On the optimal dividend strategy in a regime-switching diffusion model

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Abstract

In this paper, we consider the optimal dividend strategy under the diffusion model with regime switching. In contrast to the classical risk theory, the dividends can only be paid at the arrival times of a Poisson process. By solving an auxiliary optimal problem, we show that the optimal strategy is the modulated barrier strategy. The value function can be obtained by iteration or by solving system of differential equations. We also provide a numerical example to illustrate the effects of the restriction on the timing of the payment of dividends.

Keywords: Random discrete time; Optimal dividend strategy;Regime switching; Modulated barrier strategy; Markov decision processes

1 Introduction

Since it was proposed by De **?**, the optimization of dividend strategy has become a classical and important problem in actuarial science. This problem is usually phrased as the management's problem of determining the optimal timing and size of dividend payments in the presence of bankruptcy risk. There is a vast literature on this topic. Most of them assume that the insurer can choose any time to pay the dividends, or the dividends can be paid continuously, and the ruin (stopping the business) occurs whenever the surplus is negative.

However, in practice, it is more reasonable that the dividends can only be paid at some discrete time points rather than continuously, and an insurer with a negative surplus maybe continue her business as usual until bankruptcy takes place. To capture these features, **?** and **?** assume that the surplus process can only be observed at random times. Then ruin can only occur and the dividends can only be paid at these random discrete observation times. With the assumption that the surplus process is observed at the arrival times of a Poisson process, **?** shows that the optimal strategy is a

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band strategy if the surplus process is modeled by a general Lévy process, and the optimal strategy reduces to the barrier strategy if the surplus process is a diffusion or the compound Poisson model withe exponential claims.

Recently, **?** proposes the Gamma-Omega model which extends the diffusion model in two ways. First, if the surplus x is negative, the probability of bankruptcy within $\mathrm{d}t$ time units is $\omega(x)\mathrm{d}t$, where $\omega(x) > 0$ is the decreasing bankruptcy rate function defined on $(-\infty, 0]$. Second, the dividends can only be paid to the shareholders at the arrival times of a Poisson process with rate $\gamma > 0$. **?** studies the optimal barrier strategy, and **?** proves that the optimal barrier strategy obtained by **?** is indeed the optimal strategy among all the admissible dividend strategies under the Gamma-Omega model.

In this paper, we consider the diffusion model with regime switching. Mainly, we consider the case where the dividends can only be paid at the arrival times of a modulated Poisson process (a Cox process) as in **?**, and ruin is still defined as in the classical risk theory, i.e., the company is ruined and has to go out of business whenever the surplus is negative. In **?** and our paper, the surplus processes are observed continuously, but we restrict ourselves to the case where the dividends can only be paid at some random discrete times. From this point of view, the problem considered in our paper is similar to **?**.

Under diffusion model with regime switching, the optimal dividend strategy is studied by **?** and **?**. While the former solves this problem with two regimes by the standard method, i.e., guessing a candidate optimal solution and then verifying its optimality, the latter solves a general case by following a different method. They construct the candidate value function by directly employing a dynamic programming equation, and prove that the value function is the fixed point of a certain contraction operator which is given with the initial data, derives an explicit iterative algorithm to calculate the value function, which 'decouples' the different regimes such that at any stage one-dimensional control problems are solved. In contrast to prove the value function is the fixed point of a contraction operator, we modify the procedure of **?** by constructing a sequence of functions that converges to the value function. Then we study the functions of this sequence by an auxiliary optimal problem which depends on only one regime. With such a sequence, we do not need to find priori bounds for the value function (or the initial data of the contraction operator), which is required in **?**. The idea of introducing such a sequence is stimulated by **?** and **?** which consider the optimal control problem under piecewise deterministic processes. In fact, by this method, we reduce the original problem to a Markov decision process (MDP) * which is also used in **?**. Similar to **?** and **?**, our optimal strategy is still the modulated barrier strategy.

The remainder of the paper is organized as follows. In Section 2 we present the model and the problem. In Section 3, we introduce a sequence of functions that converges to the value function, and prove the dynamic programming equation. And the original problem is reduced to an MDP. In Section 4, in order to study the sequence constructed in Section 3, we study an auxiliary optimal problem which is the one-stage problem of the MDP. In Section 5, we go back to our original optimal problem. We show two methods to get the value function and the optimal barrier levels.

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2 The Model

Suppose that $\{J(t)\}_{t\geq 0}$ be a homogenous, irreducible continuous-time Markov chain taking values in a finite set $\mathbb{J} = \{1, 2, \cdots, K\}$ and with generator $\mathbf{Q} = \left(q_{ij}\right)_{K\times K}$ where $-q_{ii} = q_i > 0$ for $i \in \mathbb{J}$. Let $X_i(t) = \mu_i t + \sigma_i W(t)$, where $\mu_i, \sigma_i > 0$ for all $i \in \mathbb{J}$, and $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion which is independent of $\{J(t)\}_{t\geq 0}$. The surplus process of the insurer is given by

$$X(t) = x + \sum_{i=1}^{K} \int_0^t \mathbf{1}_{\{J(s)=i\}} \mathrm{d}X_i(s),$$

where $x > 0$ is the initial surplus.

When the state of the Markov chain is $i \in \mathbb{J}$, we assume that the dividends can only be paid at the arrival times of a Poisson process with rate $\gamma_i > 0$. Considering dividends, the surplus process (still denoted by $\{X(t)\}_{t\geq 0}$) is given by

$$X(t) = x + \sum_{i=1}^{K} \int_0^t \mathbf{1}_{\{J(s)=i\}} \mathrm{d}X_i(s) - D(t), \tag{2.1}$$

where $D(t)$ is the cumulative dividends until t. Let $\{N_i(t)\}_{t\geq 0}$ be a Poisson process with intensity γ_i which is assumed to be independent of $\{J(t)\}_{t\geq 0}$ and $\{W(t)\}_{t\geq 0}$. Then we can write

$$D(t) = \sum_{i=1}^{K} \int_0^t \pi(s) \mathbf{1}_{\{J(s)=i\}} \mathrm{d}N_i(s),$$

where the process $\{\pi(s)\}_{s\geq 0}$ determines the amount of dividends paid at the jump times of the Poisson processes $\{N_i(t)\}_{t\geq 0}$, $i \in \mathbb{J}$.

Suppose that all the stochastic processes mentioned above are defined on the filtered probability space $(\Omega, \mathcal{F}, \mathsf{P})$, where $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ is generated by $\{X(t)\}_{t\geq 0}$ and $\{J(t)\}_{t\geq 0}$ and satisfies the usual conditions. Denote by E_x and $\mathsf{E}_{x,i}$ the expectations conditioned on $\{X(0) = x\}$ and $\{X(0) = x, J(0) = i\}$, respectively.

We say a dividend strategy $\{\pi(s)\}_{s\geq 0}$ (for convenience, we also write π for short) is admissible, if it is $\mathcal{F}$-adapted and $0 \leq \pi(t) < X(t-)$ for $t \geq 0$. Let Π be the set of all admissible strategies. With a strategy $\pi \in \Pi$, let $\tau_\pi := \inf\{t \geq 0 : X(t) \leq 0\}$ be the time of ruin. Without loss of generality, we assume that $X(t) \equiv 0$ for $t \geq \tau_\pi$. Given the initial surplus x and initial state i, the expected value of the discounted dividends until ruin is given by

$$V_\pi(x, i) := \mathsf{E}_{x,i}\left[\sum_{k=1}^{K} \int_0^{\tau_\pi} e^{-\Lambda(s)} \mathbf{1}_{\{J(s)=k\}} \pi(s) \mathrm{d}N_k(s)\right],$$

where $\Lambda(s) = \sum_{i=1}^{K} \int_0^s \mathbf{1}_{\{J(t)=i\}} \delta_i \mathrm{d}t$ with $\delta_i > 0$ is the discount rate at state i for $i \in \mathbb{J}$. The objective functions is

$$V(x,i) = \sup_{\pi\in\Pi} V_\pi(x,i), \qquad i = 1,2,\cdots,K. \tag{2.2}$$

It is easy to see that $V(0,i) = 0$ for all $i \in \mathbb{J}$. The problem of the shareholders is to specify a dividend strategy $\pi^* \in \Pi$ such that $V(x,i) = V_{\pi^*}(x,i)$ for all $i \in \mathbb{J}$.

3 The Dynamic Programming Equation

In the following, we adopt bold-face letters to denote the vector functions in the form of

$$\mathbf{v}(x) := (v(x,1), v(x,2), \cdots, v(x,K)).$$

When we use $\le$ (or $\ge$) between two vectors (or vector functions), it means that $\le$ (or $\ge$) holds for each element. Also, we denote by $\mathbf{0}$ the zero row vector with K elements.

Let $\zeta_0 = 0$ and

$$\zeta_n := \inf\{t \ge \zeta_{n-1} : J(t) \ne J(t-)\}, \qquad n \in \mathbb{N}^+,$$

i.e., ζ_n is the n-th jump time of the Markov chain $\{J(t)\}_{t\ge0}$. For a testing function $\mathbf{v}(x)$, define the functional operator as

$$\mathcal{M}\mathbf{v}(x) := (\mathcal{M}v(x,1), \mathcal{M}v(x,2), \cdots, \mathcal{M}v(x,K)),$$

where

$$\mathcal{M}v(x,i) \;:=\; \sup_{\pi\in\Pi} \mathsf{E}_{x,i}\left[\int_0^{\tau_\pi\wedge\zeta_1} e^{-\delta_i s}\pi(s)\mathrm{d}N_i(s) + e^{-\delta_i(\tau_\pi\wedge\zeta_1)} v(X(\tau_\pi\wedge\zeta_1), J(\tau_\pi\wedge\zeta_1))\right]. \tag{3.1}$$

From the definition of $\mathcal{M}$, we have following lemmas.

