ON SIMULTANEOUS s-CORES/t-CORES

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Abstract. In this paper, the authors investigate the question of when a partition of $n \in \mathbb{N}$ is an s-core and also a t-core when s and t are not relatively prime. A characterization of all such s/t-cores is given, as well as a generating function dependent upon the polynomial generating functions for s/t-cores when s and t are relatively prime. Furthermore, characterizations and generating functions are given for s/t-cores which are self-conjugate and also for $(e,r)/(e',r)$-cores.

1. Introduction and Statement of Results

A *partition* $\Lambda = (\lambda_1, \ldots, \lambda_d)$ of size $|\Lambda| := \sum_{i=1}^{d} \lambda_i$ is any finite sequence of non-increasing positive integer *parts*. We will also use a generalization of this definition to *partitions with parts of size zero*, namely partitions $(\lambda_1, \ldots, \lambda_d, \lambda_{d+1}, \ldots, \lambda_{d+r})$, where $\lambda_{d+i} = 0$. The parts λ_{d+i} will accordingly be referred to as *parts of size zero*.

The theory of partitions has a long and storied history. Ramanujan famously found congruences for counting the number of partitions of the integers $5n+4$, $7n+5$, and $11n+6$ [18], the last of which was shown by Hardy [19]. For more examples of partition theoretic generating functions, a good source is Andrews' book [2]. The main purpose of this paper is to give a classification and generating function for s/t-core partitions, partitions which are simultaneously s-core and t-core. For s and t relatively prime, Anderson [1] has given such a classification, showing in the process that there are exactly $\frac{\binom{s+t}{s}}{s+t}$ such partitions and a bound was found for the largest integer that can be partitioned by an s/t-core in [17].

The study of t-cores has applications to representation theory when t is prime, and has been studied by a variety of authors. We will first give a brief explanation of the connection to representation theory. Partitions of n are in one-to-one correspondence with irreducible

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representations of the symmetric group S_n. Taking any such representation modulo a prime $t = p$ will result in these representations further splitting into direct products of p-modularly irreducible representations. The *Brauer graph* is constructed by forming an edge between two representations if and only if they share a p-modularly irreducible constituent. The equivalence classes of this graph are referred to as (p-)*Brauer blocks*. The isolated elements of this graph, namely the p-modularly irreducible representations, are referred to as *defect zero p-blocks*. The representations corresponding to p-core partitions are precisely these p-modularly irreducible representations (cf. [4]). Hence, Anderson's result in [1] implies that there are only finitely many n for which S_n has a Brauer block which is both defect zero p and defect zero q when $p \neq q$. Navarro and Willems conjectured in [12] that a p-Brauer block can be equal to a q-Brauer block only when both are defect zero. Although the group A_7 is a counterexample to the Navarro-Willems conjecture, the conjecture hold specifically for S_n (and more generally for principle blocks), as was recently shown by Bessenrodt, Navarro, Olsson, and Tiep [5] so that Anderson's result implies that only finitely many S_n have coinciding p and q-blocks.

In addition to their role in representation theory, t-cores have played a role in combinatorial proofs due to their generating functions being (basically) a quotient of η-series. For instance, the study of t-cores played a role in Garvan, Kim, and Stanton establishing a combinatorial proof of the above partition function congruences [6]. Garvan [7] discovered, in terms of the t-core of a partition, a new crank which proved the congruence $p(49n + 47) \equiv 0 \pmod{49}$. A remarkable result of Granville and Ono showed that every nonnegative integer n may be partitioned by a t-core partition whenever $t \geq 4$ [8], which combined with the connection to representation theory above, completed the classification of finite simple groups with defect zero Brauer p-blocks. We shall show a slight refinement of this result as a corollary for the following similar result for s/t-core partitions.

Theorem 1.1. *Consider*

$$S_{s,t} := \{n : n \text{ is partitioned by an } s/t\text{-core which is } \textbf{not} \text{ a } \gcd(s,t)\text{-core}\}.$$

Then $S_{s,t} = \{n \geq \gcd(s,t)\}$ *if* $\gcd(s,t) \geq 4$, *and otherwise* $S_{s,t}$ *has density zero.*

Then the following refinement of Granville and Ono's result [8] follows immediately by taking any $\gcd(t', s') = 1$.

Corollary 1.2. *Fix a positive integer $g \geq 4$. For every $n \geq g$ and $t' > 1$ there exists a partition of n which is a $t'g$-core but not a g-core. Moreover, this result is best possible, since partitions of $n < g$ are automatically g-cores.*

A full description of s/t-core partitions will be given in Section 2. Our classification will be given in terms of the $\gcd(s,t)$-abacus of the s/t-core partition.

