On a bivariate risk process with a dividend barrier strategy

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Abstract

In this paper, we study a continuous-time bivariate risk process in which each individual line of business implements a dividend barrier strategy. The insurance portfolios of the two insurers are correlated as they are subject to common shocks which induce dependent claims. To analyze the expected discounted dividends until the joint ruin time of the bivariate process (i.e. exit from the positive quadrant), we propose a discrete-time counterpart of the model and apply a bivariate extension of the Dickson-Waters discretization (Dickson and Waters (1991)) with the use of a bivariate Panjer type recursion (Walhin and Paris (2000)). Detailed numerical examples under different dependencies via common shocks, copulas and proportional reinsurance are discussed, and applications to optimal problems in reinsurance, capital allocation and dividends are given. It is also illustrated that the optimal pair of dividend barriers maximizing the dividend function is dependent on the initial surplus levels. A modified type of dividend barrier strategy is proposed towards the end.

Keywords: Bivariate risk process; Discretization; Dividend barrier strategy; Copulas; Capital allocation; Proportional reinsurance.

1 Introduction

In the classical compound Poisson risk model, the surplus process $\{U_*(t)\}_{t\geq 0}$ of a single line of insurance business is modelled by

$$U_*(t) = u + ct - \sum_{n=1}^{N(t)} X_n, \qquad t \geq 0,$$

where $u \geq 0$ is the insurer's initial surplus, $c > 0$ is the constant premium income per unit time, $\{N(t)\}_{t\geq 0}$ is a counting process that counts the number of claims, and X_n is the size (or severity) of the n-th claim. It is assumed that $\{N(t)\}_{t\geq 0}$ is a Poisson process with rate $\lambda > 0$, and $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables independent of $\{N(t)\}_{t\geq 0}$. The time of ruin of the process $\{U_*(t)\}_{t\geq 0}$ is defined by $T_* = \inf\{t \geq 0 : U_*(t) < 0\}$, which is the first time that the surplus process drops below zero. One requires the positive security loading condition $c > \lambda E[X_1]$ to ensure that the event of ruin $\{T_* < \infty\}$ is not certain.

A drawback of the above model is that the surplus process $\{U_*(t)\}_{t\geq 0}$ will grow to infinity in the long run, which leads to the idea of redistributing some of the surplus to the shareholders of the insurance company (de Finetti (1957)). One of the most commonly studied dividend strategies is the barrier

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strategy (see e.g. Gerber (1979), Lin et al. (2003), Dickson and Waters (2004) and Gerber et al. (2006)), in which the entire incoming premium rate is paid to the shareholders as dividend immediately whenever the surplus reaches a fixed barrier level b (as long as ruin has not occurred). Mathematically, the modified surplus process $\{U(t)\}_{t\geq 0}$ with $U(0)=u\geq 0$ and $u\leq b$ can be described by

$$\mathrm{d}U(t)=\begin{cases} c\,\mathrm{d}t-\mathrm{d}\sum_{n=1}^{N(t)}X_n, & 0\leq U(t)<b. \\ -\mathrm{d}\sum_{n=1}^{N(t)}X_n, & U(t)=b. \end{cases}$$

The quantities of interest in the literature include the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)) and the expectation or even the higher moments of the discounted dividends payable until ruin (Dickson and Waters (2004)). The study of barrier strategy is of great importance because it is known to be optimal in maximizing the expected discounted dividends until ruin when the density of X_1 is completely monotone (e.g. Loeffen (2008, Theorem 3)). In addition, for any given claim distributions, the optimal dividend barrier is independent of the initial surplus level. We refer interested readers to Albrecher and Thonhauser (2009) and Avanzi (2009) for comprehensive reviews of different dividend strategies and related optimality results in the literature.

Recently, there has been increased interest in multi-dimensional risk theory in which the surplus processes of more than one line of business are jointly analyzed. In multi-dimensional risk models, the frequencies and/or the severities of insurance claims payable by different insurers are generally correlated. Practically, such a situation arises when the insurers are subject to 'common shocks' as a result of catastrophic events (e.g. earthquakes and tsunamis) inducing large and correlated claims to them at the same time, or when an insurer transfers part of its claims to one or more reinsurers via a reinsurance contract. In this paper, we follow the former formulation, although applications to the latter situation are also possible (see Remark 1 and Section 3.3). We shall consider two lines of business, and each of them implements a dividend barrier strategy. The bivariate surplus process $\{(U_1(t),U_2(t))\}_{t\geq 0}$ with initial surplus levels $(u_1,u_2)=(U_1(0),U_2(0))$ and dividend barriers (b_1,b_2) (where $0\leq u_1\leq b_1$ and $0\leq u_2\leq b_2$) is described by, for $k=1,2$,

$$\mathrm{d}U_k(t)=\begin{cases} c_k\,\mathrm{d}t-\mathrm{d}\left(\sum_{n=1}^{N_{kk}(t)}Y_{k,n}+\sum_{n=1}^{N_{12}(t)}Z_{k,n}\right), & 0\leq U_k(t)<b_k. \\ -\mathrm{d}\left(\sum_{n=1}^{N_{kk}(t)}Y_{k,n}+\sum_{n=1}^{N_{12}(t)}Z_{k,n}\right), & U_k(t)=b_k. \end{cases} \tag{1.1}$$

Here (c_1,c_2) are the premium rates of the two lines, and $\{N_{11}(t)\}_{t\geq 0}$, $\{N_{22}(t)\}_{t\geq 0}$ and $\{N_{12}(t)\}_{t\geq 0}$ are mutually independent Poisson processes with respective parameters λ_{11}, λ_{22} and λ_{12}. Furthermore, $\{Y_{1,n}\}_{n=1}^{\infty}$, $\{Y_{2,n}\}_{n=1}^{\infty}$ and $\{(Z_{1,n},Z_{2,n})\}_{n=1}^{\infty}$ are mutually independent i.i.d. sequences, independent of the above three Poisson processes and distributed as the generic random variables Y_1, Y_2 and (Z_1,Z_2) respectively. For each $k=1,2$, the process $\{N_{kk}(t)\}_{t\geq 0}$ counts the number of claims faced by the k-th business only for claims that arise from the 'usual' claim occurrences with severity distributed as Y_k. On the other hand, $\{N_{12}(t)\}_{t\geq 0}$ counts the number of 'common shocks' which result in possibly dependent claims distributed as (Z_1,Z_2) to the two lines. It is assumed that Y_1, Y_2 and (Z_1,Z_2) are all positive continuous random variables with cumulative distribution functions (cdfs) $F_{11}(\cdot)$, $F_{22}(\cdot)$ and $F_{12}(\cdot,\cdot)$ respectively. It will be convenient to present $F_{12}(\cdot,\cdot)$ in copula form (e.g. Nelsen (2006)) as $F_{12}(z_1,z_2)=C(F_{1\bullet}(z_1),F_{\bullet 2}(z_2))$, where $C(\cdot,\cdot)$ is a copula and $F_{1\bullet}(z_1)$ and $F_{\bullet 2}(z_2)$ are the marginal cdfs of Z_1 and Z_2 respectively. For later use we also define the probability density functions (pdfs) $f_{11}(\cdot)=F_{11}'(\cdot)$, $f_{22}(\cdot)=F_{22}'(\cdot)$, $f_{1\bullet}(\cdot)=F_{1\bullet}'(\cdot)$ and $f_{\bullet 2}(\cdot)=F_{\bullet 2}'(\cdot)$. For each $k=1,2$, we assume that the loading condition $c_k>\lambda_{kk}E[Y_k]+\lambda_{12}E[Z_k]$ holds. The time of ruin of the k-th line is $T_k=\inf\{t\geq 0:U_k(t)<0\}$.

Remark 1 Suppose that (Z_1,Z_2) is the result of the splitting of a claim W between an insurer and a reinsurer via proportional reinsurance, i.e. $(Z_1,Z_2)=(s_1W,(1-s_1)W)$ for a positive continuous random variable W with cdf $F_W(\cdot)$ and a constant s_1 such that $0 < s_1 < 1$. Then one can let $F_{1\bullet}(z_1) = F_W(z_1/s_1)$ and $F_{\bullet 2}(z_2) = F_W(z_2/(1-s_1))$ and apply the comonotonicity copula $C(u, v) = \min(u, v)$ for $0 \leq u, v \leq 1$. Hence, our formulation provides a unified approach to study common shocks and proportional reinsurance. Examples concerning proportional reinsurance will be examined in Section 3.3. □

Unlike the classical univariate risk process in which ruin is defined to be the event that the surplus process ever drops below zero, there are various ways to define ruin in a bivariate risk model. Commonly studied definitions of ruin include (i) $\min(T_1, T_2) = \inf\{t \geq 0 : \min(U_1(t), U_2(t)) < 0\}$: the first time when (at least) one of the two surplus processes drops below zero (i.e. the first exit from the positive quadrant); (ii) $\inf\{t \geq 0 : \max(U_1(t), U_2(t)) < 0\}$: the first time when both processes are below zero simultaneously (i.e. the first entrance into the negative quadrant); and (iii) $\max(T_1, T_2)$: the first time when both processes have ruined (but not necessarily simultaneously). Most papers in the literature of multi-dimensional risk theory are concerned with the ruin probabilities associated with these definitions of ruin in the absence of dividends. Exact solutions are rarely available, and the existing results are mostly in the form of asymptotics (e.g. Li et al. (2007, Section 4), Chen et al. (2011), Hu and Jiang (2013), and Huang et al. (2013)), bounds (e.g. Chan et al. (2003), Picard et al. (2003, Section 4), Cai and Li (2005, 2007), Yuen et al. (2006), and Li et al. (2007, Sections 2 and 3)), and recursive approximations (e.g. Dang et al. (2009), Rabehasaina (2009, Section 5), and Gong et al. (2012)). In some special two-dimensional models involving proportional reinsurance, exact results were obtained by Avram et al. (2008a,b) and Badescu et al. (2011) via transforming the bivariate problem to simpler univariate problems. Numerical methods to evaluate ruin probabilities with particular applications in excess-of-loss and stop-loss reinsurance can be found in Kaishev and Dimitrova (2006), Kaishev et al. (2008, Section 4), Dimitrova and Kaishev (2010), and Castañer et al. (2013). We also refer interested readers to e.g. Collamore (1996, 1998), Hult et al. (2005), Hult and Lindskog (2006), Blanchet and Liu (2014), and Liu and Woo (2014) for the study of ruin-related quantities associated to the hitting of a rare set in multi-dimensional models. A comprehensive overview of multi-dimensional risk processes is given by in Asmussen and Albrecher (2010, Chapter XIII.9).

A recent work by Czarna and Palmowski (2011) took into account the effect of dividend payments in a bivariate model with proportional reinsurance. One of their proposed models involves a barrier in the form of $aU_1(t) + U_2(t) = b$, which is clearly different from our model dynamics (1.1). However, they implicitly assumed that there is a transfer of capital between the two lines of business whenever the bivariate process is on the barrier (see their Figure 1). This means that ruin (in terms of an exit from the positive quadrant) may actually occur due to capital transfer, which is practically undesirable. In this paper, we shall study the model (1.1) and define the time of ruin of the bivariate process $\{(U_1(t), U_2(t))\}_{t\geq 0}$ to be $T = \min(T_1, T_2) = \inf\{t \geq 0 : \min(U_1(t), U_2(t)) < 0\}$. The key quantity of our interest is the expected discounted dividends until the joint ruin time for each of the two lines. For each $k = 1, 2$, we aim at evaluating, for $0 \leq u_1 \leq b_1; 0 \leq u_2 \leq b_2$,

$$V_k(u_1, u_2; b_1, b_2) = c_k\, E\left[\int_0^T e^{-\delta t} I\{U_k(t) = b_k\}\, \mathrm{d}t \,\middle|\, (U_1(0), U_2(0)) = (u_1, u_2)\right], \tag{1.2}$$

where δ is the force of interest per unit time. It is instructive to note that even if there is no common shock component, the dividends of the two lines are still dependent via the joint ruin time T. As mentioned above, it is generally very difficult to derive exact results for multi-dimensional risk processes. Therefore, similar procedures as in Dickson and Waters (1991) can be applied to establish a connection between our continuous-time model and a discrete-time one which is easier to study. Then one can approximate (1.2) using its discrete counterpart.

