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TESTING FOR THE BUFFERED AUTOREGRESSIVE PROCESSES

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Abstract: This paper investigates a quasi-likelihood ratio (LR) test for the thresholds in buffered autoregressive processes. Under the null hypothesis of no threshold, the LR test statistic converges to a function of a centered Gaussian process. Under local alternatives, this LR test has nontrivial asymptotic power. A bootstrap method is proposed to obtain the critical value for the LR test. Simulation studies and an example are given to assess the performance of the test. The proof here is not standard and can be used in other non-linear time series models.

Key words and phrases: AR(p) model, bootstrap method, buffered AR(p) model, likelihood ratio test, marked empirical process, threshold AR(p) model.

1. Introduction

After the seminal work of Tong (1978), threshold autoregressive (TAR) models have achieved great success in practice; see, e.g., Tong (1990) for earlier works and Tong (2011) and the references therein for more recent ones. Generally speaking, the TAR model says that the structure of an AR model shifts among different regimes, i.e.,

$$y_t = \phi_0 + \sum_{i=1}^{p} \phi_i y_{t-i} + \Big(\psi_0 + \sum_{i=1}^{p} \psi_i y_{t-i}\Big) R_t + \varepsilon_t, \tag{1.1}$$

where $R_t = I(y_{t-d} \le r)$ is the regime indicator of y_t, r is the threshold parameter, $d(\ge 1)$ is the delay parameter, and ε_t is an uncorrelated error sequence with zero mean and variance $\sigma^2(> 0)$. There has been a lot of interest in detecting thresholds in TAR models. Chan (1990, 1991) and Chan and Tong (1990) first accomplished this by considering a likelihood ratio (LR) test. Tsay (1989) gave some novel methods in this context; Hansen (1996) studied the Wald test and Lagrange multiplier (LM) test for TAR models; Wong and Li (1997, 2000) studied the LM test for TAR-ARCH models; Li and Ling (2013) investigated the portmanteau test for threshold double AR models; see also Tsay (1998), Hansen (1999), Caner and Hansen (2001), Ling and Tong (2005), Li and Li (2008, 2011), and Zhu and Ling (2012).

Under (1.1), the regime of y_t shifts when the state of y_{t-d} changes. In practice, the regime of y_t may not shift immediately, and there could be a buffering region in which the regime of y_t depends on the regime of y_{t-d}. Li et al. (2012) first formulated this by assuming that R_t in (1.1) satisfies

$$R_t = \begin{cases} 1 & \text{if } y_{t-d} \le r_L, \\ 0 & \text{if } y_{t-d} > r_U, \\ R_{t-1} & \text{otherwise,} \end{cases} \tag{1.2}$$

where r_L and r_U are two threshold parameters such that $r_L \le r_U$. They called (1.1)−(1.2) the buffered AR (BAR) model, and the region in which y_{t-d} lies between r_L and r_U is called the buffering region. Also, they found that the BAR model is the best selected model for the sunspot series in Tong (1990) and the GNP series in Tiao and Tsay (1994); it may provide us with a new way to understand non-linear time series. However, how to test for BAR models is still unknown, and it is more challenging than testing for TAR models because the regime of y_t depends on past observations infinitely far away.

In this paper, we investigate a quasi-LR test for the thresholds in BAR models. Under the null hypothesis of no threshold, the LR test statistic converges to a function of a centered Gaussian process. Under local alternatives, this LR test has nontrivial asymptotic power. Our result contains the one in Chan (1990) as a special case, but its proof is not standard and different from the proof in that paper. A bootstrap method is proposed to obtain the critical value for the LR test. Simulation studies and an example are given to assess the performance of this LR test.

This paper is organized as follows. Section 2 states our main result on the LR test. Section 3 proposes a bootstrap procedure. The simulation results and an example are given in Section 4. The proofs are provided in the Appendix, which can be found in Zhu, Yu, and Li (2013). Throughout the paper, $|A| = (tr(A'A))^{1/2}$ is the Euclidean norm of a matrix A, $\|A\|_s = (E|A|^s)^{1/s}$ is the L^s-norm ($s \ge 1$) of a random matrix, A' is the transpose of matrix A, $o_p(1)$ ($O_p(1)$) denotes a sequence of random numbers converging to zero (bounded) in probability, $\to_d$ denotes convergence in distribution, and $\Rightarrow$ denotes weak convergence. $I(\cdot)$ is an indicator function.

