

On the Asymptotic Distribution of the Transaction Price in a Clock Model of a Multi-Unit, Oral, Ascending-Price Auction within the Common-Value Paradigm

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Abstract

Using a clock model of a multi-unit, oral, ascending-price auction, within the common-value paradigm, we analyze the behavior of the transaction price as the numbers of bidders and units gets large in a particular way. We find that even though the transaction price is determined by a fraction of losing drop-out bids, that price converges in probability to the true, but *ex ante* unknown, value. Subsequently, we demonstrate that the asymptotic distribution of the transaction price is Gaussian. Finally, we apply our methods to data from an auction of taxi license plates held in Shenzhen, China.

Key words: common value; information aggregation; multi-unit auctions; taxis; linkage principle.

JEL Classification: C20, D44, L1.

1 Motivation and Introduction

During the past half century, economists have made considerable progress in understanding the theoretical structure of equilibrium strategic behavior under market mechanisms, such as auctions; see Krishna [2010] for a comprehensive presentation and evaluation of progress.

One analytic device used to describe bidder motivation at single-object auctions is a continuous random variable that represents a bidder-specific signal concerning the object's true, but *ex ante* unknown, value. This true, but *ex ante* unknown, value will only be revealed *after* the auction has ended, when the winner has been determined and the transaction price paid. Regardless of the winner, however, the value of the object is the same to all.

The conceptual experiment involves each potential bidder's receiving a draw from a signal distribution. Conditional on his draw, a bidder is then assumed to act purposefully, using the information in his signal along with Bayes' rule to maximize either the expected profit or the expected utility of profit from winning the auction. Another frequently-made assumption is that the signal draws of bidders are independent and that the bidders are *ex ante* symmetric—their draws coming from the same distribution of signals. This framework is often referred to as the *symmetric common-value paradigm* (symmetric CVP).

Under these assumptions, a researcher can then focus on a representative agent's decision rule when characterizing equilibrium behavior. Wilson [1977] invented this framework to illustrate that, in equilibrium, the winner's curse could not obtain among rational bidders. He also demonstrated that when the number of bidders n gets large (tends to infinity) the winning bid at first-price, sealed-bid auctions converges almost surely to the true value of the object. In other words, the auction format and pricing rule play an important role in aggregating the disparate, individual pieces of information held by the bidders. Milgrom [1979] subsequently provided a precise characterization of the structure the signal distribution must possess in order for this convergence property to hold; Pesendorfer and Swinkels [1997] have referred to this as *full information aggregation*.

When several, say k , units of a good are simultaneously for sale, Weber [1983] has described a number of different multi-unit auction formats as well as pricing rules under those formats. At

least two important questions arise: specifically, who will the winning bidders be and what price(s) will those winners pay? For example, Milgrom [1981] developed a natural generalization of the Wilson [1977] model. In Milgrom's model, each bidder submits a sealed bid and the auctioneer then aggregates these demands, allocating the units to those bidders with the highest k submitted bids. The winners then pay a uniform price—specifically, the highest rejected bid.

Pesendorfer and Swinkels [1997] have built on this research by investigating a sequence of auctions $\{A_r\}$ in which both n_r and k_r increase. They demonstrated that a necessary and sufficient condition for full information aggregation is that $k_r \rightarrow \infty$ and $(n_r - k_r) \rightarrow \infty$, a condition they referred to as *double largeness*. Under the double-largeness condition, non-negligible supply can be a substitute for the strong signal structure required in Wilson [1977], Milgrom [1979, 1981], and Kremer [2002].

While it is heartening to know that conditions exist under which transaction prices will converge in probability to the true, but *ex ante* unknown, values of objects for sale, the rates at which these prices converge are also of interest. In particular, Hong and Shum [2004] asked the question “How large must n be to be large enough?” and then investigated the rates of information aggregation in common-value environments. Knowing the conditions under which the transaction price provides a potentially useful estimate of the object's unknown value is important to understanding the process some refer to as *price discovery* because neither the number of bidders nor the number of units for sale ever really gets to infinity in practice.

Of course, the pricing rule investigated in Wilson [1977] and Milgrom [1979, 1981] as well as Pesendorfer and Swinkels [1997, 2000] is not the only one that could be used under a sealed-bid auction format. For example, another pricing rule would involve allocating the k units to those bidders who tendered the highest k bids, but then having each winner pay what he bid for the unit(s) he won. In general, at multi-unit auctions, different auction formats and different pricing rules induce different equilibrium behavior and can, thus, translate into different transaction prices as well as potentially different expected revenues for sellers. Hence, as Jackson and Kremer [2004, 2006] have emphasized, understanding the effects of auction formats and pricing rules has important practical

relevance. Even small changes can have effects, as has been illustrated by Mezzetti and Tsetlin [2008, 2009].

In a companion paper to Milgrom and Weber [1982], which was published nearly two decades later, Milgrom and Weber [2000] proposed a pricing rule for multi-unit, oral, ascending-price auctions. The model studied by Milgrom and Weber [2000] is the multi-unit variant of the clock model introduced by Milgrom and Weber [1982] in order to investigate behavior at single-object, oral, ascending-price (often referred to as *English*) auctions. In the multi-unit model, bidders are assumed to demand at most one unit of the good for sale; Milgrom [2004] has referred to this as *singleton demand*. The current price for all units on sale rises continuously according to some device, such as a clock. As the price rises, the drop-out prices of losing participants are recorded when they exit the auction. The transaction price is the drop-out price of the last participant to exit the auction. Each of the remaining k participants is then allocated one unit at the transaction price.

One attractive feature of oral, ascending-price auctions *vis-à-vis* sealed-bid ones is the scope for information release at oral, ascending-price auctions. This is particularly important in informational environments with substantial common-value components. In common-value environments, by observing the actions of his competitors, a bidder can augment the information contained in his signal and, thus, may be able to reduce the uncertainty concerning the unknown value of the object for sale. Other things being equal, this reduction in uncertainty can induce participants to bid more aggressively than under sealed-bid formats, which means the revenues the seller can expect to garner can increase. The greater is the linkage between a bidder's information and what he perceives others will bid, the higher the bidding. Milgrom and Weber [1982] have referred to this as the *linkage principle*. In models of single-object auctions, Milgrom and Weber used the linkage principle to rank the revenues a seller can expect to garner under the different auction formats and pricing rules, for the same marginal distribution. In short, Milgrom and Weber [1982] demonstrated that, on average, the English auction format yields more revenue than first-price auctions, such as the oral, descending-price (Dutch) format or the first-price, sealed-bid format.

For multi-unit auctions within the CVP, as the numbers of bidders and units get large in the

Pesendorfer–Swinkels sense, we compare the behavior of transaction prices under two different combinations of auction format and pricing rule, those of Milgrom and Weber as well as Pesendorfer and Swinkels. We demonstrate that the asymptotic distributions of the transaction prices are Gaussian, but that the asymptotic variance of the transaction price under the Milgrom–Weber auction is less than that under the Pesendorfer–Swinkels auction. If the transaction prices under different auction formats and pricing rules are viewed as statistical estimators of the true, but *ex ante* unknown, value of the units for sale, then the Milgrom–Weber auction provides a more efficient estimator of the unknown value than the Pesendorfer–Swinkels auction because more information is released with ascending bids than with sealed bids. Note, however, that when the number of bidders is large, the differences in both the expected transaction prices and their asymptotic variances are small because the transaction prices converge to the same value.

From the structure of the proof in Milgrom and Weber [1982], one can deduce that the same linkage principle applies to the multi-unit auction we study in this paper. In theory, the linkage principle implies that the additional information aggregated in the price of the ascending-bid auction relative to the sealed-bid auction translates into higher expected revenues for the seller at the ascending-bid auction. Such differences in information decrease as the number of bidders increases. Whether the revenue differences induced by the information structures across auction formats and pricing rules are economically important remains an empirical issue. By estimating the variation of the signal distribution from data, one can investigate empirically the difference in the seller’s expected revenues across the multi-unit auction formats and pricing rules. To the best of our knowledge, this research represents the first attempt to quantify the value of information in multi-unit auctions.

For the data used below, we have found that the loss in expected revenues resulting from a switch to the sealed-bid format from the oral, ascending-price format is small, relative to both the transaction price and the estimated common value. Our results suggest that, on average, the Pesendorfer–Swinkels auction generates nearly as much revenue for the seller as the Milgrom–Weber auction does. In this particular application, the auctioneer could have done just as well by

selling the objects using the Pesendorfer–Swinkels auction.

We should also mention that a continuum of equilibria can exist in models of English auctions. For example, Bikhchandani et al. [2002] have characterized the symmetric, separating equilibria of a single-object English auction model. Multiple equilibria may also arise in the Milgrom–Weber model we have employed. Fortunately, the information aggregation result remains unchanged because the transaction price is determined in the final round of bidding, where the bid functions are the same across equilibria. While the bid functions in previous rounds can be different in different equilibria, provided bidders know this and use this information to invert the signals, the true signals can still be recovered, which is what matters for the information set in the final round of the auction. As noted by Bikhchandani et al. [2002], the multiplicity of equilibria affects how bids from the previous rounds of bidding are interpreted in an econometric procedure. Our empirical analysis relies on the equilibrium of the Milgrom–Weber model. Although the transaction price remains a consistent estimator of the true value, estimating the dispersion of the signal distribution is difficult in the presence of multiple equilibria.

Our paper is in six additional sections as well as an appendix. In the next, we use the Milgrom–Weber clock model to develop a theoretical framework within which to investigate the stochastic behavior of the transaction price at a multi-unit, oral, ascending-price auction within the common-value paradigm, while in section 3 we demonstrate that, as the number of bidders n and the number of units k get large in the Pesendorfer–Swinkels sense, the transaction price converges in probability to the true, but *ex ante* unknown, value. We characterize in section 4 the asymptotic distribution of the transaction price when both the number of bidders and the number of units get large, and compare the asymptotic variances of transactions prices under both the Milgrom–Weber and the Pesendorfer–Swinkels auctions. In section 5, we derive the likelihood function of observed drop-out prices, while in section 6, we apply our methods to data from an auction of taxi license plates held in Shenzhen, China. In the final section, we summarize and conclude. Any details too cumbersome to be included in the text of the paper have been collected in the appendix at the end of the paper.