Lemma 3.1. *If $\mathbf{v}_1(x) \ge \mathbf{v}_2(x)$, then it holds that $\mathcal{M}\mathbf{v}_1(x) \ge \mathcal{M}\mathbf{v}_2(x)$ for all $x \ge 0$.*

Lemma 3.2. *For all $x \ge 0$, let $\mathbf{U}_0(x) \equiv \mathbf{0}$ and $\mathbf{U}_{n+1}(x) = \mathcal{M}\mathbf{U}_n(x)$, for $n \in \mathbb{N}$. Then for each $i \in \mathbb{J}$, $\{U_n(\cdot,i)\}_{n\in\mathbb{N}}$ is an increasing sequence of functions.*

Proof. Note that $U_1(x,i) = \sup_{\pi\in\Pi} \mathsf{E}_{x,i}\left[\int_0^{\tau_\pi\wedge\zeta_1} e^{-\delta_i s}\pi(s)\mathrm{d}N_i(s)\right] \ge 0 = U_0(x,i)$, for all $x \ge 0$ and $i \in \mathbb{J}$. The result follows from Lemma 3.1. □

For $n \in \mathbb{N}$, define $\Pi_n = \{\pi \in \Pi : \pi(s) \equiv 0, \text{ for } s \ge \zeta_n\}$ be the set of all the admissible strategies that pays no dividend after the n-th jump of the Markov chain $\{J(t)\}_{t\ge0}$. Let $V_n(x,i) = \sup_{\pi\in\Pi_n} V_\pi(x,i)$.

Lemma 3.3. *For all $x \ge 0$, we have $\mathbf{V}_n(x) = \mathbf{U}_n(x)$, $\forall n \in \mathbb{N}$.*

Proof. Obviously, we have $\mathbf{V}_0(x) = \mathbf{U}_0(x) \equiv \mathbf{0}$. Let us assume that $\mathbf{V}_n(x) = \mathbf{U}_n(x)$, and show that $\mathbf{V}_{n+1}(x) = \mathbf{U}_{n+1}(x)$.

First, we will show that $\mathbf{V}_{n+1}(x) \le \mathbf{U}_{n+1}(x)$. For any $\varepsilon > 0$, there is a strategy $\pi \in \Pi_{n+1}$ such that

$$V_\pi(x,i) \ge V_{n+1}(x,i) - \varepsilon. \tag{3.2}$$

Define a strategy $\hat{\pi} \in \Pi_n$ by setting $\hat{\pi}(t) = \pi(t + \tau_\pi \wedge \zeta_1)$ for $t \geq 0$. By the strong Markov property, we have

$$\begin{aligned} V_\pi(x,i) &= \mathsf{E}_{x,i}\left[\sum_{k=1}^{K}\int_0^{\tau_\pi} e^{-\Lambda(s)}\pi(s)\mathbf{1}_{\{J(s)=k\}}\mathrm{d}N_k(s)\right] \\ &= \mathsf{E}_{x,i}\left[\int_0^{\tau_\pi\wedge\zeta_1} e^{-\delta_i s}\pi(s)\mathrm{d}N_i(s) + e^{-\delta_i(\tau_\pi\wedge\zeta_1)}V_{\hat{\pi}}(X(\tau_\pi\wedge\zeta_1), J(\tau_\pi\wedge\zeta_1))\right] \\ &\leq \mathsf{E}_{x,i}\left[\int_0^{\tau_\pi\wedge\zeta_1} e^{-\delta_i s}\pi(s)\mathrm{d}N_i(s) + e^{-\delta_i(\tau_\pi\wedge\zeta_1)}V_n(X(\tau_\pi\wedge\zeta_1), J(\tau_\pi\wedge\zeta_1))\right] \\ &= \mathsf{E}_{x,i}\left[\int_0^{\tau_\pi\wedge\zeta_1} e^{-\delta_i s}\pi(s)\mathrm{d}N_i(s) + e^{-\delta_i(\tau_\pi\wedge\zeta_1)}U_n(X(\tau_\pi\wedge\zeta_1), J(\tau_\pi\wedge\zeta_1))\right] \\ &\leq U_{n+1}(x,i). \end{aligned} \tag{3.3}$$

It follows from (3.2), (3.3) and the arbitrariness of ε that $V_{n+1}(x,i) \leq U_{n+1}(x,i)$, for all $x \geq 0$ and $i \in \mathbb{J}$.

Second, we are going to show $\mathbf{V}_{n+1}(x) \geq \mathbf{U}_{n+1}(x)$. For any $\varepsilon > 0$, there is a strategy $\pi' \in \Pi$ such that

$$U_{n+1}(x,i) \leq \mathsf{E}_{x,i}\left[\int_0^{\tau_{\pi'}\wedge\zeta_1} e^{-\delta_i s}\pi'(s)\mathrm{d}N_i(s) + e^{-\delta_i(\tau_{\pi'}\wedge\zeta_1)}U_n(X(\tau_{\pi'}\wedge\zeta_1), J(\tau_{\pi'}\wedge\zeta_1))\right] + \varepsilon,$$

and there is a strategy $\pi'' \in \Pi_n$ such that $V_n(x,i) \leq V_{\pi_2}(x,i)$ for any $x \geq 0$, and $i \in \mathbb{J}$. Now we can construct a strategy $\tilde{\pi} \in \Pi_{n+1}$ by taking the strategy π' before $\tau_{\pi'} \wedge \zeta_1$, and then following strategy π''. Thus, by the strong Markov property, we have

$$\begin{aligned} U_{n+1}(x,i) &\leq \mathsf{E}_{x,i}\left[\int_0^{\tau_{\pi'}\wedge\zeta_1} e^{-\delta_i s}\pi'(s)\mathrm{d}N_i(s) + e^{-\delta_i(\tau_{\pi'}\wedge\zeta_1)}V_n(X(\tau_{\pi'}\wedge\zeta_1), J(\tau_{\pi'}\wedge\zeta_1))\right] + \varepsilon \\ &\leq \mathsf{E}_{x,i}\left[\int_0^{\tau_{\pi'}\wedge\zeta_1} e^{-\delta_i s}\pi'(s)\mathrm{d}N_i(s) + e^{-\delta_i(\tau_{\pi'}\wedge\zeta_1)}V_{\pi''}(X(\tau_{\pi'}\wedge\zeta_1), J(\tau_{\pi'}\wedge\zeta_1))\right] + 2\varepsilon \\ &= V_{\tilde{\pi}}(x,i) + 2\varepsilon \\ &\leq V_{n+1}(x,i) + 2\varepsilon. \end{aligned}$$

Thus from the arbitrariness of ε, we have $U_{n+1}(x,i) \leq V_{n+1}(x,i)$ for all $x \geq 0$ and $i \in \mathbb{J}$, which ends our proof. □

Remark 3.4. Note that $\mathcal{M}$ can be interpreted as an MDP operator of a positive Markov decision process, and our original problem boils down to solving an MDP. The following results are standard (see e.g. **?**).

Lemma 3.5. $\lim_{n\to\infty} U_n(x,i) = V(x,i)$, *for any* $x \geq 0$ *and* $i \in \mathbb{J}$.

Proposition 3.6. *The value function* $\mathbf{V}$ *is the smallest solution of the dynamic programming equation*

$\mathbf{V} = \mathcal{M}\mathbf{V}$ *such that* $\mathbf{V} \geq \mathbf{0}$, *i.e.*

$$V(x,i) = \sup_{\pi\in\Pi} \mathsf{E}_{x,i}\left[\int_0^{\tau_\pi\wedge\zeta_1} e^{-\delta_i s}\pi(s)\mathrm{d}N_i(s) + e^{-\delta_i(\tau_\pi\wedge\zeta_1)}V(X(\tau_\pi\wedge\zeta_1), J(\tau_\pi\wedge\zeta_1))\right], \quad \forall x\geq 0, i\in\mathbb{J}. \tag{3.4}$$

Remark 3.7. In general for positive MDPs, it is not true that a maximizer of the right-hand-side in (3.4) yields the optimal strategy. Let $\tilde{\mathbf{V}}$ be the value function studied in **?**. Then there is constant $c > 0$ such that $\tilde{V}(x,i) < x + c$ for all $i \in \mathbb{J}$. Note that the set of admissible strategy Π in this paper is a subset of the one considered in **?**. It follows that $V(x,i) \leq \tilde{V}(x,i) < x+c$ for all $x \in [0,\infty)$ and $i \in \mathbb{J}$. For $i \in \mathbb{J}$, define $b(x,i) := 1 + x$ and the operator

$$\mathcal{T}_o v(x,i) := \sup_{\pi\in\Pi} \mathsf{E}_{x,i}\left[e^{-\delta_i(\tau_\pi\wedge\zeta_1)}v(X(\tau_\pi\wedge\zeta_1), J(\tau_\pi\wedge\zeta_1))\right].$$

Considering a strategy $\pi \in \Pi$, let

$$Y(t) = x + X_i(t) - \int_0^t \pi(s)\mathrm{d}N_i(s) \tag{3.5}$$

and τ_i be the time of ruin of $\{Y(t)\}_{t\geq 0}$. For any constant $\theta > 0$, denote by $\eta(\theta)$ an independent exponential random variable with mean $1/\theta$. It holds that $(Y(t), t < \tau_i \wedge \eta(q_i))$ is in distribution equal to $(X(t), J(0) = i, t < \tau_\pi \wedge \zeta_1)$. It is easy to see that

$$\begin{aligned}
\mathcal{T}_o b(x,i) &= \sup_{\pi\in\Pi} \mathsf{E}_x\left[\int_0^{\tau_i} e^{-(\delta_i+q_i)s}\sum_{j\neq i} q_{ij} b(Y(s), j)\mathrm{d}s\right] \\
&\leq \mathsf{E}_x\left[\int_0^{\infty} e^{-(\delta_i+q_i)s} q_i(1 + x + \mu_i s + \sigma_i W(s))\mathrm{d}s\right] \\
&= \frac{q_i}{\delta_i + q_i}\left(1 + x + \frac{\mu_i}{\delta_i + \mu_i}\right).
\end{aligned}$$

Thus, by iteration we have $\lim_{n\to\infty} \mathcal{T}_o^n b(x,i) = 0$, which implies the maximizer of right-hand-side in (3.4) always gives the optimal strategy (see e.g. **?**, **?**).

