Nonparametric estimate of the ruin probability in a pure-jump Lévy risk model

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Abstract

In this paper, we propose a nonparametric estimator of ruin probability in a Lévy risk model. The aggregate claims process $X = \{X_t, \geq 0\}$ is modeled by a pure-jump Lévy process. Assume that high-frequency observed data on X is available. The estimator is constructed based on Pollaczeck-Khinchine formula and Fourier transform. Risk bounds as well as a data-driven cut-off selection methodology are presented. Simulation studies are also given to show the finite sample performance of our estimator.

Keywords: Fourier (inversion) transform, Risk bound, Cut-off selection, Ruin probability.

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1. Introduction

The surplus process of an insurance company is modeled by the following process

$$U_t = u + ct - X_t, \tag{1.1}$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the constant premium rate. Here the aggregate claims process $X = \{X_t, t \geq 0\}$ with $X_0 = 0$ is a pure-jump Lévy process with characteristic function

$$\phi_{X_t}(\omega) = \mathbb{E}[exp(i\omega X_t)] = exp\left(t\int_{(0,\infty)}(e^{i\omega x} - 1)\nu(dx)\right),$$

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Preprint submitted to Insurance: Mathematics and Economics *April 8, 2013*

where ν is the Lévy measure on $(0, \infty)$ satisfying the condition $\mu_1 := \int_{(0,\infty)} x\nu(dx) < \infty$. Note that $\mu_1 < \infty$ implies that the process X has a finite mean. In fact, we have

$$\mathbb{E}X_1 = \int_{(0,\infty)} x\nu(dx) = \int_0^\infty \nu(x, \infty)dx,$$

where the second equality follows by integration by parts. The ruin probability is defined by

$$\psi(u) = \mathbb{P}\left(\inf_{0 \le t < \infty} U_t < 0 | U_0 = u\right).$$

In order to guarantee that ruin is not a certain event, we suppose the following condition holds.

Assumption *S The safety loading condition holds, i.e.* $c > \mu_1$.

In ruin theory, the study of ruin probability is one of the main topics for a long time (see e.g. Rolski et al. (1999) and Asmussen and Albrecher (2010)). The classical risk model plays the central role in the theoretical analysis in ruin theory, and lots of nice results have been obtained by actuarial researchers. However, due to the calculation complexity, it is hard to obtain exact closed-form expression for ruin probability in most specific situations. One extension of the classical risk model is the Lévy risk model, where the dynamics of the company's surplus is modeled by a Lévy process with only downward jumps. In the Lévy risk model, ruin related functions are usually expressed in terms of the scale functions, which are determined by the Laplace exponent of the process. See e.g. Section XI in Asmussen and Albrecher (2010). Note that the scale function is semi-explicit because it has to be expressed in terms of Laplace inversion.

Instead of following the analytic approach to analyze ruin probability, some researchers study it by statistical methods. See, for example, Frees (1986), Hipp (1989), Croux and Vervaerbeke (1990), Pitts (1994) and Politis (2003). Statistical methodology has some advantages over analytic and probabilistic methods. On the one hand, the model can be more general. For example, no specific structure on the claim size distribution is assumed. On the other hand, in practical situations, instead of knowing the specific model one can only obtain the data on the surplus. Thus, statistical methodology can be directly used to analyze the insurance's risk from the data. For more recent contributions on statistical estimate of the ruin probability, we refer the readers to Shimizu (2012), Masiello (2012) and Zhang et al. (2012).

In Masiello (2012) and Zhang et al. (2012), ruin probability for the classical risk model is estimated and the common key tool for estimation is the Pollaczeck-Khinchine formula. However, they use different approaches to treat the infinite sum of convolution powers in the Pollaczeck-Khinchine formula. In Masiello (2012), empirical distribution is used to estimate the convolution powers (see also Frees (1986)). Zhang et al. (2012) apply Fourier method to transform the infinite sum of convolutions to a single integral and then estimate the claim size distribution by kernel method. In this paper we will estimate the ruin probability in the pure-jump Lévy risk model (1.1) that includes the classical risk model as a special case. Note that in the Lévy risk model there may exist infinite number of jumps of small size in finite time interval. For example, consider a Lévy-Gamma risk model with $c = 50$ and $\nu(dx) = 20x^{-1}e^{-0.5x}\mathbf{1}_{(x>0)}dx$. For any $\epsilon > 0$, we have $\int_0^\epsilon \nu(dx) = \infty$, which implies that in any finite time interval the number of jumps of size less than ϵ is infinite with probability one. Figure 1 depicts a sample path of the Lévy-Gamma risk model. For an insurance company, if the surplus has lots of small fluctuations, it is not easy to capture the probability law of the inter-claim times. Hence, even if we can estimate the individual claim size distribution, it is still not convenient to estimate the ruin probability. One feasible way of dealing with this problem is to observe the surplus process (or the aggregate claims process) at some discrete time points and using the observed data to construct the estimator. Such a technique has been used by Shimizu (2009) to estimate the adjustment coefficient in a compound Poisson model with diffusion perturbation. In this paper, we assume that the premium rate c is known but the Lévy measure is unknown. Similar to Shimizu (2009), we assume that the aggregate claims process X is observed at discrete time points. We propose the estimator based on the Pollaczeck-Khinchine formula and Fourier transform.

The reminder of this paper is organized as follows. In Section 2, we present the detailed construction of the estimator $\hat{\psi}_m$ which is expressed via a function $\hat{\chi}_m$. In Section 3, we provide the risk bounds for $\hat{\chi}_m$ and $\hat{\psi}_m$. A data-driven strategy to choose the parameter m is given in Section 4. In Section 5, two simulation studies are presented to show the finite sample size performance of the estimator. Finally, some conclusions are given in Section 6.

2. The estimator

In the reminder of this paper, integrals without an indicated domain of integration are taken over the whole real line. Let L_1 and L_2 denote the classes of functions that are absolute integrable and square integrable, respectively. For $g \in L_1$ we denote its Fourier transform by

$$\phi_g(\omega) = \int e^{i\omega x} g(x)dx.$$

For a random variable Y we denote its characteristic function by $\phi_Y(\omega)$. Note that under some mild integrable conditions Fourier inversion transform gives

$$g(x) = \frac{1}{2\pi} \int e^{-i\omega x} \phi_g(\omega)d\omega.$$

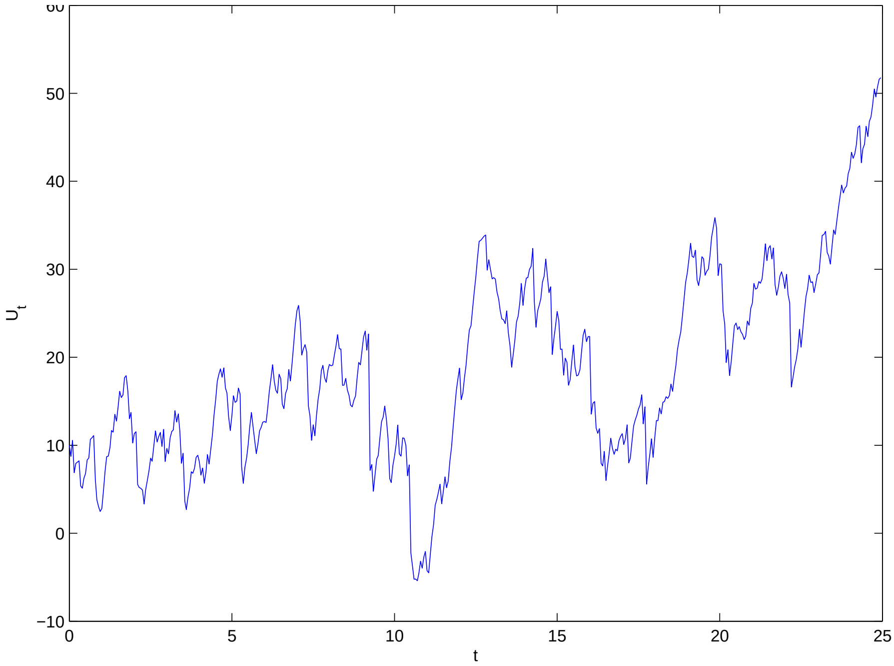

Figure 1: A sample path of the Lévy-Gamma risk model

The estimator we present is inspired by the Pollaczeck-Khinchine formula. Let

$$H(x) = \frac{1}{\mu_1} \int_0^x \nu(y, \infty) dy$$

with density $h(x) = \nu(x, \infty)/\mu_1$. Then the Pollaczeck-Khinchine type formula for ruin probability (see e.g. formula (1.3) in Huzak et al. (2004)) is given by

$$\begin{aligned}
\psi(u) &= 1 - (1-\rho) \sum_{j=0}^{\infty} \rho^j H^{j*}(u) \\
&= \rho - (1-\rho) \sum_{j=1}^{\infty} \rho^j \int_0^u h^{j*}(x) dx \\
&= \rho - (1-\rho) \int_0^u \chi(x) dx, \qquad (2.1)
\end{aligned}$$

where $\rho = \mu_1/c$, $\chi(x) = \sum_{j=1}^{\infty} \rho^j h^{j*}(x)$ and the convolutions are defined as

$$H^{j*}(x) = \int_0^x H^{(j-1)*}(x-y) H(dy), \quad h^{j*}(x) = \int_0^x h^{(j-1)*}(x-y) h(y) dy$$

with $H^{1*}(x) = H(x)$ and $h^{1*}(x) = h(x)$.