Section 2 will introduce the reader to basic facts about partitions, giving the necessary background and definition of t-cores, as well as introducing some results and techniques which we will take advantage of. Section 3 will be devoted to proving the following classification of s/t-cores.

Theorem 1.3. *A partition Λ is an s/t-core if and only if each column of the $g = \gcd(s,t)$-abacus corresponds to an $\frac{s}{g}/\frac{t}{g}$-core (possibly with parts of size zero).*

As usual, for a set S of partitions, we will define the *generating function of S* as $f(q) := 1 + \sum_{\Lambda \in S} q^{|\Lambda|}$. For ease of notation, we will sometimes refer to the generating function for sets of partitions S which may have parts of size zero. In this case, we will be referring to the generating function for the subset of S containing no parts of size zero.

Section 4 is devoted to showing the generating function for s/t-cores and to show that there are infinitely many s/t-cores which are not $\gcd(s,t)$-cores, in contrast to the relatively prime case, resolving the main remaining question of Anderson from [1]. In the process, a general result is given for determining generating functions of partitions whose columns correspond to partitions of a given type, to which our desired result as well as Theorem 1.1 are a direct corollary.

We will first fix k arbitrary sets P_i of partitions. We will say that the i-th column (modulo k) of a partition Λ *corresponds to the partition* $\Lambda_i \in P_i$ if the row numbers of the i-th column of the k-abacus of Λ containing a bead give the first column hook numbers of Λ_i, after removing parts of size zero. This concept is discussed in further detail in Section 2.

Theorem 1.4. *If the set of partitions S is defined by taking all partitions Λ with at most $k-1$ parts of size zero, such that the i-th column of Λ corresponds to a partition of type P_i (with arbitrarily many parts of size zero), then the generating function for partitions of type S is the product of the generating functions of type P_i evaluated at q^k times the generating function for g-cores.*

Moreover, we give the generating function for $(e,r)/(e',r)$-core partitions in Corollary 4.7.

Section 5 is devoted to showing some lemmata about Conjugation of partitions, which we use in Section 6 to give a classification of self-conjugate s/t-cores and a generating function in Section 7.

We conclude our discussion with a conjecture about the size of the largest s/t-core partition when $\gcd(s,t)=1$.

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2. Basic Facts and Preliminaries

2.1. **Ferrers-Young diagram.** The *Ferrers-Young diagram* of a partition $\Lambda = (\lambda_1, \lambda_2, \ldots \lambda_d)$ is formed by arranging $|\Lambda|$ nodes in rows so that the i^{th} row has λ_i nodes. The nodes are labeled by row and column coordinates as one would label the entries of a matrix. The *conjugate* of Λ, denoted Λ^C, is the partition whose Ferrers-Young diagram is the reflection along the main diagonal of Λ's diagram. The conjugate here is assumed to have no parts of size zero, but in our investigation in Section 5 of conjugate partitions we will introduce a notion of conjugation which includes partitions with parts of size zero and will be an involution. We say that Λ is *self-conjugate* if $\Lambda = \Lambda^C$. A node's *hook* consists of the node along with any other nodes directly below or to the right of the node itself. The size of a hook is its *hook number*. For partitions with parts of size zero, we will add the number of parts of size zero below each node to each first column hook number, and by convention we will define the first column hook number of a row corresponding to a part of size zero to be the number of (empty) rows strictly below this row. It will be convenient to refer to the hook number of a node in row i and column j by $H(i,j)$. A node's *rim hook* is the sequence of connected nodes on the right-hand boundary of the Ferrers-Young diagram connecting the two end points of its hook. By construction, the rim hook and the hook of a node are of equal length.

Example 2.1. *The Ferrers-Young diagram for* $\Lambda = (5,4,2,2)$ *is*

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 2 & \bullet & \bullet & \bullet & \bullet & \\ 3 & \bullet & \bullet & & & \\ 4 & \bullet & \bullet & & & \end{array}.$$

A $t-$*core* partition is a partition with no hook number divisible by t. The main concentration of this paper will is extend Anderson's [1] work to give a classification when s and t are not relatively prime. Furthermore, a generating function is obtained for these partitions, given the (polynomial) generating functions when s and t are relatively prime. For notational ease, we shall denote such partitions as s/t-cores.

