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ON MEAN VALUES OF RANDOM MULTIPLICATIVE FUNCTIONS

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Abstract. Let $\mathscr{P}$ denote the set of primes and $\{f(p)\}_{p\in\mathscr{P}}$ be a sequence of independent Bernoulli random variables taking values ± 1 with probability $1/2$. Extending f by multiplicativity to a random multiplicative function f supported on the set of squarefree integers, we prove that, for any $\varepsilon > 0$, the estimate $\sum_{n\leqslant x} f(n) \ll \sqrt{x}\,(\log\log x)^{3/2+\varepsilon}$ holds almost surely, thus qualitatively matching the law of the iterated logarithm, valid for independent variables. This improves on corresponding results by Wintner, Erdős and Halász.

1. Introduction

In many problems of an arithmetic nature, probabilistic models serve as heuristic support, sometimes leading to plain solutions. For instance, the link between the distribution of zeros of the Riemann ζ-function and random matrix theory has been extensively studied in recent years; see in particular Montgomery's pioneering article [11], and Katz and Sarnak's important monograph [9] for a more general theory.

It is well known that equivalent forms of the Riemann hypothesis (RH) may be stated in terms of mean values of multiplicative functions. The latest result in this direction, due to Soundararajan [15], states that, if μ designates the Möbius function, RH holds if, and only if, we have

$$\sum_{n\leqslant x}\mu(n) \ll_\varepsilon \sqrt{x}\,\mathrm{e}^{(\log x)^{1/2+\varepsilon}} \qquad (x\geqslant 3), \tag{1.1}$$

for all $\varepsilon > 0$. The best known estimate to date in this direction is the Korobov–Vinogradov bound

$$\sum_{n\leqslant x}\mu(n) \ll x\mathrm{e}^{-c_1(\log x)^{3/5}(\log_2 x)^{-1/5}} \qquad (x\geqslant 3),$$

where c_1 is a positive constant. Here and in the sequel, we let $\log_k$ denote the k-fold iterated logarithm.

A probabilistic approach to this question is therefore of great interest. It has been stated by many authors that RH is almost always true. However such a statement heavily depends on the nature of the random model that is chosen to

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represent the Möbius function. If one selects random independent signs ε_n, then the desired bound follows from a well-known theorem of Khintchine and Kolmogorov according to which a series $\sum_{n\geqslant 1}\varepsilon_n/n^\sigma$ is almost always convergent if, and only if, $\sigma > 1/2$. A more precise, and actually optimal, quantitative form is given by the law of the iterated logarithm which provides the exact maximal order for $|\sum_{n\leqslant x}\varepsilon_n|$ i.e. $\sqrt{\{2+o(1)\}x\log_2 x}$; see for instance [14], p. 397.

However, as observed by Lévy [10], such a model provides only limited hints from an arithmetical viewpoint since ε_n does not depend on n in a multiplicative manner. This led Wintner [17] to consider a setting that avoids Lévy's objection, thus laying the foundation for random multiplicative function theory.

Let $(\Omega, \mathscr{T}, \mathbb{P})$ be a probability space, let $\mathscr{P}$ denote the set of primes, and let $\{f(p)\}_{p\in\mathscr{P}}$ be a sequence of independent Bernoulli random variables on Ω taking ± 1 both with probability $1/2$. For each positive integer n, we may define a random variable $f(n)$ on Ω by

$$f(n) := \mu(n)^2 \prod_{p|n} f(p). \tag{1.2}$$

Clearly, $n \mapsto f(n)$ is multiplicative, so $f(n)$ is a random multiplicative function. Of course, the probability that $f = \mu$ is zero, but it may be noticed that

$$d_n := \prod_{p|n} p^{(1-f(p))/2}$$

is a random squarefree divisor of n assuming each possible value with uniform probability $1/2^{\omega(n)}$, where $\omega(n)$ denotes the total number of distinct prime factors of n, and that

$$f(n) = \mu(n)^2\mu(d_n) \qquad (n \geqslant 1). \tag{1.3}$$

The quantity

$$M_f(x) := \sum_{n\leqslant x} f(n)$$

thus measures the number of cancellations arising from the multiplicative structure of the random function f. As a heuristic support for (1.1), Wintner [17] obtained the upper bound $M_f(x) \ll x^{1/2+\varepsilon}$ for any $\varepsilon > 0$ almost surely and showed that $M_f(x) \ll x^{1/2-\varepsilon}$ is false almost surely, this latter property being shared by the Möbius function. He also noted that "the chasm between [the upper and lower bound] could perhaps be bridged by an arithmetical counterpart of Khintchine's law of the iterated logarithm".