It follows from (2.1) that we have to estimate the parameter ρ (or equivalently the mean $\mathbb{E}X_1$) and the function $\chi(x)$. Suppose that the process X can be observed at a sequence of discrete time points $\{k\Delta,\ k = 1, 2, \ldots\}$ with $\Delta > 0$ being the sampling interval. We present the solution procedure based on the following r.v.'s

$$Z_k = Z_k^{\Delta} = X_{k\Delta} - X_{(k-1)\Delta}, \quad k = 1, 2, \ldots, n.$$

Furthermore, it is assumed that the sampling interval $\Delta = \Delta_n$ tends to zero as n tends to infinity. Thus, our estimator will be presented based on high frequency data. Immediately, an unbiased estimator for ρ is given by

$$\hat{\rho} = \frac{1}{cn\Delta} \sum_{k=1}^{n} Z_k. \qquad (2.2)$$

Now we derive an alternative representation for $\chi(x)$ based on Fourier inversion transform. By integration by parts it is readily seen that

$$\phi_h(\omega) = \frac{1}{\mu_1} A(\omega),$$

where

$$A(\omega) = \int_0^\infty e^{i\omega x}\nu(x,\infty)dx = \int_0^\infty \frac{e^{i\omega x}-1}{i\omega}\nu(dx)$$

is the Fourier transform of $\nu(x,\infty)$. Standard property of Fourier transform implies that

$$\phi_\chi(\omega) = \int e^{i\omega x}\sum_{j=1}^{\infty}\rho^j h^{j*}(x)dx = \sum_{j=1}^{\infty}\rho^j\left(\phi_h(\omega)\right)^j = \frac{A(\omega)}{c-A(\omega)}.$$

Thus, Fourier inversion transform gives the following alternative representation for $\chi(x)$,

$$\chi(x) = \frac{1}{2\pi}\int e^{-i\omega x}\frac{A(\omega)}{c-A(\omega)}d\omega. \tag{2.3}$$

Remark 1. *The denominator* $c - A(\omega)$ *is bounded away from zero because by* $|e^{i\omega x}-1| \leq |\omega x|$ *we have*

$$|c-A(\omega)| \geq c - \int_0^\infty \left|\frac{e^{i\omega x}-1}{i\omega}\right|\nu(dx) \geq c-\mu_1 > 0$$

thanks to Assumption S.

It follows from (2.3) that in order to get an estimator for $\chi(x)$ we can firstly estimate $A(\omega)$. Note that $\{Z_k\}$ are i.i.d. with common characteristic function

$$\phi_Z(\omega) = exp\left(\Delta\int_0^\infty\left(e^{i\omega x}-1\right)\nu(dx)\right).$$

By inverting the above characteristic function we obtain

$$\int_0^\infty (e^{i\omega x}-1)\nu(dx) = \frac{1}{\Delta}Log\left(\phi_Z(\omega)\right),$$

where *Log* denotes the *distinguished logarithm* (see e.g. Theorem 7.6.2 in Chung(2001)). We remark that the *distinguished logarithm* is well defined because $\phi_Z(\omega)$ never vanishes (see Theorem 7.6.1 and 7.6.2 in Chung(2001)). Using the fact that $A(\omega) = \frac{1}{\Delta}\frac{Log(\phi_Z(\omega))}{i\omega}$, we know that a plausible estimator is

$$\frac{1}{\Delta}\frac{Log(\hat{\phi}_Z(\omega))}{i\omega},$$

where $\hat{\phi}_Z(\omega) = \frac{1}{n}\sum_{k=1}^n e^{i\omega Z_k}$ is the empirical characteristic function. However, on the one hand, the *distinguished logarithm* in the above formula is not well defined unless

$\hat{\phi}_Z(\omega)$ does not vanish; on the other hand, it is not preferable to deal with logarithm for numerical calculation.

In order to overcome this drawback, we follow a different approach. Write $A(\omega)$ in the following form,

$$A(\omega) = \frac{\phi_Z(\omega) - 1}{i\omega\Delta} + \frac{1}{i\omega\Delta}\left[Log(\phi_Z(\omega)) - (\phi_Z(\omega) - 1)\right].$$

Using the inequality $|e^{i\omega x} - 1| \leq |\omega x|$, we have $|\phi_Z(\omega) - 1| \leq |\omega|\Delta\mu_1$. Together with the inequality $|Log(1+z) - z| \leq |z|^2$ for $|z| < \frac{1}{2}$, we obtain

$$|Log(\phi_Z(\omega)) - (\phi_Z(\omega) - 1)| \leq (\omega\Delta\mu_1)^2, \tag{2.4}$$

provided that $\Delta|\omega|$ is small enough. Then if $\Delta|\omega| \to 0$, $[Log(\phi_Z(\omega)) - (\phi_Z(\omega) - 1)]/(i\omega\Delta)$ can be neglected, i.e.

$$A(\omega) \approx \frac{\phi_Z(\omega) - 1}{i\omega\Delta}.$$

Hence, we propose the following estimator for $A(\omega)$,

$$\hat{A}(\omega) = \frac{\hat{\phi}_Z(\omega) - 1}{i\omega\Delta}, \tag{2.5}$$

where for $\omega = 0$ (2.5) is interpreted as the limit $\hat{A}(0) := \frac{1}{n\Delta}\sum_{k=1}^{n} Z_k$.

Write $E_n(\omega) = \{|c - \hat{A}(\omega)| \geq (n\Delta)^{-\frac{1}{2}}\}$. Replacing $A(\omega)$ in (2.3) by $\hat{A}(\omega)$ gives the following estimator

$$\hat{\chi}(x) = \frac{1}{2\pi}\int e^{-i\omega x}\frac{\hat{A}(\omega)}{c - \hat{A}(\omega)}\mathbf{1}_{E_n(\omega)}d\omega, \tag{2.6}$$

where the indicator function $\mathbf{1}_{E_n(\omega)}$ is used to guarantee that the denominator is bounded away from zero. There is still no guarantee that the integral in (2.6) is finite. To deal with this problem, we consider the following cut-off modification of (2.6)

$$\hat{\chi}_m(x) = \frac{1}{2\pi}\int_{-m\pi}^{m\pi} e^{-i\omega x}\frac{\hat{A}(\omega)}{c - \hat{A}(\omega)}\mathbf{1}_{E_n(\omega)}d\omega, \tag{2.7}$$

where m is a positive cur-off parameter.

Finally, combining (2.1), (2.2) and (2.7) yields the following estimator for ruin probability

$$\begin{aligned} \hat{\psi}_m(u) &= \hat{\rho} - (1-\hat{\rho})\int_0^u \hat{\chi}_m(x)dx \qquad (2.8) \\ &= \hat{\rho} - \frac{1-\hat{\rho}}{2\pi}\int_{-m\pi}^{m\pi}\frac{1 - e^{-i\omega u}}{i\omega}\frac{\hat{A}(\omega)}{c - \hat{A}(\omega)}\mathbf{1}_{E_n(\omega)}d\omega, \end{aligned}$$

where the second step follows from Fubini's theorem.

3. Risk bounds

Throughout this paper we denote by $\overline{v}$ the complex conjugate of v . For $v, v_1, v_2 \in L_1 \cap L_2$ let

$$\| v \|^2 = \int |v(x)|^2 dx, \quad \langle v_1, v_2 \rangle = \int v_1(x)\overline{v}_2(x)dx,$$

where $|v|^2 = v\overline{v}$. Note that $\langle v_1, v_2 \rangle = \frac{1}{2\pi}\langle \phi_{v_1}, \phi_{v_2} \rangle$. In particular, Parseval identity states that $\|v\|^2 = \frac{1}{2\pi}\|\phi_v\|^2$. Let C be a generic positive constant that can take different values from line to line.

To continue with, we need the following moment condition.

Assumption *H(k)* For integer k, $\mu_k := \int_0^\infty x^k \nu(dx) < \infty$.

We present a useful lemma that will be used frequently in the reminder of this paper.

Lemma 1. *Let $p \geq 1$ be an integer. Suppose that $n\Delta \to \infty$ and assumptions S and H(2p) hold. Then we have*

$$\mathbb{E}\left| \frac{\mathbf{1}_{E_n(\omega)}}{c - \hat{A}(\omega)} - \frac{1}{c - \mathbb{E}\hat{A}(\omega)} \right|^{2p} \leq \frac{C(n\Delta)^{-p}}{|c - \mathbb{E}\hat{A}(\omega)|^{4p}},$$

where the constant C does not depend on ω.

Proof. Firstly, we have

$$\begin{aligned} &\mathbb{E}\left| \frac{\mathbf{1}_{E_n(\omega)}}{c - \hat{A}(\omega)} - \frac{1}{c - \mathbb{E}\hat{A}(\omega)} \right|^{2p} \\ = &\frac{1}{\left|c - \mathbb{E}\hat{A}(\omega)\right|^{2p}} \mathbb{P}\left(E_n(\omega)^c\right) + \mathbb{E}\left[\mathbf{1}_{E_n(\omega)} \frac{|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{2p}}{|c - \hat{A}(\omega)|^{2p}|c - \mathbb{E}\hat{A}(\omega)|^{2p}}\right]. \end{aligned} \quad (3.1)$$

It is easily seen that the following equalities hold (see e.g. Proposition 2.2 in Comte and Genon-Catalot (2009)),

$$\mathbb{E}Z_1 = \Delta\mu_1, \quad \mathbb{E}Z_1^2 = \Delta\mu_2 + \Delta^2\mu_1^2. \quad (3.2)$$

By assumption S we have

$$|c - \mathbb{E}\hat{A}(\omega)| \geq c - \left|\frac{\phi_Z(\omega) - 1}{i\omega\Delta}\right| \geq c - \frac{1}{\Delta}\mathbb{E}|Z_1| = c - \mu_1 > 0,$$

where in the equality we have used the fact $Z_1 \geq 0$ a.s. because X is a subordinator. Thus, when n large enough we have $|c - \mathbb{E}\hat{A}(\omega)| > 2(n\Delta)^{-\frac{1}{2}}$. For such n we have

$$\begin{aligned}
& \mathbb{P}\left(E_n(\omega)^c\right) \\
\leq \;& \mathbb{P}\left(|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)| > |c - \mathbb{E}\hat{A}(\omega)| - (n\Delta)^{-\frac{1}{2}}\right) \\
\leq \;& \mathbb{P}\left(|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)| > \frac{1}{2}|c - \mathbb{E}\hat{A}(\omega)|\right) \\
\leq \;& \frac{2^{2p}}{|c - \mathbb{E}\hat{A}(\omega)|^{2p}} \mathbb{E}|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{2p},
\end{aligned} \tag{3.3}$$

where the last step follows from Markov's inequality.

