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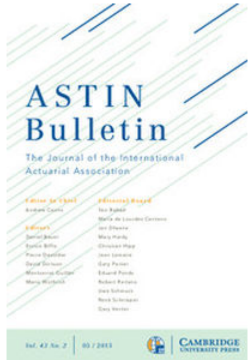

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SOME DISTRIBUTIONAL PROPERTIES OF A CLASS OF COUNTING DISTRIBUTIONS WITH CLAIMS ANALYSIS APPLICATIONS

BY

GORDON E. WILLMOT AND JAE-KYUNG WOO

ABSTRACT

We discuss a class of counting distributions motivated by a problem in discrete surplus analysis, and special cases of which have applications in stop-loss, discrete Tail value at risk (TVaR) and claim count modelling. Explicit formulas are developed, and the mixed Poisson case is considered in some detail. Simplifications occur for some underlying negative binomial and related models, where in some cases compound geometric distributions arise naturally. Applications to claim count and aggregate claims models are then given.

KEYWORDS

Mixed Poisson, compound geometric, mixed geometric, negative binomial, logarithmic series, completely monotone, shifting and truncation, Dickson–Hipp, Gerber–Shiu, stop-loss, equilibrium distribution, Panjer recursion.

1. INTRODUCTION TO THE MODEL

Consider a counting random variable N with probability mass function (pmf) $p_j = \Pr(N = j)$, $j = 0, 1, 2, \ldots$, and probability generating function (pgf) $P(z) = \sum_{j=0}^{\infty} p_j z^j$, $|z| < z_0$. Let $t \geq 0$ be such that $P(t) < \infty$, so that $t \leq z_0$, and it is easy to derive the Dickson–Hipp (e.g. Dickson and Hipp, 2001) type of relationship

$$\frac{P(z) - P(t)}{z - t} = \sum_{j=0}^{\infty} \left(\sum_{n=j+1}^{\infty} p_n t^{n-j-1} \right) z^j, \tag{1}$$

where for $z = t$, (1) is to be interpreted in the limiting sense. Define the discrete tail probabilities by $\overline{P}_j = \sum_{n=j+1}^{\infty} p_n$, $j = 0, 1, \ldots$, and for $t = 1$, (1) becomes (e.g. Feller, 1968, p. 265), $\{P(z)-1\}/(z-1) = \sum_{j=0}^{\infty} \overline{P}_j z^j$. Thus, for $0 \le t < z_0$,

$$P_1(z;t) = \sum_{j=0}^{\infty} p_{j,1}(t) z^j = \left(\frac{1-t}{z-t}\right) \frac{P(z)-P(t)}{1-P(t)} = \frac{P(z)-P(t)}{\left(\sum_{j=0}^{\infty} \overline{P}_j t^j\right)(z-t)}, \quad (2)$$

and

$$p_{j,1}(t) = \frac{1-t}{1-P(t)} \sum_{n=j+1}^{\infty} p_n t^{n-j-1}.$$

Note that (2) is a pgf, and $P_1(z;z_0)$ is a pgf if $\{1-P(z_0)\}/\{1-z_0\} < \infty$. In particular, (2) is a pgf if $0 \le t \le 1$ and $E(N) < \infty$. We remark that if $t=1$, then (2) is the pgf of the discrete equilibrium distribution of $P(z)$, which has applications in the evaluation of stop-loss premiums (e.g. Willmot *et al.*, 2005). Also, when $t=0$, (2) is the pgf of the random variable $N-1|N \ge 1$, which will be discussed in Section 5 as a model for claim counts in the context of aggregate claims analysis. The more general pgf (2) is motivated by a surplus problem in the context of discrete Gerber–Shiu analysis (e.g. Pavlova and Willmot, 2004), and in this paper various results are derived for general t, which unify and extend results for the important special cases $t=0$ and $t=1$.

Motivated by (2), we recursively define for $k = 0,1,2,\ldots,$ the pgf $P_{k+1}(z;t) = \sum_{j=0}^{\infty} p_{j,k+1}(t) z^j$ with tail probabilities $\overline{P}_{j,k+1}(t) = \sum_{n=j+1}^{\infty} p_{n,k+1}(t)$ for $j=0,1,\ldots,$ by

$$P_{k+1}(z;t) = \frac{P_k(z;t)-P_k(t;t)}{\left(\sum_{j=0}^{\infty} \overline{P}_{j,k}(t) t^j\right)(z-t)} = \left(\frac{1-t}{z-t}\right) \frac{P_k(z;t)-P_k(t;t)}{1-P_k(t;t)}, \quad (3)$$

beginning with $P_0(z;t) = P(z)$.

In Section 2, we derive an explicit formula for $p_{j,k+1}(t)$ in terms of the original probabilities $\{p_0, p_1, \ldots\}$. In particular, when $t=1$, it reduces to the kth-order equilibrium distribution of the underlying distribution. For $t=0$, it is shown that $P_{k+1}(z;0)$ is the pgf of $N-(k+1)|N \ge k+1$, for $k=0,1,2,\ldots$. Since $N|N>k$ has the same distribution as $(k+1)+[N-(k+1)|N \ge k+1]$, $P_{k+1}(z;0)$ is of use in connection with the evaluation of "value-at-risk" type risk measures such as VaR and Tail value at risk (TVaR). This is also of interest for aggregate claims analysis in the presence of a number of claims deductible. Mixture properties are then considered, and it is also shown that $P_{k+1}(z;t)$ is a mixed Poisson pgf with a relatively straightforward mixing distribution if $P_i(z;t)$ is mixed Poisson for some $i < k+1$.

In Section 3, the case where $P(z)$ is a negative binomial pgf is considered, and in some cases simple expressions for $P_{k+1}(z;t)$ are available which in turn involve compound geometric pgfs under some conditions. Further observations are made about compound distributions, including compound geometrics and compound geometric convolutions, in Section 4. Finally, applications involving claim count data and recursive evaluation of compound distributions in aggregate claims analysis are the subject matter of Section 5.

2. Explicit representations

The definition of $P_{k+1}(z;\, t)$ in (3) is recursive, and we now give an explicit formula for $p_{j,k+1}(t)$ in terms of the original distribution with pgf $P(z)$.

Theorem 1. *For* $k = 0, 1, 2, \ldots,$ *one has*

$$p_{j,k+1}(t) = \frac{\sum_{i=0}^{\infty} \binom{i+k}{k} p_{i+j+k+1} t^i}{\sum_{i=0}^{\infty} \binom{i+k}{k} \overline{P}_{i+k} t^i}, \qquad j = 0, 1, 2, \ldots. \tag{4}$$

Proof. Let $a_{j,k+1}(t) = C_{k+1}(t) p_{j,k+1}(t)$ for $k = 0, 1, 2, \ldots,$ where $C_{k+1}(t) = \prod_{i=0}^{k} \{\sum_{j=0}^{\infty} \overline{P}_{j,i}(t) t^j\}$ for $k = 0, 1, 2, \ldots$. Then with $A_k(z;\, t) = \sum_{j=0}^{\infty} a_{j,k}(t) z^j$, (3) implies that for $k = 1, 2, \ldots,$

$$\begin{aligned} A_{k+1}(z;\, t) &= C_{k+1}(t) \frac{P_k(z;\, t) - P_k(t;\, t)}{\left(\sum_{j=0}^{\infty} \overline{P}_{j,k}(t) t^j \right) (z - t)} \\ &= \frac{C_{k+1}(t)}{C_k(t) \left(\sum_{j=0}^{\infty} \overline{P}_{j,k}(t) t^j \right)} \frac{A_k(z;\, t) - A_k(t;\, t)}{z - t}, \end{aligned}$$

i.e.