Erdős (unpublished, see [5]) investigated in greater detail how the number-theoretic dependence among the $f(n)$ affects the magnitude of $M_f(x)$. He showed that the factors x^ε and $x^{-\varepsilon}$ may be replaced, still almost surely, by $(\log x)^{c_2}$ and $(\log x)^{-c_3}$ for some positive constants c_2 and c_3.

Halász [7] made an important step forward by proving that, for suitable positive constants c_4, c_5, we have almost surely

$$M_f(x) \ll \sqrt{x}\mathrm{e}^{c_4\sqrt{\log_2 x\log_3 x}} \tag{1.4}$$

while

$$M_f(x) \ll \sqrt{x}\mathrm{e}^{-c_5\sqrt{\log_2 x\log_3 x}} \tag{1.5}$$

is false almost surely.

In a very recent paper [8], Harper improved (1.5) to the assertion that, for each $\varepsilon > 0$,

$$M_f(x) \ll \sqrt{x}/(\log_2 x)^{5/2+\varepsilon} \tag{1.6}$$

is almost surely false.

Of course, these estimates still fall short of any conjectural bound based on the law of the iterated logarithm, or on the belief that dependence actually reduces the expected size: see problem 26, due to Halász, in the appendix of Montgomery's monograph [12], where it is asked whether the bound $M_f(x) \ll \sqrt{x}$ holds almost surely.

In this paper, our aim is to investigate how close one can get to optimality for an almost sure upper bound. Improving on Halász's estimates (1.4), we show that a power of an iterated logarithm is valid, on a set of probability 1, as an upper bound for the slowly varying factor, and hence that the multiplicative structure does not disrupt statistical cancellations in a significant way. To decide whether it actually increases the amount of cancellations remains an interesting open problem. However, this could only happen in a relatively narrow range, as shown by (1.6).

We start by setting up a slightly more general probabilistic model, in which the $f(n)$ may vanish. Let $\{f(p)\}_{p\in\mathscr{P}}$ be a sequence of independent random variables on $(\Omega, \mathscr{T}, \mathbb{P})$ such that

$$\mathbb{P}(\{f(p)=1\}) = \mathbb{P}(\{f(p)=-1\}) = \tfrac{1}{2}\kappa_p, \qquad \mathbb{P}(\{f(p)=0\}) = 1-\kappa_p, \tag{1.7}$$

where $\kappa_p \in [0,1]$ fulfils the following condition, where c is a positive constant:

$$\sum_{p\leqslant x} \kappa_p \log p = x + O\big(x\mathrm{e}^{-2c\sqrt{\log x}}\big) \qquad (x \geqslant 2). \tag{1.8}$$

We obtain a random multiplicative function $f(n)$ by (1.2). Selecting $\kappa_p = 1$ for all primes p, we recover Wintner's probabilistic model. With the choice $\kappa_p = p/(p+1)$, we obtain the probabilistic model for a real primitive Dirichlet character, as defined by Granville and Soundararajan [6]—see also [18]—with the slight difference that our f has support included in the set of squarefree integers.

Theorem 1.1. *Let $\varepsilon > 0$. As $x \to \infty$, we have almost surely*

$$M_f(x) \ll \sqrt{x}\,(\log_2 x)^{3/2+\varepsilon}. \tag{1.9}$$

The special case $\kappa_p \equiv 1$ of Theorem 1.1 provides a significant improvement over the estimate (1.4). In particular, our bound now pertains to the scale predicted by the law of the iterated logarithm. In short, our result shows that random signs behave in a comparable way whether or not a multiplicative structure is imposed.

Based on Halász's method, our upper bound is obtained by following Halász's suggestion [7] for removing the $\log_3 x$ from (1.4). With some specific, new refinements, we show that this idea leads to a much larger gain than expected; compare Lemma 3(ii) of [7] to Lemma 3.1 below.

It is also valuable to note, as did Erdős and Halász, that, in the case $\kappa_p = 1$, the $f(p)$ may be realized as Rademacher functions. Thus, all results in this theory find a natural interpretation in the theory of orthogonal series.

2. Preliminary estimates

In the sequel of this work, we let c_j ($j = 0, 1, 2, \dots$) denote suitable positive absolute constants.

Recall that $\omega(n)$ denotes the number of distinct prime factors of an integer n. For real, positive numbers m, u, v, y, z, we define

$$S_m = S_m(u, v; y, z) := \sum_{u<d\leqslant v}^{y,z} \mu(d)^2 m^{\omega(d)},$$

where the symbol $\sum^{y,z}$ indicates a sum restricted to integers all of whose prime factors belong to the interval $]y, z]$.