By C_r inequality we have

$$\frac{1}{|c - \hat{A}(\omega)|^{2p}} \leq C \left(\frac{1}{\left|c - \mathbb{E}\hat{A}(\omega)\right|^{2p}} + \frac{|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{2p}}{|c - \hat{A}(\omega)|^{2p}|c - \mathbb{E}\hat{A}(\omega)|^{2p}} \right),$$

which leads to

$$\begin{aligned}
& \mathbb{E}\left[\mathbf{1}_{E_n(\omega)} \frac{|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{2p}}{|c - \hat{A}(\omega)|^{2p}|c - \mathbb{E}\hat{A}(\omega)|^{2p}}\right] \\
\leq \;& \frac{C}{|c - \mathbb{E}\hat{A}(\omega)|^{4p}} \left(\mathbb{E}|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{2p} + (n\Delta)^p \mathbb{E}|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{4p}\right).
\end{aligned} \tag{3.4}$$

Next, by Marcinkiewicz-Zygmund inequality we have

$$\begin{aligned}
\mathbb{E}|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{2p} &= \mathbb{E}\left|\frac{\hat{\phi}_Z(\omega) - \phi_Z(\omega)}{i\omega\Delta}\right|^{2p} \\
&\leq \frac{C}{(n\Delta)^{2p}} \left(\sum_{k=1}^{n} \mathbb{E}\left|\frac{e^{i\omega Z_k} - \phi_(\omega)}{\omega}\right|^2\right)^p \\
&= \frac{C}{n^p \Delta^{2p}} \left(\mathbb{E}\left|\frac{e^{i\omega Z_1} - \phi_Z(\omega)}{\omega}\right|^2\right)^p.
\end{aligned}$$

Using the inequality $|e^{i\omega x} - 1| \leq |\omega x|$ we obtain

$$\begin{aligned}
\mathbb{E}|e^{i\omega Z_1} - \phi_Z(\omega)|^2 &\leq 2\mathbb{E}|e^{i\omega Z_1} - 1|^2 + 2|1 - \phi_Z(\omega)|^2 \\
&\leq 2|\omega|^2(\mathbb{E}Z_1^2 + (\mathbb{E}Z_1)^2) = 2|\omega|^2(\mu_2\Delta + o(\Delta)),
\end{aligned}$$

where we have used (3.2). Then we have

$$\mathbb{E}|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{2p} \leq \frac{C}{(n\Delta)^p}. \tag{3.5}$$

Finally, by (3.1), (3.3)-(3.5) we have

$$\begin{aligned}
&\mathbb{E}\left|\frac{\mathbf{1}_{E_n(\omega)}}{c-\hat{A}(\omega)} - \frac{1}{c-\mathbb{E}\hat{A}(\omega)}\right|^{2p} \\
\leq\ & \frac{C}{|c-\mathbb{E}\hat{A}(\omega)|^{4p}}\left(\mathbb{E}|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{2p} + (n\Delta)^p\mathbb{E}|\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^{4p}\right) \\
\leq\ & \frac{C}{|c-\mathbb{E}\hat{A}(\omega)|^{4p}}(n\Delta)^{-p}.
\end{aligned}$$

This completes the proof. □

Now we derive risk bounds for $\hat{\chi}_m$. Write

$$\chi_m(x) = \frac{1}{2\pi}\int_{-m\pi}^{m\pi} e^{-i\omega x}\frac{A(\omega)}{c-A(\omega)}d\omega.$$

By Parseval's theorem and Pythagoras theorem we have

$$\begin{aligned}
\|\hat{\chi}_m - \chi\|^2 &= \frac{1}{2\pi}\|\phi_{\hat{\chi}_m} - \phi_\chi\|^2 \\
&= \frac{1}{2\pi}\|\phi_{\hat{\chi}_m} - \phi_{\chi_m} + \phi_{\chi_m} - \phi_\chi\|^2 \\
&= \frac{1}{2\pi}\|\phi_{\hat{\chi}_m} - \phi_{\chi_m}\|^2 + \frac{1}{2\pi}\|\phi_{\chi_m} - \phi_\chi\|^2 \\
&= \|\hat{\chi}_m - \chi_m\|^2 + \|\chi_m - \chi\|^2.
\end{aligned} \tag{3.6}$$

On the ground of (3.6) we can obtain the following result.

Proposition 1. *Suppose that assumptions S and H(4) hold. For fixed positive constant m we have*

$$\mathbb{E}\|\hat{\chi}_m - \chi\|^2 \leq \|\chi_m - \chi\|^2 + C\left(\frac{m}{n\Delta} + \frac{m}{(n\Delta)^2} + \int_{-m\pi}^{m\pi}|\mathbb{E}\hat{A}(\omega) - A(\omega)|^2 d\omega\right). \tag{3.7}$$

Proof. It follows from (3.6) that we only need to study $\mathbb{E}\|\hat{\chi}_m - \chi_m\|^2$. Note that

$$\phi_{\chi_m}(\omega) = \frac{A(\omega)}{c-A(\omega)}\mathbf{1}_{[-m\pi,m\pi]}(\omega), \quad \phi_{\hat{\chi}_m}(\omega) = \frac{\hat{A}(\omega)}{c-\hat{A}(\omega)}\mathbf{1}_{\{E_n(\omega),\ \omega\in[-m\pi,m\pi]\}}.$$

We can write

$$\hat{\chi}_m(x) - \chi_m(x) = \frac{1}{2\pi} \sum_{j=1}^{5} \int_{-m\pi}^{m\pi} e^{-i\omega x} T_j(\omega) d\omega, \tag{3.8}$$

where

$$\begin{aligned}
T_1(\omega) &= \frac{c}{[c - \mathbb{E}\hat{A}(\omega)]^2} (\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)), \\
T_2(\omega) &= (\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)) \left(\frac{\mathbf{1}_{E_n(\omega)}}{c - \hat{A}(\omega)} - \frac{1}{c - \mathbb{E}\hat{A}(\omega)} \right), \\
T_3(\omega) &= \frac{\mathbb{E}\hat{A}(\omega)}{c - \mathbb{E}\hat{A}(\omega)} (\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)) \left(\frac{\mathbf{1}_{E_n(\omega)}}{c - \hat{A}(\omega)} - \frac{1}{c - \mathbb{E}\hat{A}(\omega)} \right), \\
T_4(\omega) &= \left(\frac{\mathbb{E}\hat{A}(\omega)}{c - \mathbb{E}\hat{A}(\omega)} - \frac{A(\omega)}{c - A(\omega)} \right), \quad T_5(\omega) = -\frac{\mathbb{E}\hat{A}(\omega)}{c - \mathbb{E}\hat{A}(\omega)} \mathbf{1}_{E_n^c(\omega)}.
\end{aligned}$$

By Parseval's theorem we have

$$\|\hat{\chi}_m - \chi\|^2 \;=\; \frac{1}{2\pi} \int_{-m\pi}^{m\pi} \left| \sum_{j=1}^{5} T_j(\omega) \right|^2 d\omega \le \frac{5}{2\pi} \sum_{j=1}^{5} \int_{-m\pi}^{m\pi} |T_j(\omega)|^2 d\omega. \tag{3.9}$$

By (3.5) with $p = 1$, and the inequality $|c - \mathbb{E}\hat{A}(\omega)| \ge c - \mu_1$, we can obtain

$$\mathbb{E} \int_{-m\pi}^{m\pi} |T_1(\omega)|^2 d\omega \;\le\; C \frac{m}{n\Delta}.$$

By Cauchy-Schwarz inequality, Lemma 1 and (3.5) with $p = 2$ we can obtain

$$\begin{aligned}
\mathbb{E} \int_{-m\pi}^{m\pi} |T_2(\omega)|^2 d\omega &\le \int_{-m\pi}^{m\pi} \mathbb{E}^{\frac{1}{2}} |\hat{A}(\omega) - \mathbb{E}\hat{A}(\omega)|^4 \cdot \mathbb{E}^{\frac{1}{2}} \left| \frac{\mathbf{1}_{E_n(\omega)}}{c - \hat{A}(\omega)} - \frac{1}{c - \mathbb{E}\hat{A}(\omega)} \right|^4 d\omega \\
&\le C \frac{m}{(n\Delta)^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E} \int_{-m\pi}^{m\pi} |T_3(\omega)|^2 d\omega &\le C \frac{m}{(n\Delta)^2}, \\
\int_{-m\pi}^{m\pi} |T_4(\omega)|^2 d\omega &\le C \int_{-m\pi}^{m\pi} |\mathbb{E}\hat{A}(\omega) - A(\omega)|^2 d\omega.
\end{aligned}$$

By (3.3) and (3.5) with $p = 2$ we have

$$\mathbb{E}\int_{-m\pi}^{m\pi} |T_5(\omega)|^2 d\omega \leq C \int_{-m\pi}^{m\pi} \mathbb{P}(E_n^c(\omega))d\omega \leq C\frac{m}{(n\Delta)^2}.$$

After combining the above results we complete the proof. □

Now we study the convergence rate of the estimator $\hat{\chi}_m$ based on Proposition 1. We assume that the cut-off parameter m depending on n such that $m \to \infty$ as $n \to \infty$. We need the following assumption on the Levy measure.

Assumption *V* $A \in L_1 \cap L_2$, and for some $a, L > 0$

$$\int (1+\omega^2)^a \left|A(\omega)\right|^2 d\omega < L.$$

Proposition 2. *Suppose that assumptions S, H(4) and V hold and assume that* $\Delta \to 0$, $m\Delta \to 0$ *and* $n\Delta \to \infty$, *then*

$$\mathbb{E}\|\hat{\chi}_m - \chi\|^2 = O\left(m(n\Delta)^{-1} + m^{-2a}\right).$$

In particular, when $m = O((n\Delta)^{\frac{1}{2a+1}})$ *and* $n\Delta^{2a+2} \to 0$, *we have*

$$\mathbb{E}\|\hat{\chi}_m - \chi\|^2 = O\left((n\Delta)^{-\frac{2a}{2a+1}}\right).$$

Proof. Under the condition $m\Delta \to 0$, we know that $|\omega\Delta A(\omega)| \to 0$ uniformly for $\omega \in [-m\pi, m\pi]$. Using the inequality

$$|e^x - 1 - x| \leq x^2, \quad |x| < \frac{1}{2},$$

for n large enough we have

$$\begin{aligned}
\int_{-m\pi}^{m\pi} |\mathbb{E}\hat{A}(\omega) - A(\omega)|^2 d\omega &= \int_{-m\pi}^{m\pi} \frac{\left|e^{i\omega\Delta A(\omega)} - 1 - i\omega\Delta A(\omega)\right|^2}{|\omega\Delta|^2} d\omega \\
&\leq \Delta^2 \int_{-m\pi}^{m\pi} \omega^2 |A(\omega)|^4 d\omega \\
&\leq \mu_1^2 \Delta^2 (m\pi)^{2-2a} \int (1+\omega^2)^a |A(\omega)|^2 d\omega \\
&\leq C\Delta^2 m^{2-2a}.
\end{aligned}$$

