diff. Geom. **86** (2010), pp. 539–567] the authors have started the study of uniruled projective subvarieties, and a method was developed for characterizing certain subvarieties of rational homogeneous manifolds. The method relies on non-equidimensional Cartan-Fubini extension and a notion of parallel transport of varieties of minimal rational tangents.

In the current article we apply the notion of parallel transport to a characterization of smooth Schubert varieties of rational homogeneous manifolds of Picard number 1. Given a pair (S, S_0) consisting of a rational homogeneous manifold S of Picard number 1 and a smooth Schubert variety S_0 of S, where no restrictions are placed on S_0 when $S = G/P$ is associated to a long root (while necessarily some cases have to be excluded when S is associated to a short root), we prove that any subvariety of S having the same homology class as S_0 must be gS_0 for some $g \in \mathrm{Aut}(S)$.

We reduce the problem first of all to a characterization of local deformations S_t of S_0 as a subvariety of S. By Kodaira stability, S_t is uniruled by minimal rational curves of S lying on S_t. We establish a biholomorphism between S_t and S_0 which extends to a global automorphism by reconstructing S_t by means of a repeated use of parallel transport of varieties of minimal rational tangents along minimal rational curves issuing from a general base point. Our method is applicable also to the case of singular Schubert varieties provided that there exists a minimal rational curve on the smooth locus of the variety.

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1. Introduction

In this article we consider rational homogeneous manifolds over the complex field. A rational homogeneous manifold $S = G/P$ is a projective manifold where a connected semisimple (complex) Lie group G acts transitively. For any submanifold Z of S, by applying $g \in G$ we can deform Z to a submanifold gZ of S, all having the same homology class as Z in S. The question is whether these are the only subvarieties of S having the same homology class as Z.

The answer is indeed positive for sub-Grassmannians of Grassmannians. The Grassmannian $S = Gr(k, V)$ of k-planes of a complex vector space V is a rational homogeneous manifold on which $G = \mathbb{P}SL(V)$ acts transitively. By a sub-Grassmannian S_0 of S we mean a subvariety of $Gr(k, V)$ consisting of k-planes of V which contain one fixed subspace of V and which are contained in another fixed subspace of V. By a result of Bryant [4] and Walters [28] based on differential-geometric methods, any subvariety of the Grassmannian S having the same homology class as a sub-Grassmannian S_0 is again a sub-Grassmannian. Equivalently, there is only one irreducible component in the Chow variety where the homology class (of any member) agrees with the homology class of S_0, and it consists precisely of one closed orbit of G.

In the current article we consider the general situation of a pair (S, S_0) consisting of a rational homogeneous manifold S of Picard number 1 and a smooth Schubert variety S_0 of S. We say that homological rigidity holds for the pair (S, S_0) whenever any subvariety X of S having the same homology class as S_0 must be gS_0 for some $g \in \mathrm{Aut}_0(S)$. By an entirely different method we determine completely whether homological rigidity holds for (S, S_0) in the case where $S_0 \subset S$ is a homogeneous submanifold associated to a sub-diagram of the marked Dynkin diagram corresponding to S.

Theorem 1.1. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and let $S_0 = G_0/P_0$ be a homogeneous submanifold associated to a subdiagram $\mathcal{D}(G_0)$ of the marked Dynkin diagram $\mathcal{D}(G)$ of S. Then any subvariety of S having the same homology class as S_0 is induced by the action of $\mathrm{Aut}_0(S)$, excepting when (S, S_0) is given by*

(a) $S = (C_n, \{\alpha_k\})$, $\Lambda = \{\alpha_{k-1}, \alpha_b\}$, $2 \leq k < b \leq n$;
(b) $S = (F_4, \{\alpha_3\})$, $\Lambda = \{\alpha_1, \alpha_4\}$ *or* $\{\alpha_2, \alpha_4\}$;
(c) $S = (F_4, \{\alpha_4\})$, $\Lambda = \{\alpha_2\}$ *or* $\{\alpha_3\}$,

where Λ denotes the set of simple roots in $\mathcal{D}(G)\backslash\mathcal{D}(G_0)$ which are adjacent to the subdiagram $\mathcal{D}(G_0)$.

This covers all smooth Schubert subvarieties $S_0 \subset S$ whenever $S = G/P$ is associated to a long simple root (see Proposition 3.7). We settle the same problem for non-homogeneous smooth Schubert varieties S_0 when S is a symplectic Grassmannian which is a prototype for the short-root case. (For notation, see Section 4.2.1.)

Theorem 1.2. *Let S be the symplectic Grassmannian $Gr_\omega(k, V)$ and let S_0 be a smooth Schubert variety of the form $Gr_\omega(k, V; F_a, F_{2n-1-a})$ for $0 \leq a \leq k-2$. Then any subvariety of S having the same homology class as S_0 is induced by the action of $\mathrm{Aut}_0(S) = \mathbb{P}Sp(V, \omega)$.*

Our approach is based on a method of parallel transport of varieties of minimal rational tangents along minimal rational curves developed in Mok [25] and Hong-Mok [8], in analogy with the notion of parallel transport along a geodesic in Riemannian geometry, in the framework of a theory of geometric structures of uniruled projective manifolds modeled on varieties of minimal rational tangents (see Hwang-Mok [11], Hwang [17], and Mok [25]).

Using the fact that the action of $G := \mathrm{Aut}_0(S)$ on each irreducible component of the Chow variety of S must have a closed orbit, the problem of characterizing a smooth Schubert variety $S_0 \subset S$ is first of all reduced to a problem on local deformations of S_0 in S as a subvariety. The problem is whether local deformations of S_0 are necessarily of the form gS_0 for some $g \in G$. We will call this the local characterization problem. From the local nature of the problem, an obvious approach is cohomological. More precisely, the answer to the local characterization problem is affirmative whenever $H^1(S_0, N_{S_0|S}) = 0$, and $\dim H^0(S_0, N_{S_0|S})$ is equal to the dimension of the orbit $G.[S_0]$ in $\mathrm{Chow}(S)$. Although such an approach can be implemented by means of the Borel-Weil-Bott Theorem in the case where $S_0 \subset S$ is itself a homogeneous submanifold, the cohomological method becomes inaccessible when S_0 is only almost homogeneous. A more conceptual approach is therefore desirable.

We introduce a geometric and completely different approach to the local characterization problem by reconstructing local deformations S_t of S_0 in S from their varieties of minimal rational tangents. By Kodaira stability, S_t is uniruled by minimal rational curves of S lying on S_t. For appropriate pairs (S, S_0) we establish a biholomorphism between S_t and S_0 which extends to an automorphism of S, by identifying varieties of minimal rational tangents at a general point of S_t with those of S_0, and by a repeated use of parallel transport of varieties of minimal rational tangents along minimal rational curves issuing from a general base point. Such a notion of parallel transport is made possible by the deformation theory of rational curves and, at least in the case where $S_0 \subset S$ is a homogeneous submanifold, by the compactness of the orbit of the Chow point corresponding to the variety $\mathcal{C}_x(S_0) \subset \mathbb{P}T_x(S_0)$ of minimal rational tangents at x under the action of the isotropy subgroup of G at x.

As compared to the differential-geometric method of Bryant [4] and adopted later by Hong [9] based on the use of explicit differential forms, in the method of the current article we make use of differential geometry of a different kind, which is based on notions derived from (tangents to) holomorphic families of distinguished local holomorphic curves, and as such, it is applicable to any rational homogeneous manifold of Picard number 1. Verifications of the necessary geometric hypotheses are elementary even in cases pertaining to exceptional Lie groups, a situation which would, in general, require formidable computational machinery.

In earlier works on deformation rigidity of rational homogeneous manifolds, triviality of local deformations is guaranteed by local rigidity theorems, and the focus has always been placed on the limiting behavior of varieties of minimal rational tangents. By contrast, in the current work, a homological rigidity problem on smooth Schubert varieties is first reduced to the study of local deformations of such varieties, and our approach adds a new dimension to the applicability of the geometric theory of varieties of minimal rational tangents by a further reduction of the problem to the study of local deformations of varieties of minimal rational tangents. By the very nature of our geometric approach revolving around the deformation of minimal rational curves, our method remains applicable to the case of singular Schubert varieties S_0 provided that there exists a minimal rational curve on the smooth locus of S_0, a situation which we will deal with in a forthcoming article in preparation.

In Section 2, we characterize a Schubert variety as an irreducible subvariety of $S = G/P$ such that the orbit $G.[Z]$ is closed in the Chow variety of S. We reduce the problem to prove that any local deformation of S_0 in S is induced by G (see Proposition 2.2). In Section 3 we develop a method to characterize a smooth Schubert variety by the variety of minimal rational tangents at a general point $x \in S_0$. Denoting by $P_x \subset G$ the isotropy subgroup at x, our method reduces the problem to the local non-deformability modulo the action of P_x of the variety of minimal rational tangents of S_0 at $x \in S_0$ in that of S (see Proposition 3.2(2)). This local characterization problem is solved in Section 4.

2. Schubert varieties

Let G be a connected simple group. Take a Borel subgroup B of G and a maximal torus T in B. Denote by Δ^+ the system of positive roots of G and by $\mathcal{S} = \{\alpha_1, \cdots, \alpha_\ell\}$ the system of simple roots of G. For a root α, denote by $\mathfrak{g}_\alpha$ the root space of the root α and by U_α the root subgroup of the root α. Then the unipotent part U of B is isomorphic to the product $\Pi_{\alpha\in\Delta^+}U_\alpha$ and the unipotent part U^- of the Borel subgroup B^- opposite to B is isomorphic to the product $\Pi_{\alpha\in\Delta^+}U_{-\alpha}$. For each $j = 1, \cdots, \ell$, denote by $n_j(\alpha)$ the coefficient in α_j of $\alpha = \sum_{i=1}^{\ell} n_i\alpha_i$.

Let $\mathfrak{t}$ be the Lie algebra of T. To each simple root α_k we associate a parabolic subgroup P of G, whose Lie algebra $\mathfrak{p}$ is given by $\mathfrak{p} = \mathfrak{t} + \sum_{n_k(\alpha)\geq 0}\mathfrak{g}_\alpha$. (In opposition to Hwang-Mok [10] we choose the parabolic subgroup $P \subset G$ in such a way that the tangent space $T_o(S)$ at the base point $o = eP$ of $S = G/P$ is spanned by root spaces belonging to negative roots.) The homogeneous manifold $S = G/P$ is called the rational homogeneous manifold associated to the simple root α_k. The Levi decomposition of $\mathfrak{p}$ is given by $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{l} = \mathfrak{t} + \sum_{n_k(\alpha)=0}\mathfrak{g}_\alpha$ is the reductive part of $\mathfrak{p}$ and $\mathfrak{n} = \sum_{n_k(\alpha)>0}\mathfrak{g}_\alpha$ is the nilpotent part of $\mathfrak{p}$. Put $\Delta(\mathfrak{n}) = \{\alpha \in \Delta^+ : n_k(\alpha) > 0\}$.

Let $\mathcal{W}$ be the Weyl group of G with respect to T. For each $w \in \mathcal{W}$, define a subset $\Delta(w)$ of Δ^+ by $\Delta(w) = \{\beta \in \Delta^+ : w(\beta) \in -\Delta^+\}$. Then the Borel subgroup B acts on S and has only finitely many orbits, indexed by the subset $\mathcal{W}^P$ of $\mathcal{W}$ defined by $\mathcal{W}^P := \{w \in \mathcal{W} : \Delta(w) \subset \Delta(\mathfrak{n})\}$, i.e.,

$$S = \coprod_{w\in\mathcal{W}^P} Be_w$$

where $e_w = wP, w \in \mathcal{W}^P$ are T-fixed points in S. For each $w \in \mathcal{W}^P$, $Be_w = UwP = w(w^{-1}Uw \cap U^-)P = w(\Pi_{\beta\in\Delta(w)}U_{-\beta})P$ is isomorphic to a cell $\mathbb{C}^{\ell(w)}$ of dimension $\ell(w) = |\Delta(w)|$. The closure $S(w)$ of Be_w is called *a Schubert variety of type* w. By the cell decomposition, the integral homology space $H_*(S, \mathbb{Z})$ is freely generated by the homology classes of Schubert varieties $S(w)$, where $w \in \mathcal{W}^P$. For details about the Schubert varieties of $S = G/P$, cf. Demazure [5] or Springer [26].