Lemma 2.1. *Let $\delta \in]0, 1[$. For $y \geqslant 3$, $m \geqslant 1$, $1 < u \leqslant v(1 - 1/y^{1-\delta})$, $y < z \leqslant y^2$, we have*

$$S_m \ll \frac{(v-u)m}{\log y} \sum_{u/z<r\leqslant v/y}^{y,z} \frac{m^{\omega(r)}}{r}\cdot \tag{2.1}$$

Proof. We may assume with no loss of generality that $v > y$ for S_m otherwise vanishes. Write $d = rp$, where p is the largest prime factor of d. We plainly have

$$S_m \leqslant m \sum_{u/z<r\leqslant v/y}^{y,z} m^{\omega(r)} \sum_{u/r<p\leqslant v/r} 1.$$

Since $(v-u)/r \geqslant y(v-u)/v \geqslant y^\delta$, the Brun–Titchmarsh theorem implies that the inner sum is $\ll (v-u)/(r \log y)$. □

The main aim of this section is to prove Lemma 2.3 below. For this we need to estimate moments, and so appeal to the following form of a result of Bonami [2], for which Halász [7] provided an alternate proof (Lemma 2).

Lemma 2.2. *Let $f(n)$ be defined by (1.7), (1.8) and (1.2). For $m \in \mathbb{N}^*$ and $a_j \in \mathbb{C}^{\mathbb{N}^*}$ $(1 \leqslant j \leqslant m)$, we have*

$$\left|\mathbb{E}\Big(\prod_{1\leqslant j\leqslant m} \sum_{n\geqslant 1} a_j(n) f(n)\Big)\right| \leqslant \Big(\prod_{1\leqslant j\leqslant m} \sum_{n\geqslant 1} |a_j(n)|^2 \kappa(n)^{2/m} (m-1)^{\omega(n)}\Big)^{1/2}, \tag{2.2}$$

where

$$\kappa(n) := \mu(n)^2 \prod_{p\mid n} \kappa_p. \tag{2.3}$$

Moreover, we have

$$\mathbb{E}\big(M_f(x)^2\big) \sim cx \qquad (x \to \infty) \tag{2.4}$$

with

$$c := \prod_{p\in\mathscr{P}} (1 + \kappa_p/p)(1 - 1/p),$$

and, uniformly for $v \geqslant u + 1 \geqslant 2$,

$$\mathbb{E}\big(\{M_f(v) - M_f(u)\}^4\big) \ll v^{2/3}(v-u)^{4/3}(\log v)^{52/3}. \tag{2.5}$$

Proof. When $\kappa_p = 1$ for all p, the bound (2.2) follows immediately from the induction hypothesis appearing in the proof of Lemma 2 of [7]. Letting Q denote the set of integral squares and using an asterisk to indicate that a summation is restricted to squarefree integers, it may be written as

$$\Big|\sideset{}{^*}\sum_{\substack{n_1\geqslant 1\\ n_1\cdots n_m\in Q}}\cdots\sideset{}{^*}\sum_{n_m\geqslant 1}\prod_{1\leqslant j\leqslant m} a_j(n_j)\Big| \leqslant \Big(\prod_{1\leqslant j\leqslant m}\sideset{}{^*}\sum_{n\geqslant 1}|a_j(n)|^2(m-1)^{\omega(n)}\Big)^{1/2}. \tag{2.6}$$

Consider the general case. We have

$$\mathbb{E}\Big(\prod_{1\leqslant j\leqslant m}\sum_{n\geqslant 1} a_j(n)f(n)\Big) = \sideset{}{^*}\sum_{\substack{n_1\geqslant 1\\ n_1\cdots n_m\in Q}}\cdots\sideset{}{^*}\sum_{n_m\geqslant 1}\prod_{1\leqslant j\leqslant m} a_j(n_j)\prod_{p\mid n_1\cdots n_m}\kappa_p. \tag{2.7}$$

As $0\leqslant \kappa\leqslant 1$, we plainly have $\prod_{p\mid n_1\cdots n_m}\kappa_p \leqslant \prod_{1\leqslant j\leqslant m}\kappa(n_j)^{1/m}$ for all squarefree n_j $(1\leqslant j\leqslant m)$, with equality if, and only if, $n_1=\cdots=n_m$. Thus, the modulus of the left-hand side of (2.7) does not exceed

$$\leqslant \sideset{}{^*}\sum_{\substack{n_1\geqslant 1\\ n_1\cdots n_m\in Q}}\cdots\sideset{}{^*}\sum_{n_m\geqslant 1}\prod_{1\leqslant j\leqslant m}|a_j(n_j)|\kappa(n_j)^{1/m}$$

and the bound (2.2) follows from (2.6).