2. Likelihood Ratio Test

Let $\phi = (\phi_0, \ldots, \phi_p)'$, $\psi = (\psi_0, \ldots, \psi_p)'$, $\lambda = (\phi', \psi')'$, $\gamma = (r_L, r_U)$, and $x_t = (1, y_{t-1}, \ldots, y_{t-p})'$. Then, model (1.1)−(1.2) is

$$y_t = x_t(\gamma)'\lambda + \varepsilon_t, \tag{2.1}$$

where $x_t(\gamma) = (x_t', h_t(\gamma)')'$, $h_t(\gamma) = x_t R_t(\gamma)$, and $R_t(\gamma)$ is defined as in (1.2). Here, we assume that all the roots of the characteristic equation $\phi(x) = x^p - \phi_1 x^{p-1} - \cdots - \phi_p$ lie inside the unit circle, and both p and d are known. We further assume that $d \le p$ if $p \ge 1$, because we can set $p = d$ with $\phi_{p+1} = \cdots = \phi_d = 0$ and $\psi_{p+1} = \cdots = \psi_d = 0$ in (2.1) when $d > p \ge 1$.

Suppose that $\{y_0, \ldots, y_N\}$ are $N+1$ consecutive observations from model (2.1) with the true parameters λ_0 and γ_0, where $\lambda_0 = (\phi_0', \psi_0')'$, $\phi_0 = (\phi_{00}, \ldots, \phi_{p0})'$, $\psi_0 = (\psi_{00}, \ldots, \psi_{p0})'$, and $\gamma_0 = (r_{L0}, r_{U0})$. We consider the hypotheses

$$\begin{cases} H_0 : \psi_0 = 0, \\ H_1 : \psi_0 \neq 0 \text{ for some } \gamma. \end{cases} \tag{2.2}$$

Model (2.1) is an AR(p) model under H_0 and it is a buffered AR(p) (BAR(p)) model under H_1. When $r_L = r_U$ (i.e., the buffering region is absent), (2.2) is for testing the threshold in the threshold AR(p) (TAR(p)) model, for which the likelihood ratio (LR) test was studied by Chan (1990, 1991) when $\varepsilon_t \sim N(0,1)$ is a sequence of i.i.d. random variables. When $r_L \neq r_U$, since

$$\begin{aligned} R_t(\gamma) &= I(y_{t-d} \le r_L) \\ &\quad + \sum_{j=1}^{\infty} I(y_{t-j-d} \le r_L) \prod_{i=1}^{j} I(r_L < y_{t-i+1-d} \le r_U) \ \text{a.s.}, \end{aligned} \tag{2.3}$$

we see that $R_t(\gamma)$ depends on all past observations. The $R_t(\gamma)$ in Chan (1990) only depends on y_{t-d}, so the test there is not a LR test and may be less powerful. We consider an alternative LR test for (2.2).

Let $Y = (y_p, \ldots, y_N)'$ and $Z_\gamma = (X, X_\gamma) = \big(x_p(\gamma), x_{p+1}(\gamma), \ldots, x_N(\gamma)\big)'$, where $X = (x_p, x_{p+1}, \ldots, x_N)'$ and $X_\gamma = \big(h_p(\gamma), h_{p+1}(\gamma), \ldots, h_N(\gamma)\big)'$. Let $n = N - p + 1$ be the effective number of observations. Following Chan (1990), we know that for any fixed value of γ the LR test statistic is

$$LR_n(\gamma) = \frac{n\left[\sigma_n^2 - \sigma_n^2(\gamma)\right]}{\sigma_n^2},$$

where

$$\sigma_n^2 = \frac{1}{n}\{Y'Y - (Y'X)(X'X)^{-1}(X'Y)\}, \tag{2.4}$$

$$\sigma_n^2(\gamma) = \frac{1}{n}\{Y'Y - (Y'Z_\gamma)(Z_\gamma' Z_\gamma)^{-1}(Z_\gamma' Y)\}. \tag{2.5}$$

Since the exact value of γ is unknown under H_0, it is natural to construct the LR test by using the maximum of $LR_n(\gamma)$ over the range of γ, see Davies (1977, 1987). Thus, our LR test statistic is

$$LR_n = \sup_{\gamma \in \Gamma} LR_n(\gamma),$$

where $\Gamma \equiv \{(r_L, r_U); a \leq r_L \leq r_U \leq b\}$ and $[a, b]$ is a predetermined interval. Here, we truncate the full range of γ, since LR_n may diverge to infinity in probability as $n \to \infty$, see Andrews (1993a).

Let $K_{\gamma\delta} = E[x_t(\gamma)x_t(\delta)']$. For the asymptotic theory of LR_n, we need certain technical assumptions.