4 The Solution to $\mathbf{U}_n(x)$

From the preceding section, we know that the value function can be obtained by iteration. However, to do this, we need to show what $\mathbf{U}_{n+1}$ is when $\mathbf{U}_n$ is given. This is the problem studied in this section.

4.1 An Auxiliary Optimal Problem

To solve our problem, we restrict ourselves to a special class of vector functions.

Definition 4.1. We say a vector function $\mathbf{u}(x) \in \mathbb{D}$, if
(i) $\mathbf{u}(0) = \mathbf{0}, u(\cdot, i) \in C([0,\infty))$ is increasing and concave, for each $i \in \mathbb{J}$;
(ii) for any $\theta > 0$, $\lim_{x\to\infty} e^{-\theta x} u(x,i) = 0$, for each $i \in \mathbb{J}$.

For a function $\mathbf{u} \in \mathbb{D}$, we consider the auxiliary optimal problem

$$M(x,i) := \sup_{\pi\in\Pi} M_\pi(x,i). \tag{4.1}$$

where

$$M_\pi(x,i) = \mathsf{E}_{x,i}\left[\int_0^{\tau_\pi\wedge\zeta_1} e^{-\delta_i s}\pi(s)\mathrm{d}N_i(s) + e^{-\delta_i(\tau_\pi\wedge\zeta_1)} u(X(\tau_\pi\wedge\zeta_1), J(\tau_\pi\wedge\zeta_1))\right].$$

From the general theory of stochastic control, we consider the HJB equation

$$\max_{0\le\pi\le x}\left\{\frac{\sigma_i^2}{2}m''(x,i) + \mu_i m'(x,i) - (\delta_i + q_i + \gamma_i)m(x,i) + \gamma_i[m(x-\pi,i)+\pi] + \sum_{j\ne i} q_{ij}u(x,j)\right\} = 0, \tag{4.2}$$

for the optimal problem (4.1), where m' and m'' are the first and second order partial derivatives with respect to x, respectively.

Theorem 4.2. *For $i \in \mathbb{J}$, let $m(\cdot,i) \in C^2([0,\infty))$ be an nonnegative function. Assume that $m(x,i)$ satisfies the HJB equation (4.2) for all $x \ge 0$. (i) Then it holds that $m(x,i) \ge M(x,i)$ for all $x \ge 0$; (ii) If, in addition, $m(x,i) = M_{\pi^*}(x,i)$ for some $\pi^* \in \Pi$, then π^* is an optimal dividend strategy for the problem (4.1) and $M(x,i) \equiv M_{\pi^*}(x,i)$.*

Proof. (i) Considering a strategy $\pi \in \Pi$ and recalling $\{Y(t)\}_{t\ge0}$ defined by (3.5), for any $\mathbf{u} \in \mathbb{D}$, we have

$$\begin{aligned} M_\pi(x,i) &= \sup_{\pi\in\Pi}\mathsf{E}_x\left[\int_0^{\tau_i} \mathbf{1}_{\{s<\eta(q_i)\}}e^{-\delta_i s}\pi(s)\mathrm{d}N_i(s) + \mathbf{1}_{\{\eta(q_i)<\tau_i\}}e^{-\delta_i\eta(q_i)}\sum_{j\ne i}\frac{q_{ij}}{q_i}u(Y(\eta(q_i)),j)\right] \\ &= \sup_{\pi\in\Pi}\mathsf{E}_x\left[\int_0^{\tau_i} e^{-(\delta_i+q_i)s}\pi(s)\mathrm{d}N_i(s) + \int_0^{\tau_i} e^{-(\delta_i+q_i)s}\sum_{j\ne i} q_{ij}u(Y(s),j)\mathrm{d}s\right]. \end{aligned}$$

Let a and b be real numbers satisfying $0 < a < Y(0) = x < b < \infty$. Define $\tau_a := \inf\{t \ge 0 : Y(t) \le a\}$, $\tau_b := \inf\{t \ge 0 : Y(t) \ge b\}$ and $\tau_{ab} = \tau_a \wedge \tau_b$. Applying the Itô formula to $e^{-\delta_i t}m(Y(t),i)$ yields that

$$\begin{aligned} & e^{-(\delta_i+q_i)(t\wedge\tau_{ab})}m(Y(t\wedge\tau_{ab}),i) - m(Y(0),i) \\ = & \int_0^{t\wedge\tau_{ab}} e^{-(\delta_i+q_i)s}\left[-(\delta_i+q_i)m(Y(s),i) + \mu_i m'(Y(s),i) + \frac{1}{2}\sigma_i^2 m''(Y(s),i)\right]\mathrm{d}s \\ & + \int_0^{t\wedge\tau_{ab}} e^{-(\delta_i+q_i)s}\left[m(Y(s-)-\pi(s),i) - m(Y(s-),i)\right]\mathrm{d}N_i(s) \\ & + \int_0^{t\wedge\tau_{ab}} e^{-(\delta_i+q_i)s}\sigma_i m'(Y(s),i)\mathrm{d}W(s), \qquad \text{for all } t \ge 0. \end{aligned}$$

Since $m(\cdot,i)$ satisfies (4.2), we have

$$\begin{aligned}
&\int_0^{t\wedge\tau_{ab}} e^{-(\delta_i+q_i)s}\pi(s)\mathrm{d}N_i(s)+\int_0^{t\wedge\tau_{ab}} e^{-(\delta_i+q_i)s}\sum_{j\neq i} q_{ij}u(Y(s),j)\mathrm{d}s \\
\leq\ & -e^{-(\delta_i+q_i)(t\wedge\tau_{ab})}m(Y(t\wedge\tau_{ab}),i)+m(Y(0),i)+Z_1(t\wedge\tau_{ab})+Z_2(t\wedge\tau_{ab}),
\end{aligned} \tag{4.3}$$

where $\{Z_1(t)\}_{t\geq 0}$ and $\{Z_2(t)\}_{t\geq 0}$ are local martingales defined as

$$\begin{aligned}
Z_1(t) &= \int_0^t e^{-(\delta_i+q_i)s}\sigma_i m'(Y(s),i)\mathrm{d}W(s), \\
Z_2(t) &= \int_0^t e^{-(\delta_i+q_i)s}\left[m(Y(s-)-\pi(s),i)+\pi(s)-m(Y(s-),i)\right](\mathrm{d}N_i(s)-\gamma_i\mathrm{d}s).
\end{aligned}$$

However, the stopped processes $\{Z_1(t\wedge\tau_{ab})\}_{t\geq 0}$ and $\{Z_2(t\wedge\tau_{ab})\}_{t\geq 0}$ are martingales. Recall that $m(\cdot,i)$ is nonnegative. Taking conditional expectation on both sides of (4.3) yields that

$$m(x,i)\geq \mathsf{E}_x\left[\int_0^{t\wedge\tau_{ab}} e^{-(\delta_i+q_i)s}\pi(s)\mathrm{d}N_i(s)+\int_0^{t\wedge\tau_{ab}} e^{-(\delta_i+q_i)s}\sum_{j\neq i} q_{ij}u(Y(s),j)\mathrm{d}s\right].$$

Letting $a\to 0$ and $b\to\infty$, we get $\tau_a\to\tau_i$ and $\tau_b\to\infty$. Then, $\tau_{ab}\to\tau_i$. Also, letting $t\to\infty$ and applying dominated convergence theorem yield that

$$m(x,i)\geq \mathsf{E}_x\left[\int_0^{\tau_i} e^{-(\delta_i+q_i)s}\pi(s)\mathrm{d}N_i(s)+\int_0^{\tau_i} e^{-(\delta_i+q_i)s}\sum_{j\neq i} q_{ij}u(Y(s),j)\mathrm{d}s\right]=M_\pi(x,i).$$

From the arbitrariness of the strategy π and the definition of $M(\cdot,i)$, we conclude that $m(x,i)\geq M(x,i)$.

(ii) It is obvious from (i) and the definition of $M(\cdot,i)$. □

4.2 *The Modulated Barrier Strategy*

Motivated by **?** and **?**, we consider the modulated barrier strategy. Let $\{T_1,T_2,\cdots\}$ be the times at which the dividends can be paid. Given the barrier level $\mathbf{b}=(b_1,b_2,\cdots,b_K)$, the modulated barrier strategy $\{\pi^{\mathbf{b}}(t)\}_{t\geq 0}$ is an $\mathcal{F}$-adapted process such that $\pi^{\mathbf{b}}(T_i)=(X(T_i)-b_{J(T_i)})^+$, for $i=1,2,\cdots$.

To easy the notations, let $M_{\mathbf{b}}(x,i)=M_{\pi^{\mathbf{b}}}(x,i)$. We have the following propositions.