2 Theoretical Model

Consider an oral, ascending-price auction at which k identical units of an object are for sale to a total of n bidders, each of whom wants at most one unit. Assume that k is strictly less than n . Focus on the Milgrom and Weber [2000] pricing rule described in the introduction. Assume that, conditional on the true (but unknown) value v^0 , each bidder draws an independently- and identically-distributed signal X . Denote the cumulative distribution and probability density functions of X , conditional on v , by $F_{X|V}(x|v)$ and $f_{X|V}(x|v)$, respectively, and assume $f_{X|V}(x|v)$ satisfies the monotone likelihood ratio condition in Milgrom [1981]. Denote the prior distribution of the unknown value V by $f_V(v)$

Consider the vector of signals (X_1, X_2, \dots, X_n) , a random sample of n draws from $F_{X|V}(x|v^0)$. Because this environment is symmetric, without loss of generality, we focus below on bidder 1. Denote by Y_i the i^{th} ordered signal of the opponents of bidder 1, so

$$Y_1 \geq Y_2 \geq \dots \geq Y_{n-1}.$$

Denote by Z_i the i^{th} order statistic for all of the X_i s, so

$$Z_1 \geq Z_2 \geq \dots \geq Z_n.$$

The auction proceeds in rounds $m = n, n - 1, \dots, k + 1$. In round m , m bidders continue to participate in the auction. The auction ends in round $(k + 1)$ when the $(k + 1)^{\text{st}}$ bidder exits the auction. Without loss of generality, suppose that bidders are ordered in the reverse order of exit from the auction.

Let Ω_m denote the information that has already been revealed in round m by all the bidders who have already left the auction. Hence, Ω_m equals $\{z_n, z_{n-1}, \dots, z_{m+1}\}$, where Ω_n is the empty set \emptyset . According to Milgrom and Weber [2000], the symmetric equilibrium bidding rule in round m can

be written as

$$\beta_m(x) = \mathbb{E} [V|X_1 = Y_k = \dots = Y_{m-1} = x, \Omega_m] \quad (1)$$

where \mathbb{E} denotes the expectation operator. Here, Y_k, \dots, Y_{m-1} denote the k^{th} through $(m-1)^{\text{st}}$ order statistics among the bidders who remain competing with bidder 1. On the other hand, the order statistics in the event Ω_m denote the order statistics for *all* the bidders who have exited the auction. For completeness, we describe below the reasoning behind a characterization of the equilibrium; in their paper, Milgrom and Weber [2000] presumably omitted an argument like this because they found it obvious.

At price p , bidder 1 is concerned with the event that Y_k, \dots, Y_{m-1} all drop-out simultaneously at $\beta_m^{-1}(p)$. Here, $\beta_m^{-1}(p)$ is the inverse bid function. In this event, bidder 1 will be one of the winners of the auction, together with his remaining $(k-1)$ competitors. Bidder 1 should remain active in the auction at price level p if and only if

$$\mathbb{E} [V|X_1 = x, Y_k = \dots = Y_{m-1} = \beta_m^{-1}(p), \Omega_m] > p.$$

In equilibrium, $p = \beta_m(x)$ or $x = \beta_m^{-1}(p)$, so the price at which bidder 1 should exit must satisfy the following:

$$p = \mathbb{E} [V|X_1 = x, Y_k = \dots = Y_{m-1} = \beta_m^{-1}(p), \Omega_m] \quad (2)$$

as long as the function in equation (1) is increasing in x . Hence, the functional form of the bid function.

The winning price corresponds to the bid submitted by the bidder with the $(k+1)^{\text{st}}$ order statistic of the signals during round $(k+1)$. Thus,

$$\hat{p} = \mathbb{E} [V|Z_k = Z_{k+1} = z_{k+1}, \Omega_{k+1}]. \quad (3)$$

3 Limiting Information in the Transaction Price

In this section and the next, we have two goals: first, to derive the convergence rate of the transaction price \hat{p} to the true, but *ex ante* unknown, value v^0 ; and, second, to characterize the limiting distribution of the transaction price \hat{p} . In both of these endeavors, we assume that both k and n get large, tends to infinity, in the Pesendorfer–Swinkels sense.

In this regard, we make the following assumption concerning k , the number of units for sale relative to n , the number of bidders at the auction.

Assumption 1 $\hat{\tau} \equiv [(n - k)/n] \rightarrow \tau$, where τ is strictly between 0 and 1.

In words, the proportion of demand met has a stable limit as the number of bidders gets large. Were this not the case, then as Pesendorfer and Swinkels [1997] have noted, the transaction price will not have a stable limit.

With regard to our goals, we proceed in two steps. In the first, we define \hat{v} , the maximum-likelihood estimator (MLE) of v^0 , based on the unobserved (to the researcher, but known to the participants) order statistics z_{k+1}, \dots, z_n , and then we investigate the rate at which \hat{v} converges to v^0 . In the second, we investigate the rate at which \hat{p} converges to \hat{v} . In the next section, we demonstrate formally that the rate of convergence of the price \hat{p} to the true common value v^0 will be driven (dominated) by the convergence rate of \hat{v} to v^0 . In other words, $(\hat{p} - v^0)$ is $o_p(\hat{v} - v^0)$. Therefore, to understand the rate of information aggregation, it suffices to focus on how \hat{v} approaches v^0 as the “sample size” n gets large.

Under our assumptions, the MLE \hat{v} is defined as

$$\hat{v} = \underset{v}{\operatorname{argmax}} \log \left[\binom{n}{k} \mathcal{L}_n(z_{k+1}, \dots, z_n | v) \right]$$

where the joint likelihood function of all the signals revealed under the Milgrom–Weber auction is proportional to

$$\mathcal{L}_n(z_{k+1}, \dots, z_n | v) = [1 - F_{X|V}(z_{k+1} | v)]^k f_{X|V}(z_{k+1} | v) f_{X|V}(z_{k+2} | v) \cdots f_{X|V}(z_n | v). \quad (4)$$

Here, the term $[1 - F_{X|V}(z_{k+1}|v)]^k$ represents the fact that only limited information is known concerning the signal values of the k winners, specifically, their signals are greater than z_{k+1} . Also, $\binom{n}{k}$ represents the fact that there are many ways in which the k highest order statistics of signals could exceed z_{k+1} . Equation (4) is the joint likelihood of the lowest $(n - k)$ order statistics—those from z_{k+1} to z_n .

3.1 Convergence of Transaction Price to the True, but *Ex Ante* Unknown, Value

Given equation (4), the sample average of the logarithm of the likelihood (log-likelihood) function will be a function of the lowest $(n - k)$ order statistics. A general function of order statistics can be difficult to analyze because of the potentially complex correlation structure among order statistics. When investigating the convergence properties of functions of order statistics, one possibility is to appeal to the theory of L -statistics. Fortunately, this particular sample-averaged log-likelihood function is more tractable than an L -statistic because it can be written as a function of the entire sample as well as the sample τ^{th} quantile. Specifically,

$$\begin{aligned}\hat{Q}_n(v) &= \frac{1}{n} \log \mathcal{L}_n(z_{k+1}, \dots, z_n|v) \\ &= \frac{k}{n} \log \left(1 - F_{X|V} \left[\hat{F}_n^{-1}(\hat{\tau}) | v \right] \right) + \frac{1}{n} \sum_{i=1}^n \log f_{X|V}(z_i|v) \mathbf{1} \left[X_i \leq \hat{F}_n^{-1}(\hat{\tau}) \right]\end{aligned}\tag{5}$$

where $\hat{F}_n(\cdot)$ and $\hat{F}_n^{-1}(\tau)$ denote the empirical distribution function and the quantile function; that is,

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(z_i \leq x) \quad \text{and} \quad \hat{F}_n^{-1}(\tau) = \inf\{x : \hat{F}_n(x) \geq \tau\}.$$

With this definition, when $\hat{\tau}$ equals $[(n - k)/n]$, provided $z_{k+2} < z_{k+1}$, $\hat{F}_n^{-1}(\hat{\tau})$ equals z_{k+1} with probability one. Without loss of generality, in the remainder of the paper, we assume this holds. Now, under the assumptions made formal below, and because the limit of $\hat{\tau}$ is τ , by a uniform law of large numbers, the sample percentile $\hat{F}_n^{-1}(\hat{\tau})$ converges in probability to the true population

quantile $F_{X|V}^{-1}(\tau|v^0)$, so $\hat{Q}_n(v)$ converges uniformly in the parameter space of v to a deterministic function $Q(v^0, v)$ where we define

$$Q(u, v) \equiv (1 - \tau) \log \left(1 - F_{X|V} \left[F_{X|V}^{-1}(\tau|u)|v \right] \right) + \int_{-\infty}^{F_{X|V}^{-1}(\tau|u)} f_{X|V}(x|u) \log f_{X|V}(x|v) \, dx$$

which implies

$$Q(v^0, v) \equiv (1 - \tau) \log \left(1 - F_{X|V} \left[F_{X|V}^{-1}(\tau|v^0)|v \right] \right) + \int_{-\infty}^{F_{X|V}^{-1}(\tau|v^0)} f_{X|V}(x|v^0) \log f_{X|V}(x|v) \, dx.$$

As a short-hand notation, we define $Q(u, v)$ by $Q^u(v)$ and $Q(v^0, v)$ by $Q^0(v)$.

In order for information to aggregate fully, $Q^0(v)$, as a function of v , must be uniquely maximized at $v = v^0$. As in the case of full-sample likelihood function, this can be verified using Jensen's inequality. Thus, for any v not equal to v^0 , $Q^0(v) \leq Q^0(v^0)$. One can demonstrate this by taking the sum of the following two inequalities. First, by Jensen's inequality,

$$\begin{aligned} & \int_{-\infty}^{F_{X|V}^{-1}(\tau|v^0)} f_{X|V}(x|v^0) \log f_{X|V}(x|v) \, dx - \int_{-\infty}^{F_{X|V}^{-1}(\tau|v^0)} f_{X|V}(x|v^0) \log f_{X|V}(x|v^0) \, dx \\ & \leq \tau \left[\log \int_{-\infty}^{F_{X|V}^{-1}(\tau|v^0)} f_{X|V}(x|v) \, dx - \log \tau \right]. \end{aligned}$$

Second, it is easy to demonstrate that

$$(1 - \tau) \log \left(1 - F_{X|V} \left[F_{X|V}^{-1}(\tau|v^0)|v \right] \right) + \tau \log F_{X|V} \left[F_{X|V}^{-1}(\tau|v^0)|v \right] \leq (1 - \tau) \log (1 - \tau) + \tau \log \tau$$

because the left-hand side, considered as a function of $F_{X|V} \left[F_{X|V}^{-1}(\tau|v^0)|v \right]$, is maximized at τ .

Assumption 2 For $v \neq v^0$, either $F_{X|V}^{-1}(\tau|v) \neq F_{X|V}^{-1}(\tau|v^0)$ or, with positive probability, $X \leq F_{X|V}^{-1}(\tau|v^0)$ under v^0 , $f_{X|V}(x|v) \neq f_{X|V}(x|v^0)$.