$$A_{k+1}(z;\, t) = \frac{A_k(z;\, t) - A_k(t;\, t)}{z - t}.$$

Thus, as in (1), for $k = 1, 2, \ldots,$

$$a_{j,k+1}(t) = \sum_{n=j+1}^{\infty} a_{n,k}(t) t^{n-j-1},$$

and we will now prove by induction on k that for $k = 0, 1, 2, \ldots,$

$$a_{j,k+1}(t) = t^{-j-k-1} \sum_{i=j+k+1}^{\infty} \binom{i-j-1}{k} p_i t^i. \tag{5}$$

For $k = 0$, (1) and (2) imply that $a_{j,1}(t) = C_1(t)p_{j,1}(t) = \sum_{n=j+1}^{\infty} p_n t^{n-j-1} = t^{-j-1}\sum_{i=j+1}^{\infty} p_i t^i$, and (5) holds when $k = 0$. Assuming that (5) holds for $k+1$ replaced by k, i.e. $a_{n,k}(t) = t^{-n-k}\sum_{i=n+k}^{\infty}\binom{i-n-1}{k-1}p_i t^i$, (1) and (3) yield

$$\sum_{n=j+1}^{\infty}\left\{t^{-n-k}\sum_{i=n+k}^{\infty}\binom{i-n-1}{k-1}p_i t^i\right\}t^{n-j-1}.$$

By the inductive hypothesis

$$\begin{aligned} a_{j,k+1}(t) = \sum_{n=j+1}^{\infty} a_{n,k}(t)t^{n-j-1} &= t^{-j-k-1}\sum_{i=j+k+1}^{\infty} p_i t^i \sum_{n=j+1}^{i-k}\binom{i-n-1}{k-1} \\ &= t^{-j-k-1}\sum_{i=j+k+1}^{\infty} p_i t^i \sum_{m=0}^{i-j-k-1}\binom{m+k-1}{k-1} \\ &= t^{-j-k-1}\sum_{i=j+k+1}^{\infty} p_i t^i \binom{i-j-1}{k}, \end{aligned}$$

where the substitution $m = i - n - k$ was used on the third line, and $\sum_{m=0}^{i-j-k-1}\binom{m+k-1}{k-1} = \binom{i-j-1}{k}$ is a well-known combinatorial identity (e.g. Abramowitz and Stegun, 1965, p. 822). Thus, (5) holds for $k = 0, 1, 2, \ldots$, and it may be restated as

$$a_{j,k+1}(t) = \sum_{i=0}^{\infty}\binom{i+k}{k}p_{i+j+k+1}t^i.$$

Thus, $p_{j,k+1}(t) = a_{j,k+1}(t)/C_{k+1}(t)$, and $\sum_{j=0}^{\infty} p_{j,k+1}(t) = 1$ implies that

$$C_{k+1}(t) = \sum_{j=0}^{\infty} a_{j,k+1}(t) = \sum_{i=0}^{\infty}\binom{i+k}{k}\left(\sum_{j=0}^{\infty} p_{i+j+k+1}\right)t^i = \sum_{i=0}^{\infty}\binom{i+k}{k}\overline{P}_{i+k}t^i,$$

and the result follows. ■

The formula (4) is an explicit, albeit cumbersome expression, but may be simplified in certain cases. First, if $t = 0$, (4) reduces to

$$p_{j,k+1}(0) = \frac{p_{j+k+1}}{\overline{P}_k}, \qquad j = 0, 1, 2, \ldots, \tag{6}$$

which is the distribution of $N-(k+1)|N \geq k+1$. Also, if $t=1$, (4) implies that, as a function of j,

$$p_{j,k+1}(1) \propto \sum_{i=0}^{\infty}\left\{\prod_{\ell=1}^{k}(i+\ell)\right\} p_{i+j+k+1} = \sum_{m=j+k+1}^{\infty}\left\{\prod_{\ell=1}^{k}(\ell+m-j-k-1)\right\} p_m,$$

in agreement with formula (5.10) in Willmot *et al.* (2005, p. 23).

An alternative approach to the evaluation of $P_1(z;t)$, or more generally $P_k(z;t)$, involves mixture representations. By the fundamental theorem of calculus, (1) may be expressed as

$$\frac{P(z)-P(t)}{z-t} = \int_0^1 P'\{t+\theta(z-t)\}\, d\theta. \tag{7}$$

Equation (7) may be used to express $P_1(z;t)$ as a mixture for any pgf $P(z)$. Noting that (7) holds when $z=1$, and re-expressing the argument $t+\theta(z-t)$, (2) may be expressed as

$$P_1(z;t) = \frac{\int_0^1 P'\left(\{\theta+t(1-\theta)\}\left\{\frac{t(1-\theta)}{\theta+t(1-\theta)}+\frac{\theta}{\theta+t(1-\theta)}z\right\}\right) d\theta}{\int_0^1 P'\{\theta+t(1-\theta)\}\, d\theta}. \tag{8}$$

Clearly, $P'(\gamma z)/P'(\gamma) = \sum_{j=0}^{\infty}\{(j+1)p_{j+1}\gamma^j/P'(\gamma)\}z^j$, which is a pgf. Then for $0<q<1$, $P'\{\gamma(1-q+qz)\}/P'(\gamma)$ is the pgf of the associated thinned distribution (e.g. Grandell, 1997, pp. 25–26). Let $\gamma=\theta+t(1-\theta)$ and $q=\theta/\{\theta+t(1-\theta)\}$, and (8) expresses $P_1(z;t)$ as a mixture over γ and q of pgfs of the form $P'\{\gamma(1-q+qz)\}/P'(\gamma)$, with mixing weights proportional to $P'(\gamma)$.

The following example illustrates this mixing approach.

Example 1. (Logarithmic series distribution). Suppose that

$$P(z) = \frac{\ln\left(1-\frac{\beta}{1+\beta}z\right)}{\ln\left(1-\frac{\beta}{1+\beta}\right)} = \sum_{n=1}^{\infty}\frac{\left(\frac{\beta}{1+\beta}\right)^n}{n\ln(1+\beta)}z^n, \tag{9}$$

where $\beta>0$. Then $P'(z)=\beta(1+\beta-\beta z)^{-1}/\ln(1+\beta)$, and from (7),

$$\frac{P(z)-P(t)}{z-t} = \frac{\beta}{\ln(1+\beta)}\int_0^1 \{1+\beta(1-t)(1-\theta)-\theta\beta(z-1)\}^{-1}\, d\theta.$$

To put this integral in a more recognizable form, change the variable of integration from θ to

$$y = \frac{\beta\theta}{1+\beta(1-t)(1-\theta)} = \frac{\beta\theta}{[1+\beta(1-t)]-\beta\theta(1-t)}. \tag{10}$$

Then $\beta\theta = \frac{y[1+\beta(1-t)]}{1+y(1-t)}$, implying that

$$\beta d\theta = \frac{1+\beta(1-t)}{[1+y(1-t)]^2}dy, \tag{11}$$

and also using (10) results in

$$1+\beta(1-t)(1-\theta)-\theta\beta(z-1) = \frac{\{1+\beta(1-t)\}\{1+y(1-z)\}}{1+y(1-t)}. \tag{12}$$

Thus, using (11) and (12) yields

$$\frac{P(z)-P(t)}{z-t} = \frac{1}{\ln(1+\beta)}\int_0^\beta (1+y-yt)^{-1}(1+y-yz)^{-1}\,dy.$$

Again with $z = 1$, from (2) it follows that

$$P_1(z;t) = \frac{\int_0^\beta (1+y-yt)^{-1}(1+y-yz)^{-1}dy}{\int_0^\beta (1+y-yt)^{-1}dy}, \tag{13}$$

and thus $P_1(z;t)$ is the pgf of a mixture of geometric pgfs, or equivalently a completely monotone distribution, in agreement with Steutel and Van Harn (2004, pp. 68–69). It is then a straightforward matter using (13) to verify by induction on k that (3) satisfies, for $k = 0, 1, 2, \ldots,$

$$P_{k+1}(z;t) = \frac{\int_0^\beta y^k(1+y-yt)^{-k-1}(1+y-yz)^{-1}dy}{\int_0^\beta y^k(1+y-yt)^{-k-1}dy}. \tag{14}$$

Thus, (14) implies that for the logarithmic series pgf (9), $P_{k+1}(z;t)$ is again a mixture of geometric pgfs. ■

The class of mixtures of geometrics is a subclass of the important class of mixed Poisson distributions, and (4) simplifies in the mixed Poisson case, as will be clear from the following theorem. Mixed Poisson distributions have pgf of the form $P(z) = \widetilde{a}(1-z)$, where $\widetilde{a}(s) = \int_0^\infty e^{-sx}dA(x)$ is the Laplace–Stieltjes transform (LST) of the distribution function (df) $A(x),\ x \geq 0$. They are important in insurance applications for modelling parameter uncertainty and long-tailed claim count data (e.g. Grandell, 1997; Klugman *et al.*, 2008, Section 6.10), and it is of interest to know if a distribution is of mixed Poisson form.