This paper is organized as follows. In Section 2, the evaluation of the expected discounted dividends until ruin in a discrete bivariate risk process is discussed. Despite being a stand-alone model, we demonstrate how it can be used to approximate the continuous-time model (1.2) via Dickson-Waters discretization with the help of a bivariate Panjer type recursion. The approximation is then supported by some simulations. Section 3 is concerned with comparing how dependency between the two lines affects dividends, and numerical examples involving common shocks, proportional reinsurance and the use of different copulas are given. Section 4 provides more numerical examples which focus on the pair of optimal dividend barriers maximizing the expected discounted dividends. Unlike the classical univariate case, the optimal barriers in the bivariate framework depend on the initial surplus levels of the two lines. This leads us to propose a modified type of barrier strategy. A capital allocation problem is also discussed briefly. Section 5 ends the paper with a few concluding remarks.

2 A discrete bivariate risk process with dividend barriers

2.1 The model and dividends

Under a discrete framework, we consider the bivariate process $\{(U_1^d(n), U_2^d(n))\}_{n=0}^{\infty}$ with the dividend barriers (b_1, b_2), which is defined recursively via, for $k = 1, 2$,

$$U_k^d(n) = U_k^d(n-1) + 1 - I\{U_k^d(n-1) = b_k; X_{k,n} = 0\} - X_{k,n}, \qquad n = 1, 2, \ldots, \tag{2.1}$$

with the starting capital of $U_k^d(0) = u_k$ (where $u_k = 0, 1, \ldots, b_k$). It is assumed that the premium income is 1 in each period, and the claims $\{(X_{1,n}, X_{2,n})\}_{n=1}^{\infty}$ form a sequence of i.i.d. bivariate random vectors distributed as (X_1, X_2) with common joint probability mass function (pmf) $g(\cdot, \cdot)$. In addition, X_1 and X_2 are distributed on the set of non-negative integers. The dynamics (2.1) mean that a dividend of 1 is payable to the shareholders of line k at time n if (i) the surplus of line k is at level b at time $n-1$; and (ii) line k has no claim at time n (see Dickson and Waters (2004, Section 5)). For $k = 1, 2$, let $T_k^d = \inf\{n \in \{1, 2, \ldots\} : U_k^d(n) \leq 0\}$ be the ruin time of $\{U_k^d(n)\}_{n=0}^{\infty}$ (see Remark 2). However, at time 0 we allow the individual processes to start at level zero without ruin occurring. For each $k = 1, 2$, the loading condition is given by $E[X_k] < 1$. The time of ruin for the joint bivariate surplus process $\{(U_1^d(n), U_2^d(n))\}_{n=0}^{\infty}$ is then defined as $T^d = \min(T_1^d, T_2^d)$.

Remark 2 In the study of discrete-time risk models, different researchers have adopted different definitions of ruin as to whether reaching level zero is regarded as a ruin event. But the current definition (that reaching zero leads to ruin) is expected to work better especially when one applies the discrete-time model to approximate a continuous-time one (see Dickson and Waters (1991, Section 8)). □

Assuming the force of interest to be α per period, we are interested in the expected discounted dividend payment until the joint ruin time for each of the two lines. For each $k = 1, 2$, we define, for $u_1 = 0, 1, \ldots, b_1; u_2 = 0, 1, \ldots, b_2$,

$$V_k^d(u_1, u_2; b_1, b_2) = E\left[\sum_{n=1}^{T^d} e^{-\alpha n} I\{U_k^d(n-1) = b_k; X_{k,n} = 0\} \middle| (U_1^d(0), U_2^d(0)) = (u_1, u_2)\right]. \tag{2.2}$$

In order to study the above quantity, we can condition on all possible events at time 1. Four cases need to be distinguished based on the initial capital levels.

1. For $u_1 = 0, 1, \ldots, b_1 - 1; u_2 = 0, 1, \ldots, b_2 - 1$, the premium income of 1 for both lines will be added to the respective surplus levels, and no dividends are payable at time 1. If the claims $X_{1,1}$ and $X_{2,1}$

are no larger than u_1 and u_2 respectively, then the bivariate process will continue and there will be potential future dividends; otherwise ruin occurs and no dividends will ever be paid. We arrive at, for $k = 1, 2$,

$$V_k^d(u_1, u_2; b_1, b_2) = e^{-\alpha} \sum_{i=0}^{u_1} \sum_{j=0}^{u_2} g(i, j) V_k^d(u_1 + 1 - i, u_2 + 1 - j; b_1, b_2). \tag{2.3}$$

2. For $u_1 = b_1; u_2 = 0, 1, \ldots, b_2 - 1$, line 1's premium income of 1 will be paid out as dividend if there is no claim for this line. Both lines will survive at time 1 if the claims $X_{1,1}$ and $X_{2,1}$ are no larger than $b_1 - 1$ and u_2 respectively, resulting in potential future dividends. This leads to

$$\begin{aligned} V_1^d(b_1, u_2; b_1, b_2) &= e^{-\alpha} \sum_{j=u_2+1}^{\infty} g(0, j) + e^{-\alpha} \sum_{j=0}^{u_2} g(0, j)[1 + V_1^d(b_1, u_2 + 1 - j; b_1, b_2)] \\ &\quad + e^{-\alpha} \sum_{i=1}^{b_1} \sum_{j=0}^{u_2} g(i, j) V_1^d(b_1 + 1 - i, u_2 + 1 - j; b_1, b_2) \\ &= e^{-\alpha} \sum_{j=0}^{\infty} g(0, j) + e^{-\alpha} \sum_{j=0}^{u_2} g(0, j) V_1^d(b_1, u_2 + 1 - j; b_1, b_2) \\ &\quad + e^{-\alpha} \sum_{i=1}^{b_1} \sum_{j=0}^{u_2} g(i, j) V_1^d(b_1 + 1 - i, u_2 + 1 - j; b_1, b_2), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} V_2^d(b_1, u_2; b_1, b_2) &= e^{-\alpha} \sum_{j=0}^{u_2} g(0, j) V_2^d(b_1, u_2 + 1 - j; b_1, b_2) \\ &\quad + e^{-\alpha} \sum_{i=1}^{b_1} \sum_{j=0}^{u_2} g(i, j) V_2^d(b_1 + 1 - i, u_2 + 1 - j; b_1, b_2). \end{aligned} \tag{2.5}$$

3. For $u_1 = 0, 1, \ldots, b_1 - 1; u_2 = b_2$, the analyses are identical to those in Case 2 except that the roles of line 1 and line 2 are reversed. Hence, we have

$$\begin{aligned} V_1^d(u_1, b_2; b_1, b_2) &= e^{-\alpha} \sum_{i=0}^{u_1} g(i, 0) V_1^d(u_1 + 1 - i, b_2; b_1, b_2) \\ &\quad + e^{-\alpha} \sum_{i=0}^{u_1} \sum_{j=1}^{b_2} g(i, j) V_1^d(u_1 + 1 - i, b_2 + 1 - j; b_1, b_2), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} V_2^d(u_1, b_2; b_1, b_2) &= e^{-\alpha} \sum_{i=0}^{\infty} g(i, 0) + e^{-\alpha} \sum_{i=0}^{u_1} g(i, 0) V_2^d(u_1 + 1 - i, b_2; b_1, b_2) \\ &\quad + e^{-\alpha} \sum_{i=0}^{u_1} \sum_{j=1}^{b_2} g(i, j) V_2^d(u_1 + 1 - i, b_2 + 1 - j; b_1, b_2). \end{aligned} \tag{2.7}$$

4. For $u_1 = b_1; u_2 = b_2$, each line will pay out the premium income of 1 as dividend if it has no claim, plus potential future dividends if both lines survive time 1. This results in

$$\begin{aligned}
V_1^d(b_1, b_2; b_1, b_2) &= e^{-\alpha} g(0,0) V_1^d(b_1, b_2; b_1, b_2) + e^{-\alpha} \sum_{j=0}^{\infty} g(0,j) \\
&\quad + e^{-\alpha} \sum_{j=1}^{b_2} g(0,j) V_1^d(b_1, b_2 + 1 - j; b_1, b_2) + e^{-\alpha} \sum_{i=1}^{b_1} g(i,0) V_1^d(b_1 + 1 - i, b_2; b_1, b_2) \\
&\quad + e^{-\alpha} \sum_{i=1}^{b_1} \sum_{j=1}^{b_2} g(i,j) V_1^d(b_1 + 1 - i, b_2 + 1 - j; b_1, b_2), \qquad (2.8)
\end{aligned}$$

and

$$\begin{aligned}
V_2^d(b_1, b_2; b_1, b_2) &= e^{-\alpha} g(0,0) V_2^d(b_1, b_2; b_1, b_2) + e^{-\alpha} \sum_{i=0}^{\infty} g(i,0) \\
&\quad + e^{-\alpha} \sum_{j=1}^{b_2} g(0,j) V_2^d(b_1, b_2 + 1 - j; b_1, b_2) + e^{-\alpha} \sum_{i=1}^{b_1} g(i,0) V_2^d(b_1 + 1 - i, b_2; b_1, b_2) \\
&\quad + e^{-\alpha} \sum_{i=1}^{b_1} \sum_{j=1}^{b_2} g(i,j) V_2^d(b_1 + 1 - i, b_2 + 1 - j; b_1, b_2). \qquad (2.9)
\end{aligned}$$

To conclude, for fixed b_1 and b_2, the $b_1 b_2$ equations of (2.3) at $k = 1$, b_2 equations of (2.4), b_1 equations of (2.6) and the single equation (2.8) form a system of $(b_1+1)(b_2+1)$ linear equations for $\{V_1^d(u_1, u_2; b_1, b_2) : u_1 = 0, 1, \ldots, b_1; u_2 = 0, 1, \ldots, b_2\}$ to be solved. Similarly, $\{V_2^d(u_1, u_2; b_1, b_2) : u_1 = 0, 1, \ldots, b_1; u_2 = 0, 1, \ldots, b_2\}$ can be solved from (2.3) at $k = 2$, (2.5), (2.7) and (2.9).

2.2 Deriving the approximation

Our goal is to approximate the expected discounted dividends V_k defined in (1.2) for the continuous-time model (1.1) using the quantity V_k^d defined in (2.2) for the discrete-time model (2.1). To this end, we follow similar steps to those in Dickson and Waters (1991), who studied the finite-time survival probabilities. Their approximation also proved to be useful in studying dividend problems as well (see Dickson and Waters (2004) and Cheung and Drekic (2008)). However, the above applications were all conducted under univariate risk processes. Under the present bivariate framework, there are additional complications concerning the use of copula as well as a bivariate Panjer's recursion (see Section 2.3). The derivation of the approximation consists of the following four steps.