The $t-$*core of* Λ, denoted $\Lambda_{(t)}$, is obtained by simply successively removing all rim hooks of length t from Λ. When $\Lambda_{(t)}$ is the empty partition, Λ is said to have *empty* $t-$*core*. Part of this paper is interested in $t-$core partitions with empty $r-$core where r divides t. We will refer to these partitions in the following manner.

Definition 2.2. *An* (e, r)-core partition *is a* $re-$*core partition with empty* r*-core.*

This paper will classify partitions that are simultaneous $(e, r)/(e', r)$-cores.

Denote by $Hk(\Lambda)$ the multiset of hook lengths for the Ferrers-Young diagram for Λ and $Hk(\Lambda)_t$ to be the submultiset of $Hk(\Lambda)$ consisting of hook lengths which are multiples of t. Let $t \cdot Hk(\Lambda)$ denote the multiset formed by multiplying each member of $Hk(\Lambda)$ by t. For the remainder of this paper, we shall refer to the set of first column hook numbers as *structure numbers*, denoted ST_Λ. Thus, in Example 2.1, $ST_\Lambda = \{8, 6, 3, 2\}$. If we add two parts of size zero to this partition, to obtain $\Lambda' = (5, 4, 2, 2, 0, 0)$, then $ST_{\Lambda'} = \{10, 8, 5, 4, 1, 0\}$. The number of elements in ST_Λ is necessarily equal to the number of parts in Λ, counting parts of size zero.

Example 2.3. *The structure numbers for* $\Lambda = (5, 4, 2, 2)$ *are* $ST_\Lambda = \{8, 6, 3, 2\}$*, with the full diagram of hook numbers given below.*

$$\begin{array}{ccccc} \mathbf{8} & 7 & 4 & 3 & 1 \\ \mathbf{6} & 5 & 2 & 1 & \\ \mathbf{3} & 2 & & & \\ \mathbf{2} & 1 & & & \end{array}$$

2.2. **Abaci.** The $t-$*abacus of* Λ is formed by placing beads on $t-$ runners labeled from 0 to $t - 1$ where the positions on the runners are numbered starting from zero going from left to right then down by rows. For $b = rt + c \in ST_\Lambda$, with $0 \le c < t$, a bead is placed in coordinate (r, c). Consider the characteristic function $f_t : \mathbb{Z} \times 0..t - 1 \to \{0, 1\}$ for which $f_t(r, c) = 1$ if and only if there is a bead in the (r, c) position. It will also be useful to denote the set of rows numbers which have a bead in the c-th column of the t-abacus by $RN_{\Lambda,t,c}$, since our classification of s/t-cores will be given by describing each column of the $\gcd(s,t)$-abacus. For notational ease, if $RN_{\Lambda,t,c} = ST_{\Lambda'}$, then the number of parts of size zero of Λ' will be refered to as $Z_{\Lambda,t,c}$ (or accordingly $Z_{\Lambda'}$).

Example 2.4. *The* $3-$*abacus of* $\Lambda = (5,4,2,2)$ *is*

	0	1	2
0	$\vert$	$\vert$	$\bullet_2$
1	$\bullet_3$	$\vert$	$\vert$
2	$\bullet_6$	$\vert$	$\bullet_8$
3	$\vert$	$\vert$	$\vert$

so that $RN_{\Lambda,3,0} = \{1,2\}$, $RN_{\Lambda,3,1} = \emptyset$, *and* $RN_{\Lambda,3,2} = \{0,2\}$, *with* $Z_\Lambda = Z_{\Lambda,t,0} = Z_{\Lambda,t,1} = 0$ *and* $Z_{\Lambda,t,2} = 1$.

A partition Λ is a t-core if and only if each column in its t-abacus is completely filled in from the zeroeth row because removing a rim hook is equivalent to pushing up one of the beads in the t-abacus. Note that this is equivalent to $f_t(r,c) = 1 \implies f_t(r-1,c) = 1$ for every (r,c) and $f_t(0,0) = 0$. Note that this is also equivalent to $n \in ST_\Lambda \implies n - t \in ST_\Lambda$ and $0 \notin ST_\Lambda$. This observation will be helpful later in determining relations between t' and $t'g$-abaci.

We will investigate s/t-cores via properties given to columns of the $g = \gcd(s,t)$-abacus, so in order to obtain generating functions it will be necessary to determine the size of a partition Λ given $RN_{\Lambda,g,i}$.