Proposition 4.3. *Given* $\mathbf{b}$, *it holds that*

$$\begin{aligned}
M_{\mathbf{b}}(x,i) =\ & \gamma_i W_i^{(\theta_i)}(x)\left[\int_0^{b_i} M_{\mathbf{b}}(y,i)e^{-r_iy}\mathrm{d}y+\frac{e^{-r_ib_i}}{r_i}\left(M_{\mathbf{b}}(b_i,i)+\frac{1}{r_i}\right)\right] \\
& -\gamma_i\int_0^x M_{\mathbf{b}}(y,i)W_i^{(\theta_i)}(x-y)\mathrm{d}y+W_i^{(\theta_i)}(x)\int_0^\infty e^{-r_iy}\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y \\
& -\int_0^x W_i^{(\theta_i)}(x-y)\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y,\quad 0\leq x<b_i,
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned} M_{\mathbf{b}}(x,i) &= \gamma_i W_i^{(\theta_i)}(x)\left[\int_0^{b_i} M_{\mathbf{b}}(y,i)e^{-r_iy}\mathrm{d}y + \frac{e^{-r_ib_i}}{r_i}\left(M_{\mathbf{b}}(b_i,i)+\frac{1}{r_i}\right)\right] \\ &\quad -\gamma_i\left[\int_0^{b_i} M_{\mathbf{b}}(y,i)W_i^{(\theta_i)}(x-y)\mathrm{d}y + \int_{b_i}^{x}(y-b+M_{\mathbf{b}}(b_i,i))W_i^{(\theta_i)}(x-y)\mathrm{d}y\right] \\ &\quad +W_i^{(\theta_i)}(x)\int_0^\infty e^{-r_iy}\sum_{j\neq i} q_{ij}u(y,j)\mathrm{d}y - \int_0^x W_i^{(\theta_i)}(x-y)\sum_{j\neq i} q_{ij}u(y,j)\mathrm{d}y, \quad x\geq b_i, \end{aligned} \tag{4.5}$$

where $\theta_i = \delta_i + q_i + \gamma_i$, *and*

$$W_i^{(\theta_i)}(x) = \frac{2}{\sigma_i^2}\cdot\frac{e^{r_ix}-e^{s_ix}}{r_i - s_i},$$

and $r_i > 0, s_i < 0$ *are the solutions of the equation* $\frac{\sigma_i^2}{2}r^2 + \mu_i r - \theta_i = 0$.

Proof. Denote by $\{Y_{\mathbf{b}}(t)\}_{t\geq 0}$ and $\tau_{\mathbf{b}}$ the process (3.5) and the time of ruin corresponding to the modulated barrier strategy $\mathbf{b}$, respectively. Let $Y'(t) = x + X_i(t)$ and τ be the time of ruin of $\{Y'(t)\}_{t\geq 0}$. Let T_1 be the first time at which the dividend is paid. Then $(Y'(t), t < \tau\wedge\eta(\gamma_i))$ is in distribution equal to $(Y_{\mathbf{b}}(t), t < \tau_{\mathbf{b}}\wedge T_1)$. To simplify the notations, let $f(x,i) = \sum_{j\neq i} q_{ij}u(x,j)$ and $g(y) = y-(y-b_i)^+$. Noting that $M_{\mathbf{b}}(0,i) = 0$, we have

$$\begin{aligned} M_{\mathbf{b}}(x,i) &= \mathsf{E}_x\left[\int_0^{\tau_{\mathbf{b}}\wedge T_1} e^{-(\delta_i+q_i)s} f(Y_{\mathbf{b}}(s),i)\mathrm{d}s\right] + \mathsf{E}_x\left[\mathbf{1}_{\{T_1\leq\tau_{\mathbf{b}}\}}e^{-(\delta_i+q_i)T_1}(Y_{\mathbf{b}}(T_1-)-b_i)^+\right] \\ &\quad +\mathsf{E}_x\left[\mathbf{1}_{\{T_1\leq\tau_{\mathbf{b}}\}}e^{-(\delta_i+q_i)T_1}M_{\mathbf{b}}(g(Y_{\mathbf{b}}(T_1-)),i)\right] \\ &= \mathsf{E}_x\left[\int_0^{\tau\wedge\eta(\gamma_i)} e^{-(\delta_i+q_i)s} f(Y'(s),i)\mathrm{d}s\right] + \mathsf{E}_x\left[\mathbf{1}_{\{\eta(\gamma_i)\leq\tau\}}e^{-(\delta_i+q_i)\eta(\gamma_i)}(Y'(\eta(\gamma_i))-b_i)^+\right] \\ &\quad +\mathsf{E}_x\left[\mathbf{1}_{\{\eta(\gamma_i)\leq\tau\}}e^{-(\delta_i+q_i)\eta(\gamma_i)}M_{\mathbf{b}}(g(Y'(\eta(\gamma_i))),i)\right] \\ &= \mathsf{E}_x\left[\int_0^\infty \mathbf{1}_{\{s\leq\tau\}}e^{-\theta_is}f(Y'(s),i)\mathrm{d}s\right] + \gamma_i\mathsf{E}_x\left[\int_0^\infty \mathbf{1}_{\{s\leq\tau\}}e^{-\theta_is}(Y'(s)-b_i)^+\right] \\ &\quad +\gamma_i\mathsf{E}_x\left[\int_0^\infty \mathbf{1}_{\{s\leq\tau\}}e^{-\theta_is}M_{\mathbf{b}}(g(Y'(s)),i)\right] \\ &= \int_0^\infty\left[f(y,i)+\gamma_i(y-b_i)^+ + \gamma_i M_{\mathbf{b}}(g(y),i)\right]\int_0^\infty e^{-\theta_is}\mathsf{P}_x(Y'(s)\in\mathrm{d}y, s<\tau)\mathrm{d}s. \end{aligned} \tag{4.6}$$

From Corollary 8.8 of **?** (or let $b\to\infty$ in Equation (4.4) of **?**[†]), we have

$$\int_0^\infty e^{-\theta_is}\mathsf{P}_x(Y'(s)\in\mathrm{d}y, s<\tau)\mathrm{d}s = \left[W_i^{(\theta_i)}(x)e^{-r_iy} - \mathbf{1}_{\{x\geq y\}}W_i^{(\theta_i)}(x-y)\right]\mathrm{d}y. \tag{4.7}$$

Inserting (4.7) into (4.6) yields (4.14) and (4.5). □

[†]In their paper, the left-hand side of (4.4) should be divided by θ_i.

Proposition 4.4. *The function $M_{\mathbf{b}}(x,i) \in C^2([0,\infty))$ and satisfies*

$$\frac{\sigma_i^2}{2}M_{\mathbf{b}}''(x,i)+\mu_i M_{\mathbf{b}}'(x,i)-(\delta_i+q_i)M_{\mathbf{b}}(x,i)+\sum_{j\neq i}q_{ij}u(x,j)=0, \quad 0\le x<b_i, \tag{4.8}$$

and

$$\frac{\sigma_i^2}{2}M_{\mathbf{b}}''(x,i)+\mu_i M_{\mathbf{b}}'(x,i)-\theta_i M_{\mathbf{b}}(x,i)+\gamma_i\left[M_{\mathbf{b}}(b_i,i)+x-b_i\right]+\sum_{j\neq i}q_{ij}u(x,j)=0, \quad x\ge b_i. \tag{4.9}$$

Proof. Noting that $W_i^{(\theta_i)}(x)\in C^2([0,\infty))$, we know that $M_{\mathbf{b}}(x,i)\in C^2([0,b_i))$ and $M_{\mathbf{b}}(x,i)\in C^2([b_i,\infty))$. Taking first and second order derivatives of (4.4) and (4.5), it is easy to check $M_{\mathbf{b}}''(x,i)$ is continuous at b_i. Furthermore, by the using of

$$\frac{\sigma_i^2}{2}W_i^{(\theta_i)''}(x)+\mu_i W_i^{(\theta_i)'}(x)-\theta_i W_i^{(\theta_i)}(x)=0,$$

it is easy to show (4.8) and (4.9) (for simplicity, we omit the details of calculations). □

From the above proposition, if $\mathbf{u}(x)\in\mathbb{D}$, then we have $M_{\mathbf{b}}(x,i)\in C^2([0,\infty))$ for all $i\in\mathbb{J}$. Since later we will start with $\mathbf{U}_0\equiv\mathbf{0}\in\mathbb{D}$, we can work with $\mathbf{u}\in\mathbb{D}\cap C^2([0,\infty))$ in the following.

For $x\ge b_i$, it is easy to rewrite (4.5) as

$$\begin{aligned} M_{\mathbf{b}}(x,i) &= \frac{2\gamma_i e^{s_i x}}{\sigma_i^2(r_i-s_i)}\left[\int_0^{b_i}M_{\mathbf{b}}(y,i)\left(e^{-s_i y}-e^{-r_i y}\right)\mathrm{d}y+\left(\frac{1}{s_i}e^{-s_i b_i}-\frac{1}{r_i}e^{-r_i b_i}\right)M_{\mathbf{b}}(b_i,i)\right] \\ &\quad +\frac{2\gamma_i e^{s_i x}}{\sigma_i^2(r_i-s_i)}\left(\frac{1}{s_i^2}e^{-s_i b_i}-\frac{1}{r_i^2}e^{-r_i b_i}\right)+a_i\left(x+M_{\mathbf{b}}(b_i,i)-b_i+\frac{\mu_i}{\theta_i}\right)+\Gamma_i(x), \end{aligned} \tag{4.10}$$

where $a_i=\gamma_i/\theta_i$, and

$$\Gamma_i(x)=\frac{2e^{s_i x}}{\sigma_i^2(r_i-s_i)}\int_0^x\left(e^{-s_i y}-e^{-r_i y}\right)\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y+\frac{2\left(e^{r_i x}-e^{s_i x}\right)}{\sigma_i^2(r_i-s_i)}\int_x^\infty e^{-r_i y}\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y.$$

Corollary 4.5. *For any $\mathbf{u}\in\mathbb{D}$, we have*
(i) for any $\theta>0$, $e^{-\theta x}M_{\mathbf{b}}(x,i)\to 0$, as $x\to\infty$;
(ii) $M_{\mathbf{b}}'(x,i)\to a_i+\frac{1}{\theta_i}\sum_{j\neq i}q_{ij}u'(\infty,j)$, and $M_{\mathbf{b}}''(x,i)\to 0$, as $x\to\infty$.