Theorem 2. *Suppose that*

$$P_i(z;t) = \widetilde{a}_i(1-z;t) = \int_0^\infty e^{x(z-1)}dA_i(x;t), \tag{15}$$

for some $i = 0, 1, 2, \ldots$, *where* $A_i(x; t) = 1 - \overline{A}_i(x; t)$ *is a df for* $x \geq 0$. *Then for* $k = i, i+1, i+2, \ldots$,

$$P_{k+1}(z; t) = \widetilde{a}_{k+1}(1 - z; t) = \int_0^\infty e^{x(z-1)} a_{k+1}(x; t)dx, \tag{16}$$

where $a_{k+1}(x; t)$ *is a probability density function* (*pdf*) *given by*

$$a_{k+1}(x; t) = \frac{e^{-x(t-1)} \int_x^\infty (y - x)^{k-i} e^{y(t-1)} dA_i(y; t)}{\int_0^\infty y^{k-i} e^{y(t-1)} \overline{A}_i(y; t)dy}, \qquad x > 0, \tag{17}$$

assuming that $\int_0^\infty y^{k-i} e^{y(t-1)} \overline{A}_i(y; t)dy < \infty$.

Proof. For $t = 1$, the result follows from Willmot *et al.* (2005), and thus assume that $t \neq 1$ in what follows. If $A(x) = 1 - \overline{A}(x)$ is a df, then $a_{1,r}(x)$ is a pdf where

$$a_{1,r}(x) = \frac{e^{rx} \int_x^\infty e^{-ry} dA(y)}{\int_0^\infty e^{-ry} \overline{A}(y)dy}, \tag{18}$$

and the LSTs $\widetilde{a}_{1,r}(s) = \int_0^\infty e^{-sx} a_{1,r}(x)dx$ and $\widetilde{a}(s) = \int_0^\infty e^{-sx} dA(x)$ are related by

$$\widetilde{a}_{1,r}(s) = \left(\frac{r}{s - r}\right) \frac{\widetilde{a}(r) - \widetilde{a}(s)}{1 - \widetilde{a}(r)}. \tag{19}$$

See Lin and Willmot (1999), for example. We will prove the result by induction on k. For $k = i$, it follows from (3) and (15) that

$$\begin{aligned} P_{i+1}(z; t) &= \frac{P_i(z; t) - P_i(t; t)}{1 - P_i(t; t)} \left(\frac{1 - t}{z - t}\right) \\ &= \frac{\widetilde{a}_i(1 - z; t) - \widetilde{a}_i(1 - t; t)}{1 - \widetilde{a}_i(1 - t; t)} \left\{\frac{1 - t}{1 - t - (1 - z)}\right\} \\ &= \widetilde{a}_{i+1}(1 - z; t), \end{aligned}$$

where

$$\widetilde{a}_{i+1}(s; t) = \left\{\frac{1 - t}{s - (1 - t)}\right\} \frac{\widetilde{a}_i(1 - t; t) - \widetilde{a}_i(s; t)}{1 - \widetilde{a}_i(1 - t; t)},$$

which is of the form (19) with $\widetilde{a}(s)$ replaced by $\widetilde{a}_i(s; t)$ and r by $1 - t$. Then (18) becomes (17) in this case, i.e. (16) holds when $k = i$. Assuming that $P_k(z; t) = \widetilde{a}_k(1 - z; t)$ with $a_k(y; t)$ a pdf given by (17), (3) yields again that

$$P_{k+1}(z; t) = \widetilde{a}_{k+1}(1 - z; t),$$

where

$$\widetilde{a}_{k+1}(s; t) = \left\{\frac{1 - t}{s - (1 - t)}\right\} \frac{\widetilde{a}_k(1 - t; t) - \widetilde{a}_k(s; t)}{1 - \widetilde{a}_k(1 - t; t)},$$

implying from (19) that (18) becomes

$$a_{k+1}(x;t) = \frac{e^{-x(t-1)}\int_x^\infty e^{y(t-1)}a_k(y;t)dy}{\int_0^\infty e^{y(t-1)}\overline{A}_k(y;t)dy}. \tag{20}$$

By the inductive hypothesis, using (17) with k replaced by $k-1$,

$$\begin{aligned}
&e^{-x(t-1)}\int_x^\infty e^{y(t-1)}a_k(y;t)dy \\
&= \frac{e^{-x(t-1)}\int_x^\infty e^{y(t-1)}\left\{e^{-y(t-1)}\int_y^\infty (v-y)^{k-i-1}e^{v(t-1)}dA_i(v;t)\right\}dy}{\int_0^\infty y^{k-i-1}e^{y(t-1)}\overline{A}_i(y;t)dy} \\
&= \frac{e^{-x(t-1)}\int_x^\infty\int_y^\infty (v-y)^{k-i-1}e^{v(t-1)}dA_i(v;t)dy}{\int_0^\infty y^{k-i-1}e^{y(t-1)}\overline{A}_i(y;t)dy} \\
&= \frac{e^{-x(t-1)}\int_x^\infty e^{v(t-1)}\left\{\int_x^v (v-y)^{k-i-1}dy\right\}dA_i(v;t)}{\int_0^\infty y^{k-i-1}e^{y(t-1)}\overline{A}_i(y;t)dy} \\
&= \frac{e^{-x(t-1)}\int_x^\infty (v-x)^{k-i}e^{v(t-1)}dA_i(v;t)}{(k-i)\int_0^\infty y^{k-i-1}e^{y(t-1)}\overline{A}_i(y;t)dy}.
\end{aligned}$$

Thus, replacing v by y in the numerator, (20) becomes

$$a_{k+1}(x;t) = C^*_{k+1}(t)e^{-x(t-1)}\int_x^\infty (y-x)^{k-i}e^{y(t-1)}dA_i(y;t),$$

when $C^*_{k+1}(t)$ is a constant. As $\int_0^\infty a_{k+1}(x;t)dx = 1$, it follows that

$$\left\{C^*_{k+1}(t)\right\}^{-1} = \int_0^\infty y^{k-i}e^{y(t-1)}\overline{A}_i(y;t)dy,$$

and (17) holds for $k+1$. ■

We now consider the special case of Theorem 2 when P is a Poisson pgf in the following example.