1. *Step 1: Change of monetary unit*

   First, we apply a change of monetary unit in the continuous-time model (1.1). In particular, for some positive constants β_1 and β_2 (known as scaling factors), define the random variables $Y_k^{(1)} = \beta_k Y_k$ and $Z_k^{(1)} = \beta_k Z_k$ for $k = 1, 2$. If $V_k^{(1)}(u_1, u_2; b_1, b_2)$ denotes the expected discounted dividends for line k in the continuous-time model with generic jumps $Y_1^{(1)}$, $Y_2^{(1)}$ and $(Z_1^{(1)}, Z_2^{(1)})$, Poisson rates λ_{11}, λ_{22} and λ_{12}, force of interest δ, premium rates $(\beta_1 c_1, \beta_2 c_2)$, initial surplus levels (u_1, u_2), and barrier levels (b_1, b_2), it is immediate that V_k in (1.2) satisfies

$$V_k(u_1, u_2; b_1, b_2) = \frac{1}{\beta_k} V_k^{(1)}(\beta_1 u_1, \beta_2 u_2; \beta_1 b_1, \beta_2 b_2). \qquad (2.10)$$

Note that the copula for the scaled version $(Z_1^{(1)}, Z_2^{(1)})$ is also $C(\cdot,\cdot)$, i.e. identical to that of (Z_1, Z_2) (see e.g. Denuit et al. (2005, Proposition 4.4.4(i))).

2. *Step 2: Discretization of* $Y_1^{(1)}$, $Y_2^{(1)}$, $Z_1^{(1)}$ *and* $Z_2^{(1)}$

   The random variables $Y_1^{(1)}$, $Y_2^{(1)}$, $Z_1^{(1)}$ and $Z_2^{(1)}$ defined in Step 1 are then discretized on $\{0, 1, \ldots\}$ to give the discretized versions $Y_1^{(2)}$, $Y_2^{(2)}$, $Z_1^{(2)}$ and $Z_2^{(2)}$. The 'mean preserving method' (see e.g. De Vylder and Goovaerts (1988, Section 7) and Dickson (2005, P.80)) is suggested and it is known to yield good results (see Dickson and Waters (1991, 2004) and Cheung and Drekic (2008)). The pmf of $Y_k^{(2)}$ is given by, for $k = 1, 2$,

   $$h_{kk}(i) = \beta_k \left(\int_{\frac{i}{\beta_k}}^{\frac{i+1}{\beta_k}} F_{kk}(y)\,\mathrm{d}y - \int_{\frac{i-1}{\beta_k}}^{\frac{i}{\beta_k}} F_{kk}(y)\,\mathrm{d}y \right), \qquad i = 0, 1, \ldots. \tag{2.11}$$

   For the discretized random vector $(Z_1^{(2)}, Z_2^{(2)})$, we apply the same copula $C(\cdot,\cdot)$ as the dependency structure (see e.g. Bargès et al. (2009, Section 5.2)). Therefore, the joint pmf of $(Z_1^{(2)}, Z_2^{(2)})$, namely $h_{12}(i,j)$, can be calculated from the associated joint cdf

   $$\sum_{k=0}^{i}\sum_{l=0}^{j} h_{12}(k,l) = C\left(\beta_1 \int_{\frac{i}{\beta_1}}^{\frac{i+1}{\beta_1}} F_{1\bullet}(y)\,\mathrm{d}y, \beta_2 \int_{\frac{j}{\beta_2}}^{\frac{j+1}{\beta_2}} F_{\bullet 2}(y)\,\mathrm{d}y \right), \qquad i, j = 0, 1, \ldots. \tag{2.12}$$

   Denote by $V_k^{(2)}(u_1, u_2; b_1, b_2)$ the expected discounted dividends for line k in the continuous-time model with discrete generic jumps $Y_1^{(2)}$, $Y_2^{(2)}$ and $(Z_1^{(2)}, Z_2^{(2)})$, Poisson rates λ_{11}, λ_{22} and λ_{12}, force of interest δ, premium rates $(\beta_1 c_1, \beta_2 c_2)$, initial surplus levels (u_1, u_2), and barrier levels (b_1, b_2). If $Y_1^{(2)}$, $Y_2^{(2)}$ and $(Z_1^{(2)}, Z_2^{(2)})$ are good approximations of $Y_1^{(1)}$, $Y_2^{(1)}$ and $(Z_1^{(1)}, Z_2^{(1)})$ (i.e. when β_1 and β_2 are 'large'), then

   $$V_k^{(2)}(u_1, u_2; b_1, b_2) \simeq V_k^{(1)}(u_1, u_2; b_1, b_2),$$

   and hence from (2.10) one has

   $$V_k(u_1, u_2; b_1, b_2) \simeq \frac{1}{\beta_k} V_k^{(2)}(\beta_1 u_1, \beta_2 u_2; \beta_1 b_1, \beta_2 b_2). \tag{2.13}$$

3. *Step 3: Change of time unit*

   We now change the time unit of the continuous-time model with discrete claims in Step 2 such that the premium income per time unit is 1. To achieve this, β_1 and β_2 introduced in Step 1 are chosen such that $\beta_1 c_1 = \beta_2 c_2$. The model in Step 2 is then equivalent to a model in which the discrete generic jumps are $Y_1^{(2)}$, $Y_2^{(2)}$ and $(Z_1^{(2)}, Z_2^{(2)})$, the Poisson rates are $\lambda_{11}/\beta_1 c_1$, $\lambda_{22}/\beta_1 c_1$ and $\lambda_{12}/\beta_1 c_1$, the force of interest is $\alpha = \delta/\beta_1 c_1$, the premium rates are $(1, 1)$, the initial surplus levels are (u_1, u_2), and the barrier levels are (b_1, b_2). If we denote by $V_k^{(3)}(u_1, u_2; b_1, b_2)$ the expected discounted dividends for line k under the above setting, then

   $$V_k^{(3)}(u_1, u_2; b_1, b_2) = V_k^{(2)}(u_1, u_2; b_1, b_2),$$

   and hence from (2.13) we arrive at

   $$V_k(u_1, u_2; b_1, b_2) \simeq \frac{1}{\beta_k} V_k^{(3)}(\beta_1 u_1, \beta_2 u_2; \beta_1 b_1, \beta_2 b_2). \tag{2.14}$$

4. *Step 4: Replacement of continuous-time model by discrete-time model*

   In this final step, the continuous-time model with discrete claims in Step 3 is replaced by a discrete-time one (in which the event of ruin and the payment of dividend (if any) are only monitored once per period). One can then approximate $V_k^{(3)}(u_1, u_2; b_1, b_2)$ by

$$V_k^{(3)}(u_1, u_2; b_1, b_2) \simeq V_k^d(u_1, u_2; b_1, b_2), \tag{2.15}$$

   where $V_k^d(u_1, u_2; b_1, b_2)$ is defined by (2.2) under the force of interest $\alpha = \delta/\beta_1 c_1$ and the generic discrete claims, for $k = 1, 2$,

$$X_k = \sum_{n=1}^{M_{kk}} Y_{k,n}^{(2)} + \sum_{n=1}^{M_{12}} Z_{k,n}^{(2)}. \tag{2.16}$$

   Here M_{kk} has a Poisson distribution with mean $\gamma_{kk} = \lambda_{kk}/\beta_1 c_1$ whereas M_{12} has a Poisson distribution with mean $\gamma_{12} = \lambda_{12}/\beta_1 c_1$. Moreover, $\{Y_{k,n}^{(2)}\}_{n=1}^{\infty}$ is i.i.d. with generic variable $Y_k^{(2)}$, and $\{(Z_{1,n}^{(2)}, Z_{2,n}^{(2)})\}_{n=1}^{\infty}$ is also i.i.d. with generic vector $(Z_1^{(2)}, Z_2^{(2)})$. Moreover, M_{11}, M_{22}, M_{12}, $\{Y_{1,n}^{(2)}\}_{n=1}^{\infty}$, $\{Y_{2,n}^{(2)}\}_{n=1}^{\infty}$ and $\{(Z_{1,n}^{(2)}, Z_{2,n}^{(2)})\}_{n=1}^{\infty}$ are all mutually independent. Hence X_1 and X_2 are dependent compound Poisson random variables. When β_1 (and hence β_2) is 'large', the time intervals between the points where the surplus levels are checked are small, since one time unit in the present step is equivalent to $1/\beta_1 c_1$ time unit in the original continuous-time model (1.1). Then (2.15) will be a good approximation. To conclude, one has from (2.14) that

$$V_k(u_1, u_2; b_1, b_2) \simeq \frac{1}{\beta_k} V_k^d(\beta_1 u_1, \beta_2 u_2; \beta_1 b_1, \beta_2 b_2), \tag{2.17}$$

   and one requires $\beta_1 u_1$, $\beta_2 u_2$, $\beta_1 b_1$ and $\beta_2 b_2$ to be integers.

Formula (2.17) suggests that its left-hand side, namely the expected discounted dividends V_k in (1.2) for the continuous-time model (1.1), can be approximated by its right-hand side which is in terms of V_k^d defined in (2.2) for the fully discrete model (2.1). It remains to evaluate the joint pmf of (X_1, X_2), namely $g(i, j)$, in order to apply the results in Section 2.1 to find V_k^d. This will be the subject matter of the next subsection via the use of Panjer-type recursion (see e.g. Klugman et al. (2008, Chapter 6.8)). Note also that the above approximation is different from that in Yuen et al. (2006, Section 3), who approximated a bivariate compound Poisson risk model by a bivariate compound binomial model. Their approximation does not involve Panjer's recursion for compound distribution, and in their model a common shock does not result in dependent claims in the two lines.

2.3 Bivariate Panjer's recursion for dependent compound Poisson distributions

With the components of the random vector (X_1, X_2) given by (2.16), the derivation of its joint pmf $g(i, j)$ can be done by slightly modifying the results in Walhin and Paris (2000, Section 4) who considered the case where $Z_1^{(2)}$ and $Z_2^{(2)}$ are independent and distributed as $Y_1^{(2)}$ and $Y_2^{(2)}$ respectively. Under the current setting, defining the probability generating function $\hat{g}(r, s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} r^i s^j g(i, j)$, it can be proved that

$$\hat{g}(r, s) = e^{-\gamma_{11}[1-\hat{h}_{11}(r)]-\gamma_{22}[1-\hat{h}_{22}(s)]-\gamma_{12}[1-\hat{h}_{12}(r,s)]}.$$

Here $\hat{h}_{11}(r) = \sum_{i=0}^{\infty} r^i h_{11}(i)$, $\hat{h}_{22}(s) = \sum_{j=0}^{\infty} s^j h_{22}(j)$ and $\hat{h}_{12}(r, s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} r^i s^j h_{12}(i, j)$ are the probability generating functions pertaining to the pmf's $h_{11}(\cdot)$, $h_{22}(\cdot)$ and $h_{12}(\cdot, \cdot)$ defined via (2.11) and (2.12) in Step 2; whereas $\gamma_{11} = \lambda_{11}/\beta_1 c_1$, $\gamma_{22} = \lambda_{22}/\beta_1 c_1$ and $\gamma_{12} = \lambda_{12}/\beta_1 c_1$ according to Step 4. By differentiating the above equation with respect to r, multiplying the resulting equation by r and then equating coefficients of r^i, we arrive at

$$g(i,j)=\gamma_{11}\sum_{k=1}^{i}\frac{k}{i}h_{11}(k)g(i-k,j)+\gamma_{12}\sum_{k=1}^{i}\sum_{l=0}^{j}\frac{k}{i}h_{12}(k,l)g(i-k,j-l),\qquad i=1,2,\ldots;j=0,1,\ldots. \tag{2.18}$$

Similarly,

$$g(i,j)=\gamma_{22}\sum_{l=1}^{j}\frac{l}{j}h_{22}(l)g(i,j-l)+\gamma_{12}\sum_{k=0}^{i}\sum_{l=1}^{j}\frac{l}{j}h_{12}(k,l)g(i-k,j-l),\qquad i=0,1,\ldots;j=1,2,\ldots. \tag{2.19}$$

The starting point of the recursion is

$$g(0,0)=e^{-\gamma_{11}[1-h_{11}(0)]-\gamma_{22}[1-h_{22}(0)]-\gamma_{12}[1-h_{12}(0,0)]}. \tag{2.20}$$

2.4 Numerical illustrations of the approximation

This subsection aims at demonstrating the quality of the approximation derived in Section 2.2. Since the same approximation will be used in various numerical illustrations for the rest of the paper, we highlight the procedures as far as programming work is concerned to approximate $V_k(u_1,u_2;b_1,b_2)$ for $k=1,2$.