Considering $S(w)$ as a point $[S(w)]$ in the Chow variety of S, the G-orbit of $[S(w)]$ consists of Schubert varieties with respect to Borel subgroups of G, which have the same homology classes as $S(w)$. It is clear that $gS(w)$ for $g \in G$ is a Schubert variety with respect to the Borel subgroup $B' = gBg^{-1}$ of G and $S(w)$ and $gS(w)$ have the same homology class. To see the other inclusion, take another Borel subgroup B' of G. Then there is $g \in G$ such that $B' = gBg^{-1}$. Take $P' := gPg^{-1}$ as a base point of S associated to the Borel subgroup B'. Identify the Weyl group $\mathcal{W}$ of G with respect to T with the Weyl group $\mathcal{W}'$ of G with respect to $T' = gTg^{-1}$ and let $\mathcal{W}^{P'}$ be the subset of $\mathcal{W}'$ corresponding to the subset $\mathcal{W}^P$ of $\mathcal{W}$. Then S is the disjoint union of B'-orbits $B'e'_{w'}$ of $e'_{w'} = w'P'$ for $w' \in \mathcal{W}^{P'}$ and their closures $S(w')'$ are Schubert varieties (with respect to B'). If a Schubert variety $S(w)$ for $w \in \mathcal{W}^P$ and a Schubert variety $S(w')'$ for $w' \in \mathcal{W}^{P'}$ have the same homology class, then w' corresponds to w and $S(w')' = gS(w)$. (For, otherwise, $w' \in \mathcal{W}^{P'}$ corresponds to an element $v \in \mathcal{W}^P$ which is different from w, and then the homology class of $S(w')' = gS(v)$ is different from the homology class of $S(w)$; a contradiction.)

The algebraic group G acts algebraically on (each irreducible component of) Chow(S). An algebraic subgroup $H \subset G$ is parabolic if and only if G/H is projective-algebraic. Since $S(w)$ is invariant under the action of B, the isotropy group of the action of G at $[S(w)]$ contains B, i.e., is a parabolic subgroup of G, and thus the G-orbit of $[S(w)]$ is closed. Conversely, we have the following.

Proposition 2.1. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root. Let Z be an irreducible subvariety of S. Consider Z as a point $[Z]$ in the Chow variety Chow(S) of S. If the G-orbit of $[Z]$ is closed, then Z is a Schubert variety.*

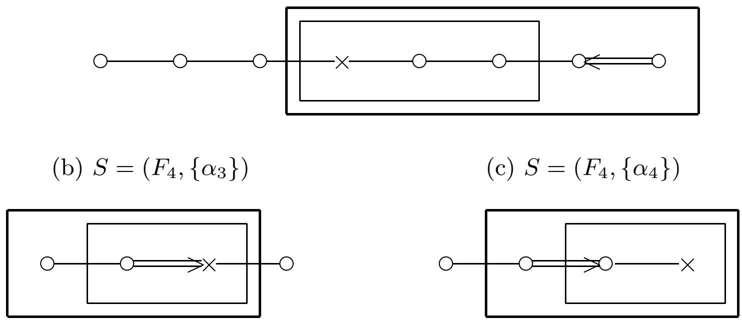
Proof. Since $G.[Z]$ is closed (and hence a projective subvariety of Chow(S)), the isotropy subgroup Q of G at $[Z]$ is a parabolic subgroup of G. Take a Borel subgroup B of G which is contained in Q. Then B acts on Z. The Borel subgroup B has only finitely many orbits in Z because it has only finitely many orbits in S. Hence, B has an open orbit in Z, which is unique because Z is irreducible. Thus Z is the closure of its open B-orbit, i.e., a Schubert variety in S. □

Proposition 2.2. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and let S_0 be a smooth Schubert variety. Suppose that any local deformation of S_0 in S is induced by the action of G. Then any subvariety of S having the same homology class as S_0 is induced by the action of G.*

Proof. Recall that the integral homology space $H_*(S, \mathbb{Z})$ is freely generated by the homology classes of Schubert varieties $S(w)$ of S (cf. the third paragraph of Section 2). Let Z be a subvariety of S having the same homology class as S_0. Since the homology class of a (complex) subvariety of S is a unique linear combination of the homology classes of Schubert varieties with non-negative integral coefficients (see Proposition 1.3.6(ii) of [2] and the remark after it), the homology class of S_0 cannot be the sum of two or more homology classes of (complex) subvarieties. Thus Z is irreducible and reduced.

Consider Z as a point $[Z]$ in the Chow variety Chow(S) of S, on which G acts algebraically. The closure of the G-orbit $G.[Z]$ contains a closed orbit $G.[Y]$ for some $[Y] \in$ Chow(S). Then Y is irreducible and has the same homology class as S_0. Since the orbit $G.[Y]$ is closed, Y is a Schubert variety by Proposition 2.1, and is of the same type as S_0 because Schubert varieties of different types cannot have the same homology class. Thus we may assume

(a) $S = (C_n, \{\alpha_k\})$



(b) $S = (F_4, \{\alpha_3\})$ (c) $S = (F_4, \{\alpha_4\})$

FIGURE 1. The small box represents the subdiagram $(\mathcal{D}(G_0), \{\alpha_k\})$ and the large box represents the subdiagram $(\mathcal{D}(G_1), \{\alpha_k\})$ containing $(\mathcal{D}(G_0), \{\alpha_k\})$ properly, whose type is (a) $(C_{n-k+1}, \{\alpha_1\})$ and (b) $(B_3, \{\alpha_3\})$ and (c) $(C_3, \{\alpha_1\})$, respectively.

that $Y = S_0$ up to the action of G. By this assumption, there is a neighborhood of $[S_0]$ which consists of G-translates of S_0. But any neighborhood of $[Y]$ intersects $G.[Z]$ because $[Y]$ lies in the closure of $G.[Z]$. Therefore Z is gS_0 for some $g \in G$. $\square$

Remarks. (a) Here and in the rest of the article, by a neighborhood, we will always mean an open neighborhood in the complex topology.

(b) As is evident from the proofs, the analogues of both Propositions 2.1 and 2.2 hold true for any rational homogeneous manifold.

Example 1. To each rational homogeneous manifold $S = G/P$ associated to a simple root α_k we associate the marked Dynkin diagram $(\mathcal{D}(G), \{\alpha_k\})$, where $\mathcal{D}(G)$ is the Dynkin diagram of G. A marked subdiagram $(\mathcal{D}(G_0), \{\alpha_k\})$ of $(\mathcal{D}(G), \{\alpha_k\})$ defines a homogeneous submanifold $S_0 = G_0/P_0$ of S, the G_0-orbit of the base point $o \in S$. We call it the homogeneous submanifold associated to the subdiagram $\mathcal{D}(G_0)$ of the marked Dynkin diagram of S. Let Λ be the set of simple roots in $\mathcal{D}(G)\backslash\mathcal{D}(G_0)$ which are adjacent to the subdiagram. Then the stabilizer of S_0 in G is the parabolic subgroup of G whose Lie algebra is $\mathfrak{t} + \sum_{\alpha \in \Delta \cap \mathbb{Z}(\mathcal{S}\backslash\Lambda)} \mathfrak{g}_\alpha + \sum_{\alpha \in \Delta^+ \backslash \mathbb{Z}(\mathcal{S}\backslash\Lambda)} \mathfrak{g}_\alpha$ (see Tits [27]). Thus S_0 is a smooth Schubert variety in S.

Remarks. The identity component $\mathrm{Aut}_0(S)$ of the automorphism group of $S = G/P$ is equal to G excepting the cases where $(G, \{\alpha_k\})$ is $(B_\ell, \{\alpha_\ell\})$,

$(C_\ell, \{\alpha_1\})$ or $(G_2, \{\alpha_1\})$. In these cases, we can think of $S = G/P$ as a rational homogeneous manifold G'/P' with $\mathrm{Aut}_0(S) = G' \supsetneq G$. By the same reason, if there is a subdiagram $(\mathcal{D}(G_1), \{\alpha_k\})$ of $(\mathcal{D}(G), \{\alpha_k\})$ containing $(\mathcal{D}(G_0), \{\alpha_k\})$ properly, whose type is $(B_\ell, \{\alpha_\ell\})$ or $(C_\ell, \{\alpha_1\})$, then, denoting by S_1 the homogeneous manifold associated to $(\mathcal{D}(G_1), \{\alpha_k\})$, G_1 is a proper subgroup of $\mathrm{Aut}_0(S_1)$ and $\mathrm{Aut}_0(S_1)$ is not contained in $\mathrm{Aut}_0(S)$. Thus there exists a non-trivial deformation of S_0 in S_1 (hence in S) which is not induced by the action of $\mathrm{Aut}_0(S)$. These are the cases where (S, S_0) are of types (a) – (c) in Theorem 1.1 (see Figure 1).

From now on we will assume that G is the connected component $\mathrm{Aut}_0(S)$ of the automorphism group of $S = G/P$.

3. Local characterizations of smooth Schubert varieties

3.1. Varieties of minimal rational tangents. Let $(X, \mathcal{L})$ be a polarized uniruled projective manifold, where $\mathcal{L}$ stands for an ample line bundle on X. By a (parametrized) rational curve on X we mean a non-constant holomorphic map $f : \mathbb{P}^1 \to X$. The latter is said to be a free rational curve if the holomorphic vector bundle f^*TX on $\mathbb{P}^1$ is semipositive, i.e., it splits holomorphically into a direct sum of holomorphic line bundles of degree ≥ 0. Let $\mathrm{Hom}(\mathbb{P}^1, X)$ be the scheme of all rational curves $f : \mathbb{P}^1 \to X$ on X, and denote by $[f\,] \in \mathrm{Hom}(\mathbb{P}^1, X)$ the member corresponding to f. Note that $\mathrm{Hom}(\mathbb{P}^1, X)$ is smooth at any $[f\,]$ corresponding to a free rational curve. A free rational curve $f : \mathbb{P}^1 \to X$ such that the degree $\deg_{\mathcal{L}}(f) := \deg(f^*\mathcal{L})$ is minimum among all free rational curves on X will be called a minimal rational curve. Let $f_0 : \mathbb{P}^1 \to X$ be a minimal rational curve (which is, by definition, necessarily free). Denote by $\mathrm{Hom}_{[f_0]}(\mathbb{P}^1, X)$ the irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)$ containing $[f_0]$, and by $\mathcal{H} \subset \mathrm{Hom}_{[f_0]}(\mathbb{P}^1, X)$ the Zariski-open subscheme consisting of all free rational curves $f : \mathbb{P}^1 \to X$ belonging to $\mathrm{Hom}_{[f_0]}(\mathbb{P}^1, X)$. Denote by $\mathcal{K} := \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1)$ the associated quotient space of unparameterized free rational curves. From minimality, it follows that $\mathrm{Aut}(\mathbb{P}^1)$ acts freely on $\mathcal{H}$, and the quotient space $\mathcal{K}$ is a quasi-projective manifold. We call $\mathcal{K}$ *a minimal rational component.* In what follows we assume that a minimal rational component $\mathcal{K}$ on X has been chosen, and by a minimal rational curve on X we will implicitly mean a minimal rational curve belonging to $\mathcal{K}$. The latter is sometimes also just referred to as a $\mathcal{K}$-curve, or as a $\mathcal{K}$-line if X is of Picard number 1 and the homology class of a $\mathcal{K}$-curve is the positive generator of

$H_z(X,\mathbb{Z}) \cong \mathbb{Z}$. For a standard reference on the deformation theory of rational curves from an algebro-geometric perspective, the reader is referred to Kollár [22].

Let $\rho : \mathcal{U} \to \mathcal{K}$ be the universal $\mathbb{P}^1$-bundle of $\mathcal{K}$-curves, and $\mu : \mathcal{U} \to X$ be the evaluation map. We write $\mathcal{U}_x := \mu^{-1}(x)$. For a member u of $\mathcal{K}_x$, represented by $f : \mathbb{P}^1 \to X, f(0) = x$, which is an immersion at the marking at x, the tangent map τ_x associates u to the tangent line $[df(T_0(\mathbb{P}^1))] \in \mathbb{P}T_x(X)$ at the marking. By Kebekus [20], any minimal rational curve passing through a general point $x \in X$ is immersed, so that the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ at x is holomorphic. Its image $\mathcal{C}_x := \tau_x(\mathcal{U}_x)$ is called the variety of minimal rational tangents at x, and by Hwang-Mok [12] and [14], the map $\tau_x : \mathcal{U}_x \to \mathcal{C}_x$ is a normalization. The closure of the union of $\mathcal{C}_x$ over general points $x \in X$ gives the fibered space $\pi : \mathcal{C} \to X$ of varieties of minimal rational tangents associated to $\mathcal{K}$. For surveys on the theory of geometric structures modeled on varieties of minimal rational tangents at various stages of its development, the reader may consult Hwang-Mok [11], Hwang [17] and Mok [25].

When we speak of the variety of minimal rational tangents of a rational homogeneous manifold S associated to a simple root, we will assume that S is equipped with the minimal rational component $\mathcal{K}$ consisting of lines $\mathbb{P}^1$ contained in S after we embed S into $\mathbb{P}^N$ by the ample generator of the Picard group of S. The variety of minimal rational tangents of a rational homogeneous manifold S was used to prove global rigidity of S in Hwang-Mok [10], [13], [15] and [16]. The article Hwang [18] contains discussions on rigidity phenomena of rational homogeneous manifolds studied by means of varieties of minimal rational tangents. In this paper, we will use the variety of minimal rational tangents of a smooth Schubert variety S_0 of S to prove that a local deformation of S_0 in S is of the form gS_0 for some $g \in G$.