This assumption mirrors a standard full-sample identification condition for likelihood analysis. While the monotone likelihood-ratio condition used by Milgrom and Weber [1982] is required to derive the equilibrium bidding strategy, conditional on the form of the equilibrium bidding strategy,

it is not strictly necessary for full-information aggregation to hold.

The first inequality will be strict under the first condition in Assumption 2 and, likewise, for the second inequality, under the second condition in Assumption 2. Thus, we have demonstrated that $Q^0(v)$ is globally and uniquely maximized at v^0 , provided the value v identifies the signal distribution $f_{X|V}(x|v)$ in the sense of Assumption 2, which is stronger than the usual full-sample identification condition whenever τ is less than one. The usual Jensen's inequality argument for full-sample likelihood function is just a special case of the above when τ is one.

Now, examine the following first-order condition at v^0 :

$$\begin{aligned}
\left. \frac{\partial Q^0(v)}{\partial v} \right|_{v=v^0} &= - \left[\frac{(1-\tau)}{1 - F_{X|V}[F_{X|V}^{-1}(\tau|v^0)|v]} \right] \left\{ \frac{\partial F_{X|V}[F_{X|V}^{-1}(\tau|v^0)|v]}{\partial v} \right\} \bigg|_{v=v^0} + \int_{-\infty}^{F_{X|V}^{-1}(\tau|v^0)} \frac{\partial f_{X|V}(x|v^0)}{\partial v} dx \\
&= - \left\{ \frac{\partial F_{X|V}[F_{X|V}^{-1}(\tau|v^0)|v]}{\partial v} \right\} \bigg|_{v=v^0} + \int_{-\infty}^{F_{X|V}^{-1}(\tau|v^0)} \frac{\partial f_{X|V}(x|v^0)}{\partial v} dx \\
&= 0.
\end{aligned}$$

Therefore, subject to the regularity conditions, which are outlined completely in the next section, \hat{v} is a consistent estimator of v^0 . To wit, $(\hat{v} - v^0) \xrightarrow{p} 0$.

Given that $Q^0(v)$ is a properly-defined sample-averaged log-likelihood function that depends linearly on the observed sample up to a given sample quantile and that the central sample quantiles are \sqrt{n} -consistent as well as distributed asymptotically normal, the information equality then holds for v , and is related to the asymptotic variance of \hat{v} . Given the form of $Q^0(v)$, the expected Hessian is $\left. \frac{\partial^2}{\partial v^2} Q^0(v) \right|_{v=v^0}$ where

$$\begin{aligned}
\frac{\partial^2}{\partial v^2} Q^0(v) &= - \frac{\partial^2}{\partial v^2} F_{X|V}[F_{X|V}^{-1}(\tau|v^0)|v] - \\
&\quad \frac{1}{1-\tau} \left(\frac{\partial}{\partial v} F_{X|V}[F_{X|V}^{-1}(\tau|v^0)|v] \right)^2 + \int_0^\tau \frac{\partial^2}{\partial v^2} \log f_{X|V}[F_{X|V}^{-1}(u|v^0)|v] du.
\end{aligned} \tag{6}$$

3.2 Information-Matrix Equality

In full-sample likelihood models, the asymptotic variance of the MLE is usually calculated using an information-matrix equality. Here, we demonstrate that an analogous information-matrix equality also holds for the *partial-sample* information model that we consider, which we then use to characterize the amount of limiting information contained in the price as an estimate of the true value.

One approach to calculating the information-matrix equality is to view the limiting first-order condition at v^0 as an identity, and then totally differentiate it with respect to v^0 . Specifically, because

$$\left. \frac{\partial}{\partial v} Q(v^0, v) \right|_{v=v^0} = 0,$$

for all possible values of v^0 , the derivative of this relation with respect to v^0 should also be zero.

$$\frac{\partial}{\partial v^0} \left[\left. \frac{\partial}{\partial v} Q(v^0, v) \right|_{v=v^0} \right] = 0.$$

This can be written as

$$\left. \frac{\partial^2}{\partial v^2} Q(v^0, v) \right|_{v=v^0} + \frac{\partial}{\partial v^0} \left[\left. \frac{\partial}{\partial v} Q(v^0, v) \right] \right|_{v=v^0} = 0. \quad (7)$$

In the next section, we demonstrate that the second term on the left-hand side of equation (7), which is the negative of the Hessian given in equation (6), equals the asymptotic variance of the score function. The following provides a direct calculation of the second term in equation (7), which independently verifies equation (7) and facilitates the comparison with the variance of the score function in the next section.

To compute this term, we need to calculate

$$\frac{\partial}{\partial v} F_{X|V}^{-1}(\tau|v)$$

as well as

$$\frac{\partial}{\partial v} F_{X|V} \left[F_{X|V}^{-1}(\tau|v) |v^0 \right] \Big|_{v=v^0} = \frac{\partial}{\partial v} F_{X|V}^{-1}(\tau|v) f_{X|V} \left[F_{X|V}^{-1}(\tau|v^0) |v \right].$$

Both can be found by totally differentiating the identity

$$\int_{-\infty}^{F_{X|V}^{-1}(\tau|v)} f_{X|V}(x|v) dx = \tau$$

with respect to v , which leads to

$$\frac{\partial}{\partial v} F_{X|V}^{-1}(\tau|v) = -\frac{\frac{\partial}{\partial v} F_{X|V} \left[F_{X|V}^{-1}(\tau|v^0) |v \right]}{f_{X|V} \left[F_{X|V}^{-1}(\tau|v^0) |v \right]}, \quad \frac{\partial}{\partial v} F_{X|V} \left[F_{X|V}^{-1}(\tau|v) |v^0 \right] \Big|_{v=v^0} = -\frac{\partial}{\partial v} F_{X|V} \left[F_{X|V}^{-1}(\tau|v^0) |v \right] \Big|_{v=v^0}.$$

Using these relations,

$$\frac{\partial}{\partial u} \left[\frac{\partial}{\partial v} Q^u(v) \right] \Big|_{u=v=v^0} = \frac{1}{1-\tau} \left(\frac{\partial}{\partial v} F_{X|V} \left[F_{X|V}^{-1}(\tau|v^0) |v \right] \right)^2 + \int_{-\infty}^{F_{X|V}^{-1}(\tau|v^0)} \left[\frac{\partial}{\partial v} \log f_{X|V}(x|v) \right]^2 f_{X|V}(x|v^0) dx \Big|_{v=v^0}.$$

In the next section, we demonstrate formally that the log-likelihood function of the partially-observed sample in our model has a similar statistical behavior to the usual full-sample log-likelihood function, so $\sqrt{n}(\hat{v} - v^0)$ will converge in distribution to a normal random variable whose asymptotic variance is the inverse of either $\frac{\partial}{\partial u} \left[\frac{\partial}{\partial v} Q^u(v) \right] \Big|_{u=v=v^0}$ or equivalently $\frac{\partial^2}{\partial v^2} Q^0(v) \Big|_{v=v^0}$. We now need to demonstrate that $\sqrt{n}(\hat{p} - \hat{v})$ is $o_p(1)$ because, then, these will also represent the asymptotic variance of $\sqrt{n}(\hat{p} - v^0)$.

For this purpose, we employ Bayesian asymptotic analysis. First, note that

$$\hat{p} = \beta_{k+1}(z_{k+1}) = \int v \frac{f_{Z|V}(Z_k = Z_{k+1} = z_{k+1}, \Omega_{k+1}|v) f_V(v)}{\int f_{Z|V}(Z_k = Z_{k+1} = z_{k+1}, \Omega_{k+1}|u) f_V(u) du} dv$$

where the likelihood of the conditioning event in the bid function is proportional to

$$f_{Z|V}(Z_k = Z_{k+1} = z_{k+1}, \Omega_{k+1}|v) = [1 - F_{X|V}(z_{k+1}|v)]^{k-1} f_{X|V}(z_{k+1}|v)^2 f_{X|V}(z_{k+2}|v) \cdots f_{X|V}(z_n|v). \quad (8)$$

Recall the definition in equation (4)

$$\begin{aligned} f_{Z|V}(Z_k = Z_{k+1} = z_{k+1}, \Omega_{k+1}|v) &= \mathcal{L}_n(z_{k+1}, \dots, z_n|v) \frac{f_{X|V}(z_{k+1}|v)}{[1 - F_{X|V}(z_{k+1}|v)]} \\ &= \exp[n\hat{Q}_n(v)] \frac{f_{X|V}(z_{k+1}|v)}{[1 - F_{X|V}(z_{k+1}|v)]}, \end{aligned}$$

which we can write, using a change of variables,

$$\sqrt{n}(\hat{p} - \hat{v}) = \int h p_n(h) dh, \quad (9)$$

where

$$p_n(h) = \frac{\exp\left(n\left[\hat{Q}_n(\hat{v} + h/\sqrt{n}) - \hat{Q}_n(\hat{v})\right]\right) f_V(\hat{v} + h/\sqrt{n}) \frac{f_{X|V}(z_{k+1}|\hat{v}+h/\sqrt{n})}{[1-F_{X|V}(z_{k+1}|\hat{v}+h/\sqrt{n})]}}{\int \exp\left(n\left[\hat{Q}_n(\hat{v} + u/\sqrt{n}) - \hat{Q}_n(\hat{v})\right]\right) f_V(\hat{v} + u/\sqrt{n}) \frac{f_{X|V}(z_{k+1}|\hat{v}+u/\sqrt{n})}{[1-F_{X|V}(z_{k+1}|\hat{v}+u/\sqrt{n})]}} du.$$

In the next section, we demonstrate that the above renormalized posterior distribution is asymptotically normal. Intuitively, $\sqrt{n}(\hat{p} - \hat{v}) \xrightarrow{p} 0$ because the mean of the above renormalized posterior distribution is asymptotically zero. Clearly, the single-unit model of the English auction investigated by Milgrom and Weber [1982] is a special case of this result, when τ is one. This corresponds to the conventional full-sample maximum-likelihood analysis and Bayesian posterior distribution. At a typical English auction, where τ is one, the only difference from full-sample maximum-likelihood analysis is that the maximum order statistic is unobserved. However, a single order statistic is asymptotically negligible. Likewise, the conditioning event in the bid function in equation (8) differs from the corresponding partial-sample likelihood in equation (4) only by a single order statistic and the difference is asymptotically negligible.