By Parseval's theorem

$$\begin{aligned}
\|\chi_m - \chi\|^2 &= \frac{1}{2\pi} \parallel \phi_{\chi_m} - \phi_\chi \parallel^2 = \frac{1}{2\pi} \int_{|\omega|>m\pi} \left|\frac{A(\omega)}{c - A(\omega)}\right|^2 d\omega \\
&\leq \frac{\int_{|\omega|>m\pi} |A(\omega)|^2 d\omega}{2\pi(c-\mu_1)^2} \leq \frac{L}{2\pi(m\pi)^{2a}(c-\mu_1)^2}.
\end{aligned}$$

Thus, Proposition 1 gives

$$\mathbb{E}\|\hat{\chi}_m - \chi\|^2 = O\left(m(n\Delta)^{-1} + \Delta^2 m^{2-2a} + m^{-2a}\right) = O(m(n\Delta)^{-1} + m^{-2a}).$$

The remainder of the proof follows immediately. □

Remark 2. *Let us consider the convergence rate presented in Proposition 2. If X is a compound Poisson subordinator, then*

$$|A(\omega)| \leq \int_0^\infty \left|\frac{e^{i\omega x} - 1}{i\omega}\right| \nu(dx) \leq \int_0^\infty (x \wedge \frac{2}{|\omega|})\nu(dx) \leq \mu_1 \wedge \frac{2\nu(0,\infty)}{|\omega|},$$

which implies that in Assumption V we have to choose $0 < a < \frac{1}{2}$. When X is a Lévy-Gamma process with with parameters (α, β), then we have

$$\phi_Z(\omega) = \left(\frac{\alpha}{\alpha - i\omega}\right)^{\beta\Delta}.$$

It follows from the construction of the distinguish logarithm *(see Theorem 7.6.2 in Chung (2001)), we have*

$$|A(\omega)| = \left|\frac{1}{\Delta}\frac{Log(\hat{\phi}(\omega))}{i\omega}\right| \leq C(1 \wedge \frac{|\log|\omega||}{|\omega|})$$

for some constant $C > 0$. Again, we have to choose $0 < a < \frac{1}{2}$ in Assumption V. For these two examples, by Proposition 2 we have

$$\mathbb{E}\|\hat{\chi}_m - \chi\|^2 \leq C(n\Delta)^{-\frac{1}{2}(1-\epsilon)}$$

for any $0 < \epsilon < 1$.

We return to study the estimator $\hat{\psi}_m$. It follows from (2.1) and (2.8) that

$$\hat{\psi}_m(u) - \psi(u) \quad = \quad (\hat{\rho} - \rho)\left(1 - \int_0^u \chi(x)dx\right) - (1 - \hat{\rho})\int_0^u (\hat{\chi}_m(x) - \chi(x))dx.$$

Thus, we have

$$\begin{aligned}
\mathbb{E}|\hat{\psi}_m(u) - \psi(u)| &\leq \left(1 - \int_0^u \chi(x)dx\right)\mathbb{E}|\hat{\rho} - \rho| \\
&\quad + \mathbb{E}\left|(1 - \hat{\rho})\int_0^u (\hat{\chi}_m(x) - \chi(x))dx\right| \\
&\leq \left(1 - \int_0^u \chi(x)dx\right)\mathbb{E}^{\frac{1}{2}}(\hat{\rho} - \rho)^2 \\
&\quad + \mathbb{E}^{\frac{1}{2}}(1 - \hat{\rho})^2 \cdot \mathbb{E}^{\frac{1}{2}}\left(\int_0^u (\hat{\chi}_m(x) - \chi(x))dx\right)^2,
\end{aligned}$$

where Cauchy-Schwarz inequality is used in the second step. It follows from Theorem 2.1 in Conte and Genon-Catalot (2009) that

$$\mathbb{E}(\hat{\rho}-\rho)^2 = \frac{1}{c^2}\mathbb{E}\left(\frac{1}{n\Delta}\sum_{k=1}^{n}(Z_k - \mathbb{E}Z_1)\right)^2 \leq \frac{C}{n\Delta}.$$

Then

$$\mathbb{E}(1-\hat{\rho})^2 \leq 2(1-\rho)^2 + 2\mathbb{E}(\hat{\rho}-\rho)^2 \leq 2(1-\rho)^2 + \frac{C}{n\Delta}.$$

By Jensen's inequality,

$$\left(\int_0^u (\hat{\chi}_m(x) - \chi(x))dx\right)^2 \leq u\int_0^u (\hat{\chi}_m(x)-\chi(x))^2 dx \leq u\|\hat{\chi}_m - \chi\|^2.$$

Finally, combining above results gives

$$\mathbb{E}|\hat{\psi}_m(u) - \psi(u)| \leq C\left(\frac{1}{\sqrt{n\Delta}} + \sqrt{u}\mathbb{E}^{\frac{1}{2}}\|\hat{\chi}_m - \chi\|^2\right). \tag{3.10}$$

Theorem 1. *Suppose that the assumptions in Proposition 2 hold. Then for* $m = O((n\Delta)^{\frac{1}{2a+1}})$ *we have*

$$\mathbb{E}|\hat{\psi}_m(u) - \psi(u)| \leq C\sqrt{u}(n\Delta)^{-\frac{a}{2a+1}}. \tag{3.11}$$

Proof. The result follows from (3.10) and Proposition 2 with $m = O((n\Delta)^{\frac{1}{2a+1}})$, □

By Remark 2, when X is either a compound Poisson subordinator or a Lévy-Gamma subordinator, we can choose $0 < a < \frac{1}{2}$ in (3.11). Hence, in these two examples, we have

$$\mathbb{E}|\hat{\psi}_m(u) - \psi(u)| \leq C\sqrt{u}(n\Delta)^{-\frac{1}{4}(1-\epsilon)}, \quad 0 < \epsilon < 1. \tag{3.12}$$

Note that one drawback of the above risk bound is that it is an increasing function of u. However, simulation studies given in Section 5 show that the estimator also performs well for large initial surplus.

4. Cut-off selection

From Section 3 we know that the estimator depends heavily on the cut-off parameter m. In this section, we propose a data-driven strategy to choose m.

First, introduce

$$\zeta(x) = \frac{\sin(\pi x)}{\pi x}, \quad (with\ \zeta(0) = 1),$$

which has fourier transform $\mathbf{1}_{[-\pi,\pi]}(\omega)$. Define the following closed subset of L_2

$$S_m = \{v \in L_2, \quad supp(\phi_v) \subset [-m\pi, m\pi]\}.$$

It is well known that $\{\zeta_{m,j}\}_{j\in\mathbb{Z}}$, defined by

$$\zeta_{m,j}(x) = \sqrt{m}\zeta(mx - j), \quad \phi_{\zeta_{m,j}}(\omega) = \frac{e^{i\omega j/m}}{\sqrt{m}}\mathbf{1}_{[-m\pi,m\pi]}(\omega),$$

is an orthonormal basis of the space S_m. For $v \in L_2$, let v_m denote its orthogonal projection on S_m. Obviously, we have $\phi_{v_m} = \phi_v\mathbf{1}_{[-m\pi,m\pi]}$ and

$$v_m(x) = \sum_{j\in\mathbb{Z}} \langle v_m, \zeta_{m,j}\rangle \zeta_{m,j}(x),$$

where the inner product is given by

$$\langle v_m, \zeta_{m,j}\rangle = \frac{1}{2\pi}\int \phi_{\zeta_{m,j}}(-\omega)\phi_{v_m}(\omega)d\omega.$$

An alternative formula for $\hat{\chi}_m$ is given by

$$\hat{\chi}_m = \sum_{j\in\mathbb{Z}} \hat{a}_{m,j}\zeta_{m,j}, \tag{4.1}$$

where

$$\hat{a}_{m,j} = \frac{1}{2\pi}\int \phi_{\zeta_{m,j}}(-\omega)\frac{\hat{A}(\omega)}{c - \hat{A}(\omega)}\mathbf{1}_{E_n(\omega)}d\omega.$$

Note that Parseval's theorem gives

$$\| \hat{\chi}_m \|^2 = \sum_{j\in\mathbb{Z}} |\hat{a}_{m,j}|^2. \tag{4.2}$$

Also, we have

$$\chi_m = \sum_{j\in\mathbb{Z}} a_{m,j}\zeta_{m,j}$$

with

$$a_{m,j} = \frac{1}{2\pi}\int \phi_{\zeta_{m,j}}(-\omega)\frac{A(\omega)}{c - A(\omega)}d\omega.$$

For $v \in S_m$, define

$$\begin{aligned}
\gamma_n(v) &= \| v \|^2 - \frac{1}{\pi}\int \phi_v(-\omega)\frac{\hat{A}(\omega)}{c - \hat{A}(\omega)}\mathbf{1}_{E_n(\omega)}d\omega \\
&= \| v \|^2 - 2\langle \hat{\chi}_m, v\rangle \\
&= \| v - \hat{\chi}_m \|^2 - \| \hat{\chi}_m \|^2 .
\end{aligned}$$

Then we have

$$\hat{\chi}_m = \arg\min_{v\in S_m} \gamma_n(v)$$

and $\gamma_n(\hat{\chi}_m) = -\|\hat{\chi}_m\|^2$. Now consider a collection $(S_m, m = 1, 2, \ldots, m_n)$ where m_n is restricted to satisfy $m_n \le n\Delta$. Here we remak that the parameter m need not be integers and can be taken from a discrete set with a finer or larger step than 1.

We select adaptively the parameter m as follows:

$$\hat{m} = \arg\min_{m\in\{1,2,\ldots,m_n\}} \{\gamma_n(\hat{\chi}_m) + pen(m)\}, \tag{4.3}$$

where the penalty function pen is defined as

$$pen(m) = 96c^2 \frac{\mathbb{E}(Z_1^2/\Delta)}{(c - \mathbb{E}[Z_1/\Delta])^4} \frac{m}{n\Delta}.$$

The motivation of the above selection criterion is as follows. It follows from (3.6) that

$$\|\hat{\chi}_m - \chi\|^2 = \|\hat{\chi}_m - \chi_m\|^2 + \|\chi_m - \chi\|^2 = \|\hat{\chi}_m - \chi_m\|^2 + \|\chi\|^2 - \|\chi_m\|^2.$$

We estimate the bias term $-\|\chi_m\|^2$ (up to a constant $\|\chi\|^2$) by $\gamma_n(\hat{\chi}_m)$. Then a compromise is made between this term and the variance term $\mathbb{E}\|\hat{\chi}_m - \chi_m\|^2$ that is estimated by the penalty function $pen(m)$. Here we remark that the exact formula for the variance term is hard to obtain and the penalty function is only constructed by approximating the leading order of the variance.