Assumption 1. *y_t is strictly stationary, ergodic and absolutely regular with mixing coefficients $\beta(m) = O(m^{-A})$ for some $A > v/(v-1)$ and $r > v > 1$; $E|y_t|^{4r} < \infty$, $E|\varepsilon_t|^{4r} < \infty$, and $K_{\gamma\gamma}$ is positive definite.*

Assumption 2. *y_t has a bounded and continuous density function.*

Assumption 3. *There exists an $A_0 > 1$ such that $2A_0 rv/(r-v) < A$.*

Assumptions 1−2 are from Hansen (1996), where the weak convergence of the empirical process was derived by using the method in Doukhan, Massart, and Rio (1995). When $\sum_{i=1}^p |\phi_i| < 1$ and $\sum_{i=1}^p |\phi_i + \psi_i| < 1$, Li et al. (2012) showed that model (2.1) is strictly stationary and ergodic. When $A > v/(v-1)$, a sufficient condition for Assumption 3 is that $v < 3r/(2r+1)$, which is stronger than $v < r$ as required in Assumption 1. Particularly, when ε_t is a sequence of i.i.d. random variables with a bounded and continuous density function, $\beta(m)$ decays exponentially under H_0 as shown in Pham and Tran (1985). Thus, Assumptions 1−3 hold in this case.

We state two key lemmas, under which a uniform expansion of $LR_n(\gamma)$ can be derived.

Lemma 1. *If Assumptions 1−3 hold, then*

(i) $$\sup_{\gamma\in\Gamma}\left|\left\{\frac{X_\gamma' X_\gamma}{n} - \frac{X_\gamma' X}{n}\left(\frac{X'X}{n}\right)^{-1}\frac{X'X_\gamma}{n}\right\}^{-1} - (\Sigma_\gamma - \Sigma_\gamma\Sigma^{-1}\Sigma_\gamma)^{-1}\right| = o_p(1);$$

(ii) *under H_0,*
$$\sup_{\gamma\in\Gamma}\left|T_\gamma - \big(-\Sigma_\gamma\Sigma^{-1}, I\big)\frac{1}{\sqrt{n}}Z_\gamma'\varepsilon\right| = o_p(1),$$
where $\varepsilon = (\varepsilon_p, \ldots, \varepsilon_N)'$, $T_\gamma = n^{-1/2}\left\{X_\gamma' - X_\gamma' X(X'X)^{-1}X'\right\}Y$, $\Sigma = E(x_t x_t')$, and $\Sigma_\gamma = E[x_t x_t' R_t(\gamma)]$.

Proof. See the Appendix in Zhu, Yu, and Li (2013).

Lemma 2. *If Assumptions 1−3 hold, then*
$$\frac{1}{\sqrt{n}}Z_\gamma'\varepsilon \Rightarrow \sigma G_\gamma$$
as $n \to \infty$, where G_γ is a Gaussian process with zero mean function and covariance $K_{\gamma\delta}$.

Proof. See the Appendix in Zhu, Yu, and Li (2013).

Note that

$$\frac{1}{\sqrt{n}} Z_\gamma' \varepsilon = \frac{1}{\sqrt{n}} \sum_{t=p}^{N} (x_t', x_t' R_t(\gamma))' \varepsilon_t.$$

We call $\{n^{-1/2} Z_\gamma' \varepsilon\}$ a marked empirical process, as in Stute (1997), where each y_{t-i-d} in $R_t(\gamma)$ is a marker. In view of (2.3), $\{n^{-1/2} Z_\gamma' \varepsilon\}$ involves infinitely many markers, as Ling and Tong (2005) studied the LR test for TMA models. Their method seems hard to implement here. Compared with Chan (1990) and Ling and Tong (2005), the proofs of Lemmas 1−2 in the Appendix are not standard; they may be useful in other non-linear time series models.

Theorem 1. *If Assumptions* 1−3 *hold, then under* H_0*,*

$$LR_n \to_d \sup_{\gamma \in \Gamma} G_\gamma' \Omega_\gamma G_\gamma$$

as $n \to \infty$*, where* $\Omega_\gamma = \left(-\Sigma_\gamma \Sigma^{-1}, I\right)' \left(\Sigma_\gamma - \Sigma_\gamma \Sigma^{-1} \Sigma_\gamma\right)^{-1} \left(-\Sigma_\gamma \Sigma^{-1}, I\right)$.

Proof. By (2.4)−(2.5) and a direct calculation,

$$n\left[\sigma_n^2 - \sigma_n^2(\gamma)\right] = T_\gamma' \left\{ \frac{X_\gamma' X_\gamma}{n} - \frac{X_\gamma' X}{n} \left(\frac{X'X}{n}\right)^{-1} \frac{X' X_\gamma}{n} \right\}^{-1} T_\gamma. \tag{2.6}$$

By Lemmas 1−2, the conclusion follows directly from the argument for Theorem 2.3 in Chan (1990).

Remark 1. Note that

$$G_\gamma' \Omega_\gamma G_\gamma = \xi_\gamma' \left(\Sigma_\gamma - \Sigma_\gamma \Sigma^{-1} \Sigma_\gamma\right)^{-1} \xi_\gamma,$$

where $\xi_\gamma = \left(-\Sigma_\gamma \Sigma^{-1}, I\right) G_\gamma$. Then, by a direct calculation, we can show that, for each $\gamma \in \Gamma$, $G_\gamma' \Omega_\gamma G_\gamma$ follows a χ^2 distribution. That is, for fixed γ, the test statistic $LR_n(\gamma)$ is asymptotically pivotal under H_0.