- Specify the parameters and distributional assumptions of the continuous-time model (1.1), which include the premium rates (c_1,c_2), the Poisson rates $(\lambda_{11},\lambda_{22},\lambda_{12})$, the cdfs $F_{11}(\cdot)$, $F_{22}(\cdot)$ and $F_{12}(\cdot,\cdot)$, and the copula $C(\cdot,\cdot)$. Specify the force of interest δ for the dividend function (1.2).
- Select the scaling factors (β_1,β_2) such that β_1u_1, β_2u_2, β_1b_1 and β_2b_2 are integers, and $\beta_1c_1=\beta_2c_2$.
- Apply (2.11) and (2.12) to find $h_{11}(\cdot)$, $h_{22}(\cdot)$ and $h_{12}(\cdot,\cdot)$.
- With $\gamma_{11}=\lambda_{11}/\beta_1c_1$, $\gamma_{22}=\lambda_{22}/\beta_1c_1$ and $\gamma_{12}=\lambda_{12}/\beta_1c_1$, evaluate $g(\cdot,\cdot)$ recursively using (2.18) and (2.19) subject to the starting point (2.20).
- Set $\alpha=\delta/\beta_1c_1$. Apply (2.3)-(2.9) (with b_1 and b_2 replaced by β_1b_1 and β_2b_2 respectively) to calculate $\{V_1^d(u_1,u_2;\beta_1b_1,\beta_2b_2):u_1=0,1,\ldots,\beta_1b_1;u_2=0,1,\ldots,\beta_2b_2\}$ and $\{V_2^d(u_1,u_2;\beta_1b_1,\beta_2b_2):u_1=0,1,\ldots,\beta_1b_1;u_2=0,1,\ldots,\beta_2b_2\}$.
- Apply (2.17) to approximate $V_k(u_1,u_2;b_1,b_2)$ for $k=1,2$.

It is instructive to note that as the scaling factors β_1 and β_2 increase (such that $\beta_1c_1=\beta_2c_2$), the approximation of the continuous-time bivariate process by a discrete-time one gets more accurate because (i) the discretization in Step 2 in Section 2.2 gets finer (i.e. continuous claims are better approximated by discrete ones); and (ii) the approximating discrete-time process in Step 4 is checked more frequently (and becomes closer to the continuous-time model). Note that two opposing sources of errors always occur when one approximates the dividend function V_k by V_k^d. First, dividends are paid immediately in the continuous-time model once the surplus of an individual line reaches its barrier; whereas in the discrete-time model, reaching the barrier does not immediately result in a dividend unless there is no claim in the next period. In this aspect, V_k^d tends to underestimate the true value V_k due to discounting. In contrast, the discrete-time model tends to survive longer due to the protection from delayed dividend payments, which means that there can be more potential future dividends. This may cause V_k^d to overestimate V_k. In the following example, we shall gradually increase the scaling factors in computing the approximated dividend values. Simulations are also conducted to verify the accuracy of the approximations and check whether one of the afore-mentioned effects is always more dominant.

Example 1 In this example, we assume the Poisson rates $\lambda_{11} = \lambda_{22} = \lambda_{12} = 1$ and the premium rates $c_1 = 2.8$ and $c_2 = 4.2$. Line 1 is subject to claims with pdf $f_{11}(y) = f_{1\bullet}(y) = 0.8e^{-0.8y}$; whereas the claims of line 2 have pdf $f_{22}(y) = f_{\bullet 2}(y) = 0.5e^{-0.5y}$. Their means are 1.25 and 2 respectively, and they both have coefficient of variation of 1. In the case of a common shock, it is assumed that Z_1 and Z_2 are independent of each other, i.e. the independence copula $C(u, v) = uv$ for $0 \leq u, v \leq 1$ is used. The loading conditions $c_1 = 2.8 > 2.5 = \lambda_{11}E[Y_1] + \lambda_{12}E[Z_1]$ and $c_2 = 4.2 > 4 = \lambda_{22}E[Y_2] + \lambda_{12}E[Z_2]$ are satisfied. The force of interest is assumed to be $\delta = 0.05$, and the barrier values are fixed to be $b_1 = b_2 = 2$ (see Section 5 for an explanation regarding the choice of low barriers). According to the computational procedures outlined at the beginning of this subsection, we require $\beta_1 c_1 = \beta_2 c_2$, or equivalently $\beta_1 = 1.5\beta_2$. The approximated values of the expected discounted dividends for the two lines using different sets of (β_1, β_2) are given in Tables 2.1(a)&(b).

V_1	$(\beta_1, \beta_2) = (3, 2)$			
	u_1 \ u_2	0	1	2
	0	0.425	0.486	0.507
	1	0.832	0.971	1.024
	2	1.526	1.721	1.805
	$(\beta_1, \beta_2) = (6, 4)$			
	u_1 \ u_2	0	1	2
	0	0.417	0.473	0.491
	1	0.826	0.958	1.003
	2	1.524	1.716	1.790
	$(\beta_1, \beta_2) = (15, 10)$			
	u_1 \ u_2	0	1	2
	0	0.413	0.467	0.482
	1	0.824	0.951	0.991
	2	1.526	1.715	1.782
	$(\beta_1, \beta_2) = (30, 20)$			
	u_1 \ u_2	0	1	2
	0	0.412	0.465	0.479
	1	0.823	0.949	0.987
	2	1.527	1.715	1.780
	$(\beta_1, \beta_2) = (60, 40)$			
	u_1 \ u_2	0	1	2
	0	0.411	0.464	0.478
	1	0.823	0.948	0.985
	2	1.527	1.715	1.778

V_2	$(\beta_1, \beta_2) = (3, 2)$			
	u_1 \ u_2	0	1	2
	0	0.957	1.493	2.268
	1	1.193	1.869	2.717
	2	1.267	2.003	2.886
	$(\beta_1, \beta_2) = (6, 4)$			
	u_1 \ u_2	0	1	2
	0	0.956	1.496	2.276
	1	1.187	1.869	2.725
	2	1.253	1.990	2.882
	$(\beta_1, \beta_2) = (15, 10)$			
	u_1 \ u_2	0	1	2
	0	0.958	1.501	2.285
	1	1.184	1.870	2.733
	2	1.245	1.984	2.881
	$(\beta_1, \beta_2) = (30, 20)$			
	u_1 \ u_2	0	1	2
	0	0.958	1.503	2.289
	1	1.184	1.871	2.736
	2	1.243	1.982	2.881
	$(\beta_1, \beta_2) = (60, 40)$			
	u_1 \ u_2	0	1	2
	0	0.959	1.504	2.291
	1	1.184	1.872	2.738
	2	1.242	1.981	2.881

Tables 2.1(a)&(b): Approximated dividends in the two lines for various sets of (β_1, β_2)

	u_1 \ u_2	0	1	2
V_1	0	0.411	0.463	0.476
	1	0.823	0.945	0.983
	2	1.527	1.715	1.777

	u_1 \ u_2	0	1	2
V_2	0	0.959	1.506	2.292
	1	1.182	1.870	2.739
	2	1.240	1.979	2.880

Tables 2.2(a)&(b): Simulated dividends in the two lines

From Table 2.1, it is clear that for a given set of (β_1, β_2), the dividend functions for both lines are increasing in the initial surplus levels u_1 and u_2 as they must be. When we increase the values of (β_1, β_2) for each fixed pair of initial surplus levels (u_1, u_2), the dividend values of line 1 always decrease (except when $(u_1, u_2) = (2, 0)$); whereas those of line 2 either increase or decrease. In all cases, the dividends for both lines appear to be converging as (β_1, β_2) increases. To further verify the results, we have also run some simulations in the continuous-time risk model and obtained Tables 2.2(a)&(b) for the dividend functions of the two lines. In Table 2.2, each pair of (V_1, V_2) is calculated using $1,000,000$ sample paths generated up to the joint ruin time. Comparing Table 2.1 with Table 2.2, it can be seen that scaling factors of $(\beta_1, \beta_2) = (60, 40)$ produce very good results: the dividend values are always the same up to at least two decimal places. More interestingly, the results obtained by smaller scaling factors of, say $(\beta_1, \beta_2) = (15, 10)$, are indeed still comparable to those by simulations. However, the simulation results are sometimes a bit larger than and sometimes a bit smaller than the discrete-time approximations. Therefore, one cannot conclude whether our approximation tends to underestimate or overestimate the true value of dividends, i.e. neither the effect of delayed dividends nor the effect of prolonged survival is always more dominant. Nonetheless, for each fixed pair of initial surplus levels (u_1, u_2), the approximated values in Table 2.1 approach the corresponding simulated values in Table 2.2 either from above or below as (β_1, β_2) increase. (Indeed, we have separately run the Dickson-Waters type of algorithm in Cheung and Drekic (2008) for a single line dual risk process with a dividend barrier. It was found that the approximated dividend values either increase or decrease to the true value as the scaling factor increases, depending on the initial surplus and the barrier level.) □

3 Three different types of dependencies

In this section, we examine the bivariate risk process (1.1) in which the two lines of business are subject to different types of dependencies via some numerical examples. These include (i) common shocks; (ii) copulas; and (iii) proportional reinsurance. Throughout this entire section, the barrier values $b_1 = b_2 = 2$ are applied because high scaling factors (β_1, β_2) will be used (see concluding remarks in Section 5).