2.3. **Determining the size of a Partition based on the columns of its t-abacus.** Let a partition Λ be given. In [9], James and Kerber obtain a natural correspondence between the partition Λ and a $(t+1)$-tuple of partitions

$$(\Lambda_{(t)}, \Lambda_0, \Lambda_1, \ldots, \Lambda_{t-1})$$

such that $\Lambda_{(t)}$ is the t-core of Λ and $ST_{\Lambda_i} = RN_{\Lambda,t,i}$.

This correspondence has the properties that

$$(1)\ |\Lambda| = |\Lambda_{(t)}| + \sum_{i=0}^{t-1} t|\Lambda_i|$$

$$(2)\ \text{If } \ n \in Hk(\Lambda_{(t)}), \text{ then } \ t \nmid n$$

$$(3)\ Hk(\Lambda)_t = \bigcup_{i=0}^{t-1} t \cdot Hk(\Lambda_i)$$

Furthermore, under this correspondence Λ^C corresponds with the $(t+1)$-tuple

$$(\Lambda^C_{(t)}, \Lambda^C_{t-1}, \Lambda^C_{t-2}, \ldots, \Lambda^C_0)$$

3. The General Structure of s/t-cores with $\gcd(s,t)>1$

3.1. **A Mapping between a k-abacus and a kn-abacus.** We first show a connection between the k and kn-abaci, which will be used to show our main result.

Lemma 3.1. *For a fixed partition Λ, $f_k(an+b,c)=1$ if and only if $f_{kn}(a,bk+c)=1$ for $0\le b<n$ and $0\le c<k$.*

Proof. $f_k(an+b,c)=1$ if and only if $ank+(bk+c)=(an+b)k+c\in ST_\Lambda$. But this occurs if and only if $f_{kn}(a,bk+c)=1$. □

3.2. **General Structure Theorem.** We now have developed the necessary tools to show Theorem 1.3. Set $g:=\gcd(s,t)$, $s':=\frac{s}{g}$ and $t':=\frac{t}{g}$.

Theorem 3.2. *A partition Λ (with $Z_\Lambda=0$) is an s/t-core if and only if for every $0\le i<g$, $RN_{\Lambda,g,i}=ST_{\Lambda'}$ for some s'/t'-core partition Λ', with $Z_{\Lambda,g,0}=0$.*

Proof. Let an s/t-core Λ and an integer $i<g$ be given. If $k\in RN_{\Lambda,g,i}$, then $kg+i\in ST_\Lambda$. Since Λ is an s-core, $(k-s')g+i=kg+i-s'g=kg+i-s\in ST_\Lambda$ and similarly $(k-t')g+i\in ST_\Lambda$ because Λ is a t-core. Hence, $k-t',k-s'\in RN_{\Lambda,g,i}$. But Λ' is an s'-core (possibly with $Z_{\Lambda'}\neq 0$) if and only if $k\in ST_{\Lambda'}$ implies that $k-s'\in ST_{\Lambda'}$. We see that $RN_{\Lambda,g,i}=ST_{\Lambda'}$ for Λ' an s'/t'-core, and we see immediately that $Z_{\Lambda,g,0}=0$, since $Z_\Lambda=0$.

Assume for the converse that for every $0\le i<g$, $RN_{\Lambda,g,i}=ST_{\Lambda_i}$ for Λ_i an s'/t'-core partition. Assume that $f_s(a,bg+c)=1$. Then $f_g(as'+b,c)=1$ by Lemma 3.1, so that $as'+b\in ST_{\Lambda_i}$. Since Λ_i is an s'-core, we know that $as'+b-s'\in ST_{\Lambda_i}$. Therefore, $f_g((a-1)s'+b,c)=1$, and hence $f_s(a-1,bg+c)=1$ by Lemma 3.1. It thus follows that Λ is an s-core, and the fact that Λ is a t-core follows analogously. □

4. Generating Functions based on the columns of a k-abacus

4.1. **Generating Function with a Given Structure in each Column.** Let k sets of partitions (possibly with zero part sizes) P_i be given, with the property that if $\Lambda\in P_i$ then if Λ' is obtained from Λ by adding parts of size zero, then $\Lambda'\in P_i$. Consider the set S of partitions such that for every $\Lambda\in S$ there exist $\Lambda_i\in P_i$ with $RN_{\Lambda,k,i}=ST_{\Lambda_i}$ and at least one of the $Z_{\Lambda_i}=0$.