Remark 2. Although Theorem 1 has Theorem 2.3(ii) of Chan (1990) as a special case, there has some difference between our LR test and that in Chan (1990). First, the denominator of $LR_n(\gamma)$ is different from that in Chan (1990), but the two are asymptotically equivalent; see also Ling and Tong (2005). Second, since our Γ is larger than that in Chan (1990), our LR test needs more computational efforts.

Remark 3. As Chan (1990), we only obtained the result under the condition that $Var(\varepsilon_t) = \sigma^2$. The case that the threshold effect is in the variance of ε_t needs further study.

Next, we study the asymptotical local power of LR_n by considering the local alternative hypothesis

$$H_{1n} : \psi_0 = \frac{h}{\sqrt{n}} \quad \text{for a constant vector } h \in \mathcal{R}^{p+1}.$$

Theorem 2. *If Assumptions* 1−3 *hold, then under* H_{1n}*,*

$$LR_n \to_d \sup_{\gamma\in\Gamma} \left\{ G'_\gamma \Omega_\gamma G_\gamma + h' \mu_{\gamma\gamma_0} h \right\},$$

as $n \to \infty$*, where* $M_{\gamma\gamma_0} = E[x_t x'_t R_t(\gamma) R_t(\gamma_0)]$ *and*

$$\mu_{\gamma\gamma_0} = \frac{1}{\sigma^2} \left(M_{\gamma\gamma_0} - \Sigma_\gamma \Sigma^{-1} \Sigma_\gamma \right)' \left(\Sigma_\gamma - \Sigma_\gamma \Sigma^{-1} \Sigma_\gamma \right)^{-1} \left(M_{\gamma\gamma_0} - \Sigma_\gamma \Sigma^{-1} \Sigma_\gamma \right).$$

Proof. Note that $Y = X\phi_0 + X_{\gamma_0} h/\sqrt{n} + \varepsilon$ under H_{1n}. Thus,

$$\begin{aligned} T_\gamma &= \frac{1}{\sqrt{n}} \left\{ X'_\gamma - X'_\gamma X (X'X)^{-1} X' \right\} \varepsilon + \frac{1}{n} \left\{ X'_\gamma - X'_\gamma X (X'X)^{-1} X' \right\} X_{\gamma_0} h \\ &= \frac{1}{\sqrt{n}} \big(-(X'_\gamma X)(X'X)^{-1}, I \big) Z'_\gamma \varepsilon + \frac{1}{n} \left\{ X'_\gamma - X'_\gamma X (X'X)^{-1} X' \right\} X_{\gamma_0} h. \end{aligned}$$

By (2.6) and Lemmas 1−2, the conclusion follows directly from the argument for Theorem 2.3 in Chan (1990).

In practice, the values of a and b can be set to empirical quantiles of $\{y_t\}_{t=0}^N$ as in Chan (1991) and Andrews (1993b), although how to choose the optimal a, b remains unclear. In this case, we can always find a smallest $n_0 \geq p$ such that y_{n_0-d} stays outside the region $[a, b]$, where the integer n_0 depends on data sample $\{y_0, \ldots, y_N\}$. This means that we can observe $R_{n_0}(\gamma)$, and then further calculate $\{R_t(\gamma)\}_{t=n_0+1}^N$ iteratively as

$$R_t(\gamma) = I(y_{t-d} \leq r_L) + R_{t-1} I(r_L < y_{t-d} \leq r_U).$$

For the remaining observations $\{y_t\}_{t=0}^{n_0-1}$ whose regions are not well identified, we set their regions to be 0. Thus, we can only use $\tilde{R}_t(\gamma)$ rather than $R_t(\gamma)$ in practice, where

$$\tilde{R}_t(\gamma) = \begin{cases} 0 & \text{for } t = 0, \ldots, n_0 - 1, \\ R_t(\gamma) & \text{for } t = n_0, \ldots, N. \end{cases} \tag{2.7}$$

Let $\tilde{LR}_n$ be defined in the same way as LR_n with $R_t(\gamma)$ being replaced by $\tilde{R}_t(\gamma)$. The following corollary shows that $\tilde{LR}_n$ and LR_n have the same asymptotic property.

Corollary 1. *If Assumptions* 1–3 *hold, then* (i) *under* H_0,

$$\tilde{LR}_n \to_d \sup_{\gamma\in\Gamma} G'_\gamma \Omega_\gamma G_\gamma \quad as \quad n\to\infty;$$

(ii) *under* H_{1n},

$$\tilde{LR}_n \to_d \sup_{\gamma\in\Gamma} \left\{G'_\gamma \Omega_\gamma G_\gamma + h'\mu_{\gamma\gamma_0} h\right\} \quad as \quad n\to\infty.$$

Proof. See the Appendix in Zhu, Yu, and Li (2013).