Proof. (i) Since $\Gamma_i(x)\ge 0$, it follows from (4.10) that $M_{\mathbf{b}}(x,i)\to\infty$ as $x\to\infty$. Recall that if $\mathbf{u}\in\mathbb{D}$, then for any $\theta>0$ and $i\in\mathbb{J}$, $e^{-\theta x}u(x,i)\to 0$, as $x\to\infty$. It holds that

$$\begin{aligned} \Gamma_i(x) &= \frac{2e^{s_i x}}{\sigma_i^2(r_i-s_i)s_i}\int_0^x e^{-s_i y}\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y+\frac{2e^{r_i x}}{\sigma_i^2(r_i-s_i)r_i}\int_x^\infty e^{-r_i y}\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y \\ &\quad -\frac{2e^{s_i x}}{\sigma_i^2(r_i-s_i)r_i}\int_0^\infty e^{-r_i y}\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y+\frac{1}{\theta_i}\sum_{j\neq i}q_{ij}u(x,j). \end{aligned}$$

Note that $\int_0^x e^{-s_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y \to \infty$ and $\int_x^\infty e^{-r_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y \to 0$ as $x\to\infty$. Hence, by the de'l Hopital's rule,

$$\Gamma_i(x) \to \frac{\mu_i}{\theta_i^2}\sum_{j\neq i} q_{ij}u'(\infty,j) + \frac{1}{\theta_i}\sum_{j\neq i} q_{ij}u(\infty,j), \qquad \text{as } x\to\infty.$$

Thus by (4.10), for any $\theta>0$, $e^{-\theta x}M_{\mathbf{b}}(x,i)\to 0$, as $x\to\infty$.

(ii) Similarly, for any $\mathbf{u}\in\mathbb{D}$, by the de'l Hopital's rule,

$$\begin{aligned}\Gamma_i'(x) &= \frac{2e^{s_i x}}{\sigma_i^2(r_i-s_i)}\int_0^x e^{-s_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y + \frac{2e^{r_i x}}{\sigma_i^2(r_i-s_i)}\int_x^\infty e^{-r_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y\\ &\quad -\frac{2s_i e^{s_i x}}{\sigma_i^2(r_i-s_i)r_i}\int_0^\infty e^{-r_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y\\ &\to \frac{1}{\theta_i}\sum_{j\neq i} q_{ij}u'(\infty,j), \qquad \text{as } x\to\infty.\end{aligned}$$

Thus by (4.10), it is easy to see that $M_{\mathbf{b}}'(x,i)\to a_i+\frac{1}{\theta_i}\sum_{j\neq i} q_{ij}u'(\infty,j)$, as $x\to\infty$.

Also,

$$\begin{aligned}\Gamma_i''(x) &= \frac{2s_i e^{s_i x}}{\sigma_i^2(r_i-s_i)}\int_0^x e^{-s_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y + \frac{2r_i e^{r_i x}}{\sigma_i^2(r_i-s_i)}\int_x^\infty e^{-r_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y\\ &\quad -\frac{2s_i^2 e^{s_i x}}{\sigma_i^2(r_i-s_i)r_i}\int_0^\infty e^{-r_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y\\ &\to 0, \qquad \text{as } x\to\infty.\end{aligned}$$

□

From Section 2.1.1 of **?**, the solution of (4.8) is given by

$$M_{\mathbf{b}}(x,i) = A_i e^{\alpha_i x} + B_i e^{\beta_i x} - \int_0^x \frac{2\left(e^{\alpha_i(x-y)}-e^{\beta_i(x-y)}\right)}{\sigma_i^2(\alpha_i-\beta_i)}\sum_{j\neq i} q_{ij}u(y,j)\mathrm{d}y, \tag{4.11}$$

where A_i and B_i are constants to be determined, and $\alpha_i>0,\beta_i<0$ are the solutions of the equation

$$\frac{\sigma_i^2}{2}r^2+\mu_i r-(\delta_i+q_i)=0.$$

The solution of (4.9) is given by

$$M_{\mathbf{b}}(x,i) = C_i e^{r_i(x-b_i)} + D_i e^{s_i(x-b_i)} - \int_{b_i}^x \frac{2\left(e^{r_i(x-y)}-e^{s_i(x-y)}\right)}{\sigma_i^2(r_i-s_i)}\sum_{j\neq i} q_{ij}u(y,j)\mathrm{d}y + a_i x + c_i, \tag{4.12}$$

where C_i and D_i are constants to be determined, and

$$c_i = \frac{\mu_i a_i + \gamma_i [m(b_i-, i) - b_i]}{\delta_i + q_i + \gamma_i}.$$

From (4.12), we have

$$\begin{aligned}
M_{\mathbf{b}}''(x,i) &= C_i r_i^2 e^{r_i(x-b_i)} + D_i s_i^2 e^{s_i(x-b_i)} - \frac{2}{\sigma_i^2}\sum_{j\neq i} q_{ij}u(x,j) \\
&\quad - \int_{b_i}^{x} \frac{2\left(r_i^2 e^{r_i(x-y)} - s_i^2 e^{s_i(x-y)}\right)}{\sigma_i^2 (r_i - s_i)} \sum_{j\neq i} q_{ij}u(y,j)\mathrm{d}y \\
&= \left[C_i r_i^2 - \frac{2r_i}{\sigma_i^2 (r_i - s_i)}\Xi_i(b_i)\right] e^{r_i(x-b_i)} + \left[D_i s_i^2 + \frac{2s_i}{\sigma_i^2 (r_i - s_i)}\sum_{j\neq i} q_{ij}u(b_i,j)\right] e^{s_i(x-b_i)} \\
&\quad + \frac{2}{\sigma_i^2 (r_i - s_i)}\left[s_i e^{s_i x}\int_0^x e^{-s_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y + r_i e^{r_i x}\int_x^\infty e^{-r_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y\right],
\end{aligned}$$

where

$$\Xi_i(b_i) = \sum_{j\neq i} q_{ij}u(b_i,j) + \int_{b_i}^\infty e^{r_i(b_i-y)}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y.$$

Since

$$s_i e^{s_i x}\int_0^x e^{-s_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y + r_i e^{r_i x}\int_x^\infty e^{-r_i y}\sum_{j\neq i} q_{ij}u'(y,j)\mathrm{d}y \to 0, \qquad \text{as } x\to\infty,$$

it follows from Corollary 4.5 (ii) that

$$C_i = \frac{2}{\sigma_i^2 (r_i - s_i) r_i}\Xi_i(b_i).$$

Since $M_{\mathbf{b}}(0,i) = 0$, from (4.11) we know that $B_i = -A_i$. From the smooth-fit conditions

$$\begin{cases} M_{\mathbf{b}}(b_i-, i) = M_{\mathbf{b}}(b_i+, i), \\ M_{\mathbf{b}}'(b_i-, i) = M_{\mathbf{b}}'(b_i+, i), \end{cases}$$

we have

$$\begin{aligned}
A_i &= \frac{1}{s_i \frac{\delta_i + q_i}{\theta_i} h_i(b_i) - h_i'(b_i)}\left[\Lambda_i(b_i) - \frac{2}{\sigma_i^2 r_i}\Xi_i(b_i) + s_i\frac{\mu_i a_i}{\theta_i} - a_i\right], \\
D_i &= \frac{\delta_i + q_i}{\theta_i}\left[A_i h_i(b_i) - \int_0^{b_i} W_i^{(\delta_i + q_i)}(b_i - y)\sum_{j\neq i} q_{ij}u(y,j)\mathrm{d}y\right] - C_i - \frac{\mu_i a_i}{\theta_i},
\end{aligned}$$

where $h_i(b_i) = e^{\alpha_i b_i} - e^{\beta_i b_i}$, and

$$\Lambda_i(b_i) = s_i \frac{\delta_i + q_i}{\theta_i} \int_0^{b_i} W_i^{(\delta_i+q_i)}(b_i - y) \sum_{j \neq i} q_{ij} u(y, j) \mathrm{d}y - \int_0^{b_i} W_i^{(\delta_i+q_i)'}(b_i - y) \sum_{j \neq i} q_{ij} u(y, j).$$

Now we consider the optimal modulated barrier strategy, i.e., we want to find the b_i that maximizes A_i. For convenience, we define the function

$$A_i(b) = \frac{1}{s_i \frac{\delta_i+q_i}{\theta_i} h_i(b) - h_i'(b)} \Delta_i(b),$$

where

$$\Delta_i(b) = \Lambda_i(b) - \frac{2}{\sigma_i^2 r_i} \Xi_i(b) + s_i \frac{\mu_i a_i}{\theta_i} - a_i.$$

Then the first order condition$A_i'(b) = 0$ implies that

$$\Delta_i'(b) \left[s_i \frac{\delta_i + q_i}{\theta_i} h_i(b) - h_i'(b) \right] = \Delta_i(b) \left[s_i \frac{\delta_i + q_i}{\theta_i} h_i'(b) - h_i''(b) \right]. \tag{4.13}$$

In the Appendix, we show that equation (4.13) admits a root in $(0, \infty)$. Note that, for any $x \geq 0$,

$$\frac{\sigma_i^2}{2} h_i''(x) + \mu_i h_i'(x) - (\delta_i + q_i) h_i(x) = 0. \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$\left[(\delta_i + q_i) + \frac{\sigma_i^2}{2} \cdot s_i \frac{\delta_i + q_i}{\theta_i} \cdot \frac{\Delta_i'(b)}{\Delta_i(b)} \right] h_i(b) = \left[\frac{\sigma_i^2}{2} \cdot s_i \frac{\delta_i + q_i}{\theta_i} + \mu_i + \frac{\sigma_i^2}{2} \cdot \frac{\Delta_i'(b)}{\Delta_i(b)} \right] h_i'(b). \tag{4.15}$$

Proposition 4.6. *Let $b_i^* > 0$ be a solution of equation (4.13), then $M'_{\mathbf{b}^*}(b_i^*, i) = 1$ and $M''_{\mathbf{b}^*}(b_i^*, i) \leq 0$, where the i-th element of $\mathbf{b}^*$ is b_i^*.*