Theorem 4.1. *If the generating function for partitions* $\Lambda \in P_i$ *with* $Z_\Lambda = 0$ *is* $\phi_i(q)$*, then the generating function for partitions of type* S *is*

$$\left(\prod_{i=0}^{k-1} \phi_i(q^k)\right) \cdot \prod_{n=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}$$

Specifically, when $P_i = P$*, with generating function* $\psi(q)$*, we obtain*

$$\big(\psi(q^k)\big)^k \cdot \prod_{n=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}.$$

Proof. Consider the set

$$T = \{(\Lambda'_{(k)}, \Lambda'_0, \ldots, \Lambda'_{k-1}) | \Lambda'_{(k)} \text{ is a } k\text{-core and for } 0 \le i < k,\ \Lambda_i \in P_i\}.$$

We will first show a 1-1 correspondence between $\Lambda \in S$ and $t \in T$.

Define $\varphi : T \to S$ by the following construction. Let $t = (\Lambda'_{(k)}, \Lambda'_0, \ldots, \Lambda'_{k-1}) \in T$ be given. Consider first the partition $\Lambda_{0,t}$ such that $RN_{\Lambda_{0,t},k,i} = ST_{\Lambda'_i}$. Now choose $m \in 0..k-1$ such that $\#ST_{\Lambda'_m}$ is the largest. Add parts of size zero to each Λ'_i to obtain Λ''_i such that $\#ST_{\Lambda''_i} = \#ST_{\Lambda'_m}$. Now consider the partition $\Lambda_{1,t}$ (possibly with $Z_{\Lambda_{1,t}} \neq 0$) such that $RN_{\Lambda_{1,t},k,i} = ST_{\Lambda''_i}$. Notice that $\Lambda_{1,t}$ has empty k-core. Now, if the k-abacus of the k-core $\Lambda'_{(k)}$ is $(0, a_1, \ldots, a_{k-1})$, where a_i is the number of beads in the i-th column, then add a_i parts of size zero to Λ''_i to obtain Λ'''_i. Finally consider the partition $\Lambda_{2,t}$ where $RN_{\Lambda_{2,t},k,i} = ST_{\Lambda'''_i}$. Now remove parts of size zero from $\Lambda_{2,t}$ to obtain a partition $\varphi(t) = \Lambda_t$ with $Z_{\Lambda_t} = 0$. Clearly, $\Lambda_t \in S$. Notice further that since rotations to remove parts of size zero do not change the k-core of a partition, the k-core of Λ is $\Lambda_{(k)}$. Thus, under the James-Kerber correspondence, Λ is $(\Lambda'_{(k)}, \Lambda'_j, \ldots, \Lambda'_{j+k-1})$, where $j+i$ is taken modulo k, and our rotations to remove zero part sizes have rotated the j-th column into the 0-th column. We know from [9] that the size of Λ_t is

$$k \cdot \left(\sum_{i=0}^{k-1} \lambda_i\right) + \lambda_k$$

where λ_i is the size of the partition Λ'_i.

Lemma 4.2. φ *is a bijection.*

Proof. Following Theorem 2.7.30 of James and Kerber [9], a tedious but straightforward calculation, keeping careful track of $\#RN_{\Lambda_{j,t},k,i}$ at each step of the construction, gives the desired bijection. Further details may be found in the second author's Master's thesis [10]. $\square$

Therefore, for each $t = (\Lambda'_{(k)}, \Lambda'_0, \ldots, \Lambda'_{k-1}) \in T$ there is a corresponding $\Lambda \in S$ with size

$$k \cdot \left(\sum_{i=0}^{k-1} \lambda_i\right) + \lambda_k$$

where λ_i is the size of the partition Λ'_i. Hence, going through each element of T to get an element of S, the generating function for partitions in S is

$$\left(\prod_{i=0}^{k-1} \phi_i(q^k)\right) \cdot \prod_{n=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}.$$

This is a generalization of the proof given by Nakamura, based on Olsson's work, for the generating function for (e, r)-cores, in [11, 13]. □

4.2. The Generating Function for s/t-core Partitions.

Corollary 4.3. *Set $g := \gcd(s,t)$, $s' := \frac{s}{g}$, and $t' := \frac{t}{g}$. If $\psi_{s',t'}(q)$ is the generating function for s'/t'-core partitions, then the generating function for s/t-core partitions is*

$$(\psi_{s',t'}(q^g))^g \cdot \prod_{n=1}^{\infty} \frac{(1-q^{gn})^g}{1-q^n}.$$

Proof. Using $P_i = P$ as the set of all s'/t'-cores by the classification in Theorem 3.2, this follows immediately from Theorem 4.1. □

The bijection given by φ along with Theorem 3.2 allows us to address the main question left in [1].