Example 2. (Poisson distribution). Suppose that

$$P(z) = P_0(z;t) = e^{\beta(z-1)},$$

and by Theorem 2 with $i=0$ where $A_0(x;t)=0$ for $z<\beta$ and $A_0(x;t)=1$ for $z\geq\beta$, $P_{k+1}(z)$ for $k=0,1,2,\ldots,$ is a mixed Poisson pgf. Therefore, from (17) it follows that

$$a_{k+1}(x;t) = \frac{(\beta-x)^k e^{x(1-t)}}{\int_0^\beta (\beta-y)^k e^{y(1-t)}dy}, \qquad 0<x<\beta,$$

which is the uniform pdf with a parameter β when $k = 0$ and $t = 1$, i.e. $a_1(x; 1) = 1/\beta$ for $0 < x < \beta$. Also, using (17) with k replaced by $k-1$ yields

$$P_k(z; t) = \frac{\int_0^\beta e^{x(z-1)}(\beta - x)^{k-1} e^{x(1-t)} dx}{\int_0^\beta (\beta - x)^{k-1} e^{x(1-t)} dx} = \frac{\int_0^\beta e^{xz}(\beta - x)^{k-1} e^{-xt} dx}{\int_0^\beta (\beta - x)^{k-1} e^{x(1-t)} dx},$$

and in turn, by equating the coefficient of z^n yields

$$p_{n,k}(t) = \frac{\int_0^\beta x^n (\beta - x)^{k-1} e^{-xt} dx}{n! \int_0^\beta (\beta - x)^{k-1} e^{x(1-t)} dx}, \qquad n = 0, 1, 2, \ldots.$$

A change in the variable of integration from x to $y = 1 - x/\beta$ in the integrals in the numerator and denominator results in

$$p_{n,k}(t) = \left(\frac{\beta^n e^{-\beta}}{n!}\right) \frac{\int_0^1 y^{k-1}(1-y)^n e^{\beta t y} dy}{\int_0^1 y^{k-1} e^{\beta(t-1)y} dx}, \qquad n = 0, 1, 2, \ldots.$$

In terms of the confluent hypergeometric function

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 y^{a-1}(1-y)^{b-a-1} e^{zy} dy,$$

(e.g. Abramowitz and Stegun, 1965, p. 505), it follows that

$$p_{n,k}(t) = \frac{\frac{\beta^n e^{-\beta}}{n!} M(k, n+k+1, \beta t)}{\binom{n+k}{n} M[k, k+1, \beta(t-1)]},$$

for $t \neq 1$, and for $t = 1$, $M(k, k+1, 0) = 1$. ■

Theorem 2 implies that if $P(z) = P_0(z; t)$ is a mixed Poisson pgf, then $P_{k+1}(z; t)$ is a mixed Poisson pgf for $k = 0, 1, 2, \ldots$. But the following example shows that fairly generally $P_{k+1}(z; t)$ can be of mixed Poisson form even if $P(z)$ is not of mixed Poisson form.

Example 3. Suppose that

$$P(z) = \alpha + (1-\alpha) \frac{\phi\{\beta(1-z)\} - \phi(\beta)}{1 - \phi(\beta)} \tag{21}$$

is a pgf for $0 \leq \alpha < 1$, $0 < \beta < \beta_\infty$ where $\beta_\infty \leq \infty$, and $\phi(x)$ is a known function. This pgf is discussed in detail by Willmot (2013), where it is demonstrated that if $\phi(x) = 1 + \ln(1+x)$ then $P(z)$ is the zero-modified logarithmic series pgf (9) but $P(z)$ is not of mixed Poisson form. Similarly, if $\phi(x) = (1+x)^\sigma$ where $0 < \sigma < 1$ then $P(z)$ is a zero-modified extended truncated negative binomial (ETNB) pgf, but again $P(z)$ is not a mixed Poisson pgf.

If (21) holds, then $P'(z) = \beta(1-\alpha)\phi'\{\beta(1-z)\}/\{\phi(\beta)-1\}$, and from (7),

$$\begin{aligned}\frac{P(z)-P(t)}{z-t} &= \frac{\beta(1-\alpha)}{\phi(\beta)-1}\int_0^1 \phi'\{\beta(1-t)+\theta\beta(t-z)\}\,d\theta \\ &= \frac{\beta(1-\alpha)}{\phi(\beta)-1}\int_0^1 \phi'\{[1+\beta(1-t)(1-\theta)]-\theta\beta(z-1)-1\}d\theta.\end{aligned}$$

With $x = \beta\theta/\{1+\beta(1-t)(1-\theta)\}$ as in (10), then using (11) and (12) results in

$$\begin{aligned}&\frac{P(z)-P(t)}{z-t} \\ &= \frac{1-\alpha}{\phi(\beta)-1}\int_0^\beta \phi'\left\{\frac{(1+\beta-\beta t)(1+x-xz)}{1+x-xt}-1\right\}\frac{(1+\beta-\beta t)}{(1+x-xt)^2}dx. \quad (22)\end{aligned}$$

Thus with $z = 1$, (2) becomes

$$P_1(z;t) = \frac{\int_0^\beta \phi'\{(\beta-x)(1-t)+x(1-z)\}\,dx}{\int_0^\beta \phi'\{(\beta-x)(1-t)\}\,dx}. \quad (23)$$

We note that from Grandell (1997, p. 26) (23) is a mixed Poisson pgf iff $P_1\{1-\frac{1}{q}(1-z);t\}$ is a pgf for all $0<q<1$. It is not hard to see using (23) that

$$P_1\left\{1-\frac{1}{q}(1-z);t\right\} = \frac{\int_0^{\beta/q}\phi'\left\{\left(\frac{\beta}{q}-x\right)(1-\{1-q(1-t)\})+x(1-z)\right\}dx}{\int_0^{\beta/q}\phi'\left\{\left(\frac{\beta}{q}-x\right)(1-\{1-q(1-t)\})\right\}dx},$$

which is of the same form as (23), but with β replaced by β/q and t by $1-q(1-t)$. This is a pgf for q arbitrarily close to 0 as long as $\beta_\infty = \infty$.

To identify the mixing distribution for $0 \le t \le 1$, note that from Equation (2.4) of Willmot (2013), one must have (assuming the mean $P'(1) = \beta(1-\alpha)\phi'(0)/\{\phi(\beta)-1\}$ is finite) for $s \ge 0$ that $\phi'(s) = \phi'(0)\widetilde{u}(s)$ where $\widetilde{u}(s) = \int_0^\infty e^{-sy}dU(y)$, and $U(y)$ is a df for $y \ge 0$. Then,

$$\frac{\phi'(\mu+xs)}{\phi'(\mu)} = \frac{\int_0^\infty e^{-(\mu+xs)y}dU(y)}{\int_0^\infty e^{-\mu y}dU(y)}$$

is the LST of an Esscher transformed version of $U(y)$, and with $\mu = (\beta-x)(1-t)$, (23) implies that the mixing LST $P_1(1-s;t)$ is a mixture over μ of LSTs of the form $\phi'(\mu+xs)/\phi'(\mu)$, with mixing weights proportional to μ.

For particular choices of $\phi(x)$, the representation (22) typically allows for identification of the mixing distribution for $t > 1$ as well. For the ETNB distribution with $\phi(x) = (1+x)^\sigma$ where $0<\sigma<1$, (21) is not a mixed Poisson pgf, but (22) becomes

$$\frac{P(z)-P(t)}{z-t}=\frac{\sigma(1-\alpha)(1+\beta-\beta t)^{\sigma}}{(1+\beta)^{\sigma}-1}\int_0^{\beta}\frac{(1+x-xz)^{\sigma-1}}{(1+x-xt)^{\sigma+1}}dx.$$

Thus, (2) becomes

$$P_1(z;t)=\frac{\int_0^{\beta}(1+x-xt)^{-\sigma-1}(1+x-xz)^{\sigma-1}\,dx}{\int_0^{\beta}(1+x-xt)^{-\sigma-1}\,dx}, \tag{24}$$

which is a mixture of the negative binomial pgfs $(1+x-xz)^{\sigma-1}$ for any t such that $0<t<1+\beta^{-1}$. Clearly, (24) is a mixed Poisson pgf, and by Theorem 2 with $i=1$, $P_{k+1}(z;t)$ is then a mixed Poisson pgf for $k=0,1,2,\ldots$.

Also, we remark that in the logarithmic series case with $\phi(x)=1+\ln(1+x)$, (13) follows directly from (22). ■

3. Negative binomial and related distributions

In this section, we consider the evaluation of pgfs of the type (3) when the underlying distribution is of negative binomial form, or a closely related distribution. While the mixture approach of Example 2 applies to the negative binomial, more convenient forms for some applications may be derived.