Selecting $m=2$ and $a_1(n)=a_2(n)=1$ in (2.7), we get

$$\mathbb{E}\big(M_f(x)^2\big) = \sum_{n\leqslant x}\kappa(n).$$

The asymptotic formula (2.4) is hence an immediate consequence of a classical theorem on multiplicative functions with values in $[0,1]$; see for instance [16], Theorem I.3.12.

Finally, since $0\leqslant\kappa\leqslant 1$, relation (2.2) with $m=4$ and Hölder's inequality imply that

$$\mathbb{E}\big(\{M_f(v)-M_f(u)\}^4\big)\leqslant\Big(\sum_{u<n\leqslant v}3^{\omega(n)}\Big)^2\leqslant\Big(\sum_{n\leqslant v}27^{\omega(n)}\Big)^{2/3}\Big(\sum_{u<n\leqslant v}1\Big)^{4/3}.$$

Therefore, the required bound (2.5) follows from the classical estimate

$$\sum_{n\leqslant v}27^{\omega(n)}\ll v(\log v)^{26}. \qquad \square$$

Remark. As mentioned in [7], Bonami [2] proved the following variant of (2.2):

$$\mathbb{E}\Big|\sum_{n\geqslant 1}a(n)f(n)\Big|^m\leqslant\Big(\sum_{n\geqslant 1}|a(n)|^2(m-1)^{\omega(n)}\Big)^{m/2} \tag{2.8}$$

in which $a\in\mathbb{C}^{\mathbb{N}^*}$ and m may assume any real value $\geqslant 2$. We shall not need such a generalization in this work.

According to (2.4), the expected order of $M_f(x)$ is $\sqrt{x}$. The next lemma, essentially identical to Lemma 1 of [7] (and the proof of which we provide for mere convenience), shows that, almost surely, this quantity fluctuates moderately in appropriate short intervals; in other words, the problem of bounding $M_f(x)$ everywhere may be reduced to doing so at suitable test-points.

Lemma 2.3. *Let $f(n)$ be defined by* (1.7), (1.8) *and* (1.2). *For any fixed constant $A > 0$, there is a suitable constant $c_6 = c_6(A) \in \,]0,1[$ such that, for*

$$x_i := \left\lfloor e^{i^{c_6}} \right\rfloor \qquad (i \geqslant 1), \tag{2.9}$$

we have almost surely

$$\max_{x_{i-1}<x\leqslant x_i} |M_f(x) - M_f(x_{i-1})| \ll_{A,f} \frac{\sqrt{x_i}}{(\log x_i)^A} \qquad (i \geqslant 1).$$

Proof. Assume that

$$\max_{x_{i-1}<x\leqslant x_i} \big|M_f(x) - M_f(x_{i-1})\big| > 2\sqrt{x_i}/(\log x_i)^A \tag{2.10}$$

and that the maximum is attained at some integer

$$x = x_{i-1} + \sum_{1\leqslant j\leqslant h} 2^{\nu_j},$$

where $\{\nu_j\}_{j=1}^h \in \mathbb{N}^h$ and $\nu_1 > \cdots > \nu_h \geqslant 0$. We split $[x_{i-1}, x]$ into a disjoint union of subintervals with limit points

$$u_k = x_{i-1} + \sum_{1\leqslant j\leqslant k} 2^{\nu_j} \qquad (0 \leqslant k \leqslant h).$$

Thus, there exists a pair $\{u_k,\, u_{k+1}\}$ such that

$$\big|M_f(u_{k+1}) - M_f(u_k)\big| > \sqrt{x_i}/(\log x_i)^{A+1}, \tag{2.11}$$

for the number of these subintervals does not exceed $1 + \{\log(x_i - x_{i-1})\}/\log 2 < 2\log x_i$. Note that

$$u_k = x_{i-1} + (\ell-1)2^m, \qquad u_{k+1} = x_{i-1} + \ell 2^m, \tag{2.12}$$

with $\ell = \sum_{1\leqslant j\leqslant k} 2^{\nu_j-\nu_k}$ and $m = \nu_k$.

Next, we bound the total probability of the occurrence of (2.11) when (2.12) holds for some ℓ and m; this clearly dominates the probability of (2.10).