3. Bootstrapped Critical Value

In this section, we use a bootstrap method to obtain the critical value for our LR test; see also Hansen (1996) and Li and Li (2008, 2011). First, we let

$$\hat{\varepsilon}_t = y_t - x_t(\gamma)'\lambda_n(\gamma) \tag{3.1}$$

with

$$\lambda_n(\gamma) \equiv \arg\min_{\lambda\in\Lambda} \sum_{t=p}^{N} \varepsilon_t^2(\lambda,\gamma) = \left[Z'_\gamma Z_\gamma\right]^{-1}\left[Z'_\gamma Y\right],$$

where Λ is a compact parametric space of λ, and $\varepsilon_t(\lambda,\gamma) = y_t - x_t(\gamma)'\lambda$. Next, we set

$$\hat{LR}_n(\gamma) = \frac{\hat{Z}'_n(\gamma)(X_{1n}(\gamma), I)'[X_{2n}(\gamma)]^{-1}(X_{1n}(\gamma), I)\hat{Z}_n(\gamma)}{\sigma_n^2}, \tag{3.2}$$

where $\hat{\varepsilon} = (\hat{\varepsilon}_p v_p, \ldots, \hat{\varepsilon}_N v_N)'$, $\{v_t\}_{t=p}^N$ is a sequence of i.i.d. $N(0,1)$ random variables, and

$$\hat{Z}_n(\gamma) = \frac{1}{\sqrt{n}} Z'_\gamma \hat{\varepsilon}, \quad X_{1n}(\gamma) = -\frac{X'_\gamma X}{n}\left(\frac{X'X}{n}\right)^{-1},$$

$$X_{2n}(\gamma) = \frac{X'_\gamma X_\gamma}{n} - \frac{X'_\gamma X}{n}\left(\frac{X'X}{n}\right)^{-1}\frac{X'X_\gamma}{n}.$$

Define

$$\hat{LR}_n \equiv \sup_{\gamma\in\Gamma} \hat{LR}_n(\gamma). \tag{3.3}$$

The asymptotic theory of $\hat{LR}_n$ is stated in the following theorem:

Theorem 3. *If Assumptions* 1–3 *hold, then under* H_0 *or* H_{1n},

$$\hat{LR}_n | y_0, \ldots, y_N \to_d \sup_{\gamma\in\Gamma} G'_\gamma \Omega_\gamma G_\gamma \quad in\ probablity\ as\ n\to\infty.$$

Proof. See the Appendix in Zhu, Yu, and Li (2013).

Remark 4. In practice, $\hat{LR}_n$ is calculated with $R_t(\gamma)$ being replaced by $\tilde{R}_t(\gamma)$. However, by using the argument for Corollary 1, we can show that it does not affect the asymptotic property of $\hat{LR}_n$.

Note that the conditional limiting distribution in Theorem 3 is the same as the null distribution in Theorem 1. Then, conditional on the data sample $\{y_0, \ldots, y_N\}$ and for given significance level α, we use a bootstrap procedure to obtain our critical value:

(i) generate i.i.d. N(0,1) samples $\{v_t\}_{t=p}^N$, and calculate $\hat{LR}_n$ via (3.1)−(3.3);

(ii) repeat step (i) J times to get $\{\hat{LR}_n^{(1)}, \ldots, \hat{LR}_n^{(J)}\}$;

(iii) choose $c_{n,\alpha}^J$ as the α-th upper percentile of $\{\hat{LR}_n^{(1)}, \ldots, \hat{LR}_n^{(J)}\}$.

We choose $c_{n,\alpha}^J$ as the critical value for our LR test and shorten $c_{n,\alpha}^J$ to c_n for brevity. In the end, we give a critical corollary as follows:

Corollary 2. *If Assumptions* 1−3 *hold, then* (i) *under* H_0,

$$\lim_{n\to\infty} \lim_{J\to\infty} P\left(LR_n \geq c_{n,\alpha}^J\right) = \alpha;$$

(ii) *under* H_{1n},

$$\lim_{h\to\infty} \lim_{n\to\infty} \lim_{J\to\infty} P\left(LR_n \geq c_{n,\alpha}^J\right) = 1.$$

Proof. See the Appendix in Zhu, Yu, and Li (2013).

Corollary 3.1 guarantees that our bootstrapped critical value $c_{n,\alpha}^J$ is asymptotically valid, and our LR test has power to detect H_{1n}. The method can also produce the critical value for the LR test in Chan (1990) by setting $\gamma_L \equiv \gamma_U$. Since $\hat{LR}_n(\gamma)$ is a step-function, the amount of computation on $c_{n,\alpha}^J$ depends only on the effective sample size n and the bootstrapped sample size J. This reduces the computational burden significantly.