Theorem 2. *Suppose that the assumptions S and H(4) hold. Then*

$$\begin{aligned} \mathbb{E} \parallel \hat{\chi}_{\hat{m}} - \chi \parallel^2 \quad \le \quad & \inf_{m\in\{1,2,\ldots,m_n\}} \big(3 \parallel \chi_m - \chi \parallel^2 + 4pen(m)\big) + \frac{C}{n\Delta} \\ & + C \int_{-m_n\pi}^{m_n\pi} |A(\omega) - \mathbb{E}\hat{A}(\omega)|^2 d\omega, \end{aligned}$$

where the constant C does not depend on n.

Proof. For $v_1, v_2 \in S_m$, we have

$$\begin{aligned} & \gamma_n(v_1) - \gamma_n(v_2) \\ = \ & \parallel v_1 - \chi \parallel^2 - \parallel v_2 - \chi \parallel^2 + 2\langle v_1 - v_2, \chi - \hat{\chi}_m\rangle \\ = \ & \parallel v_1 - \chi \parallel^2 - \parallel v_2 - \chi \parallel^2 - 2\sum_{j=1}^{5} R_{n,j}(v_1 - v_2), \end{aligned} \tag{4.4}$$

where, for $j = 1, 2, \ldots, 5$, $R_{n,j}(v) = \frac{1}{2\pi} \int \phi_v(-\omega) T_j(\omega) d\omega$ with T_j's defined in the proof of Proposition 1. Note that we have used the fact that $\langle v, \chi \rangle = \langle v, \chi_m \rangle$ for $v \in S_m$.

By the definition of $\hat{m}$, we have

$$\gamma_n(\hat{\chi}_{\hat{m}}) + pen(\hat{m}) \leq \gamma_n(\hat{\chi}_m) + pen(m) \leq \gamma_n(\chi_m) + pen(m).$$

Thus, using (4.4) we obtain

$$\begin{aligned}
& \|\hat{\chi}_{\hat{m}} - \chi\|^2 \\
\leq \quad & \|\chi_m - \chi\|^2 + pen(m) - pen(\hat{m}) + 2 \sum_{j=1}^{5} R_{n,j}(\hat{\chi}_{\hat{m}} - \chi_m) \\
\leq \quad & \|\chi_m - \chi\|^2 + pen(m) - pen(\hat{m}) + 2 \|\hat{\chi}_{\hat{m}} - \chi_m\| \sum_{j=1}^{5} \sup_{v \in B(m,\hat{m})} R_{n,j}(v),
\end{aligned}$$

where, for all m, m', $B(m, m') = \{v \in S_{m \vee m'}, \| \ v \ \| = 1\}$. Employing the inequalities $2xy \leq x^2/4 + 4y^2$ and $\|\hat{\chi}_{\hat{m}} - \chi_m\|^2 \leq 2\|\hat{\chi}_{\hat{m}} - \chi\|^2 + 2\|\chi_m - \chi\|^2$, we have

$$\begin{aligned}
2 \parallel \hat{\chi}_{\hat{m}} - \chi_m \parallel \sum_{j=1}^{5} \sup_{v \in B(m,\hat{m})} R_{n,j}(v) \quad \leq \quad & \frac{1}{2} \parallel \hat{\chi}_{\hat{m}} - \chi \parallel^2 + \frac{1}{2} \parallel \chi_m - \chi \parallel^2 \\
& + 24 \sum_{j=1}^{5} \sup_{v \in B(m,\hat{m})} R_{n,j}^2(v).
\end{aligned}$$

Combining above results we find

$$\begin{aligned}
\|\hat{\chi}_{\hat{m}} - \chi\|^2 \quad \leq \quad & 3 \parallel \chi_m - \chi \parallel^2 + 2pen(m) - 2pen(\hat{m}) + 48 \sum_{j=1}^{5} \sup_{v \in B(m,\hat{m})} R_{n,j}^2(v) \\
\leq \quad & 3 \parallel \chi_m - \chi \parallel^2 + 2pen(m) - 2pen(\hat{m}) \\
& + 48 \left(\sup_{v \in B(m,\hat{m})} R_{n,1}^2(v) - p(m, \hat{m}) \right)_+ \\
& + 48 \sum_{j=2}^{5} \sup_{v \in B(m,\hat{m})} R_{n,j}^2(v) + 48 p(m, \hat{m}),
\end{aligned}$$

where the functions $p(\cdot, \cdot)$, $j = 1, 2$, are defined in Lemma 2 in Appendix A. It is easily seen that

$$24 p(m, m') \leq pen(m) + pen(m') \tag{4.5}$$

for all m and m'. Then we have

$$\begin{aligned}
\mathbb{E}\|\hat{\chi}_{\hat{m}} - \chi\|^2 &\leq 3 \parallel \chi_m - \chi \parallel^2 + 4pen(m) \\
&\quad + 48\mathbb{E}\left(\sup_{v \in B(m,\hat{m})} R_{n,1}^2(v) - p(m,\hat{m}) \right)_+ \\
&\quad + 48 \sum_{j=2}^{5} \mathbb{E} \sup_{v \in B(m,\hat{m})} R_{n,j}^2(v). \qquad (4.6)
\end{aligned}$$

We will study the terms on the right hand side of (4.6) one by one. By Lemma 2 we have

$$\mathbb{E}\left(\sup_{v \in B(m,\hat{m})} R_{n,1}^2(v) - p(m,\hat{m}) \right)_+ \leq \frac{C}{n\Delta}.$$

For $j = 2, 3, 4, 5$, by Cauchy-Schwarz inequality and Parseval's theorem we obtain

$$\begin{aligned}
\sup_{v \in B(m,\hat{m})} R_{n,j}^2(v) &\leq \frac{1}{4\pi^2} \left(\sup_{v \in B(m,\hat{m})} \int |\phi_v(-\omega)|^2 d\omega \right) \left(\int_{-m_n\pi}^{m_n\pi} |T_j(\omega)|^2 d\omega \right) \\
&= \frac{1}{2\pi} \int_{-m_n\pi}^{m_n\pi} |T_j(\omega)|^2 d\omega.
\end{aligned}$$

Then using the results given in the proof of Proposition 1 we have for $j = 2, 3, 5$

$$\mathbb{E}\left(\sup_{v \in B(m,\hat{m})} R_{n,j}^2(v) \right) \leq \mathbb{E}\frac{1}{2\pi} \int_{-m_n\pi}^{m_n\pi} |T_j(\omega)|^2 d\omega \leq C\frac{m_n}{(n\Delta)^2} \leq \frac{C}{n\Delta}$$

thanks to $m_n \leq n\Delta$. Finally, for $j = 4$ we have

$$\mathbb{E}\left(\sup_{v \in B(m,\hat{m})} R_{n,4}^2(v) \right) \leq \mathbb{E}\frac{1}{2\pi} \int_{-m_n\pi}^{m_n\pi} |T_4(\omega)|^2 d\omega \leq C \int_{-m_n\pi}^{m_n\pi} |A(\omega) - \mathbb{E}\hat{A}(\omega)|^2 d\omega.$$

Thus, the proof is complete. $\square$

We can not use (4.3) directly to determine m because the penalty function is still unknown. To this end, we replace the theoretical penalty function by a empirical type

$$pem^*(m) = \begin{cases} 96c^2 \frac{\frac{1}{n\Delta}\sum_{j=1}^n Z_j^2}{\left(c - \frac{1}{n\Delta}\sum_{j=1}^n Z_j\right)^4} \frac{m}{n\Delta}, & \text{if } \left| c - \frac{1}{n\Delta}\sum_{j=1}^n Z_j \right| \geq \epsilon_n, \\ \frac{m}{n\Delta}, & \text{if } \left| c - \frac{1}{n\Delta}\sum_{j=1}^n Z_j \right| < \epsilon_n, \end{cases} \qquad (4.7)$$

where $0 < \epsilon_n < 1$ and $\epsilon_n \to 0$ as $n \to \infty$. We select adaptively the parameter m as follows:

$$\hat{m}^* = \arg \min_{m \in \{1,2,\ldots,m_n\}} \{\gamma_n(\hat{\chi}_m) + pen^*(m)\}. \qquad (4.8)$$

The threshold parameter ϵ_n is just used to study the risk bounds (see the proof of Theorem 3). In practical applications, we can set it to be a fixed constant as small as enough by hand.

Theorem 3. *Suppose that assumptions S and H(8) hold. Then*

$$\begin{aligned}\mathbb{E} \parallel \hat{\chi}_{\hat{m}^*} - \chi \parallel^2 \quad &\leq \quad \inf_{m \in \{1,2,\ldots,m_n\}} \big(3 \parallel \chi_m - \chi \parallel^2 + 4.05 pen(m)\big) + \frac{C}{n\Delta} \\ &\quad + C \int_{-m_n \pi}^{m_n \pi} |A(\omega) - \mathbb{E}\hat{A}(\omega)|^2 d\omega,\end{aligned}$$

where the constant C does not depend on n.

Proof. Let

$$\Omega_1 = \left\{ \left| \frac{c - \frac{1}{n\Delta} \sum_{j=1}^n Z_j}{c - \mathbb{E}[Z_1/\Delta]} - 1 \right| \leq a_1 \right\}, \quad \Omega_2 = \left\{ \left| \frac{\frac{1}{n} \sum_{j=1}^n Z_j^2}{\mathbb{E} Z_1^2} - 1 \right| \leq a_2 \right\},$$
$$\Omega_3 = \left\{ \left| c - \frac{1}{n\Delta} \sum_{j=1}^n Z_j \right| \geq \epsilon_n \right\},$$

where $0 < a_1, a_2 < 1$. Set $\Omega = \cap_{j=1}^3 \Omega_j$. Define

$$f(x, y) = \frac{96 c^2 y}{(c - x)^4}, \quad x, y > 0.$$

Then we have

$$pen(m) = f(\mathbb{E}[Z_1/\Delta],\ \mathbb{E}[Z_1^2/\Delta]) \frac{m}{n\Delta},$$

and on Ω,

$$pen^*(m) = f(\frac{1}{n\Delta} \sum_{j=1}^n Z_j,\ \frac{1}{n\Delta} \sum_{j=1}^n Z_j^2) \frac{m}{n\Delta}.$$

Note that on Ω_1

$$(1 - a_1)(c - \mathbb{E}[Z_1/\Delta]) \leq c - \frac{1}{n\Delta} \sum_{j=1}^n Z_j \leq (1 + a_1)(c - \mathbb{E}[Z_1/\Delta]),$$

and on Ω_2

$$(1 - a_2)\mathbb{E}[Z_1^2/\Delta] \leq \frac{1}{n\Delta} \sum_{j=1}^n Z_j^2 \leq (1 + a_2)\mathbb{E}[Z_1^2/\Delta].$$

Using these results with $a_1 = a_2 = 0.01$ we find that on Ω

$$0.94pen(m) \leq pen^*(m) \leq 1.08pen(m) \tag{4.9}$$

for all $m > 0$.