By Markov's inequality for the fourth moment and (2.5), we may write

$$\mathbb{P}\Big(\big|M_f(v) - M_f(u)\big| > \sqrt{x_i}/(\log x_i)^{A+1}\Big) \ll \left(\frac{v-u}{x_i}\right)^{4/3} (\log x_i)^{4A+64/3}. \tag{2.13}$$

Let $u = x_{i-1} + (\ell-1)2^m$ and $v = x_{i-1} + \ell 2^m$, where $\ell \geqslant 1$, $m \geqslant 0$ and $\ell 2^m \leqslant x_i - x_{i-1}$. Then by (2.13), the probability that (2.11) holds for some u_k, u_{k+1} of the form (2.12) is

$$\begin{aligned}
&\ll \sum\sum_{\ell 2^m\leqslant x_i-x_{i-1}} \left(\frac{2^m}{x_i}\right)^{4/3} (\log x_i)^{4A+64/3} \\
&\ll (\log x_i)^{4A+64/3} \sum_{2^m\leqslant x_i-x_{i-1}} \left(\frac{2^m}{x_i}\right)^{4/3} \frac{x_i - x_{i-1}}{2^m} \\
&\ll \left(\frac{x_i - x_{i-1}}{x_i}\right)^{4/3} (\log x_i)^{4A+64/3}.
\end{aligned} \tag{2.14}$$

Set $c_6 := 1/(272+48A)$. As $(\log x_i)^{4A+64/3} \leqslant i^{(4A+64/3)c_6}$ and $(x_i - x_{i-1})/x_i \ll 1/i^{1-c_6}$, we deduce that (2.14) is $\ll i^{-5/4}$ and hence that the same bound holds for the probability of the event (2.10). The proof is completed by the Borel-Cantelli lemma; see, e.g., [4], Theorem 4.2.1. □

Remark. Chatterjee and Soundararajan ([3], Prop. 3.1) evaluate

$$\mathbb{E}(\{M_f(v) - M_f(u)\}^4)$$

for short intervals $]u, v]$. However, inserting this estimate into the above proof would only yield to an improvement on the value of the constant c_6, with no influence on the final exponent $3/2$ appearing in Theorem 1.1.

3. Proof of Theorem 1.1

We first establish an average estimate improving significantly over the corresponding bound obtained by Halász; see [7], formula (2).

Lemma 3.1. *Let $f(n)$ be defined by* (1.7), (1.8) *and* (1.2) *and let $\{x_i\}_{i\geqslant 1}$ be given by* (2.9)*. Then, for any $\varepsilon > 0$, we have almost surely*

$$\frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} M_f(x)\,\mathrm{d}x \ll_{f,\varepsilon} \sqrt{x_i}\,(\log_2 x_i)^{3/2+\varepsilon} \qquad (i \geqslant 1).$$

Proof. We show that large values of the integral occur with small probability and conclude by the Borel-Cantelli lemma.

We have $x_i = \lfloor e^{i^{c_6}} \rfloor$. Given a large constant ℓ_0, we put $X_\ell := \mathrm{e}^{2^\ell}$ for $\ell \geqslant \ell_0$, so that $X_{\ell-1} = X_\ell^{1/2}$. We consider those x_i lying in $]X_{\ell-1}, X_\ell]$, and for $\alpha \in]0, \frac{1}{2}[$, write

$$y_0 := \exp\left\{\frac{c_6 2^\ell}{4\ell}\right\}, \quad y_j := y_{j-1}^{\mathrm{e}^\alpha} = y_0^{\mathrm{e}^{\alpha j}} \quad (j \geqslant 1), \tag{3.1}$$

and observe that, if J is minimal under the constraint $y_J \geqslant X_\ell$, then

$$J \leqslant 1 + \frac{\log\{4\ell/c_6\}}{\alpha} \ll \frac{\log \ell}{\alpha}\cdot$$

Employing the notation

$$\Psi_f(x, y) := \sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} f(n) \qquad (x \geqslant 0,\, y \geqslant 1),$$

we split the sum $M_f(x)$ according to the size of the largest prime factor $P(n)$ of the summation variable n. For $x \in [x_{i-1}, x_i]$, we thus obtain

$$\begin{aligned} M_f(x) &= \sum_{\substack{n\leqslant x\\ P(n)\leqslant y_0}} f(n) + \sum_{1\leqslant j\leqslant J} \sum_{y_{j-1}<d\leqslant x_i}^{y_{j-1},y_j} \sum_{\substack{m\leqslant x/d\\ P(m)\leqslant y_{j-1}}} f(dm) \\ &= \Psi_f(x, y_0) + \sum_{1\leqslant j\leqslant J} \sum_{y_{j-1}<d\leqslant x_i}^{y_{j-1},y_j} f(d)\Psi_f(x/d, y_{j-1}). \end{aligned}$$