By the homogeneity of $S = G/P$, a G-orbit in $\mathcal{K}$ is uniquely determined by a P-orbit of the varieties of minimal rational tangents $\mathcal{C}_o$ at the base point $o = eP$ and vice versa. Thus $\mathcal{K}$ has an open dense G-orbit, and consists of one G-orbit if S is associated to a long root, cf. Hwang-Mok [13], Proposition 1, for the long-root case, and Hwang-Mok [11], end of the first paragraph in p. 380, for the short-root cases. (For a more explicit description of the short-root cases, the reader may consult Hwang-Mok [15] and [16]; see also Theorem 4.3 of [23].)

Example 2. Let $S = G/P$ be a rational homogeneous manifold associated to a long root α_k. Let L be the reductive part of the isotropy of G at a base point $x \in S$. Then the variety $\mathcal{C}_x(S)$ of minimal rational tangents of S at $x \in S$ is a rational homogeneous manifold L/R with the marked Dynkin diagram $(\mathcal{D}(L), \Upsilon)$, where Υ is the set of simple roots α_j with $\alpha_j \neq \alpha_k$ and

$\langle \alpha_j, \alpha_k \rangle \neq 0$. In particular, $\mathcal{C}_x(S)$ is a Hermitian symmetric space of compact type. The following is the list of $\mathcal{C}_x(S)$ and its embedding into $\mathbb{P}(T_x S)$ (p. 176 of Hwang-Mok [13]):

(I) $\mathcal{A} \subset \mathbb{P}(V)$, an irreducible Hermitian symmetric space of the compact type in the first canonical embedding,

(II) $\mathbb{P}(E_1) \times \mathbb{P}(E_2) \subset \mathbb{P}(E_1 \otimes E_2)$, the Segre embedding of the product of two projective spaces $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$,

(III) $\nu_2(\mathbb{P}(E)) \subset \mathbb{P}(S^2 E)$, the second Veronese embedding of a projective space $\mathbb{P}(E)$,

(IV) $\mathbb{P}(E_1) \times \mathcal{A}_2 \subset \mathbb{P}(E_1 \otimes E_2)$, the Segre embedding of the product of a projective space $\mathbb{P}(E_1)$ and an irreducible Hermitian symmetric space of the compact type, $\mathcal{A}_2 \subset \mathbb{P}(E_2)$, in the first canonical embedding,

(V) $\mathbb{P}(E_1) \times \nu_2(\mathbb{P}(E_2)) \subset \mathbb{P}(E_1 \otimes S^2(E_2))$, the Segre embedding of the product of a projective space $\mathbb{P}(E_1)$ and a projective space, $\nu_2(\mathbb{P}(E_2)) \subset \mathbb{P}(S^2(E_2))$, in the second Veronese embedding,

(VI) $\mathbb{P}(E_1) \times \mathbb{P}(E_2) \times \mathbb{P}(E_3) \subset \mathbb{P}(E_1 \otimes E_2 \otimes E_3)$, the Segre embedding of the product of three projective spaces $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$ and $\mathbb{P}(E_3)$,

(VII) $\nu_3(\mathbb{P}(E)) \subset \mathbb{P}(S^3 E)$, the third Veronese embedding of a projective space $\mathbb{P}(E)$.

If S_0 is a homogeneous submanifold associated to a subdiagram $(\mathcal{D}(G_0), \alpha_k)$ of the marked Dynkin diagram $(\mathcal{D}(G), \alpha_k)$ of $S = G/P$, then the variety $\mathcal{C}_x(S_0)$ of minimal rational tangents of S_0 at $x \in S_0$ is a homogeneous submanifold of $\mathcal{C}_x(S)$ associated to the subdiagram $(\mathcal{D}(G_0) \cap \mathcal{D}(L), \Upsilon)$ of the marked Dynkin diagram $(\mathcal{D}(L), \Upsilon)$ of X.

For an irreducible projective variety M, by a general point on M in the standard usage, we mean a point lying outside some subvariety $E \subsetneq M$, equivalently a point lying on the subset $U = M - E$ which is open and dense in the Zariski topology. In some situations it is convenient to slightly modify the usage of the term "general" and to speak of a general point with an implicit understanding that a specific Zariski open subset $U \subset M$ has been chosen, by spelling out the conditions defining the set U.

In the sequel, for a projective submanifold $Z \subset S$ uniruled by lines lying on Z, by a general point of Z we will mean a point $z \in Z$, such that all lines on Z passing through z are free rational curves. Endowing Z with a minimal rational component $\mathcal{K}(Z)$ consisting of projective lines on S lying on Z, and denoting by $\rho : \mathcal{U}(Z) \to \mathcal{K}(Z)$ the associated universal $\mathbb{P}^1$-bundle over $\mathcal{K}(Z)$, and by $\mu : \mathcal{U}(Z) \to Z$ the evaluation map, for a general point $z \in Z$ the fiber $\mathcal{U}_z(Z) = \mu^{-1}(z)$ of the evaluation map is projective due to minimality (as follows from Mori's Breaking-up Lemma) and smooth because of the freeness of every $\mathcal{K}(Z)$-curve passing through z. (Here $\mathcal{K}(Z)$ is taken to be any connected component of the (not necessarily connected) moduli space of free rational curves on Z which are projective lines of S, thus the reference to 'a minimal rational component' in place of 'the rational component'.)

For any line C lying on Z, the tangent bundle $TZ|_C$ restricted to C is a subbundle of $T\mathbb{P}^N = \mathcal{O}(2) \oplus \mathcal{O}(1)^{N-1}$. Thus, if C is a free rational curve in Z, the non-negative holomorphic vector bundle $TZ|_C \subset T\mathbb{P}^N|_C$ must be of the form $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$; in other words, C is a standard rational curve in Z. Thus, for a general point $z \in Z$, every $\mathcal{K}(Z)$-line passing through z is standard, and the tangent map $\tau_z : \mathcal{U}_z(Z) \to \mathcal{C}_z(Z) \subset \mathbb{P}T_z(Z)$ is a biholomorphic embedding (cf. Lemma 3 of Mok [25]) of the smooth projective manifold $\mathcal{U}_z(Z)$ onto the variety of minimal rational tangents $\mathcal{C}_z(Z) \subset \mathbb{P}T_z(Z)$. Thus $\mathcal{C}_z(Z) \subset \mathbb{P}T_z(Z)$ is a projective submanifold.

Concerning smooth Schubert varieties $S_0 \subset S$ we have the following result regarding projective lines and associated varieties of minimal rational tangents on S_0.

Proposition 3.1. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and let S_0 be a smooth Schubert variety. Then S_0 is of Picard number* 1 *and it is uniruled by lines of S lying on S_0. Furthermore, for a general point x of S_0, the variety $\mathcal{C}_x(S_0)$ of minimal rational tangents of S_0 at x is a smooth linear section of the variety $\mathcal{C}_x(S)$ of minimal rational tangents of S at x.*

Proof. The map $\mathrm{Pic}(S) \to \mathrm{Pic}(S_0)$ defined by restricting holomorphic line bundles from S to S_0 is a surjective homomorphism (see Brion [2]). Since S is of Picard number 1, so is S_0.

For $w \in \mathcal{W}^P$ we have $w^{-1}Be_w = (\Pi_{\beta \in \Delta(w)} U_{-\beta})P$ (see Section 2). Let $w = s_{\beta_r} \cdots s_{\beta_1}$ be a reduced expression of $w \in \mathcal{W}^P$. Here, s_{β_i} denotes the reflection relative to a simple root β_i for $i = 1, \cdots, r$. Then we have $\Delta(w) = \{\beta_1, s_{\beta_1}(\beta_2), \cdots, s_{\beta_1} \cdots s_{\beta_{r-1}}(\beta_r)\}$ (see Lemma 8.3.2 of Springer [26]). Thus, for any $w_1 \in \mathcal{W}^P$ such that $w = s_{\gamma_t} \cdots s_{\gamma_1} w_1$ for some simple roots γ_j $(j = 1, \cdots, t)$ with $\ell(w) = \ell(w_1) + t$, $\Delta(w_1)$ is contained in $\Delta(w)$ and hence ${w_1}^{-1}Be_{w_1}$ is contained in $w^{-1}Be_w$. (For, if $w_1 = s_{\beta'_s} \cdots s_{\beta'_1}$ is a reduced expression of w_1, then $w = s_{\gamma_t} \cdots s_{\gamma_1} s_{\beta'_s} \cdots s_{\beta'_1}$ is a reduced expression of w because $\ell(w) = \ell(w_1) + t$.) In particular, for $w = s_{\beta_r} \cdots s_{\beta_1} \in \mathcal{W}^P$ such that $S_0 = S(w)$ and for $w_1 = s_{\beta_1}$, we have $w_1^{-1}Be_{w_1} \subset w^{-1}Be_w$, or equivalently, $ww_1^{-1}(Be_{w_1}) \subset Be_w$. The Schubert variety $S(w_1)$ is a line $\mathbb{P}^1$. Put $g = ww_1^{-1} \in G$. Then the line $gS(w_1)$ intersects $B.e_w$ in $\mathbb{C} \subset \mathbb{P}^1$. By letting B act on the line $gS(w_1)$ we get a family of lines covering an open dense subset of $S(w)$.

Since S_0 itself is a linear section of S (see Brion [2] and Jantzen [19]), any line in S tangent to S_0 is contained in S_0. Thus $\mathcal{C}_x(S_0)$ is equal to the intersection of $\mathcal{C}_x(S)$ with $\mathbb{P}(T_xS_0)$. Since S_0 is uniruled by projective lines, the tangent map $\tau_x : \mathcal{U}_x(S_0) \to \mathcal{C}_x(S_0)$ at x is a biholomorphism of the smooth projective manifold $\mathcal{U}_x(S_0)$ onto the variety of minimal rational tangents $\mathcal{C}_x(S_0)$ (cf. the last paragraph preceding Proposition 3.1). □

We note that any point x in the open B-orbit of $S_0 \subset S$ is a general point on S_0, to which the second half of Proposition 3.1 applies.

3.2. Reduction to local characterizations of varieties of minimal rational tangents. Recall the minimal rational component $\mathcal{K}$ on $S = G/P$ fixed in Section 3.1. By a $\mathcal{K}$-line we will mean a member of $\mathcal{K}$. In the sequel, by a general $\mathcal{K}$-line we will mean a member of the open G-orbit in $\mathcal{K}$. Let $S_0 \subset S$ be a smooth Schubert cycle, and $Q \subset G$ be the (parabolic) subgroup of G preserving S_0 as a set. S_0 is almost homogeneous under the action of some Borel subgroup $B \subset G$ and *a fortiori* under the action of Q, hence it contains a unique Zariski open Q-orbit $W(S_0)$. A point $x \in W(S_0)$ is necessarily a general point of S_0 (in the sense of the paragraph preceding Proposition 3.1).

By Proposition 3.1, S_0 is uniruled by projective lines. In general, S_0 may not contain any general $\mathcal{K}$-line. In the case where S_0 contains a general $\mathcal{K}$-line, then for $x \in W(S_0)$, there exists $\alpha \in \mathcal{C}_x(S_0)$ such that α is tangent to a general $\mathcal{K}$-line. In fact, the set of such elements α constitutes a dense Zariski open subset of $\mathcal{C}_x(S_0)$, and α will be called a general point of $\mathcal{C}_x(S_0)$. When S_0 does not contain a general $\mathcal{K}$-line, any point in $\mathcal{C}_x(S_0)$ for $x \in W(S_0)$ will be called a general point of $\mathcal{C}_x(S_0)$. (For those $S_0 \subset S$ discussed in the current article, the latter occurs only for certain homogeneous submanifolds $S_0 \subset S$ such that the isotropy at any point $x \in S_0$ acts transitively on $\mathcal{C}_x(S_0)$; cf. the proof of Proposition 3.5.)

We will use the same notation for $g \in G$ and for the differential of the action $g : S \to S$ at $x \in S$ on T_xS, mapping to $T_{gx}S$, or its projectivization $\mathbb{P}T_x(S) \to \mathbb{P}T_{gx}(S)$, e.g., $h\mathcal{C}_x(S_0)$ for $h \in P_x$ as in the following proposition.

Proposition 3.2. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and let $S_0 \subset S$ be a smooth Schubert variety. Let x be a general point of S_0 belonging to $W(S_0)$ and denote by P_x the isotropy of G at x. Assume that*

(I) *at a general point $\alpha \in \mathcal{C}_x(S_0)$, for any $h \in P_x$ sufficiently close to the identity element $e \in P_x$ and satisfying $\alpha \in h\mathcal{C}_x(S_0)$ and $T_\alpha(h\mathcal{C}_x(S_0)) = T_\alpha(\mathcal{C}_x(S_0))$ we must have $h\mathcal{C}_x(S_0) = \mathcal{C}_x(S_0)$, and*

(II) *any local deformation of $\mathcal{C}_x(S_0)$ in $\mathcal{C}_x(S)$ is induced by the action of P_x.*

Let Z be a smooth subvariety of S which is uniruled by lines of S lying on Z. If x is a general point of Z and $\mathcal{C}_x(Z)$ equals $\mathcal{C}_x(S_0)$, then S_0 is contained in Z.