Corollary 4.4. *There are infinitely many simultaneous gt/gs-cores which are not g-cores, with $g \in \mathbb{N}, g > 1$.*

Proof. This follows directly from Theorem 3.2 for gt/gs-cores, taking $RN_{\lambda,g,0} = \{1\}$ and $Z_{\lambda,g,i}$ arbitrarily large for $i > 0$. □

Proof of Theorem 1.1. Consider

$$S_g := \{n : n \text{ is partitioned by an } g\text{-core}\}$$

and define $S_g^{(m)}$ to be the set S_g translated by adding m to every element. Granville and Ono have shown that S_g is all nonnegative integers when $g \geq 4$ and density zero otherwise [8]. Corollary 4.3 along with the fact that there are only finitely many s'/t'-cores [1] shows us that $S_{s,t}$ is the union of $S_g^{(mg)}$ for finitely many m. For $g < 4$ the result now follows immediately, and for $g \geq 4$ we note that (1) is always an s'/t'-core, so that the translation with $m = 1$ gives

$$S_{s,t} \supseteq S_g^{(g)}.$$

The result of Granville and Ono shows that $S_g^{(g)}$ is the desired set. Since partitions of every integer less than g are automatically a g-core, this inclusion implies equality and is the best possible. □

Example 4.5. *The only 2/3-core partition is* (1)*, so the generating function for $2k/3k$-core partitions is*

$$(1+q^k)^k \cdot \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n} = \left(\sum_{i=0}^{k} \binom{k}{i} q^{ik}\right) \cdot \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}$$

4.3. **Generating Functions based on the columns of a k-abacus of partitions with empty k-core.** Under the same assumptions as section 4.1, assume further that each $\Lambda \in S$ has empty k-core.

Theorem 4.6. *The generating function for partitions of this type is*

$$\prod_{i=0}^{k-1} \phi_i(q^k)$$

where $\phi_i(q)$ is the generating function for partitions in $\Lambda \in P_i$ with $Z_\Lambda = 0$.

Additionally, it should be noted that if each column has the same generating function $\psi(q)$, this becomes

$$(\psi(q^k))^k$$

Proof. The proof follows the proof of Theorem 4.1 by determining the 1-1 correspondence between $\Lambda \in S$ and $k+1$-tuples $(\emptyset, \Lambda'_0, \ldots, \Lambda'_{k-1})$ with $\Lambda'_i \in P_i$, since the k-core is always empty. □

4.4. **Generating Function for simultaneous $(e,r)/(e',r)$ cores.**

Corollary 4.7. *If $\psi_{e,e'}(q)$ is the generating function for e/e' cores, then the generating function for $(e,r)/(e',r)$ cores is*

$$\left(\psi_{e,e'}(q^r)\right)^r.$$

Furthermore, it should be noted that when e and e' are relatively prime that Anderson has shown that this is a polynomial [1]*. The number of $(e,r)/(e',r)$ cores in this case would be $\left(\frac{\binom{e+e'}{e}}{e+e'}\right)^r$, due to the count of e/e' cores as shown in* [1]*.*

Proof. Using $P_i = P$ to be the set of all e/e'-cores, this follows immediately from Theorem 4.6. □

5. Conjugation

We will first define a notion of conjugation which includes partitions with parts of size zero. Let a partition Λ and an integer $m \geq \max ST_\Lambda$ be given. Then we define the partition Λ^C_m to be Λ^C plus $m - \max ST_\Lambda$ parts of size zero. Note that $\Lambda^C_{\max ST_\Lambda} = \Lambda^C$ and $\left(\Lambda^C_m\right)^C_m = \Lambda$.

5.1. **Conjugation Theorem for Structure Numbers.**

Theorem 5.1. *Let Λ and m be given as above. Then*

$$i \notin ST_\Lambda \text{ if and only if } m - i \in ST_{\Lambda^C_m} \ \forall 0 \leq i \leq m.$$

In particular, $i \notin ST_\Lambda$ if and only if $\max ST_\Lambda - i \in ST_{\Lambda^C}$.