3.3 Simple Example

Consider the following example, which can be solved in closed-form. Suppose that the conditional distribution of X is exponential, having mean v , so

$$f_{X|V}(x|v) = \frac{1}{v} \exp(-x/v) \quad \text{for } x \geq 0, v > 0.$$

The posterior distribution needed to compute the bid function in equation (1) is proportional to

$$\begin{aligned} & \left(\frac{1}{v}\right)^{n-m} \exp\left(-\frac{1}{v} \sum_{j=n}^{m+1} z_j\right) \left(\frac{1}{v}\right)^{m-k+1} \exp(-z/v)^{m-k+1} \exp(-z/v)^{k-1} f_V(v) = \\ & \left(\frac{1}{v}\right)^{n-k+1} \exp\left[-\frac{1}{v} \left(\sum_{j=n}^{m+1} z_j + mz\right)\right] f_V(v). \end{aligned}$$

Suppose $f_V(v)$ is a diffuse prior. In this case, the above posterior distribution is then an inverse gamma distribution having parameters $(n - k + 1)$ and $\left(\sum_{j=n}^{m+1} z_j + mz\right)$, which has mean

$$\mathbb{E}[V|X_1 = Y_k = \dots = Y_{m-1} = z, \Omega_m] = \frac{\sum_{j=n}^{m+1} z_j + mz}{n - k},$$

which is also the bid function at round m . Therefore, the transaction price is given by the bid function with m equal $(k + 1)$ and z equal z_{k+1} :

$$\hat{p} = \frac{\sum_{j=n}^{k+2} z_j + (k + 1)z_{k+1}}{n - k}.$$

To see why \hat{p} converges to the true v^0 , note that in this example, $Z_{k+1} \xrightarrow{p} F_{X|V}^{-1}(\tau)$ which equals $-v^0 \log(1 - \tau)$. Also, by invoking a law of large numbers,

$$\frac{\sum_{j=n}^{k+2} z_j}{n} \xrightarrow{p} v^0 \left[\frac{\log(1 - \tau)}{1 - \tau} + \tau \right].$$

Therefore,

$$\hat{p} \xrightarrow{P} v^0 \frac{1}{\tau} [(\log(1-\tau)(1-\tau) + \tau)] - v^0 \log(1-\tau) \frac{1-\tau}{\tau} = v^0.$$

The maximum-likelihood estimator \hat{v} , which is the mode of the posterior distribution, is

$$\hat{v} = \frac{\sum_{j=n}^{k+2} z_j + (k+1)z_{k+1}}{n-k+2}.$$

Hence,

$$\hat{v} = \frac{n-k}{n-k+2} \hat{p}.$$

It can then be verified that

$$\sqrt{n}(\hat{p} - \hat{v}) \xrightarrow{P} 0.$$

4 Asymptotic Distribution of Transaction Price

In this section, we provide formal conditions to justify the claims made in the previous section. Our analysis is broken into two parts: in the first, we derive the asymptotic distribution of $\sqrt{n}(\hat{v} - v^0)$, while in the second we demonstrate that $\sqrt{n}(\hat{p} - \hat{v})$ is $o_p(1)$. As Newey and McFadden [1994] as well as Chernozhukov and Hong [2003] have noted, both parts depend on the stochastic equicontinuity properties of the sample-averaged log-likelihood function $\hat{Q}_n(v)$.

To begin, we state assumptions sufficient to the task. Instead of striving for the weakest possible set of assumptions, we are content with sufficient conditions that illustrate the main results. Note, too, that in theoretical models of auctions the monotone likelihood-ratio condition is typically imposed, which restricts how weak the conditions for equicontinuity can be.

Assumption 3 *The true common value v^0 is contained in the interior of the support of the prior distribution which is continuously distributed at the point of v^0 .*

Assumption 4 *The support of $f_{X|V}(x|v)$ is independent of v and bounded, while $\log f_{X|V}(x|v)$ is uniformly bounded, having bounded continuous third derivatives in both arguments on its support.*

These two assumptions are regularity conditions required to demonstrate uniform convergence and stochastic equicontinuity.

Theorem 1 *Under Assumptions 1 to 4, if $f_V(v)$ is continuous at v^0 with a finite mean, then*

$$\sqrt{n}(\hat{v} - v^0) \xrightarrow{d} \mathcal{N}\left(0, \Sigma(\tau) = \frac{\partial}{\partial u} \left[\frac{\partial}{\partial v} Q^u(v) \right] \Big|_{u=v=v^0}^{-1}\right),$$

and

$$\sqrt{n}(\hat{p} - \hat{v}) \xrightarrow{p} 0,$$

so

$$\sqrt{n}(\hat{p} - v^0) \xrightarrow{d} \mathcal{N}[0, \Sigma(\tau)].$$

Remark 1: In the Pesendorfer–Swinkels model, under the same assumption $(k/n) \rightarrow (1 - \tau)$, only the signal of a single last-losing bidder is revealed, instead of the signals of all the losing bidders. Therefore, intuitively, the transaction price in the Pesendorfer–Swinkels model should aggregate less information than that in the Milgrom–Weber model. In fact, this turns out to be true. While the transaction prices in both the Milgrom–Weber and the Pesendorfer–Swinkels models converge to v^0 at rate \sqrt{n} , the asymptotic variance of the Pesendorfer–Swinkels price is greater than the Milgrom–Weber price. We demonstrate this result formally using the influence function representation of the asymptotic variance. First, we note from the proof of the theorem that $\Sigma(\tau)$ equals $\text{Var}[\psi_1(X, \tau)]^{-1}$, where the *influence function* $\psi_1(X, \tau)$ is given by

$$\begin{aligned} \psi_1(X, \tau) &\equiv \frac{\partial}{\partial v} \log f_{X|V}(X|v^0) \mathbf{1}[X \leq F_{X|V}^{-1}(\tau|v^0)] - \\ &\quad \mathbb{E} \left(\frac{\partial}{\partial v} \log f_{X|V}(X|v^0) \mathbf{1}[X \leq F_{X|V}^{-1}(\tau|v^0)] \right) + \\ &\quad \frac{1}{1 - \tau} \left(\frac{\partial}{\partial v} F_{X|V} [F_{X|V}^{-1}(\tau|v) | v] \right) (\mathbf{1}[X \leq F_{X|V}^{-1}(\tau|v) | v] - \tau). \end{aligned}$$

Next, we characterize the sample-averaged log-likelihood function as well as the score and influence functions in the Pesendorfer–Swinkels model, and demonstrate that they imply a variance larger than $\Sigma(\tau)$.

The sample-averaged log-likelihood of the Pesendorfer–Swinkels model, which depends only on a single order statistic $z_{k+1} = \hat{F}_n^{-1}(\hat{\tau})$, is given by

$$\begin{aligned}\tilde{Q}_n(v) &= \frac{k}{n} \log[1 - F_{X|V}(z_{k+1}|v)] + \left(1 - \frac{k}{n}\right) \log F_{X|V}(z_{k+1}|v) \\ &= (1 - \hat{\tau}) \log\left(1 - F_{X|V}\left[\hat{F}_n^{-1}(\hat{\tau})|v\right]\right) + \hat{\tau} \log F_{X|V}\left[\hat{F}_n^{-1}(\hat{\tau})|v\right].\end{aligned}$$

Its corresponding score function is

$$\frac{\partial}{\partial v} \tilde{Q}_n(v) = -\left(\frac{1-\hat{\tau}}{1-F_{X|V}[\hat{F}_n^{-1}(\hat{\tau})|v]} - \frac{\hat{\tau}}{F_{X|V}[\hat{F}_n^{-1}(\hat{\tau})|v]}\right) \frac{\partial}{\partial v} F_{X|V}[\hat{F}_n^{-1}(\hat{\tau})|v].$$

If we evaluate the first-order approximation of the score function with respect to $\hat{F}_n^{-1}(\hat{\tau})$ as it approaches $F_{X|V}^{-1}(\tau|v^0)$ at v^0 , and make use of the well-established asymptotic approximation of the sample quantile

$$\sqrt{n} \left[\hat{F}_n^{-1}(\tau) - F_{X|V}^{-1}(\tau|v^0) \right] = -\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\mathbf{1}[X_i \leq F_{X|V}^{-1}(\tau|v^0)] - \tau}{f_{X|V}[F_{X|V}^{-1}(\tau|v^0)]} + o_p(1), \quad (10)$$

then we find the following influence function representation for the Pesendorfer–Swinkels score function:

$$\sqrt{n} \frac{\partial}{\partial v} \tilde{Q}_n(v^0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_2(X_i, \tau) + o_p(1)$$

where

$$\psi_2(X, \tau) \equiv \frac{1}{\tau(1-\tau)} \frac{\partial}{\partial v} F_{X|V}[F_{X|V}^{-1}(\tau|v)|v] \left(\mathbf{1}[X \leq F_{X|V}^{-1}(\tau|v^0)] - \tau \right).$$

Letting \tilde{p} denote the transaction price in the Pesendorfer–Swinkels auction model, we have

$$\sqrt{n}(\tilde{p} - v^0) \xrightarrow{d} \mathcal{N}\left(0, \text{Var}[\psi_2(X, \tau)]^{-1}\right). \quad (11)$$

In order to demonstrate that $\text{Var}[\psi_2(X, \tau)] \leq \text{Var}[\psi_1(X, \tau)]$, we compute

$$\begin{aligned} \psi_1(X, \tau) - \psi_2(X, \tau) &= \frac{\partial}{\partial v} \log f_{X|V}(X|v^0) \mathbf{1}\left[X \leq F_{X|V}^{-1}(\tau|v^0)\right] - \\ &\quad \mathbb{E}\left(\frac{\partial}{\partial v} \log f_{X|V}(X|v^0) \mathbf{1}\left[X \leq F_{X|V}^{-1}(\tau|v^0)\right]\right) - \\ &\quad \frac{1}{\tau} \left(\frac{\partial}{\partial v} F_{X|V}\left[F_{X|V}^{-1}(\tau|v^0)\right]\right) \left(\mathbf{1}\left[X \leq F_{X|V}^{-1}(\tau|v^0)\right] - \tau\right). \end{aligned}$$

We can then verify easily that

$$\text{Cov}[\psi_1(X, \tau) - \psi_2(X, \tau), \psi_2(X, \tau)] = 0.$$

Hence,

$$\text{Var}[\psi_2(X, \tau)] \leq \text{Var}[\psi_1(X, \tau)].$$

Furthermore, this inequality can be strengthened to a strict one, $\text{Var}[\psi_2(X, \tau)] < \text{Var}[\psi_1(X, \tau)]$ provided $\text{Var}[\psi_1(X, \tau) - \psi_2(X, \tau)]$ is strictly positive. This in turn holds when

$$\text{Var}\left(\frac{\partial}{\partial v} \log f_{X|V}(X|v^0) \mathbf{1}\left[X \leq F_{X|V}^{-1}(\tau|v^0)\right]\right) > 0,$$

or when $\frac{\partial}{\partial v} \log f_{X|V}(X|v^0)$ is not a constant in the range of $X \leq F_{X|V}^{-1}(\tau|v^0)$. Intuitively, when this holds, the variation of the likelihood in this range provides more information in the Milgrom–Weber auction that is not revealed in the Pesendorfer–Swinkels auction.