Setting $\delta_i := x_i - x_{i-1}$, we express accordingly the integral to be bounded as follows:

$$\frac{1}{\delta_i} \int_{x_{i-1}}^{x_i} M_f(x)\,\mathrm{d}x = N_{i0}(f) + \sum_{1\leqslant j\leqslant J} N_{ij}(f), \tag{3.2}$$

where

$$N_{i0}(f) := \frac{1}{\delta_i}\int_{x_{i-1}}^{x_i} \Psi_f(x, y_0)\,\mathrm{d}x, \tag{3.3}$$

$$N_{ij}(f) := \sum_{y_{j-1}<d\leqslant x_i}^{y_{j-1},y_j} b_{ij}(d;f)f(d) \quad (j \geqslant 1) \tag{3.4}$$

with

$$b_{ij}(d;f) := \frac{1}{\delta_i}\int_{x_{i-1}}^{x_i} \Psi_f(x/d, y_{j-1})\,\mathrm{d}x.$$

We first establish an upper bound for the probability of the event

$$\mathscr{A} = \mathscr{A}_\ell(R) := \bigcup_{X_{\ell-1}<x_i\leqslant X_\ell} \left\{\frac{1}{\delta_i}\left|\int_{x_{i-1}}^{x_i} M_f(x)\,\mathrm{d}x\right| \geqslant 2\sqrt{x_i}R\right\}. \tag{3.5}$$

To this end, we define

$$\begin{aligned} \mathscr{B}_0 = \mathscr{B}_0(R;\ell) &:= \bigcup_{X_{\ell-1}<x_i\leqslant X_\ell} \left\{ |N_{i0}(f)| \geqslant \sqrt{x_i}R\right\}, \\ \mathscr{B}_1 = \mathscr{B}_1(R;\ell) &:= \bigcup_{X_{\ell-1}<x_i\leqslant X_\ell} \left\{ \sum_{1\leqslant j\leqslant J} |N_{ij}(f)| \geqslant \sqrt{x_i}R\right\}. \end{aligned} \tag{3.6}$$

Clearly $\mathscr{A} \subset \mathscr{B}_0 \cup \mathscr{B}_1$, so

$$\mathbb{P}(\mathscr{A}) \leqslant \mathbb{P}(\mathscr{B}_0) + \mathbb{P}(\mathscr{B}_1). \tag{3.7}$$

We first estimate $\mathscr{B}_1$. Following Halász, we consider the filtration $\{\mathscr{T}(y)\}_{y\geqslant 1}$, where $\mathscr{T}(y)$ denotes the σ-algebra generated by the variables $f(p)$ with $p \leqslant y$. Since $0 \leqslant \kappa \leqslant 1$, we may deduce from Lemma 2.2 that, for any integer $m \geqslant 1$, we have

$$\mathbb{E}\big(\big|N_{ij}(f)\big|^{2m} \mid \mathscr{T}(y_{j-1})\big) \leqslant \big(x_i D_{ij}(f)\big)^m, \tag{3.8}$$

where

$$D_{ij}(f) := \frac{1}{x_i}\sum_{y_{j-1}<d\leqslant x_i}^{y_{j-1},y_j} b_{ij}(d;f)^2\mu(d)^2(2m-1)^{\omega(d)}.$$

From the Cauchy-Schwarz inequality, we see that

$$b_{ij}(d;f)^2 \leqslant \frac{1}{\delta_i}\int_{x_{i-1}}^{x_i} \Psi_f(x/d, y_{j-1})^2\,\mathrm{d}x \leqslant \frac{x_i}{\delta_i}\int_{x_{i-1}/d}^{x_i/d} \frac{\Psi_f(t, y_{j-1})^2}{t}\,\mathrm{d}t,$$

where we made the change of variable $t := x/d$ and used the inequality $d = x/t \leqslant x_i/t$. Therefore

$$D_{ij}(f) \leqslant D_{ij}^*(f) := \frac{1}{\delta_i}\int_1^{x_i/y_{j-1}} \sum_{x_{i-1}/t<d\leqslant x_i/t}^{y_{j-1},y_j} \mu(d)^2(2m-1)^{\omega(d)}\frac{\Psi_f(t, y_{j-1})^2}{t}\,\mathrm{d}t. \tag{3.9}$$