Proof. Let $a < b \leq m \notin ST_\Lambda$ be given such that for all $a < i < b$, $i \in ST_\Lambda$. Then there are $b - a - 1$ parts of Λ with the same size, say of size $\ell \geq 1$. Thus, the length of the $(\ell+1)^{st}$ column of the Ferrers-Young Diagram is $b - a - 1$ less than the length of the ℓ-th column. Thus, if the ℓ-th largest structure number of Λ^C_m is c, then the $\ell + 1$-th largest structure number is $c - 1 - (b - a - 1) = c - (b - a)$. From symmetry of the largest structure number for conjugation of partitions without parts of size zero, the largest structure number of Λ^C is $\max ST_\Lambda - Z_\Lambda$. Since $Z_{\Lambda^C_m} = m - \max ST_\Lambda$, the largest structure number of Λ^C_m is

$$\max ST_\Lambda - Z_\Lambda + m - \max ST_\Lambda = m - Z_\Lambda.$$

We then simply note that $a = Z_\Lambda$ is the smallest natural number not contained in ST_Λ and inductively show that the ℓ-th largest structure number is $c = m - a$.

□

5.2. **Conjugation of a t-core.**

In the following exposition it will be helpful to define the *(t-th) pivot column* of a partition Λ as the column of the t-abacus containing the largest structure number, namely $\max ST_\Lambda \pmod t$. We will omit the prefix (t-th) when it is clear from the context. For a t-core partition Λ, the pivot column is the rightmost column of the t-abacus for which $Z_{\Lambda,t,i}$ is maximal. For example, the 4-core partition $(4, 3, 2, 2, 2, 1, 1, 1)$ with 4-abacus

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ 0 & | & \bullet & \bullet & \bullet \\ 1 & | & \bullet & \bullet & \bullet \\ 2 & | & \bullet & | & \bullet \end{array}$$

has pivot column 3, since columns $i = 1$ and $i = 3$ both satisfy the maximal condition $Z_{\Lambda,t,i} = 3$, but the 3-rd column is rightmost with this property.

Corollary 5.2. *Let a t-core partition Λ with pivot column k be given. Then*

$$Z_{\Lambda^C,t,i} = Z_{\Lambda,t,k} - Z_{\Lambda,t,k-i} - \delta_{i>k}.$$

where $k-i$ is taken modulo t and $\delta_{i>k} = 1$ if $i > k$ and 0 otherwise.

Proof. Theorem 5.1 gives the process which, taken modulo t, will give us this result. □

5.3. **Self-Conjugate Properties.**

Corollary 5.3. *Λ is self-conjugate if and only if for every $0 \leq i < \max ST_\Lambda$, exactly one of i and $\max ST_\Lambda - i$ is in ST_Λ.*

Proof. Since $i \in ST_{\Lambda^C}$ if and only if $\max ST_\Lambda - i \notin ST_\Lambda$, this follows directly from Theorem 5.1. □

Corollary 5.4. *Let a t-core partition Λ be given with pivot column k. Then Λ is self-conjugate if and only if*

$$Z_{\Lambda,t,i} + Z_{\Lambda,t,k-i} = Z_{\Lambda,t,k} - \delta_{i>k}.$$

and $k-i$ is taken modulo t.

Proof. This follows directly from Corollary 5.2. □

6. General Structure Theorem for Self-Conjugate s/t-cores With $\gcd(s,t) > 1$

Set $g := \gcd(s,t)$, $s' := \frac{s}{g}$ and $t' := \frac{t}{g}$.

Theorem 6.1. *Let a partition Λ with (g-th) pivot column k be given. Denote the row containing the largest structure number $m = \left\lfloor \frac{\max ST_\Lambda}{g} \right\rfloor$.*

Then Λ is a self-conjugate s/t-core if and only if $RN_{\Lambda,g,i} = ST_{\Lambda'_i}$ for Λ'_i an s'/t'-core and $\Lambda'_{k-i} = (\Lambda'_i)^C_{m-\delta_{i>k}}$, where $k-i$ is taken modulo g.

Proof. By Theorem 3.2 we know that Λ is an s/t-core if and only if $RN_{\Lambda,g,i} = ST_{\Lambda'_i}$ with Λ'_i an s'/t'-core. Corollary 5.3 taken modulo g shows that the partition is self-conjugate if and only if $\Lambda'_{k-i} = (\Lambda'_i)^C_{m-\delta_{i>k}}$.

□

7. Generating Function for Self-Conjugate s-cores/t-cores With $\gcd(s,t) > 1$

Set $g := \gcd(s,t)$, $s' := \frac{s}{g}$ and $t' := \frac{t}{g}$. Denote the generating function for self-conjugate g-cores by $\gamma_g(q)$ (given explicitly in [3, 11]),the generating function for simultaneous s'/t'-cores by $\psi_{s',t'}(q)$, and the generating function for self-conjugate simultaneous s'/t'-cores by $\zeta_{s',t'}(q)$.