Remark 2: Above, we have indexed the asymptotic variance by τ , the proportion of losing bidders. Intuitively, the larger the fraction of losing bidders, the more information revealed at the auction.

Therefore, we expect $\Sigma(\tau)$ to be a monotonically decreasing function of τ , in a matrix sense. In other words, for any $0 < \tau_1 \leq \tau_2 < 1$,

$$\Sigma(\tau_1) \geq \Sigma(\tau_2).$$

In fact, this is true. That is, when τ_1 is less than τ_2 , $\text{Var}[\psi_1(X, \tau_1)] \leq \text{Var}[\psi_1(X, \tau_2)]$, which follows from

$$\text{Cov}[\psi_1(X, \tau_1) - \psi_1(X, \tau_2), \psi_1(X, \tau_1)] = 0. \quad (12)$$

Verifying equation (12) is tedious, but straightforward: it depends on the following two key relations. First,

$$\mathbb{E} \left(\frac{\partial}{\partial v} \log f_{X|V}(X|v^0) \mathbf{1} \left[X \leq F_{X|V}^{-1}(\tau|v^0) \right] \right) = \frac{\partial}{\partial v} F_{X|V} \left[F_{X|V}^{-1}(\tau|v^0) | v \right]$$

and, second, that, for $\tau_1 > \tau_2$,

$$\mathbf{1} \left[X \leq F_{X|V}^{-1}(\tau_1|v^0) \right] \mathbf{1} \left[X \leq F_{X|V}^{-1}(\tau_2|v^0) \right] = \mathbf{1} \left[X \leq F_{X|V}^{-1}(\tau_2|v^0) \right].$$

Generically, $\text{Var}[\psi_1(X, \tau_1)]$ is strictly less than $\text{Var}[\psi_1(X, \tau_2)]$. This is true whenever

$$\text{Var}[\psi_1(X, \tau_1) - \psi_1(X, \tau_2)] > 0,$$

or whenever $[\psi_1(X, \tau_1) - \psi_1(X, \tau_2)]$ does not equal a constant with probability one. A generic sufficient condition is that

$$F_{X|V}^{-1}(\tau_1|v^0) < X < F_{X|V}^{-1}(\tau_2|v^0)$$

with strictly positive probability less than one, or that variation exists in $\frac{\partial}{\partial v} \log f_{X|V}(X|v^0)$ on the set

$$F_{X|V}^{-1}(\tau_1|v^0) < X < F_{X|V}^{-1}(\tau_2|v^0).$$

Hence, under the assumptions made above, especially the common support Assumption 4, for $\tau \in (0, 1)$, the larger is τ , the more information is aggregated in the Milgrom–Weber model, in the sense of having a smaller variance despite that the rate of convergence stays the same. It can also be demonstrated that this conclusion continues to hold without the support Assumption 4. When the upper support is increasing in v , while the condition still holds, the rate of convergence can improve beyond \sqrt{n} when τ equals one. On the other hand, if the lower support is also increasing in v , then it is possible that the convergence rate is faster than \sqrt{n} even when τ is zero.

This desirable monotonicity property of information aggregation in the Milgrom–Weber model is in contrast to the Pesendorfer–Swinkels model. The amount of information aggregated asymptotically in the transaction price of the Pesendorfer–Swinkels model is not monotonic in τ . For example, when $f_{X|V}(x|v^0)$ is uniform in X , the worst τ for information aggregation is one-half in the Pesendorfer–Swinkels model because this involves the worst balance between the winner’s curse and the loser’s curse. In general, the optimal τ in the Pesendorfer–Swinkels model depends on the shape of this conditional density. Intuitively, in the Pesendorfer–Swinkels model, a different τ selects a different information set, while in the Milgrom–Weber model, a larger τ always selects a larger information set.

5 Deriving Likelihood Function of Observed Drop-Out Prices

In section 2, we derived the bid function of a representative bidder as well as characterized the transaction price; see equations (1) and (3). In sections 3 and 4, we then demonstrated that the transaction price converged in probability to the true, but *ex ante* unknown, value v^0 and derived the asymptotic distribution of that price. In this section, in order to provide a framework within which to conduct our empirical analysis in section 6, we derive the likelihood function of the bid data observed by an econometrician. We highlight the fact that the sampling variability of the econometrician’s estimate of the true, but unknown, value v^0 will depend on nuisance parameters unknown to the econometrician.

We first introduce some additional notation. Denote the j^{th} drop-out price by \hat{p}_j where $j = 1, 2, \dots, n - k$. For example, in our empirical application, we have n equal forty bidders and k equal twenty units, so there are twenty drop-out prices, the last being the transaction price, which we denoted above by \hat{p} , but now denote as \hat{p}_{n-k} . Thus, our observables are $(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_{n-k-1}, \hat{p}_{n-k})$. Now, from equation (2), we can recover the signal consistent with the first bidder's drop-out price—viz.,

$$\tilde{z}_n = \beta_n^{-1}(\hat{p}_1).$$

Likewise, for each of $j = 2, 3, \dots, n - k$, we can recursively recover \tilde{z}_j , the signals of the $(n - k - 1)$ losing bidders, so

$$\tilde{z}_{n-j+1} = \beta_{n-j+1}^{-1}(\hat{p}_j; \Omega_{n-j+1}).$$

For the k bidders who win the auction, all we know is that Z_j exceeds $\beta_{n-(n-k-1)}^{-1}(\hat{p}_{n-k}; \Omega_{k+1})$.

In the general case, the bid function $\beta_m(x)$ in equation (1) takes the following form:

$$\beta_m(x; \Omega_m) = \int v \frac{f_V(v)g(X_1 = Y_k = \dots = Y_{m-1} = x, \Omega_m|v)}{\int f_V(u)g(X_1 = Y_k = \dots = Y_{m-1} = x, \Omega_m|u)} dv$$

where

$$g(X_1 = Y_k = \dots = Y_{m-1} = x, \Omega_m|v) = [1 - F_{X|V}(x|v)]^{k-1} f_{X|V}(x|v)^{m-k+1} \prod_{j=n}^{m+1} f_{X|V}(z_j|v).$$

If we assume that $f_V(v)$ is diffuse and that, conditional on v , X is distributed normally, having mean v and variance σ^2 , then we can write

$$g(X_1 = Y_k = \dots = Y_{m-1} = x, \Omega_m|v) = \left[1 - \Phi\left(\frac{x-v}{\sigma}\right)\right]^{k-1} \frac{1}{\sigma^{m-k+1}} \phi\left(\frac{x-v}{\sigma}\right)^{m-k+1} \frac{1}{\sigma^{n-m}} \prod_{j=n}^{m+1} \phi\left(\frac{z_j-v}{\sigma}\right).$$

To summarize, under the assumptions of normality as well as a diffuse prior,

$$\beta_m(x; \Omega_m) = \int v \frac{\left[1 - \Phi\left(\frac{x-v}{\sigma}\right)\right]^{k-1} \cdot \phi\left(\frac{x-v}{\sigma}\right)^{m-k+1} \prod_{j=n}^{m+1} \phi\left(\frac{z_j-v}{\sigma}\right)}{\int \left[1 - \Phi\left(\frac{x-u}{\sigma}\right)\right]^{k-1} \phi\left(\frac{x-u}{\sigma}\right)^{m-k+1} \prod_{j=n}^{m+1} \phi\left(\frac{z_j-u}{\sigma}\right) du} dv.$$

Consider $(\tilde{z}_n, \tilde{z}_{n-1}, \dots, \tilde{z}_{k+1})$, the vector of $(n - k)$ signals consistent with the observed drop-out prices as well as the transaction price. The joint likelihood function of all the signals consistent with the drop-out prices revealed under the Milgrom–Weber auction is

$$\begin{aligned} \tilde{\mathcal{L}}_n(\tilde{z}_{k+1}, \dots, \tilde{z}_n | v, \boldsymbol{\theta}) &= [1 - F_{X|V}(\tilde{z}_{k+1} | v, \boldsymbol{\theta})]^k f_{X|V}(\tilde{z}_{k+1} | v, \boldsymbol{\theta}) \left| \frac{\partial \beta_{n-k}^{-1}(\hat{p}_{n-k})}{\partial \hat{p}_{n-k}} \right| \\ & \quad f_{X|V}(\tilde{z}_{k+2} | v, \boldsymbol{\theta}) \left| \frac{\partial \beta_{n-k-1}^{-1}(\hat{p}_{n-k-1})}{\partial \hat{p}_{n-k-1}} \right| \cdots f_{X|V}(\tilde{z}_n | v, \boldsymbol{\theta}) \left| \frac{\partial \beta_1^{-1}(\hat{p}_1)}{\partial \hat{p}_1} \right|. \end{aligned} \quad (13)$$

Here, $\boldsymbol{\theta}$ denotes a vector of unknown parameters and captures the fact the probability density and cumulative distribution functions of signals can depend on parameters known to the bidders, but unknown to the econometrician.

The econometrician's MLE \tilde{v} is defined as

$$\tilde{v} = \underset{v}{\operatorname{argmax}} \log \left[\binom{n}{k} \tilde{\mathcal{L}}_n(\tilde{z}_{k+1}, \dots, \tilde{z}_n | v, \tilde{\boldsymbol{\theta}}) \right]$$

where $\tilde{\boldsymbol{\theta}}$ denotes the MLE of $\boldsymbol{\theta}^0$. Knowing the true nuisance parameters in $\boldsymbol{\theta}^0$ is unimportant when demonstrating that the transaction price converges in probability to the true value v^0 because the parameters contained in $\boldsymbol{\theta}^0$ are of second-order importance. The nuisance parameters are, however, critical when calculating an estimate of the sampling variation in \tilde{v} , the econometrician's estimate of the true value v^0 .

In this section, we have employed a parametric distribution (the normal) to model the conditional distribution of signals. In general, without imposing shape restrictions, it is difficult to identify the distribution of signals, $F_{X|V}(\cdot | v^0)$ nonparametrically. First, even if the signals $z_n, z_{n-1}, \dots, z_{k+1}$ were directly observed and n were large, the observed signals would only identify the lower por-

tion of the signal distribution $F_{X|V}(x|v^0)$ for $x \leq F_{X|V}^{-1}(\tau|v^0)$. Second, the signals are not directly observed and must be inferred indirectly from the observed price sequence. The observed distribution of the price sequence implicitly depends on the signal distribution $F_{X|V}(\cdot|v^0)$ nonlinearly, and cannot be analytically inverted to recover the signal distribution. Third, the bid function, which relates the signal to the observed price in equation (8), depends only on the lower portion of the signal distribution up to $F_{X|V}^{-1}(\tau|v^0)$. This implies that the observed price distribution does not contain more information than the lower portion of the signal distribution up to $F_{X|V}^{-1}(\tau|v^0)$. Parametric functional forms incorporate shape restrictions, such as the symmetry assumption implied by the normal distribution, that can be used to extrapolate information concerning the lower percentiles of the signal distribution to its upper percentiles.