To begin, define the negative binomial pgf

$$Q(z;r)=\sum_{j=0}^{\infty}q_j(r)z^j=(1+\beta-\beta z)^{-r}, \tag{25}$$

where

$$q_j(r)=\binom{r+j-1}{j}\left(\frac{1}{1+\beta}\right)^{r}\left(\frac{\beta}{1+\beta}\right)^{j};\qquad j=0,1,2,\ldots,$$

with parameters $\beta>0$ and $r>0$. In the special case when $r=1,2,3,\ldots$, the Pascal distribution results, and it is convenient notationally to note the dependence on r explicitly. Then define for $k=0,1,2,\ldots$, the pgf

$$Q_{k+1}(z;t,r)=\sum_{j=0}^{\infty}q_{j,k+1}(t,r)z^j=\sum_{j=0}^{\infty}\left\{\frac{\sum_{i=0}^{\infty}\binom{i+k}{k}q_{i+j+k+1}(r)t^i}{\sum_{i=0}^{\infty}\binom{i+k}{k}\overline{Q}_{i+k}(r)t^i}\right\}z^j, \tag{26}$$

where $\overline{Q}_n(r)=\sum_{j=n+1}^{\infty}q_j(r)$, and (26) is motivated by (4). We have the following simple expression for r, a positive integer.

Theorem 3. *For* $r = 1, 2, 3, \ldots,$ *and* $k = 0, 1, 2, \ldots,$

$$Q_{k+1}(z; t, r) = \sum_{n=1}^{r} w_{n,k+1}(t, r)\, Q(z; n) \tag{27}$$

and

$$q_{j,k+1}(t, r) = \sum_{n=1}^{r} w_{n,k+1}(t, r) q_j(n), \qquad j = 0, 1, 2, \ldots, \tag{28}$$

where

$$w_{n,k+1}(t, r) = \frac{\binom{k+r-n}{k} (1 + \beta - \beta t)^n}{\sum_{i=1}^{r} \binom{k+r-i}{k} (1 + \beta - \beta t)^i}, \qquad n = 1, 2, \ldots, r. \tag{29}$$

Proof. Note that (25) is the mixed Poisson pgf $Q(z, r) = \widetilde{e}_r(1-z)$, where $\widetilde{e}_r(s) = (1 + \beta s)^{-r} = \int_0^\infty e^{-sx} e_r(x) dx$ with

$$e_r(x) = \frac{\beta^{-r} x^{r-1} e^{-x/\beta}}{(r-1)!}, \qquad x > 0, \tag{30}$$

the Erlang$-r$ pdf. Thus, by Theorem 2, $Q_{k+1}(z; t, r) = \widetilde{a}_{k+1}(1 - z; t)$, where

$$\begin{aligned} a_{k+1}(x; t) &= C(t) e^{-x(t-1)} \int_x^\infty (y - x)^k e^{y(t-1)} e_r(y) dy \\ &= C(t) \int_0^\infty y^k e^{-y(1-t)} e_r(x + y) dy. \end{aligned}$$

But (30) satisfies $e_r(x + y) = \beta \sum_{n=1}^{r} e_n(x) e_{r+1-n}(y)$ (e.g. Willmot, 2007, equation 3.26), and thus

$$\begin{aligned} a_{k+1}(x; t) &= C(t) \int_0^\infty y^k e^{-y(1-t)} \left\{ \beta \sum_{n=1}^{r} e_n(x) e_{r+1-n}(y) \right\} dy \\ &= \beta C(t) \sum_{n=1}^{r} \left\{ \int_0^\infty y^k e^{-y(1-t)} e_{r+1-n}(y) dy \right\} e_n(x). \end{aligned}$$

Also, from (30) with r replaced by $r + 1 - n$,

$$\begin{aligned} \int_0^\infty y^k e^{-y(1-t)} e_{r+1-n}(y) dy &= \int_0^\infty \frac{\beta^{-(r+1-n)} y^{r+k-n} e^{-y\left(1-t+\frac{1}{\beta}\right)}}{(r-n)!} dy \\ &= \frac{(r + k - n)!}{(r - n)!} \beta^{n-r-1} \left(1 - t + \frac{1}{\beta}\right)^{n-r-k-1}. \end{aligned}$$

Thus,

$$a_{k+1}(x;\, t) = \frac{C(t)\beta^{k+1}(k!)}{(1+\beta-\beta t)^{r+k+1}} \sum_{n=1}^{r} \binom{r+k-n}{k} (1+\beta-\beta t)^n\, e_n(x).$$

As $C(t)$ is determined by $\int_0^\infty a_{k+1}(x;\, t)dx = 1$, it follows that

$$a_{k+1}(x;\, t) = \sum_{n=1}^{r} w_{n,k+1}(t, r) e_n(x), \tag{31}$$

where $w_{n,k+1}(t, r)$ is given by (29). Then (27) follows from $Q_{k+1}(z;\, t, r) = \widetilde{a}_{k+1}(1-z;\, t)$, and equating coefficients of z^j gives (28). ■

When $t = 0$, (6) and (28) yield the mixture representation

$$\frac{q_{j+k+1}(r)}{\overline{Q}_k(r)} = \sum_{n=1}^{r} w_{n,k+1}(0;\, r) q_j(n), \tag{32}$$

where $w_{n,k+1}(0, r)$ is given by (29) with $t = 0$. Also, when $t = 1$, $\sum_{i=1}^{r}\binom{k+r-i}{k} = \sum_{m=0}^{r-1}\binom{k+m}{k} = \binom{k+r}{k+1}$, implying that (29) reduces to $w_{n,k+1}(1, r) = \binom{k+r-n}{k}/\binom{k+r}{k+1}$, in agreement with Willmot *et al.* (2005, p. 17).

The representation (27) or its special case (32) is convenient for recursive computational procedures for the associated compound distribution if r is not too large, as compound negative binomial distributions are straightforward to compute (e.g. Klugman *et al.*, 2008, Chapter 6). Also, the mixing pdf (31) for the mixed Poisson representation for $Q_{k+1}(z;\, t, r)$ is a finite mixture of Erlangs (e.g. Willmot and Woo, 2007).

We now consider other values of r, i.e. excluding positive integers. First note that for any pgf $P(z)$, a simple geometric series argument yields

$$\sum_{j=0}^{n-1} \{P(t)\}^{n-1-j}\{P(z)\}^j = \frac{\{P(z)\}^n - \{P(t)\}^n}{P(z) - P(t)},$$

which for $z = 1$ implies that

$$\sum_{j=0}^{n-1} \{P(t)\}^{n-1-j} = \frac{1 - \{P(t)\}^n}{1 - P(t)}.$$

Therefore, using (2),

$$\left(\frac{1-t}{z-t}\right)\frac{\{P(z)\}^n-\{P(t)\}^n}{1-\{P(t)\}^n} = P_1(z;t)\left\{\frac{\sum_{j=0}^{n-1}\{P(t)\}^{n-1-j}\{P(z)\}^j}{\sum_{i=0}^{n-1}\{P(t)\}^{n-1-i}}\right\}$$

$$= P_1(z;t)\left(w_0(t)+\sum_{j=1}^{n-1}w_j(t)\{P(z)\}^j\right), \qquad (33)$$

where

$$w_j(t)=\frac{\{P(t)\}^{-j}}{\sum_{i=0}^{n-1}\{P(t)\}^{-i}}, \qquad j=0,1,2,\ldots,n-1.$$

The left-hand side of (33) is (2) when $P(z)$ is replaced by $\{P(z)\}^n$, and the right-hand side expresses this as a mixture, with weights $w_j(t)$, of pgfs of the form $P_1(z;t)\{P(z)\}^j$. Thus, if r is any positive rational number, say $r=n/m$ where n and m are positive integers (we exclude the case $m=1$ in light of Theorem 3), substitution of $Q(z;\frac{1}{m})$ from (25) for $P(z)$ in (33) yields

$$Q_1\left(z;t,\frac{n}{m}\right)=Q_1\left(z;t,\frac{1}{m}\right)\left\{w_0(t)+\sum_{j=1}^{n-1}w_j(t)\,(1+\beta-\beta z)^{-\frac{j}{m}}\right\}, \qquad (34)$$

where $w_j(t)\propto(1+\beta-\beta t)^{j/m}$, and $\sum_{j=1}^{0}=0$ in (34) when $n=1$. Therefore, (34) allows for identification of $Q_1(z;t,r)$ for rational r as a mixture (as the rational numbers are dense in the real numbers, this is sufficient in principle for the analysis of $Q_1(z;t,r)$ for any $r>0$ by the continuity theorem for pgfs). It remains to consider $Q_1(z;t,\frac{1}{m})$ for $m=2,3,\ldots$.