Proof. From (4.11), we know that

$$\begin{aligned} M'_{\mathbf{b}^*}(b_i^*, i) &= A_i(b_i^*) h_i'(b_i^*) - \int_0^{b_i^*} W_i^{(\delta_i+q_i)'}(b_i^* - y) \sum_{j \neq i} q_{ij} u(y, j) \mathrm{d}y \\ &= \frac{h_i'(b_i^*)}{s_i \frac{\delta_i+q_i}{\theta_i} h_i(b_i^*) - h_i'(b_i^*)} \Delta_i(b_i^*) - \int_0^{b_i^*} W_i^{(\delta_i+q_i)'}(b_i^* - y) \sum_{j \neq i} q_{ij} u(y, j) \mathrm{d}y. \end{aligned}$$

It follows from (4.15) that

$$\frac{h_i'(b_i^*)}{s_i \frac{\delta_i+q_i}{\theta_i} h_i(b_i^*) - h_i'(b_i^*)} = \frac{\theta_i \Delta_i(b_i^*) + \frac{\sigma_i^2}{2} s_i \Delta_i'(b_i^*)}{\Delta_i(b_i^*)(s_i \mu_i - \theta_i) a_i}.$$

The above equation yields that

$$\begin{aligned}
M'_{\mathbf{b}^*}(b_i^*,i) &= \frac{\theta_i\Delta_i(b_i^*)+\frac{\sigma_i^2}{2}s_i\Delta_i'(b_i^*)}{(s_i\mu_i-\theta_i)a_i} - \int_0^{b_i^*} W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y \\
&= 1-\frac{2\theta_i}{\sigma_i^2 s_i^2 a_i}\left[\Lambda_i(b_i^*)-\frac{2}{\sigma_i^2 r_i}\Xi_i(b_i^*)\right]-\frac{1}{s_i a_i}\left[\Lambda_i'(b_i^*)-\frac{2}{\sigma_i^2 r_i}\Xi_i'(b_i^*)\right] \\
&\quad -\int_0^{b_i^*} W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y \\
&= 1,
\end{aligned}$$

where the last equality follows from

$$\frac{\sigma_i^2}{2}W_i^{(\delta_i+q_i)''}(x)+\mu_i W_i^{(\delta_i+q_i)'}(x)-(\delta_i+q_i)W_i^{(\delta_i+q_i)}(x)=0, \quad \text{for } x\geq 0. \tag{4.16}$$

Thus, it is easy to see that

$$A_i(b_i^*)=\frac{1}{h_i'(b_i^*)}\left[1+\int_0^{b_i^*} W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y\right].$$

Consequently, from (4.11) we have

$$\begin{aligned}
M''_{\mathbf{b}^*}(b_i^*,i) &= A_i(b_i^*)h_i''(b_i^*)-\int_0^{b_i^*} W_i^{(\delta_i+q_i)''}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y-\frac{2}{\sigma_i^2}\sum_{j\neq i}q_{ij}u(b_i^*,j) \\
&= \frac{h_i''(b_i^*)}{h_i'(b_i^*)}\left[1+\int_0^{b_i^*} W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y\right] \\
&\quad -\int_0^{b_i^*} W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y.
\end{aligned}$$

Noting that $h_i'(b_i^*)\geq 0$,

$$\int_0^{b_i^*} W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y\geq 0, \text{ and } \int_0^{b_i^*} W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y\geq 0,$$

it is sufficient to show that $h_i''(b_i^*)\leq 0$. From (4.14) and (4.15), we have that

$$h_i''(b_i^*) = h_i'(b_i^*)\left[\frac{s_i\frac{\delta_i+q_i}{\theta_i}+\frac{2}{\sigma_i^2}\mu_i+\frac{\Delta_i'(b_i^*)}{\Delta_i(b_i^*)}}{1+\frac{\sigma_i^2}{2}\cdot\frac{s_i}{\theta_i}\cdot\frac{\Delta_i'(b_i^*)}{\Delta_i(b_i^*)}}-\frac{2}{\sigma_i^2}\mu_i\right]$$

$$= \quad h_i'(b_i^*)\frac{s_i\frac{\delta_i+q_i}{\theta_i}\Delta_i(b_i^*)-\frac{\sigma_i^2}{2}\cdot\frac{s_i^2}{\theta_i}\Delta_i'(b_i^*)}{\Delta_i(b_i^*)+\frac{\sigma_i^2}{2}\cdot\frac{s_i}{\theta_i}\Delta_i'(b_i^*)}.$$

Noting that $\Delta_i(b_i^*)\le 0$, $\Delta_i'(b_i^*)\le 0$, and

$$\begin{aligned}
&\Delta_i(b_i^*)+\frac{\sigma_i^2}{2}\cdot\frac{s_i}{\theta_i}\Delta_i'(b_i^*)\\
= \;& s_i\frac{\delta_i+q_i}{\theta_i}\int_0^{b_i^*}W_i^{(\delta_i+q_i)}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)\mathrm{d}y-\int_0^{b_i^*}W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)\\
&+\frac{\sigma_i^2}{2}s_i^2\frac{\delta_i+q_i}{\theta_i^2}\int_0^{b_i^*}W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)-\frac{\sigma_i^2}{2}\cdot\frac{s_i}{\theta_i}\int_0^{b_i^*}W_i^{(\delta_i+q_i)''}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)\\
&-\left(\frac{2}{\sigma_i^2 r_i}+\frac{s_i}{\theta_i}\right)\Xi_i(b_i^*)+\frac{a_i}{\theta_i}(\mu_i s_i-\theta_i)\\
= \;& -\frac{a_i}{\theta_i}\cdot\frac{\sigma_i^2}{2}s_i^2\left[\int_0^{b_i^*}W_i^{(\delta_i+q_i)'}(b_i^*-y)\sum_{j\neq i}q_{ij}u(y,j)+1\right]\\
< \;& 0,
\end{aligned}$$

where the second equality follows from (4.16) and $\mu_i s_i-\theta_i=-\sigma_i^2 s_i^2/2$. Therefore, we have $h_i''(b_i^*)\le 0$ which completes the proof. □

Proposition 4.7. *The function $M_{\mathbf{b}^*}(x,i)$ is increasing and concave on* $[0,\infty)$.

Proof. Define $\xi(x,i)=M_{\mathbf{b}^*}''(x,i)$. Note that $\xi(x,i)\in C^1([0,\infty))$, $\xi(x,i)\in C^2([0,\infty)\backslash\{b_i^*\})$ and satisfies

$$\begin{cases}\frac{\sigma_i^2}{2}\xi''(x,i)+\mu_i\xi'(x,i)-(\delta_i+q_i)\xi(x,i)+\sum_{j\neq i}q_{ij}u''(x,j)=0, & 0\le x\le b_i^*,\\ \frac{\sigma_i^2}{2}\xi''(x,i)+\mu_i\xi'(x,i)-(\delta_i+q_i+\gamma_i)\xi(x,i)+\sum_{j\neq i}q_{ij}u''(x,j)=0, & x\ge b_i^*.\end{cases}$$

Recall that $Y'(t)=x+X_i(t)$. If $Y'(0)=x\in(0,b_i^*)$, define $\tau_{0,b_i^*}:=\inf\left\{t\ge 0: Y'(t)\notin(0,b_i^*)\right\}$. From (4.8), we know that $M_{\mathbf{b}^*}''(0,i)\le 0$. Thus, from Proposition 4.6, we have $\xi(Y'(\tau_{0,b_i^*}),i)\le 0$. Applying Itô formula to $e^{-(\delta_i+q_i)t}\xi(Y'(t),i)$ yields that

$$\xi(x,i)=\mathsf{E}_x\left[e^{-(\delta_i+q_i)\tau_{0,b_i^*}}\xi(Y'(\tau_{0,b_i^*}),i)+\int_0^{\tau_{0,b_i^*}}\sum_{j\neq i}q_{ij}u''(Y'(s),j)\right]\le 0.$$

If $Y'(0)=x\in(b_i^*,\infty)$, define $\tau_{b_i^*}:=\inf\left\{t\ge 0: Y'(t)\notin(b_i^*,\infty)\right\}$. Since $\mu_i>0$, we know that $Y'(\infty)=\infty$. From Corollary 4.5 (ii) and Proposition 4.6, we have $\xi(Y'(\tau_{b_i^*}),i)\le 0$. Similarly, applying Itô formula to $e^{-(\delta_i+q_i+\gamma_i)t}\xi(Y'(t),i)$ yields that $\xi(x,i)\le 0$. Hence, we proved the concavity of $M_{\mathbf{b}^*}(x,i)$.

It follows from Corollary 4.5 (ii) that $M_{\mathbf{b}^*}'(\infty,i)>0$. Therefore, the concavity of $M_{\mathbf{b}^*}(x,i)$ implies that $M_{\mathbf{b}^*}'(x,i)>0$ for all $x\ge 0$, i.e., $M_{\mathbf{b}^*}(x,i)$ is increasing on $[0,\infty)$. □

4.3 Verification of $M_{\mathbf{b}^*}(x,i)$

In this subsection, we are going to verify the modulated barrier strategy $\pi^{\mathbf{b}^*}$ is optimal for the auxiliary problem (4.1).

From Proposition 4.3, it is easy to see $M_{\mathbf{b}^*}(0,i) = 0$. It follows from Propositions 4.4 and 4.7 that $M_{\mathbf{b}^*}(x,i) \in C^2([0,\infty))$ and it is nonnegative.

From Proposition 4.6 and the concavity of $M_{\mathbf{b}^*}(x,i)$ (see Proposition 4.7), it is easy to see that $M'_{\mathbf{b}^*}(x,i) \geq 1$ for $x \in [0,b_i^*)$ and $M'_{\mathbf{b}^*}(x,i) \leq 1$ for $x \in [b_i^*,\infty)$. Thus the maximum

$$\max_{0\leq y\leq x} \{M_{\mathbf{b}^*}(x-y,i)+y\}$$

is attained at $y = 0$ if $x \in [0,b_i^*)$ and at $y = x - b_i^*$ if $x \in [b_i^*,\infty)$. Now, it follows from Proposition 4.4 that $M_{\mathbf{b}^*}(x,i)$ satisfies the HJB equation (4.2).