By a local deformation of $\mathcal{C}_x(S_0)$ in $\mathcal{C}_x(S)$ we mean the germ of a regular family $\pi : \mathcal{A} \to \Delta$ of projective submanifolds of $\mathcal{C}_x(S)$ over the germ of unit disk Δ at 0 such that the fiber $\mathcal{A}_0$ over $0 \in \Delta$ is $\mathcal{C}_x(S_0)$. Thus, $\mathcal{A} \subset \mathcal{C}_x(S) \times \Delta$, $\pi : \mathcal{A} \to \Delta$ is the restriction of the canonical projection $\mathrm{pr}_\Delta : \mathcal{C}_x(S) \times \Delta \to \Delta$ onto the second factor, and $\pi : \mathcal{A} \to \Delta$ is abstractly a regular family of compact complex manifolds.

Proof. We will adapt the arguments in the proof of Theorem 1.2 of Hong-Mok [8] to show that S_0 is contained in Z.

By the assumption that $\mathcal{C}_x(Z)$ equals $\mathcal{C}_x(S_0)$, the locus of $\mathcal{K}$-lines in S_0 passing through x is contained in Z. Take a general $\mathcal{K}$-line C though x contained in S_0, and let C' be the subset of C consisting of all points y which are general points on Z and which belong to the open Q-orbit $W(S_0)$, where Q is the subgroup of G preserving S_0 as a set. Hence each $y \in C'$ is *a fortiori* also a general point of S_0. $C' \subset C$ is a Zariski open subset containing x. We will interpret $\mathcal{C}_y(Z)$ as a deformation of $\mathcal{C}_y(S_0)$ in $\mathcal{C}_y(S)$, as follows.

Let $H \subset Q$ be the isotropy subgroup at $x \in S_0$, and define $V := \{(y, \gamma) \in C' \times Q : \gamma x = y\}$ and denote by $\nu : V \to C'$ the canonical projection. For $y \in C' \subset W(S_0)$, there exists, by definition, $\gamma \in Q$ such that $\gamma x = y$. An element $\delta \in H$ acts on V by $\delta(y, \gamma) = (y, \gamma \circ \delta^{-1})$. Thus, $\nu : V \to C'$ is in fact a holomorphic principal bundle with structure group H. Given $y \in C'$, let $D \subset C'$ be a domain on C' biholomorphic to the unit disk Δ and containing both x and y. The holomorphic H-principal bundle $\nu : V \to C'$ is topologically trivial over D, hence holomorphically trivial on D since D is a Stein manifold, by the Oka-Grauert Principle (see Grauert [6]). Consequently, $\nu|_D : V|_D \to D$ admits a holomorphic section $\Phi : D \to V$ such that $\Phi(x) = \mathrm{id}$. Write $\Phi_y := \Phi(y)$. From $\Phi_y \in V_y$ it follows that $\Phi_y^{-1}\mathcal{C}_y(S_0) = \mathcal{C}_x(S_0)$. On the other hand, $\Phi_y^{-1}\mathcal{C}_y(Z) \subset \mathbb{P}T_x(S)$, and $\Phi_x^{-1}\mathcal{C}_x(Z) = \mathcal{C}_x(Z) = \mathcal{C}_x(S_0)$ and the regular family $\xi : \mathcal{A} \to D$, where $\mathcal{A}_w := \Phi_w^{-1}\mathcal{C}_w(Z) \subset \mathcal{C}_x(S)$ for $w \in D$, gives a deformation of $\mathcal{C}_x(S_0)$ to $\Phi_y^{-1}\mathcal{C}_y(Z)$ inside $\mathcal{C}_x(S)$. Equivalently, over y the regular family $\eta : \mathcal{B} \to D$ with fibers $\mathcal{B}_w := \Phi_y\mathcal{A}_w$ for $w \in D$ gives a deformation of $\mathcal{B}_x = \mathcal{C}_y(S_0)$ to $\mathcal{B}_y = \mathcal{C}_y(Z)$, which gives our interpretation.

By the hypothesis (II), for any point $y \in C$ sufficiently near to x, there is h in the isotropy subgroup P_y of G at y such that $\mathcal{C}_y(Z) = h\mathcal{C}_y(S_0)$. Let $\alpha \in T_yC$. Then α lies both in the affine cone $\widetilde{\mathcal{C}}_y(S_0)$ and in the affine cone $\widetilde{\mathcal{C}}_y(Z)$. By the deformation theory of rational curves, the tangent space $T_y\Sigma$ can be identified with the tangent space $T_\alpha(\widetilde{\mathcal{C}}_y(S_0))$ and also with the tangent space $T_\alpha(\widetilde{\mathcal{C}}_y(Z))$ (see Lemma 2.8 of [8]). By the hypothesis (I), we have

$\mathcal{C}_y(S_0) = \mathcal{C}_y(Z)$ and thus the locus of lines in S_0 passing through y is contained in Z.

Since S_0 is uniruled and is of Picard number 1, we have a sequence of locally closed submanifolds $\mathcal{V}^0 = \{x\} \subsetneq \mathcal{V}^1 \subsetneq \cdots \subsetneq \mathcal{V}^m$ of S_0 such that $\dim \mathcal{V}^m = \dim S_0$ and any point in $\mathcal{V}^k$ can be connected to a point in $\mathcal{V}^{k-1}$ by a line (see Section 4.3 of [10] and Section 3 of [25]). Applying the arguments repeatedly we get that a dense open subset of S_0 wiped out by lines is contained in Z. Therefore, S_0 is contained in Z. □

Remarks. (a) If S is associated to a long root (cf. Example 2 in Section 3.1), then the unipotent part of the isotropy subgroup P_x acts on $\mathcal{C}_x(S)$ trivially so that we may replace the isotropy P_x by its reductive part in Proposition 3.2.

(b) In the interpretation of $\mathcal{C}_y(Z)$ as a deformation of $\mathcal{C}_y(S_0)$ it is enough to consider points $y \in C$ sufficiently close to x in the complex topology. Since the existence of a germ of holomorphic section at x of $\nu : \Phi \to C'$ with $\Phi(0) = \mathrm{id}$ already follows from the description of the latter as an H-principal bundle, the use of the Oka-Grauert Principle is not strictly necessary.

The hypothesis (I) in Proposition 3.2 says that for a general point $\alpha \in \mathcal{C}_x(S_0)$, $h\mathcal{C}_x(S_0)$ is completely determined by $T_\alpha(h\mathcal{C}_x(S_0))$ for $h \in P_x$ sufficiently close to the identity $e \in P_x$. In the case where $S_0 \subset S$ is a homogeneous submanifold associated to a subdiagram of the marked Dynkin diagram of S, we will verify the hypothesis by using the compactness of the orbit of the Chow point corresponding to the variety $\mathcal{C}_x(S_0) \subset \mathbb{P}T_x(S_0)$ of minimal rational tangents at x under the action of P_x, as follows.

Proposition 3.3. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and $S_0 \subset S$ be a Schubert variety which is a homogeneous submanifold associated to a subdiagram of the marked Dynkin diagram of S. Let $x \in S_0$ be a point and P_x be the isotropy of the action of G at x. Let $\alpha \in \mathcal{C}_x(S_0)$ be a general point and $h \in P_x$ be a group element sufficiently close to the identity element $e \in P_x$ such that α is contained in $h\mathcal{C}_x(S_0)$ and such that $h\mathcal{C}_x(S_0)$ is tangent to $\mathcal{C}_x(S_0)$ at α. Then, $h\mathcal{C}_x(S_0) = \mathcal{C}_x(S_0)$.*

The special case of Proposition 3.3, where S is associated to a long simple root, was established in Proposition 3.6 of [8] by the determination of isotropy subgroups by means of calculations in terms of root spaces. The proof there does not apply to the short root case. Here, for the general case where $S = G/P$ associated to any simple root, we give a more geometric and conceptual proof of Proposition 3.3 based on the following lemma arising from the theory of exceptional sets of Grauert [7].

Lemma 3.4. *Let T be an irreducible and reduced compact complex space, and $\pi : \mathcal{X} \to T$ be a regular family of compact complex manifolds $X_t := \pi^{-1}(t)$*

over T equipped with a holomorphic section $\sigma : T \to \mathcal{X}$. Let Y be a complex manifold and $f : \mathcal{X} \to Y$ be a holomorphic map such that for every $t \in T$, the restriction $f\big|_{X_t} : X_t \to Y$ is an embedding. Suppose for any $t \in T$, $f(\sigma(t))$ is the same point $y_0 \in Y$ and $df\left(T_{\sigma(t)}(X_t)\right) \subset T_{y_0}(Y)$ is the same vector subspace. Then, $f(X_t)$ is the same subvariety $Y_0 \subset Y$ independent of $t \in T$.

Proof. We prove by contradiction. Suppose otherwise. Then, $Z := f(\mathcal{X}) \subset Y$ is a compact complex-analytic subvariety which contains $Z_t := f(X_t)$ as a proper subset for any $t \in T$. Writing $0 \in T$ for any base point, let g be a holomorphic function defined on some neighborhood U of y_0 on Y such that $g|_{U\cap Z_0} \equiv 0$ and such that the germ of $g|_{U\cap Z}$ at y_0 is non-zero. The latter means equivalently that the germ of $g|_{U\cap Z_t}$ at $y_0 \in Z_t$ is non-zero for a general point $t \in T$. Suppose $g|_{U\cap Z_t}$ vanishes precisely to the order $k \geq 1$ for a general point $t \in T$. Consider the holomorphic function $h = f^*g = g \circ f$ defined on $\mathcal{U} := f^{-1}(U)$. Then h vanishes on $S := \sigma(T)$. Moreover, for a general point $t \in T$, $h|_{X_t\cap\mathcal{U}}$ vanishes exactly to the order k at $x_t := \sigma(t) \in X_t$.

Fix $t \in T$. For any k tangent vectors $\xi_1, \cdots, \xi_k \in T_{x_t}(X_t)$ the k-th order partial derivative $\partial_{\xi_1}\cdots\partial_{\xi_k} h_t$ of $h_t := h|_{\mathcal{U}\cap X_t}$ at x_t is defined as follows. Taking a coordinate system $(z_1, \cdots, z_m)$ of X_t at x_t, we write $\xi_i = \sum_{j=1}^m \xi_i^j \frac{\partial}{\partial z_j}|_{x_t}$. For each i, extend ξ_i to a vector field $\widetilde{\xi_i} := \sum_{j=1}^m \xi_i^j \frac{\partial}{\partial z_j}$ defined in a neighborhood of x_t. Then the value of $\widetilde{\xi_1}\cdots\widetilde{\xi_k} h_t$ at x_t is $\sum_{j_1,\cdots j_k} \xi_1^{j_1}\cdots \xi_k^{j_k} \frac{\partial^k h_t}{\partial z_{j_1}\cdots\partial z_{j_k}}(x_t)$ because h_t vanishes to the order k at x_t. Thus $\widetilde{\xi_1}\cdots\widetilde{\xi_k} h_t(x_t)$ is independent of the choice of a coordinate system. Define $\partial_{\xi_1}\cdots\partial_{\xi_k} h_t$ by $\widetilde{\xi_1}\cdots\widetilde{\xi_k} h_t(x_t)$.

For any $t \in T$, $df|_{T_{x_t}(X_t)} : T_{x_t}(X_t) \to T_{y_0}(Z_t)$ is a linear isomorphism. and $T_{y_0}(Z_t)$ is the same as $T_{y_0}(Z_0)$. For any k vectors $\xi_1, \cdots, \xi_k \in T_{y_0}(Z_0)$, writing $\xi_i(t) \in T_{x_t}(X_t)$ for the unique tangent vector such that $df(\xi_i(t)) = \xi_i$ for $i = 1, \cdots, k$. Then $s(t) := \partial_{\xi_1(t)}\cdots\partial_{\xi_k(t)} h_t$ varies holomorphically in t. Since T is compact, s must be a constant by the maximum principle. Since $g|_{U\cap Z_0} \equiv 0$, for $x \in \mathcal{U} \cap X_0$ we have $h_0(x) = g(f(x)) = 0$. Hence, $s(0) = 0$ and $s \equiv 0$. Since $\xi_1, \cdots, \xi_k \in T_{y_0}(Z_0)$ are arbitrary, h_t must vanish to the order $\geq k+1$ at any x_t for any $t \in T$; a plain contradiction. □

We are ready to prove Proposition 3.3.