In our empirical analysis, we used data from a single auction at which the number of bidders was large, forty. The theoretical model requires that, conditional on the true value of the object, the signals of bidders be independently and identically distributed. The transaction price from a single auction having a large number of bidders identifies the true common value for this auction. To identify how that value relates to observed auction characteristics would require data from several auctions; such data were unavailable to us. If data from several auctions with large numbers of bidders are available, then observed auction characteristics could be incorporated into the analysis by relating the prices from each auctions to the observed characteristics, either parametrically or nonparametrically. Given the true value, the distribution of the signal can also be modeled to depend on the observed auction characteristics to improve estimation efficiency. The relationship between the valuation and the observed auction characteristics can then be consistently estimated when both the number of auctions and the number of bidders get large. The convergence rate and the inference distribution for the effect of the observed auction characteristics, however, will depend on the relative size of the number of auctions and the number of bidders.

6 Empirical Application

We have applied the methods described above to data from an auction of taxi license plates held in Shenzhen, China in October 2007. At this auction, the municipal transportation bureau sold 2,000 additional *red taxi* license plates. Red taxis are special in Shenzhen because they can operate both inside and outside the Special Economic Zone (SEZ), unlike *yellow taxis* which can operate only inside the SEZ, and *green taxis* which can only operate outside the SEZ.

By 2007, the city of Shenzhen had not issued any new license plates for red taxis since 1993. Rapid growth in Shenzhen's population meant that patrons were experiencing a shortage of taxis, leading to an increase in the number of illegally-operated taxis. In 2007, the per capita number of taxis in Shenzhen was low when compared to other parts of China: only 10,305 taxis were licensed in a city of 7.5 million permanent residents, about 13.74 taxis for every 10,000 residents. The Ministry of Construction in China recommended that cities should have 21 taxis for every 10,000 residents.

Before the auction, the authorities reviewed the qualifications of all those who had applied to participate at the auction. Potential bidders could be individual taxi companies or groups formed by different companies. While fifty-one 'firms' apparently requested to participate, only forty potential bidders were certified to participate at the auction. Thus, n was 40.

In written documentation, potential bidders were reminded to be aware of the risks involved. For example, consider a translation of the text from one document:

Following this auction, more taxi license plates will be issued through auction or other ways over the next four years. The number of taxis in Shenzhen will reach about 20,000 by 2011. The issuance of a great number of license plates might have much impact on the taxi industry.

Despite these warnings, representatives of taxi companies in the city showed great interest in the auction, perhaps because operating a taxi has been one of the highest profit margins in the transportation industry. Also, historically, taxis have provided a stable return against investment.

Before the auction, 53 out of 73 taxi companies in Shenzhen owned between 50 to 200 taxis each. In short, the majority of the city's taxi companies were small- and medium-sized ones. Some incumbent taxi drivers expressed concern that entry would erode profits. One was quoted in the local newspaper (our translation) as saying that

Actually we are not earning much nowadays. If more taxis were on the road, we would have a hard time making ends meet.

In contrast, local residents supported the issuance of additional license plates. One was quoted (again our translation from the local newspaper) as saying

The sooner new taxis hit the road the better. It's too hard to hail a taxi during peak hours and holidays.

This anecdotal evidence, along with casual observation, suggests to us that the value of a red-taxi license plate in Shenzhen has a large common-value component. Before the auction, however, this common value was unknown to potential bidders. Using whatever means at their disposal, potential bidders formed estimates of the unknown common value which they then used during bidding at the auction.

The auction in Shenzhen proceeded according to the rules described in Milgrom and Weber [2000]. In written rules announced before the auction, the authorities informed potential bidders that the 2,000 license plates on sale would be distributed evenly among the final twenty highest bidders; each winner would be required to buy 100 license plates.

The auctioneer, Tian Tao, was a registered member of China's auction industry association. The reserve price was set at 150,000 yuan per license plate, but the price rose to 500,000 yuan in fourth minute of bidding. During the auction, Tian reminded bidders repeatedly to be aware of the risks involved. In fact, Tao took a break for ten minutes to allow the bidders "to cool their enthusiasm." We have translated one of his comments as "this is one of the most intensive auctions I've experienced in my career as an auctioneer." At the close of the auction, the price of a red-taxi license plate was 542,500 yuan, around US\$80,000.

In table 1, we present the prices called out during the auction along with the number of bidders who exited the auction at those prices, while in figure 1 we depict the empirical survivor function of prices. The units in this and other tables are in 10,000 yuan.

Zhang Hongzhi, a manager of Shenzhen Xilie Taxi Company, was reported in the newspaper to have said that he “felt very excited after we won a bid.” Before his attending the auction, his company had decided on 550,000 yuan as the highest they would pay for a red-taxi license plate.

To implement equation (4), we assumed that, conditional on v^0 , X is distributed normally, having variance σ^2 , so

$$F_{X|V}(x|v^0) = \int_{-\infty}^x \frac{1}{\sigma \sqrt{2\pi}} \exp \left[-\frac{(u - v^0)^2}{\sigma^2} \right] du \equiv \Phi \left[\frac{(x - v^0)}{\sigma} \right]$$

with

$$f_{X|V}(x|v^0) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[-\frac{(x - v^0)^2}{\sigma^2} \right] \equiv \frac{1}{\sigma} \phi \left[\frac{(x - v^0)}{\sigma} \right].$$

We also assumed that $f_V(v)$ is a diffuse prior. In table 2, we present the maximum-likelihood estimates of v^0 and σ as well as their standard errors; the logarithm of the likelihood function for this empirical specification is -55.98 . Again, the units of the parameters estimates are in 10,000 yuan.

Our theoretical analysis suggests that, despite the same rate of convergence, the asymptotic variance of the transaction price is smaller in the Milgrom–Weber auction than in the Pesendorfer–Swinkels auction. At the estimated parameters, for normally distributed signals, the probability of a signal’s being less than zero is very small. Consequently, back-of-the-envelope estimates of the asymptotic variances can be calculated using the estimated variance of the normal distribution. For a standard normal Z , the asymptotic variance of the Milgrom–Weber transaction price given in Theorem 1 and equation (7) is

$$\left(\frac{1}{1 - \tau} \frac{1}{\sigma^2} \phi \left[\Phi^{-1}(\tau) \right]^2 + \frac{1}{\sigma^2} \mathbb{E} \left[Z^2 \mathbf{1}(Z \geq 0) \right] \right)^{-1},$$

which, when τ is one half, is $\frac{2\pi\sigma^2}{\pi+2}$. On the other hand, the asymptotic variance for the Pesendorfer–Swinkels transaction price in equation (11) is given by

$$\text{Var}\left(\frac{1}{\tau(1-\tau)}\frac{1}{\sigma}\phi\left[\Phi^{-1}(\tau)\right]\left(\mathbf{1}\left[Z\leq\Phi^{-1}(\tau)\right]-\tau\right)\right)^{-1}=\tau(1-\tau)\sigma^2\phi\left(\Phi^{-1}(\tau)\right)^{-2}.$$

When τ is one half, this equals $\frac{2\pi\sigma^2}{4}$, which is twenty-nine percent larger than the Milgrom–Weber variance. The extent to which this difference in the asymptotic variance translates into a difference in the expected seller revenues, however, depends on the variance of the prior value density function.

In order to understand the implications of these parameter estimates better, we used them to simulate the differences between transaction prices at a Milgrom–Weber auction and at a Pesendorfer–Swinkels auction. Some of these results are reported in table 3. Each entry in the table records the difference in the expected revenues between the Milgrom–Weber auction and the Pesendorfer–Swinkels auction, again measured in units of 10,000 yuan. In calculating table 3, we need three parameters: the prior mean and variance of the common value distribution as well as the variance of the signal distribution conditional on the common value. We used the estimate of v^0 to specify the prior mean, the estimate of σ to specify the variance of the signal distribution, and we varied the prior variance of the value distribution as a proportion of the signal variance.

As predicted by the linkage principle of Milgrom and Weber [1982], the Milgrom–Weber auction always generates an higher expected revenue than the Pesendorfer–Swinkels auction. However, as table 3 illustrates, the difference in the expected revenues is relatively small when compared to both the selling price and the estimated common value. Table 3 also reveals that the difference in expected revenues at the estimated parameters is decreasing in the number of objects for a given number of bidders: as the number of losing bidders decreases, relatively less information is being revealed at an Milgrom–Weber auction relative to an Pesendorfer–Swinkels auction. As the prior variance of the value distribution increases relative to the variance of the signal distribution (indicating a larger variation of the common value component), the revenue difference also increases.

7 Summary and Conclusions

Using a clock model of a multi-unit, oral, ascending-price auction, within the common-value paradigm, we have analyzed the behavior of the transaction price as the numbers of bidders and units gets large. We have demonstrated that even though the transaction price is determined by a (potentially small) fraction of losing drop-out bids, that price converges in probability to the true, but *ex ante* unknown, value. Subsequently, we have also demonstrated that the asymptotic distribution of the transaction price is Gaussian, and that the asymptotic variance of the transaction price under the Milgrom–Weber pricing rule is less than that under the pricing rule used by Pesendorfer and Swinkels for a sealed-bid auction format. Thus, if the transaction prices under different auction formats and pricing rules are viewed as statistical estimators of the true, but *ex ante* unknown, value of the units for sale, then the Milgrom–Weber auction is a more efficient estimator of the unknown value than the Pesendorfer–Swinkels auction because more information is released under the Milgrom–Weber auction than under the Pesendorfer–Swinkels auction. Note, however, that when the number of bidders is large, the differences both in the expected transaction prices and in their asymptotic variances are small because both transaction prices converge to the same value. Finally, we applied our methods to data from an auction of taxi license plates held in Shenzhen, China. We have found that the loss in the expected revenue by switching to the sealed-bid auction from the ascending-bid auction is small, relative to both the transaction price and the estimated common value. Our research suggests that the Pesendorfer–Swinkels auction can generate nearly as much revenues for the seller as the Milgrom–Weber auction does, at least in our particular application.

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Appendix

To reduce clutter in the text of the paper, in this appendix, we collect the proofs of the results claimed in the text.

Proof of Main Theorem

The proof involves verifying two high-level conditions in Newey and McFadden [1994] as well as Chernozhukov and Hong [2003]. The first condition delivers consistency, while the second delivers asymptotic normality of \hat{v} and the relation that $\sqrt{n}(\hat{p} - \hat{v})$ is $o_p(1)$. We first state these conditions within the context of our notation.