4. Simulation and One Real Example

In this section, we first compare the performance of our LR test (LR_n) and Chan's (1990) LR test (LR_n^*) in the finite sample. We generate 1,000 replications of sample size $n = 200$ from the BAR model

$$y_t = y_{t-1} - 0.09y_{t-2} + (\psi_1 y_{t-1} + \psi_2 y_{t-2})R_t(\gamma) + \varepsilon_t, \tag{4.1}$$

where $R_t(\gamma)$ is defined as in (1.2) with $d = 1$, ε_t is $N(0, 1)$, and $y_0 = y_1 = R_1(\gamma) = 0$. We choose $\gamma = (0, 0)$, $(0, 0.5)$, $(0, 1.5)$ or $(0, 2)$, and use the significance level

Table 1. Rejection rates.

ψ		γ			LR_n^*	
ψ_1	ψ_2	r_L	r_U	LR_n	LR_{1n}^*	LR_{2n}^*
0.0	0.0	—	—	4.9	4.9	3.4
0.1	-0.09	0.0	0.0	7.7	7.7	3.8
		0.0	0.5	7.5	7.4	3.7
		0.0	1.5	7.6	6.5	3.2
		0.0	2.0	7.5	7.0	5.4
0.3	-0.27	0.0	0.0	31.9	34.2	14.3
		0.0	0.5	30.6	30.3	16.5
		0.0	1.5	33.4	29.6	15.4
		0.0	2.0	32.0	27.1	15.6
0.5	-0.45	0.0	0.0	64.7	69.1	54.0
		0.0	0.5	76.0	79.6	55.2
		0.0	1.5	76.1	75.5	56.0
		0.0	2.0	75.2	72.6	53.9
0.7	-0.63	0.0	0.0	95.8	97.1	86.4
		0.0	0.5	89.4	90.1	89.5
		0.0	1.5	96.0	96.0	87.8
		0.0	2.0	95.9	95.9	89.9

$\alpha = 0.05$. Since the pair of characteristic roots is $(0.1, 0.9)$ in the regime of $R_t(\gamma) = 0$, we choose $(\psi_1, \psi_2) = (0, 0)$, $(0.1, -0.09)$, $(0.3, -0.27)$, $(0.5, -0.45)$ or $(0.7, -0.63)$ such that the pair of characteristic roots in the regime of $R_t(\gamma) = 1$ is $(0.1, 0, 9)$, $(0.2, 0.9)$, $(0.4, 0.9)$, $(0.6, 0.9)$ or $(0.8, 0.9)$, respectively. For each replication, the value of a and b for the interval $[a, b]$ are set as the empirical 10th and 90th quantiles of the data sample, the critical value for LR_n is calculated by the bootstrap method in Section 3 with $J = 1{,}000$, and the critical value for LR_n^* is either calculated in the same way as the one for LR_n or taken as 15.18 according to Table 2 in Chan (1991).

Table 1 lists the rejection rates of LR_n and LR_n^* with different values of ψ and γ. The results for LR_n^* based on the bootstrapped critical value and Chan's (1991) critical value are denoted by LR_{1n}^* and LR_{2n}^*, respectively. The sizes of these tests correspond to the case $(\psi_1, \psi_2) = (0, 0)$. From Table 1, we find that the sizes of LR_n and LR_{1n}^* are close to their nominal levels, but the size of LR_{2n}^* is conservative. Although the power of all tests becomes larger as the two regimes for $R_t(\gamma) = 0$ and $R_t(\gamma) = 1$ are more distinguishing, the power of LR_{2n}^* is less than that of LR_n or LR_{1n}^* in all cases. This suggests that the bootstrapped critical values may be more precise than the critical values in Chan (1991) for the LR_n^* test. When the distance between r_L and r_U is small, LR_n is less powerful than LR_{1n}^*, and this power advantage grows as the distance between r_L and r_U becomes large. As we expected, this is because LR_n (or LR_n^*) is the LR test

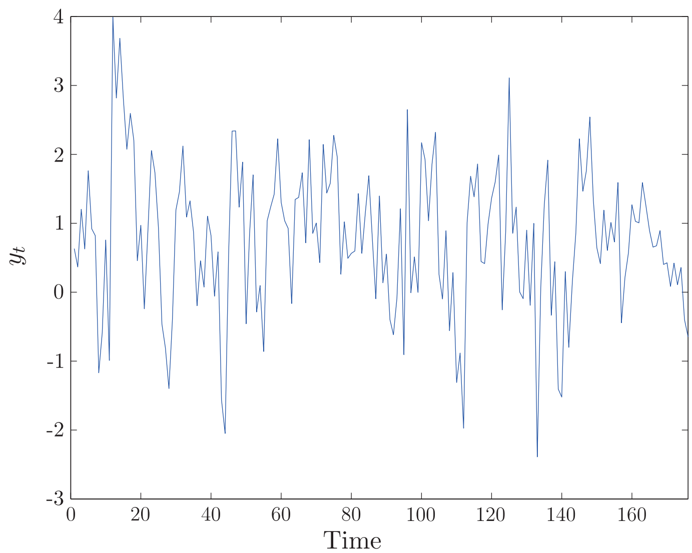

Figure 1. 100 times log-return of quarterly U.S. real GNP (in 1982 dollars) from the first quarter of 1947 to the first quarter of 1991.

when r_L and r_U are far from (or closed to) each other. The simulation results show that LR_n performs well, especially when the buffering region is wide.