Proof of Proposition 3.3. Let $Q \subset G$ be the stabilizer subgroup of S_0 as a set. Then, $Q \subset G$ is a parabolic subgroup, i.e., $R := G/Q$ is compact (cf. Example 1 after Proposition 2.2). Identify R as a subset of $\mathrm{Chow}(S)$ containing $[S_0]$. Let $x \in S_0$, P_x be the isotropy subgroup of G at x, and define $R_x := \{[gS_0] : x \in gS_0\} \subset R$. Given $[T] \in R$ and $y \in T$, there exists $g \in G$ such that $T = gS_0$ and $g(x) = y$ since the stabilizer $Q \subset G$ of S_0 acts transitively on S_0. Specializing to the cases where $y = x$, we conclude that the compact subvariety $R_x \subset R$ is $\{[gS_0] : g \in P_x\}$. In other words, the P_x-orbit of S_0 in $\mathrm{Chow}(S)$ is compact. (It does not hold if S_0 is not homogeneous. It is here that we make use of the assumption that S_0 is homogeneous.)

Let X be the variety $\mathcal{C}_x(S)$ of minimal rational tangents of S at x and let X_0 be the variety $\mathcal{C}_x(S_0)$ of minimal rational tangents of S_0 at x. Note that neither X nor X_0 need be a rational homogeneous manifold (when $S = G/P$ is associated to a short simple root). But both X and X_0 are smooth (Proposition 3.1). The isotropy action of P_x on $\mathbb{P}T_x(S)$ leaves X invariant. Thus, P_x acts on X and the latter action naturally induces an action of P_x on the Chow variety $\mathrm{Chow}(X)$ of X. Then the P_x-orbit $\mathfrak{R}$ of $[X_0]$ in $\mathrm{Chow}(X)$ is compact because it is the image of the holomorphic mapping $[gS_0] \in R_x \mapsto \mathcal{C}_x(gS_0) = gX_0$.

Now consider the double fibration $\mathfrak{R} \xleftarrow{\pi} \mathfrak{U} \xrightarrow{f} X$, where $\mathfrak{U}$ is defined by $\{([gX_0], \alpha) \in \mathfrak{R} \times X : \alpha \in gX_0\}$ and π and f are natural projections. Let $\alpha \in X_0$ be a general point and let $\mathfrak{B} \subset \mathfrak{R}$ be the subset consisting of all $[gX_0]$ belonging to $\mathfrak{R}$ such that $\alpha \in gX_0$. Then $\mathfrak{B}$ is compact. The assignment $[gX_0] \mapsto T_\alpha(gX_0)$ defines a holomorphic map $\tau : \mathfrak{B} \to Gr(r, T_\alpha(X))$ of r-planes in the tangent space $T_\alpha(X)$, where $r = \dim X_0$.

If Proposition 3.3 fails for (S, S_0), then there exists a positive-dimensional compact irreducible reduced complex space $T \subset \mathfrak{B} \subset \mathfrak{R}$ such that $\tau(t_1) = \tau(t_2)$ for any $t_1, t_2 \in T$. By restricting π to $\pi^{-1}(T) \to T$, we get a regular family $\pi : \mathcal{X} \to T$ of compact complex manifolds $X_t = \pi^{-1}(t)$ equipped with a holomorphic section $\sigma : T \to \mathcal{X}$ corresponding to the common base point $\alpha \in X_0$. The restriction of f to X_t is a holomorphic embedding and $f(\sigma(t)) = \alpha$, and $df(T_{\sigma(t)}X_t) = T_\alpha(X_t) \subset T_\alpha X$ is the same vector space by the definition of the holomorphic map $\tau : \mathfrak{B} \to Gr(r, T_\alpha(X))$ and the definition of T. Thus, the hypotheses of Lemma 3.4 are satisfied, and it follows from the latter lemma that $X_t = X_0$ for every $t \in T$, a plain contradiction to the definition of $\mathfrak{R}$ as a subvariety of $\mathrm{Chow}(X)$. $\square$

Proposition 3.3 covers all pairs (S, S_0) in Theorem 1.1. In the remaining case where S is the symplectic Grassmannian $Gr_\omega(k, V)$ and S_0 is the Schubert variety $Gr_\omega(k, V; F_a, F_{2n-1-a})$ as in Theorem 1.2, we will give a direct argument in Section 4.2 (Proposition 4.3).

Concerning the verification of the hypothesis (II) in Proposition 3.2, we will show the following.

Proposition 3.5. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and S_0 be a smooth Schubert variety. Let $x \in S_0$ be a point and P_x be the isotropy of the action G at x. Then any local deformation of $\mathcal{C}_x(S_0)$ in $\mathcal{C}_x(S)$ is induced by the action of the isotropy subgroup P_x at x*

in the following cases:

(A) *S is associated to a long root and S_0 is a homogeneous submanifold associated to a subdiagram of the Dynkin diagram of the marked diagram of S;*

(B) *S is the symplectic Grassmannian $Gr_\omega(k, V)$ and S_0 is the Schubert variety of the form $Gr_\omega(k, V; F_a, F_b)$, where $2 \leq k < \frac{1}{2}\dim V$ and $0 \leq a \leq k-2$ and $2n-1-a \leq b \leq 2n-a$;*

(C) *S is of type (F_4, α_3) (respectively, of type (F_4, α_4)) and S_0 is a homogeneous submanifold associated to a subdiagram of the marked diagram of S corresponding to $\Lambda = \{\alpha_4\}$ or $\Lambda = \{\alpha_1\}$ (respectively, $\Lambda = \{\alpha_1\}$).*

Proof. In case (A) (respectively, (B) and (C)) it follows from Lemma 4.1 (respectively, Lemmas 4.4 and 4.6). □

Proposition 3.5 covers all pairs (S, S_0) in Theorems 1.1 and 1.2 except when

(B$'$) S is the symplectic Grassmannian $Gr_\omega(k, V)$ and S_0 is the Schubert variety of the form $Gr_\omega(k, V; F_a, F_b)$, where $0 \leq a \leq k-2$ and $k+1 \leq b \leq n$;

(C$'$) S is of type (F_4, α_3) and S_0 is a homogeneous submanifold associated to a subdiagram of the marked diagram of S corresponding to $\Lambda = \{\alpha_2\}$.

Recall that a rational homogeneous manifold S associated to a short root is of type (B_ℓ, α_ℓ), (C_ℓ, α_k) for $1 \leq k \leq \ell-1$, (F_4, α_3), (F_4, α_4), (G_2, α_1). We exclude the case that S is of type (B_ℓ, α_ℓ) or of type (C_ℓ, α_1) or of type (G_2, α_1) because in these cases S is isomorphic to the rational homogeneous manifold of type $(D_{\ell+1}, \alpha_{\ell+1})$ or of type $(A_{2\ell-1}, \alpha_1)$ or of type (B_3, α_1).

In the cases (B$'$) and (C$'$), S is a subvariety of another rational homogeneous manifold $\tilde{S} = \tilde{G}/\tilde{P}$ and S_0 may be considered as a Schubert variety of $\tilde{S}$:

(B$'$) $\tilde{S} =$ the Grassmanninan $Gr(k, V)$ consisting of k-subspaces in V;

(C$'$) $\tilde{S} =$ the rational homogeneous manifold of type (E_6, α_2).

The description of the embedding of S into $\tilde{S}$ is obvious in case (B$'$). For the description of the embedding of S into $\tilde{S}$ in case (C$'$) we will use the geometric realization of the rational homogeneous manifolds of type (E_6, α_2) and (F_4, α_3) in Section 6 of [23]. Let $\mathcal{J}_3(\mathbb{O})$ be the space of 3×3 $\mathbb{O}$-Hermitian symmetric matrices. Then $\tilde{S}$ is $\mathbb{P}\{A \wedge B : A, B \in \mathcal{J}_3(\mathbb{O}),\ \text{rank } A = \text{rank } B = 1\}$ and S is $\mathbb{P}\{A \wedge B : A, B \in \mathcal{J}_3(\mathbb{O}),\ \text{rank } A = \text{rank } B = 1 \text{ and } \mathrm{tr}A = \mathrm{tr}B = 0 \text{ and } AB = 0\}$. S_0 is $\mathbb{P}\{A \wedge B : A, B \in \mathcal{J}_3(\mathbb{O}),\ \text{rank } A = \text{rank } B = 1, \langle A, B\rangle \subset \langle \alpha, \beta, \gamma\rangle\}$ for some $\alpha, \beta, \gamma \in \mathcal{J}_3(\mathbb{O})$ with rank $\alpha =$ rank $\beta =$ rank $\gamma = 1$ and $\mathrm{tr}\,\alpha = \mathrm{tr}\,\beta = \mathrm{tr}\,\gamma = 0$ and $\alpha\beta = \beta\gamma = \gamma\alpha = 0$.

Proposition 3.6. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and let S_0 be a smooth Schubert variety as in Theorems* 1.1 *and* 1.2.

(1) *Let x be a general point of S_0. If a smooth subvariety Z of S is uniruled by lines and contains x as a general point with $\mathcal{C}_x(Z) = \mathcal{C}_x(S_0)$, then S_0 is contained in Z.*
(2) *Any local deformation of S_0 in S is induced by the action of G.*

Proof. (1) By Proposition 3.2 it suffices to check that (I) and (II) in Proposition 3.2 hold. By Propositions 3.3 and 4.3 and 3.5 we have the desired results in cases (A) and (B) and (C). In the remaining cases (B′) and (C′), S is a subvariety of another rational homogeneous manifold $\tilde{S} = \tilde{G}/\tilde{P}$ and S_0 (respectively, Z) may be considered as a Schubert variety (respectively, as a subvariety) of $\tilde{S}$: In these cases Proposition 3.6(1) follows from the results for $\tilde{S}$.

(2) By Proposition 3.1, S_0 is of Picard number 1 and it is uniruled by lines. Let $\{S_t \subset S\}_{t\in\Delta}$ be a local deformation of S_0 in S. By Kodaira stability [21] we get a regular family of minimal rational components $\mathcal{K}_t$ on S_t consisting of lines contained in S_t for $t \in \Delta$. In particular, S_t is uniruled by lines of S lying on S_t.

Since G acts transitively on S, for t sufficiently small, replacing S_t by $\gamma_t S_t$ with $\gamma_t \in G$ varying holomorphically in t, without loss of generality we may assume that x is a general point of S_t for any t. Then, the variety $\mathcal{C}_x(S_t)$ of minimal rational tangents of S_t at x is a deformation of $\mathcal{C}_x(S_0)$ in $\mathcal{C}_x(S)$. In cases (A) and (B) and (C), by Proposition 3.5, for t sufficiently small, we may assume that $\mathcal{C}_x(S_t)$ is equal to $g_t\mathcal{C}_x(S_0) = \mathcal{C}_x(g_tS_t)$ for some $g_t \in P_x$. By (1), S_t equals g_tS_0.

It remains to consider the cases (B′) and (C′). When S is the symplectic Grassmannian $Gr_\omega(k, V)$ and S_0 is the Schubert variety of the form $Gr_\omega(k, V; F_a, F_b)$, where $0 \le a \le k-2$ and $k+1 \le b \le n$, any local deformation S_t of S_0 in S can be thought of as a local deformation of S_0 in the Grassmannian $Gr(k, V)$, and thus, is again a sub-Grassmannian $Gr(k, V; W_{a,t}, W_{b,t}) = \{E \in Gr(k, V) : W_{a,t} \subset E \subset W_{b,t}\}$ for some subspaces $W_{a,t}$ and $W_{b,t}$ of V such that $W_{a,t} \subset W_{b,t}$ and $\dim W_{a,t} = a$ and $\dim W_{b,t} = b$. For a sub-Grassmannian $Gr(k, V; W_{a,t}, W_{b,t})$ to be contained in the symplectic Grassmannian $S = Gr_\omega(k, V)$, $W_{b,t}$ should be isotropic if $a \le k-2$. Therefore, $S_t = Gr(k, V; W_{a,t}, W_{b,t})$ is equal to $Gr_\omega(k, V; W_{a,t}, W_{b,t})$ and S_t equals g_tS_0 for some $g_t \in G$.

Similarly, when S is of type (F_4, α_3) and S_0 is a homogeneous submanifold associated to a subdiagram of the marked diagram of S corresponding to $\Lambda = \{\alpha_2\}$, any local deformation S_t of S_0 in S can be thought of as a local deformation of S_0 in the rational homogeneous manifold $\tilde{S}$ of type (E_6, α_2). Thus S_t is a linear space $\mathbb{P}^2_t = \mathbb{P}\{A \wedge B : A, B \in \mathcal{J}_3(\mathbb{O}), \mathrm{rank} A = \mathrm{rank} B = 1, \langle A, B\rangle \subset \langle \alpha_t, \beta_t, \gamma_t\rangle\}$ for some $\alpha_t, \beta_t, \gamma_t \in \mathcal{J}_3(\mathbb{O})$ with rank $\alpha_t =$ rank $\beta_t =$ rank $\gamma_t = 1$. Since S_t is contained in S, we have tr $\alpha_t =$ tr $\beta_t =$ tr $\gamma_t = 0$ and $\alpha_t\beta_t = \beta_t\gamma_t = \gamma_t\alpha_t = 0$. Therefore, S_t equals g_tS_0 for some $g_t \in G$. □

Proof of Theorems 1.1 *and* 1.2 *modulo results of Section* 4. By Proposition 2.2 it suffices to show that any local deformation of S_0 in S is induced by the action of G, which follows from Proposition 3.6(2). This completes the proof of Theorems 1.1 and 1.2. □

Remarks. Theorem 1.1 has already been proven if S is an irreducible Hermitian symmetric space of compact type by Bryant [4] and Hong [9]. The special cases of Theorem 1.1, where $S_0 \subset S$ is a line, follows from the description of the moduli space of minimal rational curves as given in Hwang-Mok (see [13], [14], and [16]) and Landsberg-Manivel (see [23]).