One has

$$\sum_{j=1}^{m}\left(\frac{1+\beta-\beta z}{1+\beta-\beta t}\right)^{\frac{j}{m}}=\frac{1-\frac{1+\beta-\beta z}{1+\beta-\beta t}}{\left(\frac{1+\beta-\beta z}{1+\beta-\beta t}\right)^{-\frac{1}{m}}-1}$$

$$=\frac{\beta(z-t)\{1+\beta-\beta t\}^{-1-\frac{1}{m}}}{(1+\beta-\beta z)^{-\frac{1}{m}}-(1+\beta-\beta t)^{-\frac{1}{m}}}.$$

That is,

$$\frac{(1+\beta-\beta z)^{-\frac{1}{m}}-(1+\beta-\beta t)^{-\frac{1}{m}}}{z-t} = \frac{\beta}{(1+\beta-\beta t)^{1+\frac{1}{m}}}\left\{\sum_{j=1}^{m}\left(\frac{1+\beta-\beta z}{1+\beta-\beta t}\right)^{\frac{j}{m}}\right\}^{-1}. \tag{35}$$

For $z=1$, (35) yields

$$\frac{1-(1+\beta-\beta t)^{-\frac{1}{m}}}{1-t} = \frac{\beta}{(1+\beta-\beta t)^{1+\frac{1}{m}}}\left\{\sum_{j=1}^{m}(1+\beta-\beta t)^{-\frac{j}{m}}\right\}^{-1}, \tag{36}$$

and division of (35) by (36) yields

$$Q_1\left(z;t,\frac{1}{m}\right) = \frac{\sum_{j=1}^{m}(1+\beta-\beta t)^{-\frac{j}{m}}}{\sum_{j=1}^{m}(1+\beta-\beta t)^{-\frac{j}{m}}(1+\beta-\beta z)^{\frac{j}{m}}}, \tag{37}$$

which holds for $m=1$ as well as $m=2,3,\ldots$. To put (37) in a recognizable form, we note that

$$G(z;r) = \frac{(1+\beta-\beta z)^{-r}-(1+\beta)^{-r}}{1-(1+\beta)^{-r}} \tag{38}$$

is the pgf of an ETNB distribution if $\beta>0$ and $r>-1$, $r\neq 0$ (e.g. Klugman *et al.,* 2008, Section 6.7), and $G(z;-1)=z$, is also a pgf. The following theorem expresses $Q_1(z;t,\frac{1}{m})$ as a compound geometric distribution.

Theorem 4. *For* $m=1,2,3,\ldots,$ $Q_1(z;t,\frac{1}{m})$ *may be expressed in compound geometric form as*

$$Q_1\left(z;t,\frac{1}{m}\right) = \left[1-\beta\left(t,\frac{1}{m}\right)\left\{F\left(z;t,\frac{1}{m}\right)-1\right\}\right]^{-1}, \tag{39}$$

where

$$\beta\left(t,\frac{1}{m}\right) = \frac{\sum_{i=1}^{m}\left\{(1+\beta)^{\frac{i}{m}}-1\right\}(1+\beta-\beta t)^{-\frac{i}{m}}}{\sum_{j=1}^{m}(1+\beta-\beta t)^{-\frac{j}{m}}}, \tag{40}$$

$$F\left(z;t,\frac{1}{m}\right) = \sum_{j=1}^{m} w_{j,1}\left(t,\frac{1}{m}\right)G\left(z;-\frac{j}{m}\right), \tag{41}$$

and

$$w_{j,1}\left(t, \frac{1}{m}\right) = \frac{\left\{(1+\beta)^{\frac{j}{m}} - 1\right\}(1+\beta-\beta t)^{-\frac{j}{m}}}{\sum_{i=1}^{m}\left\{(1+\beta)^{\frac{i}{m}} - 1\right\}(1+\beta-\beta t)^{-\frac{i}{m}}}, \qquad j = 1, 2, \ldots, m. \quad (42)$$

Proof. One has, using (38) and (42),

$$\begin{aligned}
&\sum_{j=1}^{m} (1+\beta-\beta t)^{-\frac{j}{m}} (1+\beta-\beta z)^{\frac{j}{m}} \\
&\quad = \sum_{j=1}^{m} (1+\beta-\beta t)^{-\frac{j}{m}} + \sum_{j=1}^{m} \frac{(1+\beta-\beta z)^{\frac{j}{m}} - (1+\beta)^{\frac{j}{m}} + (1+\beta)^{\frac{j}{m}} - 1}{(1+\beta-\beta t)^{\frac{j}{m}}} \\
&\quad = \sum_{j=1}^{m} (1+\beta-\beta t)^{-\frac{j}{m}} - \sum_{j=1}^{m} \frac{(1+\beta)^{\frac{j}{m}} - 1}{(1+\beta-\beta t)^{\frac{j}{m}}} \left\{G\left(z; -\frac{j}{m}\right) - 1\right\} \\
&\quad = \sum_{j=1}^{m} (1+\beta-\beta t)^{-\frac{j}{m}} - \left\{\sum_{i=1}^{m} \frac{(1+\beta)^{\frac{i}{m}} - 1}{(1+\beta-\beta t)^{\frac{i}{m}}}\right\} \sum_{j=1}^{m} w_{j,1}\left(t, \frac{1}{m}\right) \\
&\qquad \times \left\{G\left(z; -\frac{j}{m}\right) - 1\right\},
\end{aligned}$$

and division by $\sum_{j=1}^{m} (1+\beta-\beta t)^{-\frac{j}{m}}$ yields the reciprocal of (39) with the help of (40) and (41). ■

Therefore, (39) expresses $Q_1(z; t, \frac{1}{m})$ in compound geometric form, which for m not too large is convenient for the recursive computation of the associated aggregate distribution using repeated applications of Panjer's recursion (e.g. Klugman *et al.*, 2008, Section 6.8) as both the ETNB and the geometric are members of the so-called $(a, b, 1)$ class.

Note that from (38),

$$1 - G(z; r) = \frac{1 - (1+\beta-\beta z)^{-r}}{1 - (1+\beta)^{-r}},$$

and thus

$$\begin{aligned}
&\left(\frac{1-t}{z-t}\right)\frac{G(z;r)-G(t;r)}{1-G(t;r)}\\
&=\left(\frac{1-t}{z-t}\right)\frac{\{1-G(t;r)\}-\{1-G(z;r)\}}{1-G(t;r)}\\
&=\left(\frac{1-t}{z-t}\right)\frac{(1+\beta-\beta z)^{-r}-(1+\beta-\beta t)^{-r}}{1-(1+\beta-\beta t)^{-r}}.
\end{aligned}$$

That is, in an obvious notation, $G_1(z;t,r) = Q_1(z;t,r)$, i.e.

$$Q_1(z;t,r)=\left(\frac{1-t}{z-t}\right)\frac{(1+\beta-\beta z)^{-r}-(1+\beta-\beta t)^{-r}}{1-(1+\beta-\beta t)^{-r}} \tag{43}$$

is a pgf not only for $r > 0$ but also for $-1 < r < 0$, as $Q_1(z;t,r)$ also results with $G(z;r)$ as the original distribution.