Theorem 7.1. *If $g = \gcd(s,t)$ is odd, then the generating function for self-conjugate s/t-cores is*

$$\gamma_g(q)\left(\psi_{s',t'}(q^{2g})\right)^{\frac{g-1}{2}}\zeta_{s',t'}(q^g).$$

If g is even, then the generating function for self-conjugate s/t-cores is

$$\gamma_g(q)\left(\psi_{s',t'}(q^{2g})\right)^{\frac{g}{2}}.$$

Proof. Let a partition Λ be given. Recall that the correspondence of James and Kerber [9] gives a $g+1$-tuple

$$(\Lambda'_{(g)}, \Lambda'_0, , \Lambda'_{g-1})$$

and the $g+1$-tuple corresponding to Λ^C is

$$((\Lambda'_{(g)})^C, (\Lambda'_{g-1})^C, \dots, (\Lambda'_0)^C).$$

Hence Λ is self-conjugate if and only if $\Lambda'_{(g)}$ is self-conjugate and $\Lambda'_{g-i} = (\Lambda'_i)^C$. Theorems 3.2 and 4.3 show a correspondence between Λ'_i and s'/t'-cores $\widetilde{\Lambda}'_i$ of the same size.

Assume first that g is even. Since the property of being an s'/t'-core is invariant under conjugation, we have a correspondence between $\frac{g}{2}+1$-tuples $(\Lambda'_{(g)}, \widetilde{\Lambda}'_0, \dots, \widetilde{\Lambda}'_{\frac{g}{2}-1})$ and self-conjugate s/t-cores, where $\Lambda'_{(g)}$ is a self-conjugate g-core and $\widetilde{\Lambda}'_i$ is an s'/t'-core. From above, under the correspondence of James and Kerber, Λ corresponds to

$$(\Lambda'_{(g)}, \Lambda'_0, \dots, \Lambda'_{\frac{g}{2}-1}, (\Lambda'_{\frac{g}{2}-1})^C, \dots, (\Lambda'_0)^C),$$

and the size of this partition is

$$\left(\sum_{i=0}^{\frac{g}{2}-1} g\cdot 2\left|\widetilde{\Lambda}'_i\right|\right) + \left|\Lambda'_{(g)}\right|,$$

The number of choices for positions of the $\widetilde{\Lambda}'_i$'s are the multinomial coefficients. Therefore, we obtain the generating function

$$\gamma_g(q)\left(\psi_{s',t'}(q^{2g})\right)^{\frac{g}{2}}.$$

The argument for g odd follows analogously. □

7.1. **Generating Function for Self-Conjugate (e,r)-cores/(e',r)-cores.** Denote the generating function for simultaneous e/e'-cores by $\psi_{e,e'}(q)$, and the generating function for self-conjugate simultaneous e/e'-cores by $\zeta_{e,e'}(q)$.

Theorem 7.2. *If r is odd, then the generating function for self-conjugate $(e,r)/(e',r)$-cores is*

$$\left(\psi_{e,e'}(q^{2r})\right)^{\frac{r-1}{2}}\zeta_{e,e'}(q^r)$$

If r is even, then the generating function for $(e,r)/(e',r)$-cores is

$$\left(\psi_{e,e'}(q^{2r})\right)^{\frac{r}{2}}.$$

Proof. This follows immediately from Theorem 4.6 and Theorem 7.1. □

If e and e' are relatively prime, then it should be noted that this generating function will be a polynomial [1]. Additionally, if r is even, then the number of partitions of this type is $\left(\frac{\binom{e+e'}{e}}{e+e'}\right)^{\frac{r}{2}}$ [1]. If r is odd, then we have the number of partions bounded below by $\left(\frac{\binom{e+e'}{e}}{e+e'}\right)^{\frac{r-1}{2}}$ and above by $\left(\frac{\binom{e+e'}{e}}{e+e'}\right)^{\frac{r+1}{2}}$, since there are at most as many self-conjugate e/e'-cores as general e/e'-cores.

8. Further Questions

While investigating s/t-cores, the authors also investigated the case when s and t were relatively prime to attempt to understand the polynomial generating function for use in our generating functions above. Due to computational evidence, an inherent structure, and a simple proof for $2 = s < t$, we make the following conjecture.

Conjecture 8.1. *If s and t are relatively prime, then the largest size of a partition which is an s/t-core is*

$$\frac{(s^2-1)(t^2-1)}{24}.$$

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