3.1 Different levels of common shocks

Example 2 In this example, we aim at examining the impact of different levels of common shocks on the dividends by varying the values of λ_{11}, λ_{22} and λ_{12} while keeping $\lambda_{11}+\lambda_{12} = \lambda_{22}+\lambda_{12} = 2$ fixed, i.e. each individual line of business is subject to the same total claim arrival rate of 2. To illustrate the versatility of the approximation, we use the more complicated density functions $f_{11}(y) = f_{1\bullet}(y) = 8e^{-2y}\sin^2 y$, $f_{22}(y) = f_{\bullet 2}(y) = (1/4)(0.6^2 ye^{-0.6y}) + (3/4)(9^2 ye^{-9y})$. These two distributions have rational Laplace transforms, and they were used in Cheung and Drekic (2008). They both have mean 1, and their coefficients of variation are 0.50 and 1.80 respectively. While $f_{22}(y)$ represents a standard mixture of two Erlang(2) distributions, the less common $f_{11}(y)$ is the pdf of a damped squared sine distribution with low variability. In particular, the density $f_{11}(y)$ is strictly increasing starting from $f_{11}(0) = 0$ until it reaches the global maximum of 0.83152 at $\pi/4 = 0.78540$, from which $f_{11}(y)$ is strictly decreasing until zero is reached at $\pi = 3.14159$. Due to the periodicity induced by the sine function, $f_{11}(y)$ also achieves (i) global and local minimum of zero at $y = n\pi$ for $n = 1, 2, \ldots$; and (ii) local maximum at $y = n\pi + \pi/4$ for $n = 1, 2, \ldots$. Nonetheless, $f_{11}(y)$ is very very close to zero for $y > \pi$, and $f_{11}(y)$ is strictly unimodal for $y \geq 0$. The premium rates are assumed to be $c_1 = 2.2$ and $c_2 = 3.3$, so that the loading conditions $c_1 = 2.2 > 2 = \lambda_{11}E[Y_1] + \lambda_{12}E[Z_1]$ and $c_2 = 3.3 > 2 = \lambda_{22}E[Y_2] + \lambda_{12}E[Z_2]$ hold true. (Note that the premium of line 2 is assumed to have a larger loading factor because its claims have larger variance.)

The remaining model assumptions are same as those in Example 1: Z_1 and Z_2 are independent in case of a common shock; the force of interest is $\delta = 0.05$, and the barrier values are $b_1 = b_2 = 2$. Using the scaling factors $(\beta_1, \beta_2) = (60, 40)$, we follow the approximation procedures outlined at the beginning of Section 2.4 and the resulting dividend values are listed in Tables 3.1(a)&(b). The tables start with the extreme case of $\lambda_{11} = \lambda_{22} = 2$ and $\lambda_{12} = 0$ in which the two lines of business only face their own claims independently with no common shocks at all. Then as we move down the tables, λ_{11} and λ_{22} are decreased by 0.5 whereas λ_{12} is increased by 0.5 each time until we reach another extreme case where $\lambda_{11} = \lambda_{22} = 0$ and $\lambda_{12} = 2$. The last case indicates that the two lines are only subject to common shocks.

	$\lambda_{11}=\lambda_{22}=2$, $\lambda_{12}=0$			
	u_1 \ u_2	0	1	2
	0	0.260	0.303	0.308
	1	0.689	0.801	0.816
	2	1.404	1.601	1.627
V_1	$\lambda_{11}=\lambda_{22}=1.5$, $\lambda_{12}=0.5$			
	u_1 \ u_2	0	1	2
	0	0.305	0.342	0.347
	1	0.769	0.873	0.886
	2	1.503	1.688	1.712
	$\lambda_{11}=\lambda_{22}=1$, $\lambda_{12}=1$			
	u_1 \ u_2	0	1	2
	0	0.358	0.387	0.390
	1	0.857	0.950	0.962
	2	1.609	1.781	1.803
	$\lambda_{11}=\lambda_{22}=0.5$, $\lambda_{12}=1.5$			
	u_1 \ u_2	0	1	2
	0	0.419	0.437	0.439
	1	0.955	1.035	1.045
	2	1.722	1.880	1.900
	$\lambda_{11}=\lambda_{22}=0$, $\lambda_{12}=2$			
	u_1 \ u_2	0	1	2
	0	0.490	0.493	0.494
	1	1.062	1.127	1.135
	2	1.842	1.986	2.004

	$\lambda_{11}=\lambda_{22}=2$, $\lambda_{12}=0$			
	u_1 \ u_2	0	1	2
	0	0.906	1.401	2.113
	1	1.637	2.458	3.318
	2	1.946	2.910	3.846
V_2	$\lambda_{11}=\lambda_{22}=1.5$, $\lambda_{12}=0.5$			
	u_1 \ u_2	0	1	2
	0	1.041	1.538	2.260
	1	1.799	2.621	3.483
	2	2.092	3.067	4.005
	$\lambda_{11}=\lambda_{22}=1$, $\lambda_{12}=1$			
	u_1 \ u_2	0	1	2
	0	1.193	1.686	2.417
	1	1.971	2.791	3.655
	2	2.244	3.231	4.170
	$\lambda_{11}=\lambda_{22}=0.5$, $\lambda_{12}=1.5$			
	u_1 \ u_2	0	1	2
	0	1.365	1.845	2.584
	1	2.156	2.969	3.834
	2	2.402	3.401	4.341
	$\lambda_{11}=\lambda_{22}=0$, $\lambda_{12}=2$			
	u_1 \ u_2	0	1	2
	0	1.558	2.018	2.763
	1	2.352	3.155	4.022
	2	2.566	3.578	4.519

Tables 3.1(a)&(b): Approximated dividends in the two lines for different levels of common shocks

A look at Table 3.1 reveals that the dividend values for both lines increase as the rate of common shocks increases. This can be interpreted as follows. For the case where $\lambda_{11} = \lambda_{22} = 2$ and $\lambda_{12} = 0$, the surplus processes of the two lines are indeed independent, and the mean total number of claim events per unit time, namely $\lambda_{11} + \lambda_{22} + \lambda_{12}$, is 4. As we move towards the most dependent case of $\lambda_{11} = \lambda_{22} = 0$ and $\lambda_{12} = 2$ where there are common shocks only, the mean total number of claim events per unit time decreases to 2. Since each instant of a claim event can potentially be the joint ruin time $T = \min(T_1, T_2)$, the bivariate process is likely to survive longer when there are more common shocks (keeping the total claim arrival rate for each line fixed), resulting in more dividends. Note that the dividends for both lines cease once ruin has occurred in one of the two lines. Another interpretation of our results is that when there are no common shocks at all, one of the lines in fact has positive surplus at the joint ruin time T

but no further dividends are paid, i.e. the situation is not economical. In contrast, when common shocks are more frequent, it is more likely that both lines have negative surplus at the joint ruin time T anyway, i.e. there is less chance that resources are wasted. These observations complement the numerical results in Gong et al. (2012, Figure 1) who studied the joint ruin probability in the absence of dividends. $\square$

3.2 Different copulas

In this subsection, we apply different parametric copulas to describe the dependency between Z_1 and Z_2 when a common shock strikes both lines. Three copulas will be considered: (i) Ali-Mikhail-Haq (AMH) copula (e.g. Nelsen (2006, Exercises 2.14(d) and 5.10)); (ii) Farlie-Gumbel-Morgenstern (FGM) copula (e.g. Nelsen (2006, Examples 3.12 and 5.2)); and (iii) Gaussian (or normal) copula (e.g. Denuit et al. (2005, Chapter 4.3.3 and Exercise 5.4.6)). Each copula's definition and Kendall's tau τ (or Kendall's rank correlation coefficient) are summarized in Table 3.2 below.

Copula	$C(u,v)$ for $0 \leq u,v \leq 1$	Range of θ	Kendall's tau τ	Range of τ
AMH	$\frac{uv}{1-\theta(1-u)(1-v)}$	$-1 \leq \theta \leq 1$	$\frac{3\theta-2}{3\theta} - \frac{2(1-\theta)^2 \ln(1-\theta)}{3\theta^2}$	$-0.18173 \leq \tau \leq 0.33333$
FGM	$uv + \theta uv(1-u)(1-v)$	$-1 \leq \theta \leq 1$	$\frac{2}{9}\theta$	$-0.22222 \leq \tau \leq 0.22222$
Gaussian	$\Phi_\theta(\Phi^{-1}(u), \Phi^{-1}(v))$	$-1 < \theta < 1$	$\frac{2}{\pi} \arcsin \theta$	$-1 < \tau < 1$

Table 3.2: Copulas and their Kendall's rank correlation coefficients

In the definition of the Gaussian copula, $\Phi(\cdot)$ is the standard normal cdf whereas $\Phi_\theta(\cdot,\cdot)$ represents the bivariate standard normal cdf with covariance θ. Note that the Kendall's rank correlation coefficient, as a measure of dependency, is only specific to a given copula and is independent of the marginal distributions (e.g. Nelsen (2006, Theorem 5.1.3)). In all three copulas, $\theta = 0$ corresponds to the case of independence (i.e. $C(u,v) = uv$ for $0 \leq u,v \leq 1$), and the resulting Kendall's tau is zero. It is instructive to note that all these three copulas have one parameter. This means that from the point of view of calibration, if one of these copulas has been identified as suitable for a given set of data, then the parameter follows in a straightforward manner once the Kendall's rank correlation has been estimated (see e.g. McNeil et al. (2005, Section 5.5.1)).

The application of copulas as a tool for risk management in finance and insurance was discussed extensively in Embrechts et al. (2002, 2003), and we also refer interested readers to Frees and Valdez (1998), Klugman and Parsa (1999), and Trivedi and Zimmer (2005) for general actuarial applications and fitting of bivariate loss distributions using copulas. The reasons for the choice of the above three copulas are as follows. First, the AMH copula belongs to the class of Archimedean copulas, which possess nice properties and are popular for modelling (see e.g. Genest and MacKay (1986), Denuit et al. (2005, Chapter 4.5), and Nelsen (2006, Chapter 4)). See also Denuit et al. (2004) for the use of Archimedean copulas in non-life insurance. A plot of the AMH copula pdf can be found in Panjer (2006, Figure 8.8). Second, the FGM copula belongs to the class of polynomial copulas (see e.g. Drouet-Mari and Kotz (2001, Chapter 4.5.2)). It is a tractable copula which is a first order approximation of both the Plackett copula and the Frank copula (see e.g. Nelsen (2006, Exercises 3.39 and 4.9)). Due to its simplicity, the FGM copula has become increasingly popular in modelling aggregate claims in insurance risk models (see e.g. Cossette et al. (2010), Bargès et al. (2011), Woo and Cheung (2013, Section 4), and Chadjiconstantinidis and Vrontos (2014)). While AMH and FGM copulas only allow moderate dependence (which is evident from the range of Kendall's tau), stronger dependency can be modelled by the Gaussian copula which is commonly used for comparison purposes. See e.g. Denuit et al. (2005, Figure 4.4) which depicts the increasing dependency of the components of a bivariate Gaussian copula as θ increases. It is known (e.g.

Trivedi and Zimmer (2005, Chapter 2.3.3)) that the bivariate Gaussian copula attains the Fréchet lower and upper bounds respectively when θ tends to -1 and 1.

Example 3 In this example, we follow the same assumptions as in Example 2 of Section 3.1 under the Poisson rates $\lambda_{11} = \lambda_{22} = \lambda_{12} = 1$, except that the three copulas in Table 3.2 are applied to the pair (Z_1, Z_2) arising from common shocks. For a fair comparison among different copulas, we fix the value of Kendall's tau τ and then solve for the appropriate parameter θ. First, setting $\tau = 0.2$ yields $\theta = 0.71349$, $\theta = 0.9$ and $\theta = 0.30902$ respectively for AMH, FGM and Gaussian copulas. Tables 3.3(a)&(b) summarize the approximated dividend values calculated using the procedures at the beginning of Section 2.4 under the scaling factors $(\beta_1, \beta_2) = (60, 40)$. If we instead fix a negative Kendall's tau $\tau = -0.2$, it is found that $\theta = -0.9$ and $\theta = -0.30902$ for FGM and Gaussian copulas respectively; whereas the AMH copula cannot reach such a Kendall's tau according to the last column of Table 3.2. The corresponding approximated dividend values are given in Tables 3.4(a)&(b).