Condition 1 *For any $\delta > 0$, there exists an $\epsilon > 0$, such that*

$$\liminf_{n \rightarrow \infty} P_* \left\{ \sup_{|v-v^0| \geq \delta} [\hat{Q}_n(v) - \hat{Q}_n(v^0)] \leq -\epsilon \right\} = 1.$$

Condition 2 *There exists $\Delta_n(v^0)$ and J^0 such that for v in an open neighbourhood of v^0 ,*

$$i. n \left[\hat{Q}_n(v) - \hat{Q}_n(v^0) \right] = (v - v^0) \Delta_n(v^0) - \frac{1}{2} (v - v^0)^2 \left[n J^0 \right] + R_n(v),$$

ii. For any sequence $\delta_n \rightarrow 0$,

$$\sup_{|v-v^0| \leq \delta_n} \frac{|R_n(v)|}{1 + n|v - v^0|^2} = o_p(1).$$

iii. $\Delta_n(v^0) / \sqrt{n} \xrightarrow{d} \mathcal{N}(0, \Omega^0)$, where both J^0 and Ω^0 are positive definite.

Under Conditions 1 and 2, it is shown in Theorem 1 in Chernozhukov and Hong [2003] that for h and $p_n(h)$ defined in equation (9),

$$\int |h|^\alpha \left| p_n(h) - \phi(h; 0, J^{0-1} \Omega^0 J^{0-1}) \right| \xrightarrow{p} 0,$$

for any $\alpha > 0$, where $\phi(h; 0, J^{0-1} \Omega^0 J^{0-1})$ denotes a normal density with mean 0 and variance $J^{0-1} \Omega^0 J^{0-1}$. In other words, the convergence of $p_n(h)$ to a normal limiting density is in any polynomial moments and is stronger than convergence in the total variation norm. Using equation (9), this implies that $\sqrt{n}(\hat{p} - \hat{v}) = \int h p_n(h) dh \xrightarrow{p} 0$. In the following we focus on verifying Conditions 1 and 2.

Condition 1 is, in turn, implied by uniform convergence of $\hat{Q}_n(v)$ to $Q^0(v)$ and because $Q^0(v)$ is uniquely maximized at v^0 . The unique maximum of $Q^0(v)$ at v^0 is a direct consequence of the identification Assumption 2. To show that $\sup_{v \in \mathcal{V}} |\hat{Q}_n(v) - Q^0(v)|$ is $o_p(1)$, first note that the individual terms in the summand of the second term consist of the product of $\log f_{X|V}(X_i|v)$ and $\mathbf{1}(X_i \leq \xi)$, where ξ represents a generic argument that will be evaluated at $\hat{\xi} = \hat{F}_n^{-1}(\hat{\tau})$. Given Assumption 4, the first is a type II function and the second is a type I function defined in Andrews [1994]. Both satisfy Pollard's entropy condition, and are stable under multiplication. Hence,

$$\sup_{v, \xi} \left| \frac{1}{n} \sum_{i=1}^n \log f_{X|V}(X_i|v) \mathbf{1}(X_i \leq \xi) - \mathbb{E} [\log f_{X|V}(X_i|v) \mathbf{1}(X_i \leq \xi)] \right| = o_p(1).$$

Next, $\mathbb{E} [\log f_{X|V}(X_i|v) \mathbf{1}(X_i \leq \xi)]$ is a Lipschitz function in ξ and the Lipschitz constant is uniform in v . Hence, given that $\hat{F}_n^{-1}(\hat{\tau}) \xrightarrow{P} F_{X|V}^{-1}(\tau|v^0)$, we also have

$$\sup_v \left| \mathbb{E} \left(\log f_{X|V}(X_i|v) \mathbf{1} \left[X_i \leq \hat{F}_n^{-1}(\hat{\tau}) \right] \right) - \mathbb{E} \left(\log f_{X|V}(X_i|v) \mathbf{1} \left[X_i \leq F_{X|V}^{-1}(\tau|v^0) \right] \right) \right| = o_p(1).$$

Therefore, the second term of $\hat{Q}_n(v)$ converges uniformly in v to the second term of $Q^0(v)$. The first term of $\hat{Q}_n(v)$ is also a Lipschitz function of $\hat{F}_n^{-1}(\hat{\tau})$ with the Lipschitz constant being uniform in v . Therefore, by the same argument, the first term of $\hat{Q}_n(v)$ also converges uniformly in v to the first term of $Q^0(v)$. Hence, Condition 1 holds.

The second condition is more involved than the first. We define $\hat{\xi}$ to be $\hat{F}_n^{-1}(\hat{\tau})$ where $\hat{\xi}^0$ denotes $F_{X|V}^{-1}(\hat{\tau}|v^0)$ and ξ^0 denotes $F_{X|V}^{-1}(\tau|v^0)$. We rewrite $\hat{Q}_n(v)$ as $\hat{Q}_n(v, \hat{\xi})$ to emphasize its direct dependence on $\hat{\xi}$. Note that, while $\hat{Q}_n(v, \hat{\xi})$ is differentiable in v , it is not in $\hat{\xi}$, so arguments relying on stochastic continuity arguments are required. The $\Delta_n(v^0)$ and J^0 elements in Condition 2 are given by

$$\Delta_n(v^0) = n \frac{\partial}{\partial v} \hat{Q}_n(v^0, \hat{\xi}^0) + n \frac{\partial^2}{\partial v \partial \xi} Q^0(v^0, \xi^0) (\hat{\xi} - \hat{\xi}^0)$$

and

$$J^0 = - \frac{\partial^2}{\partial v^2} Q^0(v^0, \xi^0),$$

respectively. We decompose $R_n(v)$ into $R_n^1(v) + R_n^2(v)$ with

$$R_n^1(v) = n \left[\frac{\partial}{\partial v} \hat{Q}_n(v^0, \hat{\xi}) - \frac{\partial}{\partial v} \hat{Q}_n(v^0, \hat{\xi}^0) - \frac{\partial^2}{\partial v \partial \xi} Q^0(v^0, \xi^0) (\hat{\xi} - \hat{\xi}^0) \right] (v - v^0)$$

and

$$R_n^2(v) = \frac{1}{2} n (v - v^0)^2 \left[\frac{\partial^2}{\partial v^2} \hat{Q}_n(v^*, \hat{\xi}) - J^0 \right]$$

where v^* is a mean value between v^0 and v . Because $\frac{\partial^2}{\partial v^2} \hat{Q}_n(v^*, \hat{\xi}) - J^0 \xrightarrow{p} 0$, it follows that

$$\sup_{|v-v^0| \leq \delta_n} \frac{|R_n^2(v)|}{1+n|v-v^0|^2} \leq \frac{|R_n^2(v)|}{n|v-v^0|^2} = o_p(1).$$

Consider, next, $R_n^1(v)$, and define $\hat{m}(v, \xi) = \frac{\partial}{\partial v} \hat{Q}_n(v, \xi)$. As in equation (5), define

$$Q_n(v, \xi) \equiv (1 - \hat{\tau}) \log [1 - F_{X|V}(\xi|v)] + \mathbb{E} [\log f_{X|V}(z_i|v) \mathbf{1}(X_i \leq \xi)]$$

and $m(v, \xi) = \frac{\partial}{\partial v} Q_n(v, \xi)$. Because the summand terms in $\hat{m}(v, \xi)$ and $m(v, \xi)$, $\frac{\partial}{\partial v} \log f_{X|V}(z_i|v) \mathbf{1}(X_i \leq \xi)$, are the product of type I and type II functions defined in Andrews [1994], Theorems 2 and 3 in Andrews [1994] show that these terms satisfy Pollard's entropy conditions and, therefore, the stochastic equicontinuity conditions in equations (2.2) and (2.3) in Andrews [1994] hold with the parameter now double indexed by v and ξ . It follows from this stochastic equicontinuity property that

$$\sqrt{n} \left[\hat{m}(v^0, \hat{\xi}) - \hat{m}(v^0, \hat{\xi}^0) - m(v^0, \hat{\xi}) + m(v^0, \hat{\xi}^0) \right] = o_p(1).$$

Note, too, by a second-order mean-value expansion of $m(v^0, \xi)$ in ξ , that

$$\begin{aligned} & \sqrt{n} \left[m(v^0, \hat{\xi}) - m(v^0, \hat{\xi}^0) - \frac{\partial^2}{\partial v \partial \xi} Q^0(v^0, \xi^0)(\hat{\xi} - \hat{\xi}^0) \right] \\ &= \left[\frac{\partial}{\partial v} m(v^0, \xi_*^0) - \frac{\partial}{\partial v} m(v^0, \xi^0) \right] \sqrt{n}(\hat{\xi} - \hat{\xi}^0) = o_p(1) \times \sqrt{n}(\hat{\xi} - \hat{\xi}^0) = o_p(1). \end{aligned}$$

for a mean value $(\xi_*^0 - \xi^0) \xrightarrow{p} 0$, since $\sqrt{n}(\hat{\xi} - \hat{\xi}^0) = O_p(1)$. Therefore, we can write

$$\sqrt{n} \left[\hat{m}(v^0, \hat{\xi}) - \hat{m}(v^0, \hat{\xi}^0) - \frac{\partial^2}{\partial v \partial \xi} Q^0(v^0, \xi^0)(\hat{\xi} - \hat{\xi}^0) \right] = o_p(1).$$

Consequently, $R_n^1(v) = \sqrt{n}(v - v^0) = o_p(1)$. Using the relation that $x/(1+x^2) \leq 1/2$, we conclude that

$$\sup_{|v-v^0| \leq \delta_n} \frac{|R_n^1(v)|}{1+n|v-v^0|^2} \leq o_p(1) \quad \text{and} \quad \sup_{|v-v^0| \leq \delta_n} \frac{\sqrt{n}|v-v^0|}{1+n|v-v^0|^2} = o_p(1).$$