Next, we study the quarterly U.S. real GNP (in 1982 dollars) from the first quarter of 1947 to the first quarter of 1991. The 100 times log-return, denoted by $\{y_t\}$, has a total of 176 observations; see Figure 1. We apply our test LR_n and the LR test LR_n^* in Chan (1990) to this data set. The results with different values of p and d are reported in Table 2. From Table 2, we find that a marginal threshold effect can be detected at the 5% significance level in either the BAR or TAR model with $p = d = 2$. Our finding is consistent with those in Potter (1995) and Hansen (1996), in which they also detected a marginal threshold effect in the TAR model by using the sup-LM test. Hence, we fit $\{y_t\}$ by the following two specifications:

$$
\begin{cases}
y_t = \begin{cases} \underset{(0.1979)}{1.2211} + \underset{(0.1236)}{0.1597} y_{t-1} + \underset{(0.1656)}{0.4017} y_{t-2} + \varepsilon_t & \text{if } R_t = 1 \\ \underset{(0.1245)}{0.0704} + \underset{(0.0856)}{0.3754} y_{t-1} + \underset{(0.0954)}{0.3031} y_{t-2} + \varepsilon_t & \text{if } R_t = 0 \end{cases}, \\
\text{where} \\
R_t = \begin{cases} 1 & \text{if } y_{t-2} \le -0.617 \\ 0 & \text{if } y_{t-2} > 1.237 \\ R_{t-1} & \text{otherwise;} \end{cases}
\end{cases}
\tag{4.2}
$$

Table 2. Results of tests applied to data set $\{y_t\}^\dagger$.

		BAR model				TAR model			
p	d	LR_n	$c_{0.1}$	$c_{0.05}$	$c_{0.01}{}^\S$	LR_n^*	$c_{0.1}^*$	$c_{0.05}^*$	$c_{0.01}^*{}^\S$
1	1	4.29	13.66	16.51	23.29	4.29	9.69	11.79	18.58
2	1	9.08	17.97	22.07	30.76	5.83	14.57	17.75	24.92
2	2	21.08‡	18.53	21.36	29.58	13.69‡	12.47	14.52	18.82
3	1	7.18	20.88	23.93	31.63	6.46	15.60	19.10	26.02
3	2	18.15	21.34	24.62	31.70	13.84	14.59	16.70	21.92
3	3	14.38	20.07	23.67	32.50	8.16	17.02	20.83	30.15

†The value of a and b are set to be the 10th and 90th quantiles of $\{y_t\}$.

‡ The p-values for LR_n and LR_n^* are 0.053 and 0.064, respectively.

§ c_α (or c_α^*) is obtained by the bootstrap method in Section 3 with $J = 1{,}000$.

$$\begin{cases} y_t = \begin{cases} -0.4515 + 0.3924 y_{t-1} - 0.8379 y_{t-2} + \varepsilon_t & \text{if } R_t = 1 \\ \;(0.2620) \;\; (0.1400) \qquad\quad (0.2628) & \\ 0.3971 + 0.3241 y_{t-1} + 0.1822 y_{t-2} + \varepsilon_t & \text{if } R_t = 0 \\ (0.1503) \;\; (0.0845) \qquad\quad (0.1129) & \end{cases}, \\ \text{where} \\ R_t = \begin{cases} 1 & \text{if } y_{t-2} \le -0.008 \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{4.3}$$

Models (4.2) and (4.3) are estimated by the least squares method (standard errors are in parentheses, and estimated values of σ_ε^2 are 0.85 and 0.90, respectively). For model (4.2), the first 20 autocorrelations or partial autocorrelations of the residuals $\{\hat{\varepsilon}_t\}$ or $\{\hat{\varepsilon}_t^2\}$ are not significant at the 5% level; see Figure 2. Similar results hold for model (4.3), and they are not reported here. This suggests that both models are adequate to fit $\{y_t\}$. The values of log-likelihood for models (4.2) and (4.3) are -233.1 and -237.3, respectively, and hence a BAR(2) model is more suitable than TAR(2) model to fit $\{y_t\}$.