V_1	AMH		
u_1 \ u_2	0	1	2
0	0.369	0.398	0.402
1	0.881	0.975	0.987
2	1.636	1.810	1.831
	FGM		
u_1 \ u_2	0	1	2
0	0.371	0.401	0.404
1	0.885	0.980	0.991
2	1.641	1.815	1.837
	Gaussian		
u_1 \ u_2	0	1	2
0	0.372	0.401	0.405
1	0.887	0.982	0.994
2	1.643	1.818	1.840

V_2	AMH		
u_1 \ u_2	0	1	2
0	1.230	1.727	2.458
1	2.035	2.863	3.729
2	2.307	3.304	4.244
	FGM		
u_1 \ u_2	0	1	2
0	1.236	1.733	2.465
1	2.045	2.876	3.743
2	2.318	3.318	4.258
	Gaussian		
u_1 \ u_2	0	1	2
0	1.239	1.736	2.468
1	2.052	2.885	3.752
2	2.327	3.330	4.271

Tables 3.3(a)&(b): Approximated dividends in the two lines for different copulas with $\tau = 0.2$

V_1	FGM		
u_1 \ u_2	0	1	2
0	0.345	0.374	0.377
1	0.831	0.923	0.935
2	1.578	1.748	1.770
	Gaussian		
u_1 \ u_2	0	1	2
0	0.344	0.373	0.376
1	0.830	0.921	0.933
2	1.577	1.746	1.768

V_2	FGM		
u_1 \ u_2	0	1	2
0	1.153	1.640	2.371
1	1.901	2.709	3.570
2	2.173	3.148	4.085
	Gaussian		
u_1 \ u_2	0	1	2
0	1.152	1.639	2.369
1	1.901	2.708	3.569
2	2.172	3.147	4.084

Tables 3.4(a)&(b): Approximated dividends in the two lines for different copulas with $\tau = -0.2$

Within each of the four Tables 3.3(a)&(b) and 3.4(a)&(b), it is clear that the dividend values are considerably close for different copulas with the Kendall's tau being fixed. When one compares the values across Tables 3.1 (i.e. the case with $\lambda_{11} = \lambda_{22} = \lambda_{12} = 1$), 3.3 and 3.4, the dividend function for each line increases as the Kendall's tau changes from -0.2 to 0 and then to 0.2. The intuition behind is as follows. For each of the three copulas, for fixed (u,v) the value of $C(u,v)$ increases with the parameter θ, which in turn increases with the Kendall's tau. Thus, for fixed values of z_1 and z_2, the joint cdf $F_{12}(z_1,z_2) = C(F_{1\bullet}(z_1), F_{\bullet 2}(z_2))$ increases with respect to the Kendall's tau. In other words, when dependency is positive, the possibility for (Z_1,Z_2) to lie outside $(0,z_1]\times(0,z_2]$ is smaller, leading to less chance of ruin of the bivariate process from a given common shock and hence more dividends. □

3.3 Proportional reinsurance

In this subsection, we illustrate the interpretation and application of our model (1.1) in problems involving proportional reinsurance (see Remark 1). To begin, we first discuss the formulation and some notations that will be used throughout. In the absence of any reinsurance, it is assumed that line 1 of business faces two independent classes of aggregate claims with Poisson arrival rates λ_{11} and λ_{12} and generic claim severities Y_1 and W respectively. In addition, line 2 is only subject to an aggregate claims process with Poisson rate λ_{22} and generic claim Y_2. Suppose that the generic claim W is more dangerous than Y_1 (e.g. W is heavy-tail and Y_1 is light-tail), and line 1 wants to reduce its risk exposure by purchasing reinsurance from line 2 for part of the risk W. We assume a proportional reinsurance contract such that line 1 retains a proportion s_1 of each claim W (and reinsures the remaining portion of $1-s_1$) for some $0<s_1<1$. Under the reinsurance arrangements described above, the model (1.1) is applicable by letting $(Z_1,Z_2)=(s_1W,(1-s_1)W)$. With $F_W(\cdot)$ being the cdf of the positive continuous random variable W, one has $F_{1\bullet}(y_1)=F_W(y_1/s_1)$ and $F_{\bullet 2}(y_2)=F_W(y_2/(1-s_1))$. Because Z_1 and Z_2 are comonotonic, the comonotonicity copula $C(u,v)=\min(u,v)$ for $0\le u,v\le 1$ should be applied. It is assumed that line 1 imposes the security loading factors η_{11} and η_{12} to the claims Y_1 and W; whereas line 2 imposes the loadings η_{21} and η_{22} to Y_2 and Z_2. Thus, the net premium income rates c_1 and c_2 are given by

$$\begin{cases} c_1 = (1+\eta_{11})\lambda_{11}E[Y_1] + \left[(1+\eta_{12})-(1+\eta_{22})(1-s_1)\right]\lambda_{12}E[W]. \\ c_2 = (1+\eta_{21})\lambda_{22}E[Y_2] + (1+\eta_{22})(1-s_1)\lambda_{12}E[W]. \end{cases} \tag{3.1}$$

Practically, the loading factor η_{22} charged by the reinsurer is no less than the loading η_{12}. Otherwise, line 1 can simply reinsure the entire risk W to earn a risk-free profit. In addition, line 1 of business should not choose to accept the risk W unless it can generate positive expected net profit. This gives rise to the condition

$$\left[(1+\eta_{12})-(1+\eta_{22})(1-s_1)\right]\lambda_{12}E[W] > s_1\lambda_{12}E[W]. \tag{3.2}$$

The left-hand side of the above equation represents the net premium income of line 1 upon accepting the risk W and reinsuring part of it; while the right-hand side is line 1's expected net claims arising from W after reinsurance.

Example 4 In this example, we assume $\lambda_{11}=\lambda_{22}=\lambda_{12}=1$, $f_{11}(y)=8e^{-2y}\sin^2 y$, $f_W(y)=5\cdot 8^5/(y+8)^6$ and $f_{22}(y)=(1/4)(0.6^2ye^{-0.6y})+(3/4)(9^2ye^{-9y})$. Note that Y_1 and Y_2 are both light-tail with mean 1 while W is heavy tail with mean 2. As in previous examples, the force of interest is assumed to be $\delta=0.05$ and the barrier values are $(b_1,b_2)=(2,2)$. In addition, the loading factors are $\eta_{11}=0.2$, $\eta_{12}=\eta_{21}=0.5$ and $\eta_{22}=1$. Plugging in these assumptions into the inequality (3.2) yields $s_1>0.5$. Then, the values of c_1 and c_2 are calculated according to (3.1) based on different choices of s_1. We shall again apply the approximation procedures stated in Section 2.4 to produce the dividend values. Note that it is not possible to apply the same scaling factors (β_1,β_2) for different values of s_1 due to the constraint $\beta_1c_1=\beta_2c_2$. In order to make a fair comparison of the dividends across different s_1, the values of β_1 (or β_2) are chosen such that the resulting values of β_1c_1 (or β_2c_2) are comparable for different s_1, so that the approximating discrete-time processes are checked at roughly the same frequency. (Recall that one time unit in the approximating discrete-time model is equivalent to $1/\beta_1 c_1$ time unit in the continuous-time model.) Our study is performed under a set of (β_1, β_2) whose values of $\beta_1 c_1$ are all around 160. The approximated expected discounted dividends for the two lines as well as their sums for various values of s_1 are given in Tables 3.5(a),(b)&(c).

V_1			
$s_1 = 0.55$, $(\beta_1, \beta_2) = (66, 48), \beta_1 c_1 = 158.4$			
$u_1 \backslash u_2$	0	1	2
0	0.696	0.750	0.757
1	1.361	1.563	1.578
2	2.157	2.481	2.516
$s_1 = 0.6$, $(\beta_1, \beta_2) = (62, 52), \beta_1 c_1 = 161.2$			
$u_1 \backslash u_2$	0	1	2
0	0.829	0.899	0.907
1	1.553	1.778	1.796
2	2.370	2.727	2.758
$s_1 = 0.65$, $(\beta_1, \beta_2) = (58, 56), \beta_1 c_1 = 162.4$			
$u_1 \backslash u_2$	0	1	2
0	0.966	1.053	1.064
1	1.743	1.991	2.012
2	2.579	2.966	2.996
$s_1 = 0.7$, $(\beta_1, \beta_2) = (54, 60), \beta_1 c_1 = 162$			
$u_1 \backslash u_2$	0	1	2
0	1.104	1.211	1.224
1	1.931	2.202	2.227
2	2.783	3.196	3.229
$s_1 = 0.75$, $(\beta_1, \beta_2) = (50, 64), \beta_1 c_1 = 160$			
$u_1 \backslash u_2$	0	1	2
0	1.242	1.372	1.388
1	2.114	2.411	2.440
2	2.982	3.420	3.458

V_2			
$s_1 = 0.55$, $(\beta_1, \beta_2) = (66, 48), \beta_1 c_1 = 158.4$			
$u_1 \backslash u_2$	0	1	2
0	1.642	2.214	2.968
1	2.391	3.399	4.280
2	2.594	3.770	4.732
$s_1 = 0.6$, $(\beta_1, \beta_2) = (62, 52), \beta_1 c_1 = 161.2$			
$u_1 \backslash u_2$	0	1	2
0	1.523	2.085	2.828
1	2.173	3.123	3.995
2	2.342	3.453	4.389
$s_1 = 0.65$, $(\beta_1, \beta_2) = (58, 56), \beta_1 c_1 = 162.4$			
$u_1 \backslash u_2$	0	1	2
0	1.392	1.942	2.673
1	1.950	2.844	3.706
2	2.091	3.131	4.048
$s_1 = 0.7$, $(\beta_1, \beta_2) = (54, 60), \beta_1 c_1 = 162$			
$u_1 \backslash u_2$	0	1	2
0	1.252	1.788	2.506
1	1.727	2.565	3.416
2	1.843	2.809	3.713
$s_1 = 0.75$, $(\beta_1, \beta_2) = (50, 64), \beta_1 c_1 = 160$			
$u_1 \backslash u_2$	0	1	2
0	1.106	1.626	2.331
1	1.505	2.289	3.128
2	1.601	2.494	3.385

Sum			
$s_1 = 0.55$, $(\beta_1, \beta_2) = (66, 48), \beta_1 c_1 = 158.4$			
$u_1 \backslash u_2$	0	1	2
0	2.338	2.964	3.725
1	3.752	4.962	5.858
2	4.751	6.251	7.248
$s_1 = 0.6$, $(\beta_1, \beta_2) = (62, 52), \beta_1 c_1 = 161.2$			
$u_1 \backslash u_2$	0	1	2
0	2.353	2.984	3.735
1	3.725	4.901	5.790
2	4.712	6.181	7.147
$s_1 = 0.65$, $(\beta_1, \beta_2) = (58, 56), \beta_1 c_1 = 162.4$			
$u_1 \backslash u_2$	0	1	2
0	2.358	2.995	3.736
1	3.693	4.835	5.718
2	4.669	6.097	7.044
$s_1 = 0.7$, $(\beta_1, \beta_2) = (54, 60), \beta_1 c_1 = 162$			
$u_1 \backslash u_2$	0	1	2
0	2.356	2.999	3.730
1	3.657	4.768	5.643
2	4.626	6.005	6.942
$s_1 = 0.75$, $(\beta_1, \beta_2) = (50, 64), \beta_1 c_1 = 160$			
$u_1 \backslash u_2$	0	1	2
0	2.348	2.998	3.719
1	3.620	4.700	5.568
2	4.583	5.914	6.843

Tables 3.5(a),(b)&(c): Approximated dividends for different s_1 with $\beta_1 c_1 \approx 160$