Applying the same arguments as in the proof of Theorem 2, we know that on Ω

$$\begin{aligned} \| \hat{\chi}_{\hat{m}^*} - \chi \|^2 \quad \leq \quad & 3 \| \chi_m - \chi \|^2 + 2pen^*(m) - 2pen^*(\hat{m}^*) \\ & +48 \left(\sup_{v \in B(m,\hat{m}^*)} R_{n,1}^2(v) - p^*(m,\hat{m}^*) \right)_+ \\ & +48 \sum_{j=2}^{5} \sup_{v \in B(m,\hat{m}^*)} R_{n,j}^2(v) + 48p^*(m,\hat{m}^*), \end{aligned}$$

where the functions $p^*(\cdot,\cdot) = 0.94p(\cdot,\cdot)$. After applying (4.5) and (4.9) to the above inequality we find that

$$\begin{aligned} \| \hat{\chi}_{\hat{m}^*} - \chi \|^2 \mathbf{1}_\Omega \quad \leq \quad & 3 \| \chi_m - \chi \|^2 + 4.05pen(m) \\ & +48 \left(\sup_{v \in B(m,\hat{m}^*)} R_{n,1}^2(v) - p^*(m,\hat{m}^*) \right)_+ \mathbf{1}_\Omega \\ & +48 \sum_{j=2}^{5} \sup_{v \in B(m,\hat{m}^*)} R_{n,j}^2(v) \mathbf{1}_\Omega. \end{aligned}$$

Then similar to Theorem 2 we can prove that

$$\mathbb{E}[\| \hat{\chi}_{\hat{m}^*} - \chi \|^2 \mathbf{1}_\Omega] \leq \inf_{m \in \{1,2,\ldots,m_n\}} \big(3 \| \chi_m - \chi \|^2 + 4.05pen(m) \big) + \frac{C}{n\Delta}.$$

Now we bound the the expectation $\mathbb{E}[\| \hat{\chi}_{\hat{m}^*} - \chi \|^2 \mathbf{1}_{\Omega^c}]$. Firstly, note that

$$\mathbb{P}(\Omega^c) \leq \sum_{j=1}^{3} \mathbb{P}(\Omega_j^c).$$

By Markov's inequality and Theorem 2.1 in Comte and Genon-Catalot (2009) we have

$$\mathbb{P}(\Omega_2^c) \quad \leq \quad \frac{1}{\big(a_2 \mathbb{E}[Z_1^2/\Delta]\big)^{2p}} \mathbb{E} \left(\frac{1}{n\Delta} \sum_{j=1}^{n} Z_j^2 - \mathbb{E}[Z_1^2/\Delta] \right)^{2p} \leq \frac{C}{(n\Delta)^p}$$

Similarly, $\mathbb{P}(\Omega_1^c) \leq \frac{C}{(n\Delta)^p}$. Because $\epsilon_n \to 0$, we have $\Omega_3^c \subseteq \Omega_1^c$ for large n. Thus, we also have $\mathbb{P}(\Omega_3^c) \leq \frac{C}{n\Delta}$. From these results we conclude that

$$\mathbb{P}(\Omega^c) \leq \frac{C}{(n\Delta)^p}. \tag{4.10}$$

Note that

$$\| \hat{\chi}_{\hat{m}^*} - \chi \|^2 = \| \hat{\chi}_{\hat{m}^*} - \chi_{\hat{m}^*} \|^2 + \| \chi_{\hat{m}^*} - \chi \|^2 \leq \| \hat{\chi}_{\hat{m}^*} - \chi_{\hat{m}^*} \|^2 + \| \chi \|^2$$

By (4.10) with $p = 1$ we obtain $\mathbb{E}[\| \chi \|^2 \mathbf{1}_{\Omega^c}] \leq \frac{C}{n\Delta}$. By Cauchy-Schwarz inequality we have

$$\mathbb{E}[\| \hat{\chi}_{\hat{m}^*} - \chi_{\hat{m}^*} \|^2 \mathbf{1}_{\Omega^c}] \leq \mathbb{E}^{\frac{1}{2}}[\| \hat{\chi}_{\hat{m}^*} - \chi_{\hat{m}^*} \|^4]\mathbb{P}^{\frac{1}{2}}(\Omega^c). \tag{4.11}$$

By Cauchy-Schwarz inequality and Lemma 1 we have

$$\begin{aligned}
\mathbb{E} \| \hat{\chi}_{\hat{m}^*} - \chi_{\hat{m}^*} \|^4 &\leq \mathbb{E}\left(\frac{1}{2\pi}\int_{-m_n\pi}^{m_n\pi}\left|\frac{\hat{A}(\omega)}{c-\hat{A}(\omega)}\mathbf{1}_{E_n(\omega)} - \frac{A(\omega)}{c-A(\omega)}\right|^2 d\omega\right)^2 \\
&\leq \frac{m_n}{2\pi}\int_{-m_n\pi}^{m_n\pi}\mathbb{E}\left|\frac{\hat{A}(\omega)}{c-\hat{A}(\omega)}\mathbf{1}_{E_n(\omega)} - \frac{A(\omega)}{c-A(\omega)}\right|^4 d\omega \\
&\leq C\frac{m_n^2}{(n\Delta)^2} = O(1)
\end{aligned}$$

thanks to $m_n \leq n\Delta$. Then by (4.10) with $p = 2$ and (4.11) we get

$$\mathbb{E}[\| \hat{\chi}_{\hat{m}^*} - \chi_{\hat{m}^*} \|^2 \mathbf{1}_{\Omega^c}] \leq \frac{C}{n\Delta}.$$

Hence, we have proved that

$$\mathbb{E}[\| \hat{\chi}_{\hat{m}^*} - \chi \|^2 \mathbf{1}_{\Omega^c}] \leq \frac{C}{n\Delta}.$$

This completes the proof. □

Under the conditions in Proposition 2, we have

$$\int_{-m_n\pi}^{m_n\pi} |A(\omega) - \mathbb{E}\hat{A}(\omega)|^2 d\omega = o(m_n^{-2a}),$$

and consequently Theorem 2 yields

$$\mathbb{E} \| \hat{\chi}_{\hat{m}^*} - \chi \|^2 \leq \inf_{m\in\{1,2,\ldots,m_n\}} \big(3 \| \chi_m - \chi \|^2 + 4.05 pen(m)\big) + C((n\Delta)^{-1} + m_n^{-2a}).$$

5. Simulation studies

In this section, we provide two simulation studies to show the performance of our estimator with finite sample size.

We first describe the calculation procedure. Instead of using formula (2.7), we will use (4.1) to calculate $\hat{\chi}_m$. The cut-off parameter m is selected base on the strategy given in (4.8). Since we can only compute a finite number of $\hat{a}_{m,j}$'s, we truncate the infinite sum in (4.1) by a sufficiently large integer K_n, i.e. we use the following approximation

$$\hat{\chi}_m(x) \approx \sum_{|j|\leq K_n} \hat{a}_{m,j}\zeta_{m,j}(x), \tag{5.1}$$

where the coefficients $\hat{a}_{m,j}$ are calculated by IFFT. We remark that this approximation has little affect on the estimator, and at least it does not change the rate of convergence. We refer the readers to Comte et al. (2006) for theoretical arguments on such truncation. Now we summarize the solution steps as follows.

- Apply IFFT to compute the coefficients $\hat{a}_{m,j}$ for $m = 1, 2, \ldots, n\Delta$ and $j = -K_n, -K_n + 1, \ldots, 1, \ldots, K_n - 1, K_n$;
- For each $m \in \{1, 2, \ldots, n\Delta\}$, compute $\gamma_n(\hat{\chi}_m) + pen^*(m)$;
- Choose $\hat{m}^*$ according to (4.8);
- Compute $\hat{\rho}$ by (2.2);
- Apply (2.8) to obtain $\hat{\psi}_{\hat{m}^*}(u) \approx \hat{\rho} - (1-\hat{\rho})\sum_{|j|\leq K_n} \hat{a}_{\hat{m}^*,j}\int_0^u \zeta_{\hat{m}^*,j}(x)dx$.

Now, we consider two specific cases of X. One is a compound Poisson process, and the other is a Lévy-Gamma process. In the following two examples, the truncation parameter $K_n = 2^{16} - 1$.

Example 1 (Compound Poisson Process). *Assume that the Lévy measure is given by $\nu(dx) = 20e^{-x}dx$. Then X is a compound Poisson process where the Poison intensity is 20 and the individual claim sizes are exponentially distributed with mean 1. Set the premium rate $c = 25$.*

Example 2 (Lévy-Gamma Process). *Assume that the Lévy measure is given by $\nu(dx) = 20x^{-1}e^{-0.5x}\mathbf{1}_{(x>0)}dx$. Then X is a Lévy-Gamma process with $\mathbb{E}X_1 = 40$. Set the premium rate $c = 50$.*

In example 1, the true ruin probability is $\psi(u) = 0.8e^{-0.2u}$. In example 2, we use IFFT to approximate the run probability and then compare the estimator with this approximation. Firstly, for each example, 20 estimated curves are given in Figure 2 and Figure

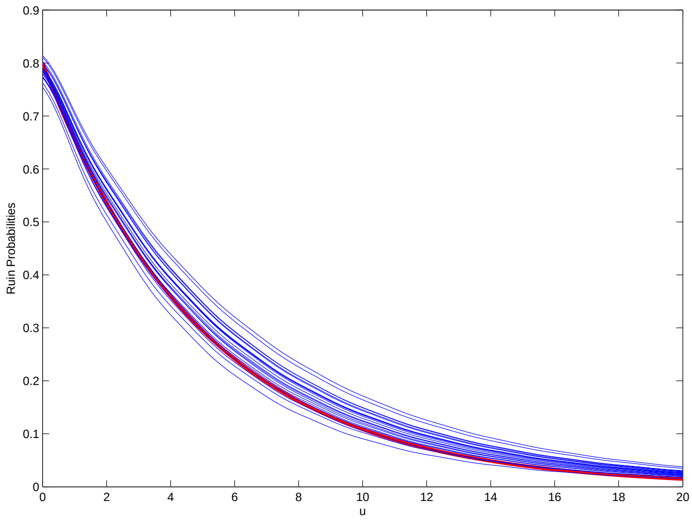

Figure 2: Estimation of the ruin probability in the compound Poisson risk model with exponential claim sizes. True ruin probability (red line) and 20 estimated curves (blue lines). Sample size $n = 40000$, sampling interval $\Delta = 0.005$.