Applying Lemma 2.1 to the right-hand side of (3.9), we obtain

$$\begin{aligned} D_{ij}^*(f) &\ll \frac{1}{\log y_{j-1}} \sum_{r\leqslant X_\ell}^{y_{j-1},y_j} \frac{(2m-1)^{\omega(r)+1}}{r} \int_{x_{i-1}/ry_j}^{x_i/ry_{j-1}} \frac{\Psi_f(t,y_{j-1})^2}{t^2}\,\mathrm{d}t \\ &\ll \frac{m}{\log y_{j-1}} \int_1^{X_\ell/y_{j-1}} \sum_{x_{i-1}/ty_j<r\leqslant x_i/ty_{j-1}}^{y_{j-1},y_j} \frac{(2m-1)^{\omega(r)}}{r} \frac{\Psi_f(t,y_{j-1})^2}{t^2}\,\mathrm{d}t. \end{aligned}$$

Now, we observe that, for $y<\min(z,w)$, $m\geqslant 1$, we have

$$\sum_{r\leqslant w}^{y,z}(2m-1)^{\omega(r)} \ll \sum_{s\leqslant w}^{y,z}(2m-2)^{\omega(s)} \sum_{d\leqslant w/s}^{y,w} 1 \ll \frac{w}{\log y} \prod_{y<p\leqslant z}\Big(1+\frac{2m-2}{p}\Big), \tag{3.10}$$

from which we deduce by partial summation that

$$\sum_{x_{i-1}/ty_j<r\leqslant x_i/ty_{j-1}}^{y_{j-1},y_j} \frac{(2m-1)^{\omega(r)}}{r} \ll \alpha \mathrm{e}^{c_6\alpha m}$$

provided $\alpha \gg 1/2^\ell \gg 1/\log y_0$. Writing

$$I_{j\ell} := \int_1^{X_\ell} \frac{\Psi_f(t,y_{j-1})^2}{t^2}\,\mathrm{d}t,$$

we thus have

$$D_{ij}^*(f)^m \leqslant \Big(\frac{c_8\alpha m \mathrm{e}^{c_7\alpha m} I_{j\ell}}{\log y_{j-1}}\Big)^m. \tag{3.11}$$

Let $T\geqslant 1$ be specified later. Defining the events

$$\mathscr{C}_j := \Big\{I_{j\ell} \leqslant T\log y_{j-1}\Big\} \quad (1\leqslant j\leqslant J), \quad \mathscr{C} := \bigcap_{1\leqslant j\leqslant J} \mathscr{C}_j, \tag{3.12}$$

we plainly have

$$\mathscr{B}_1 \subset (\mathscr{B}_1 \cap \mathscr{C}) \cup \overline{\mathscr{C}}. \tag{3.13}$$

By Lemma 2.2 and the classical estimate ([16], Th. III.5.1)

$$\Psi(x,y) := \sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} 1 \ll x^{1-1/(2\log y)} \qquad (x\geqslant y\geqslant 2),$$

we have $\mathbb{E}\big(I_{j\ell}\big) \ll \log y_{j-1}$, whence

$$\mathbb{P}(\overline{\mathscr{C}_j}) \ll \frac{1}{T}\cdot \tag{3.14}$$

Moreover, since, as first noticed by Basquin [1], $\{I_{j\ell}\}_{j=0}^J$ is a submartingale with respect to the filtration $\{\mathscr{T}(y_j) : 1\leqslant j\leqslant J\}$, we actually deduce from Doob's inequality (see, e.g., [13], Theorem II.1.7) that

$$\mathbb{P}(\overline{\mathscr{C}}) \ll \frac{\log \ell}{T}\cdot \tag{3.15}$$

Indeed,

$$\mathbb{P}\Big(\sup_{2^r<\log y_{j-1}\leqslant 2^{r+1}} \frac{I_{j\ell}}{\log y_{j-1}} > T\Big) \ll \frac{1}{T}$$

for each integer r such that $c_6 2^\ell/(4\ell\log 2) < 2^r \leqslant 2^\ell$. Since there are only $\ll \log\ell$ possible values of r, (3.15) follows.