3.3. Classification of smooth Schubert varieties. In the long root case the following result shows that for the study of smooth Schubert subvarieties it suffices to consider homogeneous submanifolds. The proof involves the deformation theory of rational curves.

Proposition 3.7. *Let $S = G/P$ be a rational homogeneous manifold associated to a long simple root. Then any smooth Schubert variety in S is a homogeneous submanifold of S associated to a subdiagram of the marked Dynkin diagram of S.*

Proof. Let $S(w) = \mathrm{cl}(B.e_w)$ be a smooth Schubert variety of S, where $w \in \mathcal{W}^P$. We claim that there is a unique homogeneous submanifold S_0 of S associated to a subdiagram of the marked Dynkin diagram of S such that $e_w \in gS_0$ for some $g \in G$ and $\mathcal{C}_{e_w}(gS_0) = \mathcal{C}_{e_w}(S(w))$. Granting this, by Proposition 3.6(1), gS_0 is contained in $S(w)$. If $\dim S_0 = \dim S(w)$, then we have $gS_0 = S(w)$. If $\dim S_0 < \dim S(w)$, then $B.e_w - gS_0 \neq \emptyset$. Take a point e' in $B.e_w - gS_0$. By the same arguments as above there is $g' \in G$ such that $e' \in g'S_0$ and $g'S_0$ is contained in $S(w)$. In this way we have a (possibly singular) integrable distribution $\mathcal{W}$ on $S(w)$ with compact leaves gS_0 where g lies in some subvariety G' of G, such that for a general point $g \in G'$ and for a general point $x \in gS_0$ any line in $S(w)$ passing through x lies in gS_0. Note that the singular locus of any meromorphic foliation on a complex manifold is necessarily of codimension ≥ 2. From the definition, at a general point $x \in S(w)$, the variety of minimal rational tangents $\mathcal{C}_x(S(w))$ is contained in $\mathbb{P}\mathcal{W}_x$. By Proposition 12 of [10], the existence of $\mathcal{W}$ contradicts with the fact that the uniruled projective manifold $S(w)$ is of Picard number 1. ($\mathcal{W}_x$ is spanned by minimal rational tangents at a general point x in [10], but the same applies when $\mathcal{W}_x$ contains the linear span of minimal rational tangents.)

By argument by contradiction, we conclude that $\dim S_0$ is equal to $\dim S(w)$, hence $S(w)$ is equal to gS_0 for some $g \in G$.

We will prove the claim by induction. Let L_w be the reductive part of the isotropy of G at e_w. Then the variety X of minimal rational tangents of S at e_w is a Hermitian symmetric space of compact type on which L_w acts transitively (see Example 2 in Section 3.1). By the smoothness of $S(w)$, the variety X_0 of minimal rational tangents at $e_w \in S(w)$ is smooth (Proposition 3.1) and $B \cap L_w$ acts on X_0 invariantly. Since w is an element of $\mathcal{W}^P$, the Borel subgroup $w(B \cap L)$ of $w(L) = L_w$ is contained in $B \cap w(L) = B \cap L_w$ and thus acts on X_0 invariantly.

If L_w is simple, then X is an irreducible Hermitian symmetric space of compact type, and X_0 is a Schubert variety of X because X_0 is smooth and the Borel subgroup $w(B \cap L)$ of $L_w = w(L)$ acts on X_0 invariantly (Proposition 2.1). The dimension of X_0 is smaller than the dimension of S and thus, by the inductive assumption, X_0 is a homogeneous submanifold of X associated to a subdiagram of the marked Dynkin diagram of X. Therefore, there is a unique homogeneous submanifold S_0 of S associated to a subdiagram of the marked Dynkin diagram of S such that $e_w \in gS_0$ for some $g \in G$ and the variety of minimal rational tangents of gS_0 at e_w is equal to X_0.

If L_w is not simple, then L_w is the product $L_1 \times L_2$ or $L_1 \times L_2 \times L_3$ of its simple factors L_i, and X is the product $X_1 \times X_2$ or $X_1 \times X_2 \times X_3$ of irreducible Hermitian symmetric spaces X_i, and $w(B \cap L)$ is the product $B_1 \times B_2$ or $B_1 \times B_2 \times B_2$ of Borel subgroups B_i of L_i. (The fact that there are at most three factors, L_i follows from a case-by-case checking using Dynkin diagrams of the varieties of minimal rational tangents as hightest weight orbits (cf. Hwang-Mok [13], Section 1) corresponding to the fact that by removing 1 simple node from the Dynkin diagram, there remains at most 3 connected maximal subdiagrams. See Example 2 of Section 3.1.) Since $w(B \cap L)$ acts on X_0 invariantly and X_0 is smooth, X_0 is the product $X_{1,0} \times X_{2,0}$ or $X_{1,0} \times X_{2,0} \times X_{3,0}$ of submanifolds $X_{i,0}$ of X_i, on which B_i acts invariantly. Applying the same arguments to each factor, we get that X_0 is the product $X_{1,0} \times X_{2,0}$ or $X_{1,0} \times X_{2,0} \times X_{3,0}$ of homogeneous submanifolds $X_{i,0}$ of X_i associated to a subdiagram of the marked Dynkin diagram of X_i. Therefore, there is a unique homogeneous submanifold S_0 of S associated to a subdiagram of the marked Dynkin diagram of S such that $e_w \in gS_0$ for some $g \in G$ and the variety of minimal rational tangents of gS_0 at e_w is equal to X_0. This completes the proof of the claim and Proposition 3.7 follows from the claim by the arguments at the beginning. □

Remarks. Proposition 3.7 is already known for the case where S is an irreducible Hermitian symmetric space of compact type (see Brion-Polo [3]).

In general, given a Schubert variety one can determine its smoothness by checking whether a reduced expression of the element in the Weyl group corresponding to it avoids a list of patterns (see Billey-Postnikov [1]). It could be possible to prove Proposition 3.7 by using this method.

4. Local characterizations of varieties of minimal rational tangents

In this section we will verify the hypothesis (II) in Proposition 3.2 for every pair (S, S_0) belonging to (A), (B) or (C) in Proposition 3.5, consisting of a rational homogeneous manifold $S = G/P$ of Picard number 1 and a smooth Schubert variety $S_0 \subset S$. Especially, we will prove that any local deformation of the variety $\mathcal{C}_x(S_0)$ of minimal rational tangents of S_0 at x in the variety $\mathcal{C}_x(S)$ of minimal rational tangents of $S = G/P$ at x is induced by the action of the isotropy subgroup P_x of G at x for a general point $x \in S_0$. The point is that the variety of minimal rational tangents of S has a smaller dimension and a simpler structure than S itself, so that we can get the desired rigidity either by using induction (long root case) or by a simpler argument (short root case).

4.1. The case where S is associated to a long root.

Lemma 4.1. *Let $S = G/P$ be a rational homogeneous manifold associated to a long simple root and let $S_0 = G_0/P_0$ be a homogeneous submanifold associated to a subdiagram of the marked Dynkin diagram of S. Let $x \in S_0$. Then any local deformation of $X_0 := \mathcal{C}_x(S_0)$ in $X := \mathcal{C}_x(S)$ is induced by the action of the isotropy subgroup of G at x.*

Proof. If S is associated to a long root, then varieties of minimal rational tangents are Hermitian symmetric spaces of lower dimension (see Example 2 in Section 3.1). Thus, Lemma 4.1 follows essentially from an inductive argument.

To be more precise, first, consider the case where X is the product $\mathbb{P}^{k-1} \times \mathbb{P}^m$. Then X_0 is the product $\mathcal{B}_1 \times \mathcal{B}_2$ of two projective spaces $\mathcal{B}_1 = \mathbb{P}^{k'-1}$ and $\mathcal{B}_2 = \mathbb{P}^{m'}$. Thus any local deformation X_t of X_0 is again a product $\mathcal{B}_{1,t} \times \mathcal{B}_{2,t}$ of two projective spaces of dimension $k' - 1$ and m', respectively. A priori, this product structure may not be the same as the product structure of $X = \mathbb{P}^{k-1} \times \mathbb{P}^m$. To prove that this is the case, let $\pi_1 : X \to \mathbb{P}^{k-1}$ and $\pi_2 : X \to \mathbb{P}^m$ be the projections. Take a holomorphic section $y_t \in \mathcal{B}_{2,t}$ over t and take a small neighborhood of $y_0 \in \mathbb{P}^m$. Then for any t in a small neighborhood of 0 we may think of π_2 restricted to $\mathcal{B}_{1,t} \times \{y_t\}$ as a holomorphic function defined on a compact manifold $\mathcal{B}_{1,t} \times \{y_t\}$, which should be constant. Similarly, for any holomorphic section $x_t \in \mathcal{B}_{1,t}$ over t, π_1 is constant on $\{x_t\} \times \mathcal{B}_{t,2}$ for small

t. So the product structure of $X_t = \mathcal{B}_{t,1} \times \mathcal{B}_{t,2}$ is the same as the product structure of $X = \mathbb{P}^{k-1} \times \mathbb{P}^m$. Therefore there is $\ell_t \in L = SL(k) \times SL(m+1)$ such that $X_t = \ell_t X_0$ for any t in a small neighborhood of 0. By Proposition 3.2(II), any local deformation of a sub-Grassmannian S_0 of a Grassmannian S is obtained by the action of the automorphism group $\mathrm{Aut}(S)$ of S.

If X is the image $\nu(\mathbb{P}(E))$ of the Veronese embedding $\nu : P(E) \to \mathbb{P}(S^2E)$ and $X_0 = \nu(\mathbb{P}(F))$ is the image of a projective space, then, since any local deformation of a projective subspace in a projective space is again a projective space, any local deformation of X_0 is of the form $\nu(\mathbb{P}(F_t))$ for some subspace F_t of E. By Proposition 3.2(II), any local deformation of a Lagrangian sub-Grassmannian S_0 of a Lagrangian Grassmannian S is obtained by the action of the automorphism group $\mathrm{Aut}(S)$ of S.

Now consider the case where X is an irreducible Hermitian symmetric space of compact type. If X is either a Grassmannian or a Lagrangian Grassmannian, then we have already proved that any local deformation of X_0 in X can be obtained by the action of $\mathrm{Aut}(X)$. Otherwise, the variety of minimal rational tangents of X at the base point is again an irreducible Hermitian symmetric space of compact type, and thus we can use induction on the dimension of X and Proposition 3.2(II) to get that any local deformation of X_0 in X is induced by the action of $\mathrm{Aut}\,(X)$. If X is the product $X_1 \times X_2$ or $X_1 \times X_2 \times X_3$, we apply the same argument as in the case where X is the product of two linear spaces to show that any local deformation of X_0 is again a product of two or three submanifolds. Finally, we apply the arguments listed above to each factor of the product to obtain the desired result. □

This completes the proof of Theorem 1.1 in the case (A) in Proposition 3.5 (cf. Section 3.2).

4.2. The case where S is associated to a short root.

4.2.1. The C_n-case. Let V be a complex vector space of dimension $2n$ equipped with a non-degenerate skew-symmetric bilinear form ω. Take a basis $\{e_1, \cdots, e_{2n}\}$ of V such that $\omega(e_{n-i}, e_{n+i+1}) = -\omega(e_{n+i+1}, e_{n-i}) = 1$ for $1 \le i \le n$ and all other $\omega(e_i, e_j)$ are zero. Define $F_j \subset V$ by the subspace generated by $e_1, \cdots, e_j$ for $1 \le j \le 2n$. Then $F_{n-i}^{\perp} = F_{n+i}$ for $1 \le i \le n$ and we get an isotropic flag $F_\bullet : 0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = V$. The subgroup of $G = \mathbb{P}Sp(V)$ consisting of elements fixing this flag is a Borel subgroup B of G.