It is not hard to see from (43) that for $0 < r < 1$,

$$Q_1(z;t,-r) = (1+\beta-\beta z)^r\, Q_1(z;t,r), \tag{44}$$

and thus from (44) with $r = 1/m$ and (37) ,

$$Q_1\left(z;t,-\frac{1}{m}\right)=\frac{\sum_{j=0}^{m-1}(1+\beta-\beta t)^{-\frac{j}{m}}}{\sum_{j=0}^{m-1}(1+\beta-\beta t)^{-\frac{j}{m}}(1+\beta-\beta z)^{\frac{j}{m}}}, \tag{45}$$

for $m = 2, 3, \ldots$. Then, we have the following corollary to Theorem 4.

Corollary 1. *For $m = 2, 3, \ldots$, $Q_1(z;t,-\frac{1}{m})$ may be expressed in compound geometric form as*

$$Q_1\left(z;t,-\frac{1}{m}\right)=\left[1-\beta\left(t,-\frac{1}{m}\right)\left\{F\left(z;t,-\frac{1}{m}\right)-1\right\}\right]^{-1}, \tag{46}$$

where

$$\beta\left(t,-\frac{1}{m}\right)=\frac{\sum_{i=1}^{m-1}\left\{(1+\beta)^{\frac{i}{m}}-1\right\}(1+\beta-\beta t)^{-\frac{i}{m}}}{\sum_{j=0}^{m-1}(1+\beta-\beta t)^{-\frac{j}{m}}},$$

$$F\left(z;t,-\frac{1}{m}\right)=\sum_{j=1}^{m-1}w_{j,1}\left(t,-\frac{1}{m}\right)G\left(z;-\frac{j}{m}\right),$$

and

$$w_{j,1}\left(t,-\frac{1}{m}\right)=\frac{\left\{(1+\beta)^{\frac{j}{m}}-1\right\}(1+\beta-\beta t)^{-\frac{j}{m}}}{\sum_{i=1}^{m-1}\left\{(1+\beta)^{\frac{i}{m}}-1\right\}(1+\beta-\beta t)^{-\frac{i}{m}}}, \qquad j=1,2,\ldots,m-1.$$

Proof. In a manner which is identical to that in the proof of Theorem 4, one has

$$\sum_{j=0}^{m-1}(1+\beta-\beta t)^{-\frac{j}{m}}(1+\beta-\beta z)^{\frac{j}{m}}$$
$$=\sum_{j=0}^{m-1}(1+\beta-\beta t)^{-\frac{j}{m}}-\sum_{j=1}^{m-1}\frac{(1+\beta)^{\frac{j}{m}}-1}{(1+\beta-\beta t)^{\frac{j}{m}}}\left\{G\left(z;-\frac{j}{m}\right)-1\right\},$$

and the result follows as in the proof of Theorem 4 using (45). ■

Again, (46) is well-suited for the application of Panjer-type recursive techniques.

4. Compound and related distributions

The compound geometric representation technique of the previous section is applicable to other situations as well. For the Sibuya distribution (e.g. Klugman *et al.*, 2008, p. 124) with pgf $1-(1-z)^{\alpha}$ with $0<\alpha<1$ for example, the "shifted" version with $\alpha=1/m$ with $m=2,3,\ldots,$ has compound geometric pgf

$$\frac{1}{z}\left\{1-(1-z)^{\frac{1}{m}}\right\}=\left[1-(m-1)\left\{\sum_{j=1}^{m-1}\frac{1-(1-z)^{\frac{j}{m}}}{m-1}-1\right\}\right]^{-1},$$

as is easily shown. Similarly, the lost games distribution (e.g. Johnson *et al.*, 2005, p. 504) has pgf

$$z^{j}\left\{\frac{1-\sqrt{1-4q(1-q)z}}{2qz}\right\}^{a},$$

where $a>0,\ 0<q<1/2$, and $j=0,1,2,\ldots$. If one reparameterizes by letting $q=\{1-(1+\beta)^{-\frac{1}{2}}\}/2$, it is not difficult to show that

$$\frac{1-\sqrt{1-4q(1-q)z}}{2qz}=\frac{1}{z}\left\{\frac{\sqrt{1+\beta-\beta z}-\sqrt{1+\beta}}{1-\sqrt{1+\beta}}\right\},$$

which is $Q_1(z; 0, -\frac{1}{2})$ from (43). Thus, the lost games distribution is a compound negative binomial distribution from (46) with negative binomial parameters $r = a$ and $\beta = \beta(0, -\frac{1}{2})$, shifted to the right by j.

For $k > 1$, the pgf $Q_k(z; t, r)$ for $k > 1$ seems to be more difficult to analyze when r is not a positive integer. First, if $P(z) = R\{K(z)\}$, then from (2), it follows that

$$
\begin{aligned}
P_1(z; t) &= \left(\frac{1-t}{z-t}\right) \frac{R\{K(z)\} - R\{K(t)\}}{1 - R\{K(t)\}} \\
&= \left\{ \frac{1 - K(t)}{K(z) - K(t)} \, \frac{R\{K(z)\} - R\{K(t)\}}{1 - R\{K(t)\}} \right\} \left\{ \frac{1-t}{z-t} \, \frac{K(z) - K(t)}{1 - K(t)} \right\}.
\end{aligned}
$$

Thus, with $R_1(z; t) = \left(\frac{1-t}{z-t}\right) \frac{R(z)-R(t)}{1-R(t)}$ and similarly for $K_1(z; t)$, one has the convolution representation

$$P_1(z; t) = R_1 \{K(z); K(t)\} \, K_1(z; t), \tag{47}$$

where the parameter t is replaced by $K(t)$ in the compound pgf $R_1\{K(z); K(t)\}$. The special case of (47) when $t = 1$ is given by Willmot *et al.* (2005).

The following theorem deals with the compound geometric case.

Theorem 5. Suppose that $P(z) = P_0(z; t) = \{1 + \beta - \beta K(z)\}^{-1}$, then $P_k(z; t)$ for $k = 1, 2, \dots,$ defined by (3) has a compound geometric convolution representation

$$P_k(z; t) = \left\{ \sum_{j=1}^{k} w_{j,k}(t) K_j(z; t) \right\} P(z), \tag{48}$$

where $\{w_{j,k}(t);\ j = 1, 2, \dots, k\}$ is a discrete probability measure.

Proof. For the compound geometric distribution with $R(z) = (1+\beta-\beta z)^{-1}$, it is clear that $R_k(z; t) = R(z)$ for $k = 0, 1, 2, \dots,$ and all t. Thus for the compound geometric, (47) reduces to

$$P_1(z; t) = P(z) K_1(z; t). \tag{49}$$

From (49) when $P(z) = \{1+\beta-\beta K(z)\}^{-1}$, one has

$$\begin{aligned}
&\frac{P_1(z;t) - P_1(t;t)}{z-t} \\
&= \frac{\{1+\beta-\beta K(z)\}^{-1} K_1(z;t) - \{1+\beta-\beta K(t)\}^{-1} K_1(t;t)}{z-t} \\
&= \frac{\{1+\beta-\beta K(t)\} K_1(z;t) - K_1(t;t)\{1+\beta-\beta K(z)\}}{z-t} P(t)P(z) \\
&= \left[\{1+\beta-\beta K(t)\} \frac{K_1(z;t)-K_1(t;t)}{z-t} + \beta K_1(t;t)\frac{K(z)-K(t)}{z-t}\right] P(t)P(z) \\
&= \left\{\frac{K_1(z;t)-K_1(t;t)}{z-t} + \beta P(t)K_1(t;t)\frac{K(z)-K(t)}{z-t}\right\} P(z).
\end{aligned}$$

With $z=1$, this implies that

$$\frac{1-P_1(t;t)}{1-t} = \frac{1-K_1(t;t)}{1-t} + \beta P(t)K_1(t;t)\frac{1-K(t)}{1-t}.$$