Having verified Conditions 2.i and 2.ii, it remains to verify Condition 2.iii. The Hessian term J^0 is obviously positive definite because the limiting likelihood function is multiple-times smoothly differentiable, and because v^0 uniquely maximizes $Q^0(v, \xi^0)$. We note, next, that $\Delta_n(v^0)/n$ takes the form

$$\begin{aligned} & -\frac{1-\tau}{1-F_{X|V}[\hat{F}_n^{-1}(\tau)|v]} \frac{\partial}{\partial v} F_{X|V}[\hat{F}_n^{-1}(\tau)|v] + \frac{1-\tau}{1-F_{X|V}[F_{X|V}^{-1}(\tau)|v]} \frac{\partial}{\partial v} F_{X|V}[F_{X|V}^{-1}(\tau)|v] + \\ & (\hat{E} - \mathbb{E}) \left[\frac{\partial}{\partial v} \log f_{X|V}(X_i|v^0) \mathbf{1}[X_i \leq F_n^{-1}(\tau)] \right] + \hat{E} \left(\frac{\partial}{\partial v} \log f_{X|V}(X_i|v^0) \mathbf{1}[X_i \leq \hat{F}_n^{-1}(\tau)] \right) - \\ & \mathbb{E} \left(\frac{\partial}{\partial v} \log f_{X|V}(X_i|v^0) \mathbf{1}[X_i \leq F_{X|V}^{-1}(\tau|v^0)] \right) + o_p\left(\frac{1}{\sqrt{n}}\right) \end{aligned}$$

where \hat{E} denotes the empirical mean. Because we have represented the influence function of $(\hat{\xi} - \xi^0)$ as equation (10), we can compute that

$$\Delta_n(v^0) = \sum_{i=1}^n \psi(X_i) + o_p(\sqrt{n})$$

where

$$\begin{aligned} \psi(X_i) &= \frac{\partial}{\partial v} \log f_{X|V}(X_i|v^0) \mathbf{1}[X_i \leq F_{X|V}^{-1}(\tau|v)] - \mathbb{E} \left(\frac{\partial}{\partial v} \log f_{X|V}(X_i|v^0) \mathbf{1}[X_i \leq F_{X|V}^{-1}(\tau|v)] \right) - \\ & \frac{1}{1-\tau} \left(\frac{\partial}{\partial v} F_{X|V}[F_{X|V}^{-1}(\tau)|v] \right) f_{X|V}[F_{X|V}^{-1}(\tau)|v] \left[\hat{F}_n^{-1}(\tau) - F_{X|V}^{-1}(\tau|v^0) \right] \\ &= \frac{\partial}{\partial v} \log f_{X|V}(X_i|v) \mathbf{1}[X_i \leq F_{X|V}^{-1}(\tau|v^0)] - \mathbb{E} \left(\frac{\partial}{\partial v} \log f_{X|V}(X_i|v) \mathbf{1}[X_i \leq F_{X|V}^{-1}(\tau|v^0)] \right) + \\ & \frac{1}{1-\tau} \left(\frac{\partial}{\partial v} F_{X|V}[F_{X|V}^{-1}(\tau|v^0)|v] \right) \mathbf{1}[X_i \leq F_{X|V}^{-1}(\tau|v^0)] - \tau. \end{aligned}$$

Direct calculation of the asymptotic variance in the last line, while accounting for the covariance between the two terms, yields

$$\text{Var} [\psi(X_i)] = \mathbb{E} \left[\frac{\partial}{\partial v} \log f_{X|V}(X_i|v) \right]^2 \mathbf{1} [X_i \leq F_{X|V}^{-1}(\tau|v)] + \frac{1}{1-\tau} \left(\frac{\partial}{\partial v} F_{X|V} [F_{X|V}^{-1}(\tau|v)|v] \right)^2.$$

By inspection, we see that its inverse coincides with the asymptotic variance given in $\Sigma(\tau)$, which has been verified to equal J^0 in the information matrix equality calculation and, hence, is also positive definite. Its inverse yields the asymptotic variance of $\sqrt{n}(\hat{p} - v^0)$ and $\sqrt{n}(\hat{v} - v^0)$.

References

- Donald W.K. Andrews. Empirical process methods in econometrics. In Robert F. Engle and Daniel L. McFadden, editors, *Handbook of Econometrics, Volume 4*. New York: Elsevier, 1994.
- Sushil Bikhchandani, Philip A. Haile, and John G. Riley. Symmetric separating equilibria in english auctions. *Games and Economic Behavior*, 38:19–27, 2002.
- Victor Chernozhukov and Han Hong. A MCMC approach to classical estimation. *Journal of Econometrics*, 115:293–346, 2003.
- Han Hong and Matthew Shum. Rates of information aggregation in common value auctions. *Journal of Economic Theory*, 116:1–40, 2004.
- Matthew O. Jackson and Ilan Kremer. The relationship between the allocation of goods and a seller’s revenue. *Journal of Mathematical Economics*, 40:371–392, 2004.
- Matthew O. Jackson and Ilan Kremer. The relevance of a choice of auction format in a competitive environment. *Review of Economic Studies*, 73:961–981, 2006.
- Ilan Kremer. Information aggregation in common value auctions. *Econometrica*, 70:1675–1682, 2002.

- Vijay Krishna. *Auction Theory*, Second Edition. San Diego: Academic Press, 2010.
- Claudio Mezzetti and Ilia Tsetlin. On the lowest-winning-bid and the highest-losing-bid auctions. *Journal of Mathematical Economics*, 44:1040–1048, 2008.
- Claudio Mezzetti and Ilia Tsetlin. On the lowest-winning-bid and the highest-losing-bid auctions. *Games and Economic Behavior*, 66:855–864, 2009.
- Paul R. Milgrom. A convergence theorem for competitive bidding with differential information. *Econometrica*, 47:670–688, 1979.
- Paul R. Milgrom. Rational expectations, information acquisition and competitive bidding. *Econometrica*, 49:921–943, 1981.
- Paul R. Milgrom. *Putting Auction Theory to Work*. New York: Cambridge University Press, 2004.
- Paul R. Milgrom and Robert J. Weber. A theory of auctions and competitive bidding. *Econometrica*, 50:1089–1122, 1982.
- Paul R. Milgrom and Robert J. Weber. A theory of auctions and competitive bidding, II. In Paul D. Klemperer, editor, *The Economic Theory of Auctions*. London: Edward Elgar, 2000.
- Whitney K. Newey and Daniel L. McFadden. Large sample estimation and hypothesis testing. In Robert F. Engle and Daniel L. McFadden, editors, *Handbook of Econometrics, Volume 4*. New York: Elsevier, 1994.
- Wolfgang Pesendorfer and Jeroen Swinkels. Efficiency and information aggregation in auctions. *American Economic Review*, 90:499–525, 2000.
- Wolfgang Pesendorfer and Jeroen M. Swinkels. The loser’s curse and information aggregation in common value auctions. *Econometrica*, 65:1247–1281, 1997.

Robert J. Weber. Multiple-object auctions. In Richard Engelbrecht-Wiggans, Martin Shubik, and Richard M. Stark, editors, *Auctions, Bidding, Contracting: Uses and Theory*. New York: New York University Press, 1983.

Robert B. Wilson. A bidding model of perfect competition. *Review of Economic Studies*, 44: 511–518, 1977.

Table 1: Announced Price, Number of Exits, and Total Exits

Price	20.00	22.00	23.00	24.00	25.00	26.00	27.00	28.00
Exits	0	2	0	0	0	0	0	0
Total Exits	0	2	2	2	2	2	2	2
Price	29.00	30.00	31.00	32.00	33.00	34.00	35.00	36.00
Exits	0	0	0	0	0	0	0	1
Total Exits	2	2	2	2	2	2	2	3
Price	37.00	38.00	39.00	40.00	41.00	42.00	43.00	44.00
Exits	0	0	0	0	1	0	2	1
Total Exits	3	3	3	3	4	4	6	7
Price	45.00	46.00	47.00	48.00	49.00	50.00	50.50	51.00
Exits	0	3	1	0	1	1	2	2
Total Exits	7	10	11	11	12	13	15	17
Price	51.20	51.40	51.50	51.60	51.70	51.80	51.90	52.00
Exits	0	0	0	0	0	0	0	0
Total Exits	17	17	17	17	17	17	17	17
Price	52.10	52.20	52.30	52.40	52.50	52.55	52.60	52.65
Exits	0	0	0	0	1	0	1	0
Total Exits	17	17	17	17	18	18	19	19
Price	52.70	52.75	52.80	52.85	52.90	52.95	53.00	53.05
Exits	0	0	0	0	0	0	0	0
Total Exits	19	19	19	19	19	19	19	19
Price	53.10	53.15	53.20	53.25	53.30	53.35	53.40	53.45
Exits	0	0	0	0	0	0	0	0
Total Exits	19	19	19	19	19	19	19	19
Price	53.50	53.55	53.60	53.65	53.70	53.75	53.80	53.85
Exits	0	0	0	0	0	0	0	0
Total Exits	19	19	19	19	19	19	19	19
Price	53.90	53.95	54.00	54.05	54.10	54.15	54.20	54.25
Exits	0	0	0	0	0	0	0	1
Total Exits	19	19	19	19	19	19	19	20

Table 2: Maximum-Likelihood Estimates of Normal Specification

Parameter	Estimate	Std.Error
ν^0	56.31	0.97
σ	19.35	3.95

Table 3: Simulated Differences in Expected Revenue

Number of Bidders	Variance Ratio*=0.25 Number of Objects			Variance Ratio=1.0 Number of Objects		
	10	20	30	10	20	30
20	0.120			0.221		
30	0.127	0.056		0.238	0.103	
40	0.108	0.070	0.042	0.208	0.136	0.079

Number of Bidders	Variance Ratio=4 Number of Objects			Variance Ratio=9 Number of Objects		
	10	20	30	10	20	30
20	0.311			0.289		
30	0.381	0.154		0.412	0.154	
40	0.359	0.233	0.126	0.426	0.274	0.136

* Variance Ratio= Prior Variance/ Signal Variance.

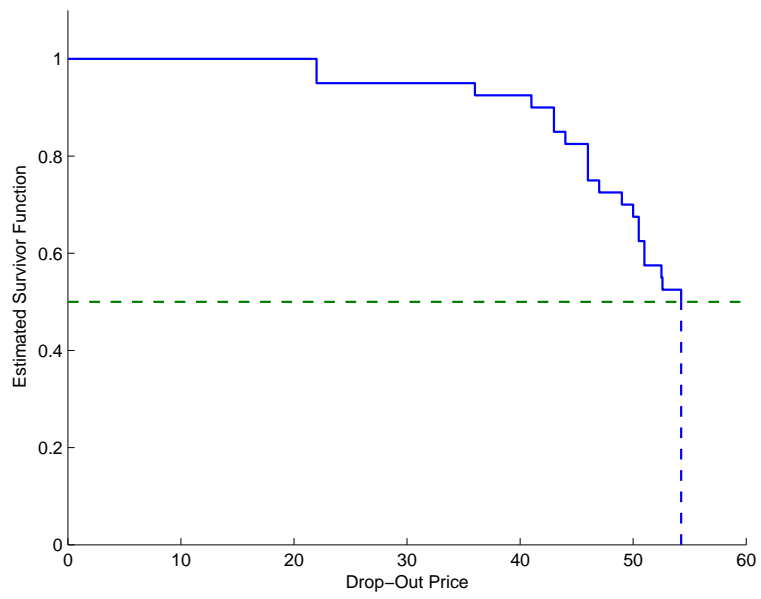


Figure 1: Estimated Survivor Function of Drop-Out Prices