Let $2 \le k \le n-1$. Consider the symplectic Grassmannian $S = Gr_\omega(k, V)$ of isotropic k-planes of V. Schubert varieties of $Gr_\omega(k, V)$ are indexed by the set of ordered subsets $\{p_1 < p_2 < \cdots < p_k\}$ of $\{1, 2, \cdots, 2n\}$ satisfying $p_i + p_j \neq 2n+1$. For such an ordered subset $\{p_1 < p_2 < \cdots < p_k\}$,

$$\{E \in Gr_\omega(k, V) : \dim(E \cap F_{p_j}) = j, \text{ for } 1 \le j \le k\}$$

is a B-orbit and its closure

$$\{E \in Gr_\omega(k, V) : \dim(E \cap F_{p_j}) \geq j, \text{ for } 1 \leq j \leq k\}$$

is a Schubert variety.

For $0 \leq a < k < b \leq 2n - a$, put $Gr_\omega(k, V; F_a, F_b) := \{E \in Gr_\omega(k, V) : F_a \subset E \subset F_b\}$. Then $Gr_\omega(k, V; F_a, F_b)$ is a homogeneous manifold associated to a subdiagram of the marked Dynkin diagram of S if and only if $k < b \leq n$ or $b = 2n - a$. If $b = 2n - a - 1$, then $Gr_\omega(k, V; F_a, F_b)$ is not homogeneous, but is a smooth Schubert variety (see Mihai [24]).

For any two vector spaces E and F, denote by $\widetilde{\mathbf{S}}(E \otimes F)$ the cone of nonzero decomposable vectors in $E \otimes F$. Let $\varpi : E \otimes F \backslash \{0\} \to \mathbb{P}(E \otimes F)$ denote the canonical projection map. Then $\varpi(\widetilde{\mathbf{S}}(E \otimes F))$, the projectivization of the cone $\widetilde{\mathbf{S}}(E \otimes F)$, is isomorphic to the product $\mathbb{P}(E) \times \mathbb{P}(F)$. Let $\mathrm{pr}_1 : \mathbb{P}(E) \times \mathbb{P}(F) \to \mathbb{P}(E)$ denote the projection map to the first factor. Then the composition $\mathrm{pr}_1 \circ \varpi : \widetilde{\mathbf{S}}(E \otimes F) \to \mathbb{P}(E)$, which maps $e \otimes f \in \widetilde{\mathbf{S}}(E \otimes F)$ to $[e] \in \mathbb{P}(E)$, can be thought of as a vector bundle on $\mathbb{P}(E)$ which is trivial.

Lemma 4.2. *Let S be the symplectic Grassmannian $Gr_\omega(k, V)$ and let S_0 be a smooth Schubert variety of the form $Gr_\omega(k, V; F_a, F_b)$, where $0 \leq a \leq k-2$ and ($k + 1 \leq b \leq n$ or $2n - 1 - a \leq b \leq 2n - a$).*

(1) *The variety of minimal rational tangents $\mathcal{C}_{[E]}(S)$ at $[E] \in S$ is the projectivization of the cone*

$$\{e^* \otimes v \in E^* \otimes (V/E) : \omega(v, \cdot\,) \in \mathbb{C}e^*\} \backslash \{0\}$$

and is the projectivization of the vector bundle $\mathcal{E} = \mathcal{F} \oplus \mathcal{L}$ on $\mathbb{P}(E^*)$, where $\mathcal{F}$ is the trivial vector bundle on $\mathbb{P}(E^*)$ defined by the cone $\widetilde{\mathbf{S}}(E^* \otimes (E^\perp/E))$ and $\mathcal{L} = \mathcal{O}(-1)$ is a choice of a direct summand of $\mathcal{F}$ in $\mathcal{E}$.

(2) *The variety of minimal rational tangents $\mathcal{C}_{[E]}(S_0)$ at a general point $[E] \in S_0$ is the projectivization of the cone*

$$\{e^* \otimes v \in (E/F_a)^* \otimes (F_b/E) : \omega(v, \cdot\,) \in \mathbb{C}e^*\} \backslash \{0\}.$$

If $k + 1 \leq b \leq n$, then $\mathcal{C}_{[E]}(S_0)$ is the projectivization of the trivial vector bundle $\mathcal{F}_0$ on $\mathbb{P}((E/F_a)^*)$ defined by the cone $\widetilde{\mathbf{S}}((E/F_a)^* \otimes (F_b/E))$. If $2n - 1 - a \leq b \leq 2n - a$, then $\mathcal{C}_{[E]}(S_0)$ is the projectivization of the vector bundle $\mathcal{E}_0 = \mathcal{F}_0 \oplus \mathcal{L}$, where $\mathcal{F}_0$ is the trivial vector bundle on $\mathbb{P}((E/F_a)^*)$ defined by the cone $\widetilde{\mathbf{S}}((E/F_a)^* \otimes ((F_b \cap E^\perp)/E))$ and $\mathcal{L} = \mathcal{O}(-1)$ is a choice of a direct summand of $\mathcal{F}_0$ in $\mathcal{E}_0$.

Proof. (1) Proposition 3.2.1 of Hwang-Mok [16].

(2) The proof is essentially the same as (1) except that we should take a general point $[E]$. Consider the case when $a = 0$ and $b = 2n-1$. For $[E] \in S_0$,

$$T_{[E]}S_0 = \{\varphi : E \to F_{2n-1}/E : \omega(\varphi(v_1), v_2) + \omega(v_1, \varphi(v_2)) = 0\}.$$

Take $[E] \in S_0$ such that $E \cap F_1 = 0$. Then the dimension of $E^\perp \cap F_{2n-1}$ is $2n - k - 1$ and $F_{2n-1}/(E^\perp \cap F_{2n-1})$ is isomorphic to E^*. Under the map

$$\psi : E^* \otimes (F_{2n-1}/E) \to E^* \otimes (F_{2n-1}/(E^\perp \cap F_{2n-1})) \simeq E^* \otimes E^*,$$

$T_{[E]}S_0$ is given by the inverse image $\psi^{-1}(S^2E^*)$ of $S^2E^* \subset E^* \otimes E^*$.

The variety $\mathcal{C}_{[E]}(S_0)$ of minimal rational tangents is the variety of decomposable vectors in $T_{[E]}S_0$. If a decomposable vector $\varphi = e^* \otimes v$ is contained in $T_{[E]}S_0$, then $\omega(v, w_2) = -\omega(e, (e^* \otimes v)(w_2)) = (-\omega(e, v))e^*(w_2)$ for all $w_2 \in E$. Conversely, if $\omega(v, \cdot\,) = \lambda e^*$ for some $\lambda \in \mathbb{C}$, then $\varphi = e^* \otimes v$ is contained in $T_{[E]}S_0$. Thus

$$\mathcal{C}_{[E]}(S_0) = \mathbb{P}\{e^* \otimes v \in E^* \otimes (F_{2n-1}/E) : \omega(v, \cdot\,) \in \mathbb{C}e^*\}.$$

By the map $e^* \otimes v \mapsto e^*$, $\mathcal{C}_{[E]}(S_0)$ becomes the projectivization of the vector bundle $\mathcal{E}_0 = \mathcal{F}_0 \oplus \mathcal{L}$ on $\mathbb{P}(E^*)$, where $\mathcal{F}_0$ is the trivial vector bundle on $\mathbb{P}(E^*)$ defined by the cone $\widetilde{\mathbf{S}}(E^* \otimes ((E^\perp \cap F_{2n-1})/E))$ and $\mathcal{L}$ is a choice of a line bundle of the form $\{e^* \otimes v_e : e^* \in E^*\}$ with $\omega(e, v_e) \neq 0$. Then we have $\mathcal{L} = \mathcal{O}(-1)$ on $\mathbb{P}(E^*)$.

In general, by the same arguments, we have

$$\mathcal{C}_{[E]}(S_0) = \mathbb{P}\{e^* \otimes v \in (E/F_a)^* \otimes (F_b/E) : \omega(v, \cdot\,) \in \mathbb{C}e^*\}.$$

□

Proposition 4.3. *Let S be the symplectic Grassmannian $Gr_\omega(k, V)$ and let S_0 be a smooth Schubert variety of the form $Gr_\omega(k, V; F_a, F_{2n-1-a})$. Let $[E]$ be a general point of S_0 and let α be a general point in $\mathcal{C}_{[E]}(S_0)$. If h is an element of the isotropy subgroup $P_{[E]}$ of $\mathbb{P}Sp(V)$ at $[E]$ such that $\alpha \in h\mathcal{C}_{[E]}(S_0)$ and if $h\mathcal{C}_{[E]}(S_0)$ is tangent to $\mathcal{C}_{[E]}(S_0)$ at α, then $h\mathcal{C}_{[E]}(S_0)$ is equal to $\mathcal{C}_{[E]}(S_0)$.*

Proof. Assume that $a = 0$. Let $\widetilde{\mathcal{C}}_{[E]}(S_0)$ be the pre-image of $\mathcal{C}_{[E]}(S_0)$ under the projection $T_{[E]}(S_0)\backslash\{0\} \to \mathbb{P}(T_{[E]}(S_0))$. Then the affine cone

$$\widetilde{\mathcal{C}}_{[E]}(S_0) = \{e^* \otimes v \in E^* \otimes (F_{2n-1}/E) : \omega(v, \cdot\,) \in \mathbb{C}e^*\}\backslash\{0\}$$

with the projection $\widetilde{\mathcal{C}}_{[E]}(S_0) \to \mathbb{P}(E^*)$ defined by $e^* \otimes v \mapsto [e^*]$ is a vector bundle $\mathcal{E}$ on $\mathbb{P}(E^*)$. The fiber $\mathcal{E}_{e^*}$ at $[e^*]$ is $(F_{2n-1} \cap E^\perp)/E + \langle v_e \rangle$, where v_e is an element in F_{2n-1} such that $\omega(v_e, \cdot\,) = e^*$ on E (v_e is unique up to $F_{2n-1} \cap E^\perp$). Let $\alpha_s = e_s^* \otimes v_s$ be a curve in $\widetilde{\mathcal{C}}_{[E]}(S_0)$ such that $e_0 = e$ and $v_0 = v$. Then $v_s \in (F_{2n-1} \cap E^\perp)/E + \langle v_{e_s} \rangle$, where v_{e_s} is an element in F_{2n-1}

such that $\omega(v_{e_s}, \cdot\,) = e_s^*$ on E. From $\omega(\frac{d}{ds}|_{s=0}v_{e_s}, \cdot\,) = \frac{d}{ds}|_{s=0}e_s^*$, it follows that the tangent space of $\widetilde{\mathcal{C}}_{[E]}(S_0)$ at $\alpha = e^* \otimes v$ is

$$\begin{cases} e^* \otimes (F_{2n-1} \cap E^\perp)/E + \{f^* \otimes v + e^* \otimes v_f : f \in E\} \\ \qquad\qquad\qquad\qquad \text{if } v \in v_e + (F_{2n-1} \cap E^\perp)/E, \\ e^* \otimes (F_{2n-1} \cap E^\perp)/E + E^* \otimes v + \langle e^* \otimes v_e \rangle \\ \qquad\qquad\qquad\qquad \text{if } v \in (F_{2n-1} \cap E^\perp)/E. \end{cases}$$

Note that F_{2n-1} is generated by $F_{2n-1} \cap E^\perp$ and $\langle v_f : f \in E\rangle$. Thus we can recover F_{2n-1} from the tangent space of $\widetilde{\mathcal{C}}_{[E]}(S_0)$ at $\alpha = e^* \otimes v$ if $v \in v_e + (F_{2n-1} \cap E^\perp)/E$, or equivalently, $\alpha = e^* \otimes v$ is a general point of $\widetilde{\mathcal{C}}_{[E]}(S_0)$.

Let $h \in P_{[E]}$ and $\alpha = e^* \otimes v \in h\widetilde{\mathcal{C}}_{[E]}(S_0)$. Then, since $h.E = E$ and $v \notin E^\perp/E$, the tangent space of

$$h\widetilde{\mathcal{C}}_{[E]}(S_0) = \{e^* \otimes v \in E^* \otimes (hF_{2n-1}/E) : \omega(v, \cdot\,) \in \mathbb{C}e^*\}\backslash\{0\}$$

at $\alpha = e^* \otimes v$ is

$$e^* \otimes (hF_{2n-1} \cap E^\perp) + \{f^* \otimes v + e^* \otimes v_f^h : f \in E\}$$

where, for $f \in E$, v_f^h is an element in hF_{2n-1} such that $\omega(v_f^h, \cdot\,) = f^*$ on E. Then hF_{2n-1} is generated by $hF_{2n-1} \cap E^\perp$ and $\langle v_f^h : f \in E\rangle$ and thus we can recover hF_{2n-1} from the tangent space of $h\widetilde{\mathcal{C}}_{[E]}(S_0)$ at $\alpha = e^* \otimes v$. Therefore, if α is a general point in $\widetilde{\mathcal{C}}_{[E]}(S_0)$ and if $h\widetilde{\mathcal{C}}_{[E]}(S_0)$ is tangent to $\widetilde{\mathcal{C}}_{[E]}(S_0)$ at α, then $h\widetilde{\mathcal{C}}_{[E]}(S_0)$ is equal to $\widetilde{\mathcal{C}}_{[E]}(S_0)$.