We have shown that $M_{\mathbf{b}^*}(x,i)$ satisfies the conditions of Theorem 4.2. Therefore, $M_{\mathbf{b}^*}(x,i)$ is the value function of the auxiliary optimal problem (4.1), and the modulated barrier strategy $\pi^{\mathbf{b}^*}$ is the optimal strategy.

Now, we can show the answer to the question raised at the beginning of this section, i.e., what $\mathbf{U}_{n+1}$ is when $\mathbf{U}_n$ is given. From Corollary 4.5, Proposition 4.7, we know that if $\mathbf{u}(x) \in \mathbb{D}$, then $\mathbf{M}_{\mathbf{b}^*}(x) \in \mathbb{D}$, where

$$\mathbf{M}_{\mathbf{b}^*}(x) = (M_{\mathbf{b}^*}(x,1), M_{\mathbf{b}^*}(x,2), \cdots, M_{\mathbf{b}^*}(x,K))$$

and $\mathbf{b}^* = (b_1^*, b_2^*, \cdots, b_K^*)$. Obviously, $\mathbf{0} \in \mathbb{D}$. Thus, from the definition of $U_n(x,i)$, it is easy to see that $\mathbf{U}_n(x) \in \mathbb{D}$, for $n = 0,1,2,\cdots$. Furthermore, when $\mathbf{U}_n(x)$ is given, $U_{n+1}(x,i)$ is given by (4.11) and (4.12) with u replaced by U_n.

5 Back to the Original Problem

5.1 The General Cases

Now, we consider the original problem (2.2). Since $\mathbf{U}_n(x) \in \mathbb{D}$, for $n = 0,1,2,\cdots$, we know $\mathbf{V}(x) \in \mathbb{D}$ as it is the point-wise limit of $\mathbf{U}_n(x)$. From the results given in the preceding section, we know that a modulated barrier strategy $\pi^{\mathbf{b}}$ at some barrier level $\mathbf{b} = (b_1, b_2, \cdots, b_K)$ will be a maximizer of the right-hand-side in (3.4). Recalling Remark 3.7, such a modulated barrier strategy is also the optimal strategy of the original problem (2.2).

There are two ways to get the value function the optimal barrier levels. The first method is iteration which is described as:

Step 1: Set $\mathbf{U}_0(x) \equiv 0$;

Step 2: Find $\mathbf{b}_{n+1}$ by equation (4.13), and find $\mathbf{U}_{n+1}(x)$ by (4.11) and (4.12);

Step 3: Stop when $\sup_{x\geq 0, i\in\mathbb{J}} |U_{n+1}(x,i) - U_n(x,i)| < \varepsilon$; otherwise, return to Step 2, where $\varepsilon > 0$ is the desirable level of accuracy.

The second method is to solve system of differential equations. From (4.8) and (4.9), the value function $\mathbf{V}(x)$ and the optimal barrier levels $\mathbf{b} = (b_1, b_2, \cdots, b_K)$ satisfy

$$\begin{cases} \frac{\sigma_i^2}{2} V''(x,i) + \mu_i V'(x,i) - (\delta_i + q_i) V(x,i) + \sum_{j\neq i} q_{ij} V(x,j) = 0, & 0 \le x < b_i, \\ \frac{\sigma_i^2}{2} V''(x,i) + \mu_i V'(x,i) - \theta_i V(x,i) + \gamma_i [V(b_i,i) + x - b_i] + \sum_{j\neq i} q_{ij} V(x,j) = 0, & x \ge b_i, \end{cases} \tag{5.1}$$

for all $i \in \mathbb{J}$. The system (5.1) can be solved with the conditions

$$\begin{cases} V(0,i) = 0, \\ V(b_i-, j) = V(b_i+, j), \\ V'(b_i-, j) = V'(b_i+, j), \\ V'(b_i-, i) = 1, \\ V''(\infty, i) = 0, \end{cases} \tag{5.2}$$

for all $i, j \in \mathbb{J}$.

5.2 The Special Case with Two Regimes

In the special case with two regimes, the first method, i.e. iteration is less efficient than solving the system of differential equations. So we consider the second method in this subsection.

Without loss of generality, let $0 \le b_1 \le b_2$. For solving the system (5.1), we have to consider the following cases: $x \in [0, b_1), x \in [b_1, b_2)$ and $x \in [b_2, \infty)$. Also, we need the following lemma. The proof is similar to Lemma 3.1 in **?** (see also **?**).

Lemma 5.1. *Let c_1 and c_2 be two strictly positive constants. The following system of equations on (r,s)*

$$\begin{cases} 0 = \frac{\sigma_1^2}{2} r^2 + \mu_1 r - (c_1 + q_1) + q_1 s, \\ 0 = \frac{\sigma_2^2}{2} r^2 + \mu_2 r - (c_2 + q_2) + q_2 / s, \end{cases} \tag{5.3}$$

has four real roots (r_i, s_i),i=1,2,3,4, and $r_1 < r_2 < 0 < r_3 < r_4$.

In the following, when we mention the roots of the system (5.3), $r_i, i = 1,2,3,4$, are sorted as $r_1 < r_2 < 0 < r_3 < r_4$.

If $x \in [0, b_1)$, the system (5.1) yields

$$\begin{cases} 0 = \frac{\sigma_1^2}{2} V''(x,1) + \mu_1 V'(x,1) - (\delta_1 + q_1) V(x,1) + q_1 V(x,2), \\ 0 = \frac{\sigma_2^2}{2} V''(x,2) + \mu_2 V'(x,2) - (\delta_2 + q_2) V(x,2) + q_2 V(x,1). \end{cases} \tag{5.4}$$

The solution of the above system of differential equation is given by

$$\begin{cases} V(x,1) = A_1 e^{r_1 x} + A_2 e^{r_2 x} + A_3 e^{r_3 x} + A_4 e^{r_4 x}, \\ V(x,2) = A_1 s_1 e^{r_1 x} + A_2 s_2 e^{r_2 x} + A_3 s_3 e^{r_3 x} + A_4 s_4 e^{r_4 x}, \end{cases}$$

where (r_i, s_i), $i = 1,2,3,4$, are the four roots of the system (5.3) with $c_i = \delta_i$, $i = 1,2$, and A_i, $i = 1,2,3,4$ are constants to be determined.

If $x \in [b_1, b_2)$, the system (5.1) yields

$$\begin{cases} 0 = \frac{\sigma_1^2}{2} V''(x,1) + \mu_1 V'(x,1) - (\delta_1 + \gamma_1 + q_1) V(x,1) + q_1 V(x,2) \\ \quad +\gamma_1 [x - b_1 + V(b_1,1)], \\ 0 = \frac{\sigma_2^2}{2} V''(x,2) + \mu_2 V'(x,2) - (\delta_2 + q_2) V(x,2) + q_2 V(x,1). \end{cases} \tag{5.5}$$

The solution of the above system is given by

$$\begin{cases} V(x,1) = B_1 e^{\hat{r}_1 (x-b_1)} + B_2 e^{\hat{r}_2 (x-b_1)} + B_3 e^{\hat{r}_3 (x-b_1)} + B_4 e^{\hat{r}_4 (x-b_1)} + k_1 x + l_1, \\ V(x,2) = B_1 \hat{s}_1 e^{\hat{r}_1 (x-b_1)} + B_2 \hat{s}_2 e^{\hat{r}_2 (x-b_1)} + B_3 \hat{s}_3 e^{\hat{r}_3 (x-b_1)} + B_4 \hat{s}_4 e^{\hat{r}_4 (x-b_1)} + k_2 x + l_2, \end{cases}$$

where $(\hat{r}_i, \hat{s}_i)$, $i = 1,2,3,4$, are the four roots of the system (5.3) with $c_1 = \delta_1 + \gamma_1, c_2 = \delta_2$, B_i, $i = 1,2,3,4$ are constants to be determined, and

$$\begin{aligned} k_1 &= \frac{(q_2 + \delta_2)\gamma_1}{(\gamma_1 + q_1 + \delta_1)(q_2 + \delta_2) - q_1 q_2}, \qquad k_2 = \frac{q_2 \gamma_1}{(\gamma_1 + q_1 + \delta_1)(q_2 + \delta_2) - q_1 q_2}, \\ l_1 &= \frac{k_1}{\gamma_1}\left[k_1 \mu_1 + \frac{\gamma_1 + q_1 + \delta_1}{q_2} \mu_2 k_2 + \gamma_1 (V(b_1,1) - b_1)\right] - \frac{\mu_2}{q_2} k_2, \\ l_2 &= \frac{k_2}{\gamma_1}\left[k_1 \mu_1 + \frac{\gamma_1 + q_1 + \delta_1}{q_2} \mu_2 k_2 + \gamma_1 (V(b_1,1) - b_1)\right]. \end{aligned}$$

If $x \in [b_2, \infty)$, the system (5.1) yields

$$\begin{cases} 0 = \frac{\sigma_1^2}{2} V''(x,1) + \mu_1 V'(x,1) - (\delta_1 + \gamma_1 + q_1) V(x,1) + q_1 V(x,2) \\ \quad +\gamma_1 [x - b_1 + V(b_1,1)], \\ 0 = \frac{\sigma_2^2}{2} V''(x,2) + \mu_2 V'(x,2) - (\delta_2 + \gamma_2 + q_2) V(x,2) + q_2 V(x,1) \\ \quad +\gamma_2 [x - b_2 + V(b_2,2)]. \end{cases} \tag{5.6}$$