From Table 3.5, it can be observed that for each fixed pair of initial capital levels under consideration, the dividend values for line 1 increase while those for line 2 decrease as s_1 increases by steps of 0.05 from 0.55 to 0.75. (We have also tested larger values of s_1 up to $s_1 = 1$ and the same pattern prevails.) If one's interest is to maximize the sum of the dividend functions of the two lines, we note that the maximum is attained at different values of s_1 depending on the initial surplus levels. Among the nine pairs of initial surplus levels, six of them have the optimal joint dividends achieved at $s_1 = 0.55$. The exceptions include the cases of $(u_1, u_2) = (0, 0)$ and $(u_1, u_2) = (0, 2)$ for which the optimal s_1 is 0.65, along with the case of $(u_1, u_2) = (0, 1)$ for which the optimal s_1 is 0.7. The results suggest that in order to maximize the joint dividends in a proportional reinsurance contract, the optimal retention level s_1 should not be chosen at the extremes of 0 or 1, i.e. the risk should be shared. $\square$

4 The optimal dividend barriers for the bivariate process

In the standard univariate compound Poisson risk process under a dividend barrier strategy, it is known (see Gerber et al. (2006, Section 4)) that the optimal dividend barrier b^* which maximizes the expected discounted dividends until ruin (with respect to the barrier level b) is independent of the initial surplus $u \geq 0$, as long as $u \leq b^*$. If $u > b$, it is typically assumed that the excess amount $u - b$ over the barrier is paid immediately as a lump sum dividend so that the process will be starting at b. Under this setting, Gerber et al. (2006, Section 5) found that the dividend function also attains a local maximum at $b = b^*$ even for $u > b^*$, and they commented that in many cases this is expected to be the global maximum as well. See also Gerber et al. (2010, Section 5) for discussion of the optimal dividend barrier in a univariate discrete-time model.

Under a bivariate risk model, we will adopt the convention that if a certain line of business has its initial surplus above its own barrier level then the excess is paid immediately as dividend. Therefore, in the continuous-time setting one has

$$V_1(u_1, u_2; b_1, b_2) = \begin{cases} V_1(u_1, b_2; b_1, b_2), & 0 \leq u_1 \leq b_1; u_2 > b_2 \\ u_1 - b_1 + V_1(b_1, u_2; b_1, b_2), & u_1 > b; 0 \leq u_2 \leq b_2 \\ u_1 - b_1 + V_1(b_1, b_2; b_1, b_2), & u_1 > b; u_2 > b_2 \end{cases}$$

for line 1. Similar definition applies to line 2 and for the discrete-time model as well. We are mostly interested in the optimal pair of dividend barriers that maximize the sum of the dividend functions of the two lines. However, the techniques used to analyze the optimal barrier in the single line case as in Gerber et al. (2006) do not apply to the bivariate continuous-time model. Consequently, we will work with the discrete-time model and apply the approximation procedures in Section 2.4 to give some numerical illustrations which will provide more insights to the problem.

4.1 Are the optimal barriers independent of the initial surplus levels?

Under the bivariate (or more generally multivariate) framework, one does not expect the pair of optimal dividend barriers to be independent of the initial surplus levels. In the following brief example, we provide a fully discrete case to justify our claim.

Example 5 We consider the discrete bivariate risk process introduced in Section 2.1. The generic claims X_1 and X_2 are assumed independent so that $g(i,j) = g_1(i)g_2(j)$ for $i,j = 1,2,\ldots$. It is assumed that X_1 and X_2 follow different zero-modified geometric distributions. For line 1, we assume $g_1(0) = 0.78$ and $g_1(k) = 0.55 \times 0.6(1-0.6)^k$ for $k = 1,2,\ldots$, and thus $E[X_1] = 0.367$. For line 2, $g_2(0) = 0.8$ and $g_2(k) = 0.4 \times 0.5(1-0.5)^k$ for $k = 1,2,\ldots$, and hence $E[X_2] = 0.4$. Let the force of interest be $\alpha = 0.05$. Using the system of equations developed in Section 2.1, we have computed the dividend values for integer values of barriers (b_1, b_2) (as the barriers can only take integer values in the fully discrete model) and searched for the optimal barriers (b_1^*, b_2^*) that maximize the total dividends $V_1^d(u_1, u_2; b_1, b_2) + V_2^d(u_1, u_2; b_1, b_2)$. For $1 \leq u_1, u_2 \leq 9$, the optimal barriers are provided in Table 4.1, and the resulting optimal total dividend values are given in Table 4.2. It is clear from Table 4.1 that although most combinations of initial surplus levels do share the same pair of optimal barriers $(b_1^*, b_2^*) = (5,6)$, in general the values of (b_1^*, b_2^*) do depend on the initial surplus levels (u_1, u_2).

u_1 \ u_2	1	2	3	4	5	6	7	8	9
1	(5,6)	(5,6)	(5,5)	(5,5)	(5,5)	(5,5)	(5,5)	(5,5)	(5,5)
2	(5,6)	(5,6)	(5,6)	(5,5)	(5,5)	(5,6)	(5,6)	(5,6)	(5,6)
3	(5,6)	(5,6)	(5,6)	(5,6)	(5,5)	(5,6)	(5,6)	(5,6)	(5,6)
4	(4,5)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)
5	(4,5)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)
6	(4,5)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)
7	(4,5)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)
8	(4,5)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)
9	(4,5)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)	(5,6)

Table 4.1: The optimal pair of barriers (b_1^*, b_2^*) for $1 \leq u_1, u_2 \leq 9$

u_1 \ u_2	1	2	3	4	5	6	7	8	9
1	11.248	12.995	14.345	15.501	16.634	17.634	18.634	19.634	20.634
2	12.918	14.898	16.404	17.660	18.823	19.889	20.889	21.889	22.889
3	14.106	16.230	17.832	19.145	20.317	21.439	22.439	23.439	24.439
4	15.187	17.324	18.981	20.329	21.521	22.652	23.652	24.652	25.652
5	16.187	18.369	20.042	21.401	22.599	23.732	24.732	25.732	26.732
6	17.187	19.369	21.042	22.401	23.599	24.732	25.732	26.732	27.732
7	18.187	20.369	22.042	23.401	24.599	25.732	26.732	27.732	28.732
8	19.187	21.368	23.042	24.401	25.599	26.732	27.732	28.732	29.732
9	20.187	22.368	24.042	25.401	26.599	27.732	28.732	29.732	30.732

Table 4.2: The optimal total dividends for $1 \leq u_1, u_2 \leq 9$

□

4.2 Examination of the table of the joint dividends

For the rest of the paper, we follow closely the model settings as in Example 2 in Section 3.1 with Poisson rates $\lambda_{11} = \lambda_{22} = \lambda_{12} = 1$, i.e. the premium rates are $c_1 = 2.2$ and $c_2 = 3.3$, the claim pdf's are $f_{11}(y) = f_{1\bullet}(y) = 8e^{-2y}\sin^2 y$ and $f_{22}(y) = f_{\bullet 2}(y) = (1/4)(0.6^2 ye^{-0.6y}) + (3/4)(9^2 ye^{-9y})$ with Z_1 and Z_2 independent, and the force of interest is $\delta = 0.05$. The approximation procedures outlined at the beginning of Section 2.4 will be applied throughout. Because we will look at larger barrier levels, the smaller scaling factors of $(\beta_1, \beta_2) = (3, 2)$ will be applied throughout (see concluding remarks in Section 5). As in the fully discrete Example 5, we have tested that the optimal barriers depend on the initial capital levels. Since we use $(\beta_1, \beta_2) = (3, 2)$, the initial surplus levels (u_1, u_2) and the barriers (b_1, b_2) (and hence the optimal barriers (b_1^*, b_2^*)) in the continuous-time model being approximated can be in the fractional form of e.g. $(9\frac{1}{3}, 10\frac{1}{2})$. However, for illustrative purposes, we only consider integer values of (u_1, u_2) and (b_1, b_2) for convenience.

b_1 \ b_2	1	2	3	4	...	9	10	11	12
1	11.741	10.912	10.045	9.139	...	6.175	5.988	5.859	5.770
2	12.596	12.158	11.670	11.083	...	7.978	7.568	7.224	6.936
3	12.638	12.583	12.537	12.396	...	10.144	9.656	9.202	8.786
4	12.175	12.391	12.704	12.973	...	12.063	11.621	11.167	10.719
...	...	...	...	...	...	...	...	...	...
8	10.174	10.567	11.221	12.014	...	14.603	**14.641**	14.556	14.374
9	10.029	10.385	11.000	11.765	...	14.500	14.596	14.567	14.435
10	9.943	10.267	10.845	11.576	...	14.336	14.467	14.473	14.374
11	9.893	10.193	10.739	11.436	...	14.155	14.303	14.329	14.251

Table 4.3: Approximated total dividends when $(u_1, u_2) = (5, 5)$

b_1 \ b_2	1	2	3	4	...	9	10	11	12
1	12.741	11.912	11.045	10.139	...	6.469	6.197	6.009	5.879
2	13.596	13.158	12.670	12.083	...	8.567	8.067	7.648	7.297
3	13.638	13.583	13.537	13.396	...	10.942	10.382	9.861	9.384
4	13.175	13.391	13.704	13.973	...	13.010	12.519	12.013	11.513
...	...	...	...	...	...	...	...	...	...
8	11.174	11.567	12.221	13.014	...	15.726	**15.763**	15.668	15.469
9	11.029	11.385	12.000	12.765	...	15.619	15.718	15.681	15.534
10	10.943	11.267	11.845	12.576	...	15.450	15.584	15.583	15.471
11	10.893	11.193	11.739	12.436	...	15.264	15.415	15.434	15.343

Table 4.4: Approximated total dividends when $(u_1, u_2) = (5, 6)$

b_1 \ b_2	1	2	3	4	...	9	10	11	12
1	13.741	12.912	12.045	11.139	...	6.881	6.485	6.212	6.023
2	14.596	14.158	13.670	13.083	...	9.248	8.641	8.131	7.704
3	14.638	14.583	14.537	14.396	...	11.786	11.150	10.556	10.012
4	14.175	14.391	14.704	14.973	...	13.960	13.422	12.864	12.310
...	...	...	...	...	...	...	...	...	...
8	12.174	12.567	13.221	14.014	...	16.794	**16.831**	16.726	16.508
9	12.029	12.385	13.000	13.765	...	16.686	16.785	16.740	16.577
10	11.943	12.267	12.845	13.576	...	16.514	16.648	16.640	16.512
11	11.893	12.193	12.739	13.436	...	16.324	16.475	16.487	16.381

Table 4.5: Approximated total dividends when $(u_1, u_2) = (5, 7)$

In order to study how the barrier values affect the total expected discounted dividends, we have fixed (u_1, u_2) to be $(5, 5)$, $(5, 6)$ and $(5, 7)$ in turn and then tabulated the approximated total dividend values for various choices of (b_1, b_2) in Tables 4.3-4.5. The bold number in each table shows the largest dividend value for its specific combination of (u_1, u_2). We first look at Table 4.3 for which $(u_1, u_2) = (5, 5)$. If one fixes $b_1 = 1$, $b_1 = 2$ or $b_1 = 3$, then the total dividend value is decreasing in b_2, i.e. $b_2 = 1$ gives the largest total dividends. If the value of b_1 is fixed to be larger, then a larger value of b_2 is required to maximize the total dividends. This is also depicted graphically in Figure 4.1, which plots the approximated total dividends against the barrier value b_2 for each fixed $b_1 = 2, 4, 6, 8, 10, 12$. Similar phenomenon is observed if one instead fixes the value of b_2 and varies b_1, and the same is true even when one looks at Tables 4.4 and 4.5. This suggests that in order to achieve high joint dividends, the barriers should be fairly close to each other. In addition, as indicated by the bold number in each table, it is found that the maximum total dividend value is achieved at the barriers $(b_1^*, b_2^*) = (8, 10)$ for all three pairs of initial surplus levels considered. This will be further discussed in Section 4.3.