Models (4.2) and (4.3) basically tell us different stories. Following Tiao and Tsay (1994), if we treat a negative growth in GNP as ‘contraction’ and a positive growth as ‘expansion’, model (4.2) shows that the region of y_t does not shift unless we have experienced a big ‘contraction’ or ‘expansion’ two years before, while model (4.3) indicates that the region of y_t almost fully relies on the kind of economic status that we have at that time. Society or government may not have a large or quick response to a moderate growth in GNP, and hence the region of y_t is most likely unchanged in this case. Thus, based on these facts, it is fair to conclude that a BAR(2) model is more reasonable than TAR(2) model to fit $\{y_t\}$.

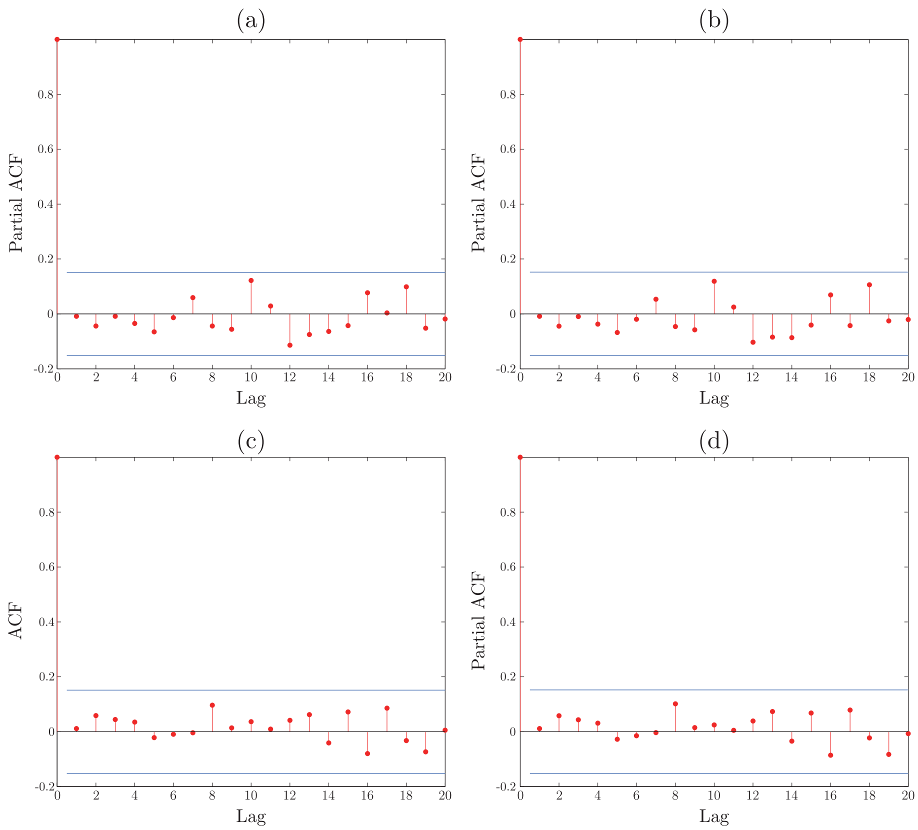

Figure 2. (a) the autocorrelations for $\{\hat{\varepsilon}_t\}$; (b) the partial autocorrelations for $\{\hat{\varepsilon}_t\}$; (c) the autocorrelations for $\{\hat{\varepsilon}_t^2\}$; and (d) the partial autocorrelations for $\{\hat{\varepsilon}_t^2\}$.

In the end, it is also of interest to fit $\{y_t\}$ by a three-regime TAR model:

$$y_t = \begin{cases} -0.4969 + 0.3735 y_{t-1} - 0.8500 y_{t-2} + \varepsilon_t \text{ if } y_{t-2} \leq -0.288 \\ \;(0.3649)\;\;(0.1399)\qquad\;(0.3193) \\ -3.3614 + 1.1691 y_{t-1} - 15.872 y_{t-2} + \varepsilon_t \text{ if } -0.288 < y_{t-2} \leq -0.058 \\ \;(1.2807)\;\;(1.0193)\qquad\;(4.3454) \\ 0.3837 + 0.3233 y_{t-1} + 0.1908 y_{t-2} + \varepsilon_t \quad \text{if } y_{t-2} > -0.058 \\ (0.1439)\;\;(0.0818)\qquad\;(0.1083) \end{cases}. \quad (4.4)$$

Model (4.4) is estimated by the least squares method (standard errors in parentheses, and the estimated value of σ_ε^2 is 0.84). Model (4.4) may also be adequate to fit $\{y_t\}$ by looking at the first 20 autocorrelations and partial autocorrelations of the residuals $\{\hat{\varepsilon}_t\}$ and $\{\hat{\varepsilon}_t^2\}$. However, the number of effective observations for these regimes from lower to upper are 25, 10, and 139, respectively. Thus, although the value of log-likelihood for model (4.4) is -231.6, greater than that for model (4.2), a model with two regimes for $\{y_t\}$ seems more likely. We prefer to fit $\{y_t\}$ by a BAR(2) model in view of this point.

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