The solution of the above system is given by

$$\begin{cases} V(x,1) = C_1 e^{\tilde{r}_1 (x-b_2)} + C_2 e^{\tilde{r}_2 (x-b_2)} + C_3 e^{\tilde{r}_3 (x-b_2)} + C_4 e^{\tilde{r}_4 (x-b_2)} + \tilde{k}_1 x + \tilde{l}_1, \\ V(x,2) = C_1 \tilde{s}_1 e^{\tilde{r}_1 (x-b_2)} + C_2 \tilde{s}_2 e^{\tilde{r}_2 (x-b_2)} + C_3 \tilde{s}_3 e^{\tilde{r}_3 (x-b_2)} + C_4 \tilde{s}_4 e^{\tilde{r}_4 (x-b_2)} + \tilde{k}_2 x + \tilde{l}_2, \end{cases}$$

where $(\tilde{r}_i, \tilde{s}_i)$, $i = 1,2,3,4$, are the four roots of the system (5.3) with $c_i = \gamma_i + \delta_i, i = 1,2$, and C_i,

$i = 1,2,3,4$, are constants to be determined, and

$$\tilde{k}_1 = \frac{q_1\gamma_2 + \gamma_1(\gamma_2+q_2+\delta_2)}{(\gamma_1+q_1+\delta_1)(\gamma_2+q_2+\delta_2)-q_1q_2}, \qquad \tilde{k}_2 = \frac{q_2\gamma_1 + \gamma_2(\gamma_1+q_1+\delta_1)}{(\gamma_1+q_1+\delta_1)(\gamma_2+q_2+\delta_2)-q_1q_2},$$

$$\tilde{l}_1 = \frac{q_1\mu_2\tilde{k}_2 + (\gamma_2+q_2+\delta_2)\mu_1\tilde{k}_1 + q_1\gamma_2(V(b_2,2)-b_2) + \gamma_1(\gamma_2+q_2+\delta_2)(V(b_1,1)-b_1)}{(\gamma_1+q_1+\delta_1)(\gamma_2+q_2+\delta_2)-q_1q_2},$$

$$\tilde{l}_2 = \frac{q_2\mu_1\tilde{k}_1 + (\gamma_1+q_1+\delta_1)\mu_2\tilde{k}_2 + q_2\gamma_1(V(b_1,1)-b_1) + \gamma_2(\gamma_1+q_1+\delta_1)(V(b_2,2)-b_2)}{(\gamma_1+q_1+\delta_1)(\gamma_2+q_2+\delta_2)-q_1q_2}.$$

The constants $A_i, B_i, C_i, i = 1,2,3,4$, and the barrier levels b_1 and b_2 can be obtained from the condition (5.2).

Example 5.2. We choose all the parameters except γ_i as in **?** which are listed in Table 5.1.

i	μ_i	σ_i	q_i	δ_i
1	0.06	0.24	2	0.04
2	0.08	0.30	3	0.05

Table 5.1: The parameter-set

By the using of the function FindRoot of Mathematica, we calculate the optimal barrier levels for different $\gamma_i, i = 1,2$. The result is given in Table 5.2[‡]. The value $(1.050, 1.070)$ for $\gamma_1 = \gamma_2 = \infty$ is taken from **?**. We can see that both of the optimal barrier levels monotonically increase when $\gamma_i, i = 1,2$ increase, and they convergence to the case with $\gamma_1 = \gamma_2 = \infty$. This is consistent with the arguments of **?** (see Page 50).

		γ_1					
		10	50	100	200	500	∞
γ_2	10	(0.9959, 1.0059)	-	-	-	-	-
	50	(1.0062, 1.0338)	(1.0264, 1.0405)	(1.0323, 1.0417)	(1.0367, 1.0422)	(1.0408, 1.0426)	-
	100	(1.0081, 1.0418)	(1.0274, 1.0480)	(1.0333, 1.0490)	(1.0376, 1.0496)	(1.0417,1.0499)	-
	200	(1.0090, 1.0477)	(1.0279, 1.0535)	(1.0337, 1.0545)	(1.0381, 1.0551)	(1.0421, 1.0554)	-
	500	(1.0096, 1.0532)	(1.0282, 1.0586)	(1.0340, 1.0600)	(1.0383, 1.0602)	(1.0424, 1.0605)	-
	∞	-	-	-	-	-	(1.050, 1.070)

Table 5.2: The optimal (b_1, b_2) for different $\gamma_i, i = 1,2$

Appendix

In this appendix, we show that the equation (4.13) admits a root in $(0,\infty)$. Since $h'(x) > 0$, (4.13) is equivalent to

$$\Delta_i'(b)\left[s_i\frac{\delta_i+q_i}{\theta_i}\cdot\frac{h_i(b)}{h_i'(b)} - 1\right] = \Delta_i(b)\left[s_i\frac{\delta_i+q_i}{\theta_i} - \frac{h_i''(b)}{h_i'(b)}\right]. \tag{A.1}$$

[‡]When $\gamma_2 = 10$ and $\gamma_1 = 50, \cdots, 500$, the results show that $b_1 > b_2$. So we do not list them here.

Let

$$f(b) = \Delta_i'(b)\left[s_i\frac{\delta_i+q_i}{\theta_i}\cdot\frac{h_i(b)}{h_i'(b)}-1\right]-\Delta_i(b)\left[s_i\frac{\delta_i+q_i}{\theta_i}-\frac{h_i''(b)}{h_i'(b)}\right].$$

Obviously, $f(b)$ is continuous. From Sections 7 and 8 of **?**, we know that

$$\frac{h_i''(b_0)}{h_i'(b_0)} = s_i\frac{\delta_i+q_i}{\theta_i},$$

where

$$b_0 = \frac{1}{\alpha_i-\beta_i}\ln\frac{\beta_i^2}{\alpha_i^2}+\frac{1}{\alpha_i-\beta_i}\ln\frac{r_i-\alpha_i}{r_i-\beta_i} > 0.$$

Noting that the left-hand-side of (A.1) is positive, $h_i''(b)/h_i'(b)$ is increasing and $\Delta_i(b) < 0$, we have $f(0) > 0$. To estimate $f(\infty)$, we can write

$$f(b) = F_1(b)-F_2(b)-F_3(b)-F_4(b)$$

where

$$F_1(b) = \frac{2\left(s_i\frac{\delta_i+q_i}{\theta_i}-\alpha_i\right)}{\sigma_i^2(\alpha_i-\beta_i)}e^{\alpha_i b}\left[\left(s_i\frac{\delta_i+q_i}{\theta_i}\cdot\frac{h_i(b)}{h_i'(b)}-1\right)-\frac{1}{\alpha_i}\left(s_i\frac{\delta_i+q_i}{\theta_i}-\frac{h_i''(b)}{h_i'(b)}\right)\right]\int_0^b e^{-\alpha_i y}\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y;$$

$$F_2(b) = \frac{2\left(s_i\frac{\delta_i+q_i}{\theta_i}-\beta_i\right)}{\sigma_i^2(\alpha_i-\beta_i)}e^{\beta_i b}\left[\left(s_i\frac{\delta_i+q_i}{\theta_i}\cdot\frac{h_i(b)}{h_i'(b)}-1\right)-\frac{1}{\beta_i}\left(s_i\frac{\delta_i+q_i}{\theta_i}-\frac{h_i''(b)}{h_i'(b)}\right)\right]\int_0^b e^{-\beta_i y}\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y;$$

$$F_3(b) = \frac{2}{\sigma_i^2}\left[\left(s_i\frac{\delta_i+q_i}{\theta_i}\cdot\frac{h_i(b)}{h_i'(b)}-1\right)-\frac{1}{r_i}\left(s_i\frac{\delta_i+q_i}{\theta_i}-\frac{h_i''(b)}{h_i'(b)}\right)\right]\int_b^\infty e^{r_i(b-y)}\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y;$$

$$F_4(b) = \left(s_i\frac{\mu_i a_i}{\theta_i}-a_i\right)\left(s_i\frac{\delta_i+q_i}{\theta_i}-\frac{h_i''(b)}{h_i'(b)}\right).$$

Note that $h_i(b)/h_i'(b)\to 1/\alpha_i$, $h_i''(b)/h_i'(b)\to\alpha_i$ as $b\to\infty$. Since

$$\int_0^b e^{-\alpha_i y}\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y \le \frac{1}{\alpha_i}\sum_{j\neq i}q_{ij}u'(0,j),$$

by the de'l Hopital's rule,

$$e^{\alpha_i b}\left[\left(s_i\frac{\delta_i+q_i}{\theta_i}\cdot\frac{h_i(b)}{h_i'(b)}-1\right)-\frac{1}{\alpha_i}\left(s_i\frac{\delta_i+q_i}{\theta_i}-\frac{h_i''(b)}{h_i'(b)}\right)\right]\to 0,$$

we have $F_1(b)\to 0$, as $b\to\infty$. Recalling $\mathbf{u}\in\mathbb{D}$, we have

$$e^{\beta_i b}\int_0^b e^{-\beta_i y}\sum_{j\neq i}q_{ij}u'(y,j)\mathrm{d}y \le e^{\beta_i b}\left(\sum_{j\neq i}q_{ij}u(b,j)-\sum_{j\neq i}q_{ij}u(0,j)\right)\to 0,\ \text{as } b\to\infty.$$

Thus, $F_2(b) \to 0$, as $b \to \infty$. Recalling

$$\int_b^\infty e^{r_i(b-y)} \sum_{j\neq i} q_{ij} u'(y,j)\mathrm{d}y \to 0, \text{ as } b \to \infty,$$

we have $F_3(b) \to 0$, as $b \to \infty$. Since $h_i''(b)/h_i'(b)$ is increasing, and $s_i\mu_i - \theta_i = -\sigma_i^2 s_i^2/2 < 0$, we have

$$F_4(b) \to \left(s_i \frac{\mu_i a_i}{\theta_i} - a_i\right)\left(s_i \frac{\delta_i + q_i}{\theta_i} - \alpha_i\right) > 0, \text{ as } b \to \infty.$$

Thus we have

$$f(b) \to -\left(s_i \frac{\mu_i a_i}{\theta_i} - a_i\right)\left(s_i \frac{\delta_i + q_i}{\theta_i} - \alpha_i\right) < 0, \text{ as } b \to \infty.$$

Then the continuity of $f(b)$ yields that equation (4.13) has root in $(0,\infty)$.

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References