The proof for the case where $a \neq 0$ will be similar. $\square$

Remarks. (1) There is a positive dimensional family h_t of elements in $\mathbb{P}Sp(V)$ such that $h_t.E = E$ and $(h_tF_{2n-1} \cap E^\perp)/E = (F_{2n-1} \cap E^\perp)/E$. Then $h_t\widetilde{\mathcal{C}}_{[E]}(S_0)$ is tangent to $\widetilde{\mathcal{C}}_{[E]}(S_0)$ along the hyperplane $\{e^* \otimes v : e^* \in E^*, v \in (F_{2n-1} \cap E^\perp)/E\}$. Therefore, it is necessary to assume that α is a general point.

(2) $P_{[E]}$-orbit of $\mathcal{C}_{[E]}(S_0)$ in the Chow variety of $\mathcal{C}_{[E]}(S)$ is not closed, while it is closed in the case where S_0 is a homogeneous submanifold associated to a subdiagram of the marked diagram of S (Proposition 3.3). Thus we cannot apply the arguments in the proof of Proposition 3.3 directly.

Lemma 4.4. *Let S be the symplectic Grassmannian $Gr_\omega(k, V)$ and let S_0 be a smooth Schubert variety of the form $Gr_\omega(k, V; F_a, F_b)$, where $0 \leq a \leq k-2$ and $2n-1-a \leq b \leq 2n-a$. Let $[E]$ be a general point of S_0. Then any local deformation of $\mathcal{C}_{[E]}(S_0)$ in $\mathcal{C}_{[E]}(S)$ is induced by the action of the isotropy subgroup of G at $[E]$.*

Proof. If $b = 2n - a$, then $F_b = F_a^\perp$. Let E be the isotropic subspace generated by $e_1, \cdots, e_k$ so that $F_a \subset E$. Then we have $E^\perp \subset F_a^\perp$. The variety X of minimal rational tangents of S at $[E]$ is the projectivization of the vector bundle $\mathcal{E} = \mathcal{F} \oplus \mathcal{L}$, where $\mathcal{F}$ is the trivial vector bundle on $\mathbb{P}(E^*)$ defined by the cone $\widetilde{\mathbf{S}}(E^* \otimes (E^\perp/E))$ and $\mathcal{L}$ is a choice of a direct summand of $\mathcal{F}$ in $\mathcal{E}$, and the variety X_0 of minimal rational tangents at $[E]$ is the projectivization of the vector bundle $\mathcal{E}_0 = \mathcal{F}_0 \oplus \mathcal{L}$ on $\mathbb{P}((E/F_a)^*)$, where $\mathcal{F}_0$ is the trivial bundle on $\mathbb{P}((E/F_a)^*)$ defined by the cone $\widetilde{\mathbf{S}}((E/F_a)^* \otimes (E^\perp/E))$. Then $\mathcal{F}_0$ is the restriction of the trivial bundle $\mathcal{F}$ to $\mathbb{P}((E/F_a)^*)$.

Any local deformation X_t of X_0 in X is the projectivization of a vector bundle $\mathcal{O}^{2n-2k} \oplus \mathcal{O}(-1)$ on $\mathbb{P}_t^{k-a-1} \subset \mathbb{P}(E^*)$ and thus is the projectivization of the vector bundle $\mathcal{E}_t = \mathcal{E}|_{\mathbb{P}_t^{k-a-1}}$. Hence, there is ℓ_t in the reductive part $L = SL(E^*) \times Sp(E^\perp/E)$ of the isotropy of G at $[E]$ such that $X_t = \ell_t X_0$.

If $b = 2n - a - 1$, then F_b is a hyperplane of $F_a^\perp$. We will prove the lemma for $a = 0$ (and $b = 2n - 1$). The proof will be similar to this case for $a > 0$. Let E be the isotropic subspace generated by $e_{n-k+1}, \cdots, e_n$. The variety X of minimal rational tangents of S at $[E]$ is the projectivization of the vector bundle $\mathcal{E} = \mathcal{F} \oplus \mathcal{L}$ on $\mathbb{P}(E^*)$, where $\mathcal{F}$ is the trivial vector bundle on $\mathbb{P}(E^*)$ defined by the cone $\widetilde{\mathbf{S}}(E^* \otimes (E^\perp/E))$ and $\mathcal{L} = \mathcal{O}(-1)$ is a choice of a direct summand. The variety X_0 of minimal rational tangents of S_0 at $[E]$ is the projectivization of the vector bundle $\mathcal{E}_0 = \mathcal{F}_0 \oplus \mathcal{L}$ on $\mathbb{P}(E^*)$, where $\mathcal{F}_0$ is the trivial vector bundle on $\mathbb{P}(E^*)$ defined by the cone $\widetilde{\mathbf{S}}(E^* \otimes ((F_{2n-1} \cap E^\perp)/E))$. Here, we may take the same direct summand $\mathcal{L}$.

Any local deformation X_t of X_0 is the projectivization of a subbundle $\mathcal{E}_t$ of $\mathcal{E}$ of corank 1. Since $\mathcal{E}_0 = \mathcal{O}^{2n-2k-1} \oplus \mathcal{O}(-1)$, any local deformation $\mathcal{E}_t$ is also $\mathcal{O}^{2n-2k-1} \oplus \mathcal{O}(-1)$. Let $\mathcal{F}_t$ be the trivial factor $\mathcal{O}^{2n-2k-1}$ of $\mathcal{E}_t$. Then $\mathcal{F}_t$ is a subbundle of $\mathcal{F}$ of corank one and thus $\mathcal{F}_t$ is defined by the cone $\widetilde{\mathbf{S}}(E^* \otimes ((F_t \cap E^\perp)/E))$ for some subspace F_t of V of codimension 1 which contains E. For small t there is $g_t \in Sp(V)$ such that $g_t E = E$ and $g_t F_{2n-1} = F_t$ because the rank of the symplectic form ω restricted on F_{2n-1} is maximal, and thus we have $X_t = g_t X_0$ with g_t in the isotropy of G at $[E]$.

More precisely, we have $E^\perp = \langle e_1, \cdots, e_n, e_{n+k+1}, \cdots e_{2n}\rangle$ and $F_{2n-1} \cap E^\perp = \langle e_1, \cdots, e_n, e_{n+k+1}, \cdots e_{2n-1}\rangle$. Up to the action of $Sp(E^\perp/E)$, we have $F_t \cap E^\perp = F_{2n-1} \cap E^\perp$, so that $F_t = \langle e_1, \cdots, e_n, e_{n+1,t}, \cdots, e_{n+k,t}, e_{n+k+1}, \cdots, e_{2n-1}\rangle$ and $\{e_n^* \otimes e_{n+1,t}, \cdots, e_{n-k+1}^* \otimes e_{n+k,t}\}$ defines a direct summand $\mathcal{L}_t$ of $\mathcal{F}_t$ in $\mathcal{E}_t$, where $e_{n+i,t} = e_{n+i} + c_{n+i,t} e_{2n}$ for some $c_{n+i,t} \in \mathbb{C}$ for $1 \leq i \leq k$. Put $e_{1,t} = e_1 - c_{n+1,t} e_n - \cdots - c_{n+k,t} e_{n-k+1}$. Then $\{e_{1,t}, e_2, \cdots, e_n, e_{n+1,t}, \cdots, e_{n+k,t}, e_{n+k+1}, \cdots, e_{2n-1}\}$ spans F_t and hence there is $g_t \in Sp(V)$ which fixes $e_2, \cdots, e_n, e_{n+k+1}, \cdots, e_{2n-1}$ and which sends e_1 to $e_{1,t}$ and e_{n+i} to $e_{n+i,t}$ for $1 \leq i \leq k$. □

This completes the proof of Theorems 1.1 and 1.2 in the case (B) in Proposition 3.5 (cf. Section 3.2).

4.2.2. The F_4-case. Let G be a connected simple algebraic group of type F_4 and let $S = G/P$ be a rational homogeneous manifold associated to a short root α_k. If $k = 4$, then S is a smooth hyperplane section of the rational homogeneous manifold of type (E_6, α_1). If $k = 3$, then S is the space of G-invariant lines in the rational homogeneous space of type (F_4, α_4).

Lemma 4.5. *Let G be a connected simple algebraic group of type F_4 and let $S = G/P$ be a rational homogeneous manifold associated to a short root.*

(1) *The variety $X := \mathcal{C}_x(S)$ of minimal rational tangents of S at $x \in S$ is given as follows:*

(I) *The (F_4, α_3)-case: X is the projectivization of the cone*

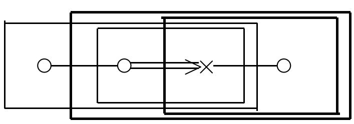
$$\{e^* \otimes q + (e_1^* \wedge e_2^*) \otimes q^2 : e \wedge e_1 \wedge e_2 = 0, e, e_1, e_2 \in E, q \in Q\}\backslash\{0\}$$

in $(E^* \otimes Q) \oplus (\bigwedge^2 E^* \otimes S^2 Q)$, where E is a complex vector space of dimension 3 and Q is a complex vector space of dimension 2. In other words, X is the Grassmannian bundle of 2-planes of the vector bundle $\mathcal{E}^*$ on $\mathbb{P}^1 = \mathbb{P}(Q)$, where $\mathcal{E}$ is a vector bundle of rank 4 which splits as $\mathcal{O}(1)^3 \oplus \mathcal{O}$.

(II) *The (F_4, α_4)-case: X is a smooth hyperplane section of the spinor variety of type (D_5, α_5) embedded in the spin representation (D_5, λ_5).*

(2) *Homogeneous submanifolds S_0 of S associated to subdiagrams of the marked diagram of S and the varieties $X_0 := \mathcal{C}_x(S_0)$ of minimal rational tangents of S_0 at $x \in S_0$ are given as follows:*

(I) *The (F_4, α_3)-case.*



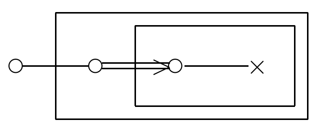
(a) $\Lambda = \{\alpha_1, \alpha_4\}$, $S_0 = \mathbb{P}^3$,
$X_0 = \mathbb{P}^2 \subset$ a fiber of $X \to \mathbb{P}(Q)$.

(b) $\Lambda = \{\alpha_4\}$, $S_0 = (B_3, \alpha_3)$,
$X_0 = Gr(2,4) =$ a fiber of $X \to \mathbb{P}(Q)$.

(c) $\Lambda = \{\alpha_2\}$, $S_0 = \mathbb{P}^2$,
$X_0 = \mathbb{P}^1 = \mathbb{P}(Q)$.

(d) $\Lambda = \{\alpha_1\}$, $S_0 = (C_3, \alpha_2)$,
$X_0 = \mathbb{P}^2$-bundle over $\mathbb{P}^1 = \mathbb{P}(Q)$
$= X \cap \mathbb{P}((F^* \otimes Q) \oplus (\bigwedge^2 F^* \otimes S^2 Q))$
for some subspace $F \subset E$ of dimension 2.

(II) *The (F_4, α_4)-case.*



(a) $\Lambda = \{\alpha_2\}$, $S_0 = \mathbb{P}^2$,
$X_0 = \mathbb{P}^1$.

(b) $\Lambda = \{\alpha_1\}$, $S_0 = \mathbb{P}^5$,
$X_0 = \mathbb{P}^4$.

Here, Λ is the set consisting of the nodes in $\mathcal{D}(G)\backslash\mathcal{D}(G_0)$ which are connected to the subdiagram $\mathcal{D}(G_0)$ by an edge.

Proof. For the description of the variety X of minimal rational tangents of S at x, see Hwang-Mok [16] in the case (I) and see [15] in the case (II). The description of the variety X_0 of minimal rational tangents of S_0 at x is straightforward. □

Lemma 4.6. *Let G be a connected simple algebraic group of type F_4 and let $S = G/P$ be a rational homogeneous manifold associated to a short root. Let S_0 be a homogeneous submanifold of S associated to a subdiagram of the marked diagram of S. Assume that S is of type (F_4, α_3) and Λ is either $\{\alpha_4\}$ or $\{\alpha_1\}$, or, S is of type (F_4, α_4) and Λ is $\{\alpha_1\}$. Then any local deformation of $X_0 := \mathcal{C}_x(S_0)$ in $X := \mathcal{C}_x(S)$ is induced by the action of the isotropy subgroup of G at $x \in S$.*

Proof. The proof is similar to the proof of Lemma 4.1 and of Lemma 4.4. □

This completes the proof of Theorem 1.1 in the case (C) in Proposition 3.5 (cf. Section 3.2).

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Department of Mathematical Sciences, Research Institute of Mathematics, Seoul National University, 599 Gwanak-ro Gwanak-gu Seoul 151-742, Korea
E-mail address: jhhong00@snu.ac.kr

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
E-mail address: nmok@hkucc.hku.hk