4, respectively, where we set $n = 40000$, $\Delta = 0.005$. In each example, we find the little variability of the estimator. Next, we study the impact of the sample size n. We consider three cases: (1) $n = 5000$, $\Delta = 0.02$ ($n\Delta = 100$); (2) $n = 15000$, $\Delta = 0.01$ ($n\Delta = 150$); (3) $n = 40000$, $\Delta = 0.005$ ($n\Delta = 200$). In each case, 1000 experiments are performed. We plot the means in Figure 3 and Figure 5 based on the 1000 estimated curves. As is expected, the results improve as the sample size increases. Finally, we compute the mean squared errors ($\widehat{MSE}$) and present some results in Table 1. The results are computed based on the above 1000 experiments. Again, we find that for fixed initial surplus the mean squared errors decrease w.r.t. the sample size. We also observe that the problem is easier for smaller or larger initial surplus. This may be due to the fact that the curve of the ruin probability has smaller curvature when the initial surplus is smaller or larger.

6. Conclusions

In this paper we present a nonparametric estimator of the ruin probability in a pure-jump Lévy risk model. By high-frequency observation of the aggregate claims process, we use Pollaczeck-Khinchine formula and Fourier transform to construct the estimator.

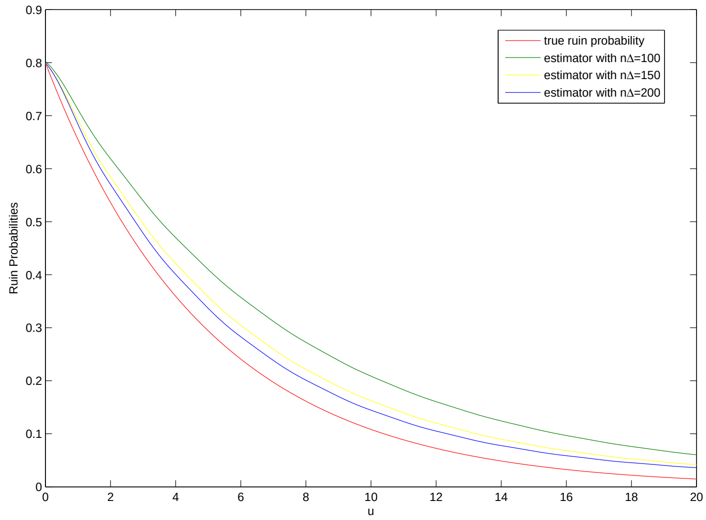

Figure 3: Estimation of the ruin probability in the compound Poisson risk model with exponential claim sizes. Sample size $n = 5000, 15000, 40000$, sampling interval $\Delta = 0.02, 0.01, 0.005$.

	$\widehat{MSE}$ (Compound Poisson)			$\widehat{MSE}$ (Lévy-Gamma)		
u	$n\Delta = 100$	$n\Delta = 150$	$n\Delta = 200$	$n\Delta = 100$	$n\Delta = 150$	$n\Delta = 200$
0	0.0002938	0.0002470	0.0001885	0.0003682	0.0001897	0.0001738
5	0.0137545	0.0046167	0.0022288	0.0131979	0.0044545	0.0024117
10	0.0077125	0.0018827	0.0007069	0.0087679	0.0025835	0.0013761
15	0.0028048	0.0005094	0.0001587	0.0036593	0.0009388	0.0005172
20	0.0006815	7.9191e-5	3.5201e-5	0.0011041	0.0002243	0.0001337

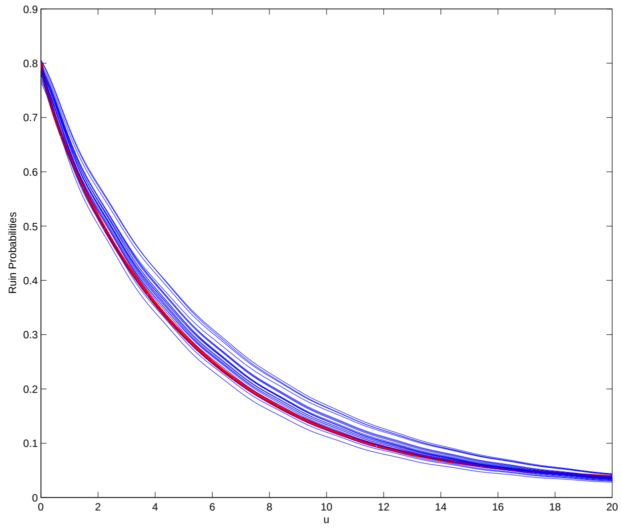

Figure 4: Estimation of the ruin probability in the Lévy-Gamma risk model. IFFT approximation to the ruin probability (red line) and 20 estimated curves (blue lines). Sample size $n = 40000$, sampling interval $\Delta = 0.005$.

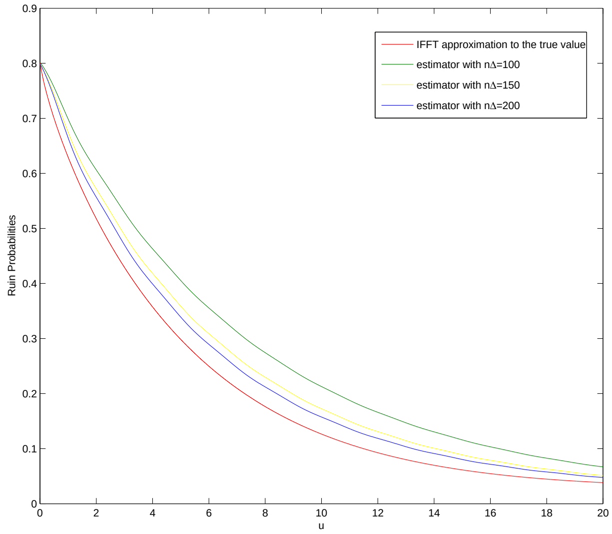

Figure 5: Estimation of the ruin probability in the compound Poisson risk model with exponential claim sizes. Sample size $n = 5000, 15000, 40000$, sampling interval $\Delta = 0.02, 0.01, 0.005$.

Risk bounds are given and an adaptive strategy to select the cut-off parameter m is also presented. Simulation studies show that the estimator performs well when the sample size is finite.

There are also some open problems for further study, for example, how to estimate the ruin probability in a more general Lévy risk model and how to estimate other risk measures such as the discounted penalty function of the surplus before ruin and the deficit at ruin.

Appendix A. Lemma and Proof

Lemma 2. *Define*

$$p(m, m') = 4c^2 \frac{\mathbb{E}(Z_1^2/\Delta)}{(c - \mathbb{E}[Z_1/\Delta])^4} \frac{m \vee m'}{n\Delta},$$

and suppose that assumptions S and H(4) hold. For $R_{n,1}$ defined in the proof of Theorem 2, we have

$$\mathbb{E}\left(\sup_{v \in B(m,\hat{m})} R_{n,1}^2(v) - p(m, \hat{m}) \right)_+ \leq \frac{C}{n\Delta}.$$

Proof. Write

$$\tau_1(\omega) = \mathbb{E}\left(\frac{e^{i\omega Z_1}-1}{i\omega\Delta}\mathbf{1}_{\{|Z_1|\le k_n\sqrt{\Delta}\}}\right), \quad \tau_2(\omega) = \mathbb{E}\left(\frac{e^{i\omega Z_1}-1}{i\omega\Delta}\mathbf{1}_{\{|Z_1|>k_n\sqrt{\Delta}\}}\right)$$

and let $\hat{\tau}_1(\omega)$ and $\hat{\tau}_2(\omega)$ be their empirical counterparts, where the constant k_n will be specified later. We decompose $R_{n,1}(v)$ as $R_{n,1}(v) = R_{n,1,1}(v) + R_{n,1,2}(v)$, where

$$R_{n,1,j}(v) = \frac{1}{2\pi}\int \frac{c\phi_v(-\omega)}{[c-\mathbb{E}\hat{A}(\omega)]^2}(\hat{\tau}_j(\omega)-\tau_j(\omega))d\omega, \quad j=1,2.$$

Then we have

$$\begin{aligned}\mathbb{E}\left(\sup_{v\in B(m,\hat{m})} R_{n,1}^2(v) - p(m,\hat{m})\right)_+ &\le \mathbb{E}\left(\sup_{v\in B(m,\hat{m})} R_{n,1,1}^2(v) - p(m,\hat{m})\right)_+ \\ &\quad + \mathbb{E}\left(\sup_{v\in B(m,\hat{m})} R_{n,1,2}^2(v)\right).\end{aligned}$$

We use Talagrand inequality to bound the first expectation on the right hand side of the above inequality. Write $R_{n,1,1}(v)$ in the following form

$$R_{n,1,1}(v) = \frac{1}{n}\sum_{j=1}^{n}\left(f_v(Z_j) - \mathbb{E}f_v(Z_j)\right),$$

where

$$f_v(z) = \frac{1}{2\pi}\int \frac{c\phi_v(-\omega)}{[c-\mathbb{E}\hat{A}(\omega)]^2}\frac{e^{i\omega z}-1}{i\omega\Delta}\mathbf{1}_{\{|z|\le k_n\sqrt{\Delta}\}}d\omega.$$

We need to specify the constants M_1, H_1, c_1 (see the Talagrand inequality in Appendix B).

By Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\sup_{|z|\in\mathbb{R}, v\in B(m,m')} |f_v(z)| &\le \frac{1}{2\pi}\left(\int |\phi_v(-\omega)|^2 d\omega\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{-m''\pi}^{m''\pi} \frac{c^2}{|c-\mathbb{E}\hat{A}(\omega)|^4}\left|\frac{e^{i\omega z}-1}{i\omega\Delta}\right|^2 \mathbf{1}_{\{|z|\le k_n\sqrt{\Delta}\}}d\omega\right)^{\frac{1}{2}} \\ &\le \frac{c}{\sqrt{2\pi}(c-\mu_1)^2}\left(\int_{-m''\pi}^{m''\pi}\left|\frac{e^{i\omega z}-1}{i\omega\Delta}\right|^2\mathbf{1}_{\{|z|\le k_n\sqrt{\Delta}\}}d\omega\right)^{\frac{1}{2}} \qquad \text{(A.1)} \\ &\le \frac{k_n\sqrt{m''}}{(c-\mu_1)^2\sqrt{\Delta}},\end{aligned}$$

where $m'' = m \vee m'$, and the last step follows from the inequality $|e^{i\omega z} - 1| \leq |\omega z|$. Thus, we can set

$$M_1 = \frac{k_n \sqrt{m''}}{(c-\mu_1)^2 \sqrt{\Delta}}.$$

Again, by Cauchy-Schwarz inequality we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{v\in B(m,m')} R^2_{n,1,1}(v)\right) &\leq \frac{1}{2\pi}\mathbb{E}\int_{-m''\pi}^{m''\pi} \frac{c^2}{\left|c - \mathbb{E}\hat{A}(\omega)\right|^4} \left|\hat{\tau}_1(\omega) - \mathbb{E}\hat{\tau}_1(\omega)\right|^2 d\omega \\
&\leq \frac{c^2}{2\pi(c-\mu_1)^4 n}\int_{-m''\pi}^{m''\pi} \mathbb{E}\left|\frac{e^{i\omega Z_1} - 1}{i\omega\Delta}\right|^2 d\omega \\
&\leq \frac{c^2 \mathbb{E}[Z_1^2/\Delta]}{(c-\mu_1)^4}\frac{m''}{n\Delta} = H_1^2.
\end{aligned}$$

Note that $\mathbb{E}[Z_1^2/\Delta]$ in the above formula is bounded.

By (A.1) we have

$$\begin{aligned}
\sup_{v\in B(m,m')} Var(f_v(Z_1)) &\leq \sup_{v\in B(m,m')} \mathbb{E}|f_v(Z_1)|^2 \\
&= \frac{c^2}{2\pi(c-\mu_1)^4}\int_{-m''\pi}^{m''\pi} \mathbb{E}\left|\frac{e^{i\omega Z_1} - 1}{i\omega\Delta}\right|^2 d\omega \\
&= \frac{c^2}{2\pi(c-\mu_1)^4}\left(\int_{|\omega|\leq \ln(m''\pi)} + \int_{\ln(m''\pi)<|\omega|\leq m''\pi} \mathbb{E}\left|\frac{e^{i\omega Z_1} - 1}{i\omega\Delta}\right|^2 d\omega\right).
\end{aligned}$$

For the first integral, using the inequality $|e^{i\omega x} - 1| \leq |\omega x|$ gives

$$\int_{|\omega|\leq \ln(m''\pi)} \mathbb{E}\left|\frac{e^{i\omega Z_1} - 1}{i\omega\Delta}\right|^2 d\omega \leq \frac{2\ln(m''\pi)\mathbb{E}(Z_1^2/\Delta)}{\Delta}.$$

While for the second integral we have

$$\begin{aligned}
\int_{\ln(m''\pi)<|\omega|\leq m''\pi} \mathbb{E}\left|\frac{e^{i\omega Z_1} - 1}{i\omega\Delta}\right|^2 d\omega &\leq \int_{\ln(m''\pi)<|\omega|\leq m''\pi} \frac{2}{\omega^2\Delta^2}\mathbb{E}|e^{i\omega Z_1} - 1| d\omega \\
&\leq \frac{4\mathbb{E}[Z_1/\Delta]}{\Delta}\int_{\ln(m''\pi)<\omega\leq m''\pi} \frac{1}{\omega} d\omega \\
&\leq \frac{4\mu_1 \ln(m''\pi)}{\Delta}.
\end{aligned}$$

Thus,

$$Var(f_v(Z_1)) \leq \frac{c^2(\mathbb{E}(Z_1^2/\Delta) + 2\mu_1)}{\pi(c-\mu_1)^4}\frac{\ln(m''\pi)}{\Delta} := c_1.$$

By Talagrand inequality with $\epsilon_1^2 = \frac{1}{2}$ we have

$$\mathbb{E}\left(\sup_{v\in B(m,m')} R_{n,1,1}^2(v) - p(m,m')\right)_+ \leq C\left(\frac{\ln m''}{n\Delta} e^{-C'm''/\ln m''} + \frac{k_n^2 m''}{n^2\Delta} e^{-C''\sqrt{n}/k_n}\right).$$

Choosing

$$k_n = \frac{C}{4}\frac{\sqrt{n}}{ln(n\Delta)}$$

and using the same arguments as in Proposition A.1 of Comte and Genon-Catalot (2009), we can obtain

$$\mathbb{E}\left(\sup_{v\in B(m,\hat{m})} R_{n,1}^2(v) - p(m,\hat{m})\right)_+ \leq \frac{C}{n\Delta}.$$

For $R_{n,1,2}(v)$ we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{v\in B(m,\hat{m})} R_{n,1,2}^2(v)\right) \\
\leq\ & \frac{1}{2\pi}\mathbb{E}\int_{-m_n\pi}^{m_n\pi} \frac{c^2}{|c-\mathbb{E}\hat{A}(\omega)|^4} |\hat{\tau}_2(\omega) - \mathbb{E}\hat{\tau}_2(\omega)|^2 d\omega \\
\leq\ & \frac{c^2}{2\pi(c-\mu_1)^4 n}\int_{-m_n\pi}^{m_n\pi} \mathbb{E}\left|\frac{e^{i\omega Z_1}-1}{i\omega\Delta}\mathbf{1}_{\{|Z_1|>k_n\sqrt{\Delta}\}}\right|^2 d\omega \\
\leq\ & \frac{c^2}{2\pi(c-\mu_1)^4 n\Delta^3 k_n^2}\left(\int_{|\omega|\leq \ln(m_n\pi)} + \int_{\ln(m_n\pi)<|\omega|\leq m_n\pi} \mathbb{E}\left|Z_1\frac{e^{i\omega Z_1}-1}{i\omega}\right|^2 d\omega\right)
\end{aligned}$$

Again, it is easily seen that

$$\int_{|\omega|\leq\ln(m_n\pi)} \mathbb{E}\left|Z_1\frac{e^{i\omega Z_1}-1}{i\omega}\right|^2 d\omega \leq 2\mathbb{E}[Z_1^4]\ln(m_n\pi)$$

and

$$\int_{\ln(m_n\pi)<|\omega|\leq m_n\pi} \mathbb{E}\left|Z_1\frac{e^{i\omega Z_1}-1}{i\omega}\right|^2 d\omega \leq 4\mathbb{E}[Z_1^3]\ln(m_n\pi).$$

Thus, with $k_n = \frac{C}{4}\frac{\sqrt{n}}{ln(n\Delta)}$, we have

$$\mathbb{E}\left(\sup_{v\in B(m,\hat{m})} R_{n,1,2}^2(v)\right) \leq \frac{c^2(\mathbb{E}[Z_1^4/\Delta] + 2\mathbb{E}[Z_1^3/\Delta])}{\pi(c-\mu_1)^4}\frac{\ln(m_n\pi)}{n\Delta^2 k_n^2} \leq C\frac{\ln^3(n\Delta)}{(n\Delta)^2} \leq \frac{C}{n\Delta}$$

thanks to $m_n \leq n\Delta$. Note that $\mathbb{E}[Z_1^3/\Delta]$ and $\mathbb{E}[Z_1^4/\Delta]$ are both bounded. This completes the proof.

□

Appendix B. Useful inequalities

Marcinkiewicz-Zygmund inequality. Let $(Y_j)_{j=1,\dots,n}$ be independent centered random variables, such that $\mathbb{E}|Y_j|^p < \infty$ for some integer $p \geq 1$. Then

$$B_p \mathbb{E}\left(\sum_{j=1}^{n} Y_j^2\right)^{p/2} \leq \mathbb{E}\left|\sum_{j=1}^{n} Y_j\right|^p \leq C_p \mathbb{E}\left(\sum_{j=1}^{n} Y_j^2\right)^{p/2}$$

where B_p and C_p are positive constants depending only on p. In particular, we can choose $C_p = (4+2p)^{p/2}$.

Talagrand Inequality. Let $(Y_j)_{j=1,\dots,n}$ be independent random variables and $\nu_{n,Y}(f) = \frac{1}{n}\sum_{j=1}^{n}[f(Y_j) - \mathbb{E}f(Y_j)]$ and let $\mathcal{F}$ be a countable class of uniformly bounded measurable functions. Then for $\epsilon_1^2 > 0$

$$\begin{aligned} & \mathbb{E}\left[\sup_{f\in\mathcal{F}} |\nu_{n,Y}(f)|^2 - 2(1+2\epsilon_1^2)H_1^2\right]_+ \\ \leq \; & \frac{4}{K_1}\left(\frac{c_1}{n} e^{-K_1\epsilon_1^2 \frac{nH_1^2}{c_1}} + \frac{98M_1^2}{K_1 n^2 Q^2(\epsilon_1^2)} e^{-\frac{2K_1 Q(\epsilon_1^2)\epsilon_1}{7\sqrt{2}} \frac{nH_1}{M_1}}\right), \end{aligned}$$

where $Q(\epsilon_1^2) = \sqrt{1+\epsilon_1^2} - 1$, $K_1 = 1/6$, and

$$\sup_{f\in\mathcal{F}} \|f\|_\infty \leq M_1, \quad \mathbb{E}\left[\sup_{f\in\mathcal{F}} |\nu_{n,Y}(f)|\right] \leq H_1, \quad \sup_{f\in\mathcal{F}} \frac{1}{n}\sum_{j=1}^{n} Var(f(Y_j)) \leq c_1.$$

Standard density arguments show that the above result can be extended to the case where $\mathcal{F}$ is a unit ball of a linear normed space. For more about the Talagrand inequality, we refer the readers to Massart (2000, 2007) and Comte and Genon-Catalot (2009).

Acknowledgements

The authors would like to express their sincere thanks to two anonymous referees for their comments and suggestions, which substantially improved the paper. Zhimin Zhang acknowledges the support from National Natural Science Foundation of China (project No. 11101451, 11171361) and the Fundamental Research Funds for the Central Universities (project No. CDJRC11100001). Hailiang Yang would like to thank the Research Grants Council of the Hong Kong Special Administrative Region (project No. HKU 706611P).

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