Applying Hölder's inequality in the form

$$\Big(\sum_{1\leqslant j\leqslant J} |N_{ij}(f)|\Big)^{2m} \leqslant J^{2m-1}\sum_{1\leqslant j\leqslant J}|N_{ij}(f)|^{2m},$$

we derive from (3.6), (3.8), (3.9) and (3.11) that

$$\begin{aligned}\mathbb{P}(\mathscr{B}_1\cap\mathscr{C}) \leqslant \mathbb{P}(\mathscr{B}_1|\mathscr{C}) &\leqslant \sum_{X_{\ell-1}<x_i\leqslant X_\ell}\ \sum_{1\leqslant j\leqslant J}\frac{\mathbb{E}\big(|N_{ij}(f)|^{2m}\mid\mathscr{C}_j\big)J^{2m-1}}{\big(x_iR^2\big)^m}\\ &\ll 2^{\ell/c_6}\Big(\frac{c_8TJ^2\alpha m\mathrm{e}^{c_7\alpha m}}{R^2}\Big)^m.\end{aligned}\tag{3.16}$$

Finally, we bound $\mathbb{P}(\mathscr{B}_0)$. Using the Cauchy-Schwarz inequality, we get, as before,

$$\begin{aligned}\mathbb{E}\big(N_{i0}(f)^2\big) &\leqslant \frac{1}{\delta_i}\int_{x_{i-1}}^{x_i}\mathbb{E}\big(\Psi_f(x,y_0)^2\big)\,\mathrm{d}x\\ &\leqslant \frac{1}{\delta_i}\int_{x_{i-1}}^{x_i}\Psi(x,y_0)\,\mathrm{d}x \ll x_i\mathrm{e}^{-(\log X_\ell)/(4\log y_0)} = x_i2^{-\ell/c_6}\end{aligned}$$

since $x_i\geqslant X_{\ell-1}=X_\ell^{1/2}$. By Markov's inequality, we then deduce that

$$\begin{aligned}\mathbb{P}\big(\mathscr{B}_0\big) &\leqslant \sum_{X_{\ell-1}<x_i\leqslant X_\ell}\mathbb{P}\big(N_{i0}(f)\geqslant\sqrt{x_i}R\big)\\ &\leqslant \sum_{X_{\ell-1}<x_i\leqslant X_\ell}\frac{\mathbb{E}\big(N_{i0}(f)^2\big)}{(\sqrt{x_i}R)^2} \ll \frac{2^{-\ell/c_6}}{R^2}\sum_{X_{\ell-1}<x_i\leqslant X_\ell}1 \ll \frac{1}{R^2},\end{aligned}\tag{3.17}$$

by our choice for x_i and X_ℓ.

Collecting our estimates (3.13), (3.15), (3.16), (3.17) and inserting back into (3.7), we get

$$\mathbb{P}(\mathscr{A}) \ll \frac{1}{R^2} + 2^{\ell/c_6}\Big(\frac{c_8TJ^2\alpha m\mathrm{e}^{c_7\alpha m}}{R^2}\Big)^m + \frac{\log\ell}{T}.\tag{3.18}$$

Selecting

$$T:=\ell^{1+\varepsilon/2},\qquad R:=\ell^{3/2+\varepsilon},\qquad \alpha:=1/\ell,\qquad m:=\ell,$$

so that $J\ll\ell\log\ell$, we conclude that

$$\mathbb{P}(\mathscr{A}) \ll_\varepsilon \frac{\log\ell}{\ell^{1+\varepsilon/2}}.\tag{3.19}$$

Thus, the Borel-Cantelli lemma implies that

$$\mathbb{P}\Big(\limsup_{\ell\geqslant 1}\mathscr{A}_\ell(R)\Big)=0.$$

This finishes the proof of Lemma 3.1. □

Now we are ready to prove Theorem 1.1. From the identity

$$\int_{x_{i-1}}^{x_i} M_f(t)\mathrm{d}t = \delta_i M_f(x) - \delta_i\{M_f(x) - M_f(x_{i-1}\} + \int_{x_{i-1}}^{x_i} \big\{M_f(t) - M_f(x_{i-1})\big\}\mathrm{d}t,$$

we deduce that, for all $i \geqslant 1$ and all $x \in \,]x_{i-1},\, x_i]$, we have

$$|M_f(x)| \leqslant \frac{1}{\delta_i}\bigg|\int_{x_{i-1}}^{x_i} M_f(t)\mathrm{d}t\bigg| + 2 \max_{x_{i-1}<t\leqslant x_i} |M_f(t) - M_f(x_{i-1})|.$$

Write

$$E_\ell := \bigg\{ \sup_{X_{\ell-1}<x\leqslant X_\ell} \frac{|M_f(x)|}{\sqrt{x}\,(\log_2 x)^{3/2+\varepsilon}} > 4\bigg\}.$$

From the above upper bound, Lemma 2.3 with $A = 1$, and (3.19), we have

$$\mathbb{P}(E_\ell) \ll (\log \ell)/\ell^{1+\varepsilon/2}.$$

It hence follows from the Borel-Cantelli lemma that $\mathbb{P}\big(\limsup_{\ell\to\infty} E_\ell\big) = 0$, as required.

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