Thus, $P_2(z;t)$ may be expressed as

$$\left(\frac{1-t}{z-t}\right)\frac{P_1(z;t)-P_1(t;t)}{1-P_1(t;t)} = \frac{\frac{K_1(z;t)-K_1(t;t)}{z-t} + \beta P(t)K_1(t;t)\frac{K(z)-K(t)}{z-t}}{\frac{1-K_1(t;t)}{1-t} + \beta P(t)K_1(t;t)\frac{1-K(t)}{1-t}} P(z).$$

That is,

$$P_2(z;t) = \left[w_{2,2}(t)K_2(z;t) + \left\{1-w_{2,2}(t)\right\}K_1(z;t)\right] P(z), \tag{50}$$

where

$$w_{2,2}(t) = \frac{\frac{1-K_1(t;t)}{1-t}}{\frac{1-K_1(t;t)}{1-t} + \beta P(t)K_1(t;t)\frac{1-K(t)}{1-t}}.$$

Then the result inductively follows. ∎

The relation (48) may be used to analyze $Q_k(z;t,-\frac{1}{m})$ or $Q_k(z;t,\frac{1}{m})$ due to the compound geometric nature of $Q_k(z;t,-\frac{1}{m})$ and $Q_k(z;t,\frac{1}{m})$, but the details are awkward even for small k and m. As an example, it follows from Corollary 1 that one may write

$$Q_2\left(z;t,-\frac{1}{m}\right) = \left\{\sum_{j=1}^{m-1} w_{j,2}\left(t,-\frac{1}{m}\right) Q_1\left(z;t,-\frac{j}{m}\right)\right\} Q_1\left(z;t,-\frac{1}{m}\right),$$

which implies for $m=2$ that one must have $Q_2(z;t,-\frac{1}{2}) = \{Q_1(z;t,-\frac{1}{2})\}^2$. Analysis of $Q_k(z;t,r)$ for $r=\pm 1/2$ may be carried out as in Willmot (2013, Section 4).

5. Insurance applications

The special cases when $t = 0$ and $t = 1$ are useful in aggregate loss analysis. The discrete equilibrium distributions which result when $t = 1$ are useful for stop-loss analysis, as discussed in Willmot *et al.* (2005).

When $t = 0$, it follows from (6) that

$$P_{j,k+1}(z) = \frac{\sum_{j=k+1}^{\infty} p_j z^{j-k-1}}{\overline{P}_k}$$

is the pgf of $N - (k+1)|N > k$, which is of interest with a number of claims deductible. In the negative binomial case, the mixture representation (27) with $t = 0$ implies that $Q_{k+1}\{H(z); 0, r\}$ is a convenient model for aggregate claims analysis if r is a positive integer. From (43),

$$Q_1(z; 0, r) = \frac{(1+\beta-\beta z)^{-r} - (1+\beta)^{-r}}{z\{1-(1+\beta)^{-r}\}}, \tag{51}$$

a valid pgf for $r > -1$, and in fact is a good candidate as a model for claim counts in its own right. Also, recursive computational techniques are available for the associated compound distribution with pgf $Q_1\{H(z); 0, r\}$ for many choices of r, as discussed in Section 3.

To illustrate the use of these distributions in claim count model fitting and aggregate claims analysis, we consider the class with pgf (51).

For parameter estimation, we note that (51) may be expressed as

$$Q_1(z; 0, r) = \frac{V\left(\frac{\beta}{1+\beta} z\right)}{V\left(\frac{\beta}{1+\beta}\right)}, \tag{52}$$

where $V(z) = \{(1-z)^{-r} - 1\}/z$. Thus, $Q_1(z; 0, r)$ is the pgf of a generalized power-series distribution, implying that for an independent and identically distributed sample, the maximum likelihood estimate (mle) of the mean is the sample mean $\overline{X}$ (e.g. Ord, 1972, pp. 117–118). It is clear from (51) that the mean $\mu = \frac{\partial}{\partial z} Q_1(z; 0, r)|_{z=1}$ is

$$\mu = \frac{r\beta}{1-(1+\beta)^{-r}} - 1. \tag{53}$$

Hence, the mle of β satisfies (for r known)

$$\overline{X} + 1 = \frac{r\hat{\beta}}{1-\left(1+\hat{\beta}\right)^{-r}}. \tag{54}$$

Thus, if $r = -1/2$, $\hat{\beta} = 4\overline{X}(\overline{X}+1)$ or if $r = 1/2$, $\hat{\beta} = \frac{1}{2}(4\overline{X} + 3 - \sqrt{8\overline{X}+9})$.

TABLE 1

CLAIM COUNT MODEL FITTING — TRÖBLIGER DATA.

j	No. of drivers	Fitted number of drivers $r = 1/2$	$r = 2$	$r = 3$
0	20,592	20,657	20,574	20,552
1	2,651	2,530	2,670	2,707
2	297	345	308	297
3	41	49	33	30
4	7	7	4	3
5	0	1	0	0
6	1	0	0	0
7+	0	0	0	0
$\hat{\beta}$		0.1952	0.0947	0.0705
χ^2		13.976	4.436	8.996
DF		3	2	2

To illustrate the count data fitting with a pgf (51) for $r = \pm 1/m$ or $r = m$ where $m = 2, 3, \ldots$, and the corresponding mle of $\hat{\beta}$ calculated with the sample mean as given above, consider the following data set of the number of claims/year (j) for automobile insurance given by Tröbliger (1961) (e.g. Klugman *et al.*, 2008, Example 15.31). The sample mean $\overline{X}$ is 0.14422, and thus the mle of β is calculated using (54) for each r. Then, the fitted pmfs with a pgf (51) for each pair of parameters $(r, \hat{\beta})$ are obtained. In turn, the fitted number of drivers (the fitted pmfs multiplied by the total number of drivers 23,589) is provided in Table 1. Some rounding errors may exist. With three degrees of freedom (DF), the critical value at 5% is 7.814 and at a 1% significant level is 11.345. With DF $= 2$, the critical value at 5% significant level is 5.991, and at a 1% significant level is 9.210. Therefore, the model with $r = 2$ is a good fit at 5% significant level, and the one with $r = 3$ is an adequate fit at a 1% significant level.

For aggregate claims analysis, note that from (46),

$$Q_1\left\{H(z); 0, -\frac{1}{2}\right\} = \left[1 - \beta\left(0, -\frac{1}{2}\right)\left(G\left\{H(z); -\frac{1}{2}\right\} - 1\right)\right]^{-1}. \tag{55}$$

Thus, for the compound distribution with pgf (55), one can first compute the compound ETNB distribution with pgf $G\{H(z); -\frac{1}{2}\}$ using a Panjer-type recursion (e.g. Klugman *et al.*, 2008, p. 129), and then use a second compound geometric recursion to compute the distribution with pgf (55) with parameter $\beta(0, -1/2) = \{\sqrt{1+\beta} - 1\}/\{\sqrt{1+\beta} + 1\}$. Similarly, for $r = 1/2$, from (39) and (40) one has

$$Q_1\left\{H(z); 0, \frac{1}{2}\right\} = \left[1 - \beta\left(0, \frac{1}{2}\right)\left(G\left\{H(z); -\frac{1}{2}\right\} - 1\right)\right]^{-1}, \tag{56}$$

where $\beta(0, 1/2) = \sum_{i=1}^{2}\{1 - (1+\beta)^{-i/2}\}/\sum_{j=1}^{2}(1+\beta)^{-j/2}$. Finally, for $r = 2$ one finds from (27) and (28)

$$Q_1\{H(z); 0, 2\} = w_{1,1}(0, 2)\, Q\{H(z); 1\} + w_{2,1}(0, 2)\, Q\{H(z); 2\}, \qquad (57)$$

where mixing weights are given by $w_{n,1}(0, 2) = (1+\beta)^n/\{\sum_{i=1}^{2}(1+\beta)^i\}$ for $n = 1, 2$.

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