INSERT FIGURE 4.1

Figure 4.1: Plot of approximated total dividends against b_2 when $(u_1, u_2) = (5, 5)$

Note that each of Tables 4.3-4.5 is divided into four sections: (i) $b_1 < u_1$ and $b_2 < u_2$; (ii) $b_1 \geq u_1$ and $b_2 < u_2$; (iii) $b_1 < u_1$ and $b_2 \geq u_2$; and (iv) $b_1 \geq u_1$ and $b_2 \geq u_2$. In the first section, the barrier level is lower than the initial surplus for each line, and therefore each line pays a dividend at time 0 and the bivariate process actually starts at (b_1, b_2). Going from Table 4.3 to Table 4.4, the only change is that line 2 possesses one more unit of initial capital, and this explains the fact that each dividend value in Table 4.4 is exactly one unit larger than the corresponding value in Table 4.3 within the first section.

The same observation applies to Tables 4.4 and 4.5 as well. For the second section, the dividend values show the same properties as those in the first section because it is line 2 that pays a dividend at time 0 and then the process starts at (u_1, b_2). In the third section, line 1 (instead of line 2) needs to pay the excess of u_1 over b_2 and then the process starts at (b_1, u_2). In the fourth section, none of the two lines pay out immediate dividends at time 0. Note that all the dividend values in Table 4.5 are higher than the corresponding ones in Table 4.4, which are in turn larger than those in Table 4.3. This is expected since the dividends must be increasing in the initial capital u_2.

4.3 Optimal barriers and restricted optimal barriers

In this section, we are interested in the pair of optimal barriers for every combination of (integer values of) (u_1, u_2) for $1 \leq u_1 \leq 9$ and $1 \leq u_2 \leq 12$. The optimal barriers and the corresponding optimal joint dividend values are given in Tables 4.6 and 4.7 respectively.

u_1 \ u_2	1	2	3	4	5	6	7	8	9	10	11	12
1	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)
2	(8,9)	(8,9)	(7,8)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)
3	(8,9)	(8,9)	(8,9)	(8,9)	(8,9)	(7,8)	(7,8)	(7,8)	(7,8)	(7,8)	(7,8)	(7,8)
4	(8,10)	(8,10)	(8,10)	(8,10)	(8,9)	(8,9)	(8,9)	(8,9)	(8,9)	(8,9)	(8,9)	(8,9)
5	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)
6	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)
7	(6,8)	(7,9)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)
8	(6,8)	(7,9)	(7,9)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(9,11)	(9,11)	(9,11)
9	(6,8)	(7,9)	(7,9)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(8,10)	(9,11)	(9,11)	(9,11)

Table 4.6: The optimal barriers for $1 \leq u_1 \leq 9$ and $1 \leq u_2 \leq 12$

u_1 \ u_2	1	2	3	4	5	6	7	8	9	10	11	12
1	4.309	5.309	6.309	7.309	8.309	9.309	10.309	11.309	12.309	13.309	14.309	15.309
2	5.839	6.752	7.644	8.622	9.622	10.622	11.622	12.622	13.622	14.622	15.622	16.622
3	7.250	8.400	9.498	10.534	11.521	12.498	13.473	14.460	15.460	16.460	17.460	18.460
4	8.371	9.701	10.959	12.133	13.241	14.295	15.315	16.318	17.316	18.316	19.316	20.316
5	9.291	10.766	12.156	13.445	14.641	15.763	16.831	17.863	18.874	19.875	20.875	21.875
6	10.083	11.672	13.164	14.539	15.807	16.986	18.098	19.160	20.189	21.198	22.198	23.198
7	11.012	12.537	14.074	15.512	16.831	18.051	19.193	20.276	21.318	22.332	23.332	24.332
8	12.012	13.537	15.044	16.467	17.810	19.049	20.206	21.300	22.348	23.374	24.383	25.383
9	13.012	14.537	16.044	17.467	18.810	20.049	21.206	22.300	23.348	24.379	25.390	26.390

Table 4.7: Approximated optimal dividends for $1 \leq u_1 \leq 9$ and $1 \leq u_2 \leq 12$

From Table 4.6, although the optimal barriers (b_1^*, b_2^*) vary with the initial surplus level (u_1, u_2), they often take the value of $(b_1^*, b_2^*) = (8, 10)$. The anomalies of lower optimal barriers usually happen when u_1 or u_2 is small. In particular, when line 1 possesses low initial surplus of $u_1 = 1$ or $u_1 = 2$, the optimal barriers (b_1^*, b_2^*) are mostly the small values of $(3, 1)$. Intuitively, when one of the two lines possesses low initial surplus, the bivariate process is likely to ruin early anyway. To optimize joint dividends, it is important to ensure that some early dividends are paid before ruin (in terms of immediate dividend at time 0 or reaching the barrier early), resulting in lower optimal barriers (b_1^*, b_2^*). However, this effect is a bit less obvious when u_2 is low. One possible explanation is that line 2 has a higher security loading (as it has higher premium rate but the same expected claim costs compared to line 1) and hence lower chance of early ruin than line 1, all else being equal. It is also instructive to note that the optimal barriers are always either both high or both low, i.e. it is not optimal for one insurer to set a high barrier if the other one has a low barrier and vice versa. This can be attributed to the fact that dividend payments for both lines cease at the joint ruin time T, and if one of the lines has large positive surplus at time T it would have been better paid as a dividend at the beginning. See Section 3.1 for similar comments.

Turning to Table 4.7, it is clear that the discounted dividend increases with respect to both initial surplus levels. The table can also help us study an optimal allocation problem as well if the criterion is to maximize the total dividends of the two lines. (See also e.g. Loisel (2005, Section 5) or Gong et al. (2012, Section 6.3) for discussion of a capital allocation that minimizes multivariate risk measures.) Suppose that both business lines belong to a larger corporation who wants to choose (u_1, u_2) to maximize $V_1(u_1, u_2; b_1, b_2)+V_2(u_1, u_2; b_1, b_2)$ subject to the constraint $u_1+u_2 = K$ (and of course $u_1 \geq 0$ and $u_2 \geq 0$) for a given total initial capital of $K > 0$. This can be regarded as a two-step procedure. First, the optimal pairs of barriers (b_1^*, b_2^*) and the resulting optimal joint dividends are determined as in Tables 4.6 and 4.7. Then, we can look at the line $u_1 + u_2 = K$ in Table 4.7 to find the optimal combination of (u_1^*, u_2^*) that gives the highest dividend value. For easy reference, we additionally plot the optimal joint dividends against the capital u_2 allocated to line 2 given that $K = 5, 6, 7, 8, 9, 10$ in Figure 4.2. For example, if $K = 7$ then $u_2^* = 3$ yields the highest joint dividends and hence $u_1^* = K - u_2^* = 4$; if $K = 10$ then $(u_1^*, u_2^*) = (5, 5)$ (and in both cases $(b_1^*, b_2^*) = (8, 10)$). It is noted that the optimal allocation (u_1^*, u_2^*) appears to occur at places where the total capital K is roughly equally split. The intuitive reason is that if the allocation is at either extreme end, then it is more likely that one of the two lines possesses positive surplus at the joint ruin time T and resources are wasted (see Section 3.1).

INSERT FIGURE 4.2

Figure 4.2: Plot of approximated optimal dividends against u_2 under $u_1 + u_2 = K$

So far, when we maximize the joint dividends we place no restrictions on whether the barriers should be below or above the respective initial surplus levels of the two lines. However, we already know from Table 4.6 that this could lead to optimal barriers that are much lower than the initial surplus levels, leading to earlier ruin than the case if higher barriers are applied. Practically, early ruin may not be desirable for risk management purposes even dividends are maximized. These lead to the idea of maximizing dividends under a penalty at ruin or a ruin probability constraint (see e.g. Dickson and Waters (2004), Dickson and Drekic (2006), Gerber et al. (2006), and Thonhauser and Albrecher (2007)). In the present context, we can delay ruin by maximizing the joint dividends under the constraint that the barrier levels should be no less than the respective initial surplus levels. The resulting barrier levels will be called 'restricted optimal barriers'. Tables 4.8 and 4.9 give the restricted optimal barriers and the resulting joint dividend values respectively.

u_1 \ u_2	1	2	3	4	5	6	7	8	9	10	11	12
1	**(3,1)**	(3,2)	(4,3)	(5,4)	(5,5)	(6,6)	(7,7)	(7,8)	(8,9)	(8,10)	(9,11)	(9,12)
2	**(8,9)**	**(8,9)**	**(7,8)**	(7,8)	(5,5)	(6,6)	(7,7)	(7,8)	(8,9)	(8,10)	(9,11)	(9,12)
3	**(8,9)**	**(8,9)**	**(8,9)**	**(8,9)**	**(8,9)**	**(7,8)**	**(7,8)**	**(7,8)**	(8,9)	(8,10)	(9,11)	(9,12)
4	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,9)**	**(8,9)**	**(8,9)**	**(8,9)**	**(8,9)**	(8,10)	(9,11)	(9,12)
5	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	(9,11)	(9,12)
6	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	(9,11)	(9,12)
7	(7,9)	**(7,9)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	(9,11)	(9,12)
8	(8,10)	(7,9)	(7,9)	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(8,10)**	**(9,11)**	**(9,11)**	(9,12)
9	(9,11)	(9,11)	(9,11)	(9,11)	(9,11)	(9,11)	(9,11)	(9,11)	(9,11)	**(9,11)**	**(9,11)**	(9,12)

Table 4.8: The restricted optimal barriers for $1 \leq u_1 \leq 9$ and $1 \leq u_2 \leq 12$

u_1 \ u_2	1	2	3	4	5	6	7	8	9	10	11	12
1	**4.309**	4.906	5.553	6.222	6.899	7.532	8.108	8.634	9.089	9.483	9.812	10.092
2	**5.839**	**6.752**	**7.644**	8.505	9.397	10.354	11.231	12.037	12.736	13.345	13.854	14.289
3	**7.250**	**8.400**	**9.498**	**10.534**	**11.521**	**12.498**	**13.473**	**14.460**	15.359	16.146	16.804	17.369
4	**8.371**	**9.701**	**10.959**	**12.133**	**13.241**	**14.295**	**15.315**	**16.318**	**17.316**	18.245	19.026	19.699
5	**9.291**	**10.766**	**12.156**	**13.445**	**14.641**	**15.763**	**16.831**	**17.863**	**18.874**	**19.875**	20.751	21.511
6	**10.083**	**11.672**	**13.164**	**14.539**	**15.807**	**16.986**	**18.098**	**19.160**	**20.189**	**21.198**	22.143	22.970
7	10.932	**12.537**	**14.074**	**15.512**	**16.831**	**18.051**	**19.193**	**20.276**	**21.318**	**22.332**	23.324	24.199
8	11.680	13.395	14.998	**16.467**	**17.810**	**19.049**	**20.206**	**21.300**	**22.348**	**23.374**	**24.383**	25.288
9	12.267	14.068	15.747	17.281	18.678	19.960	21.151	22.273	23.344	**24.379**	**25.390**	26.306

Table 4.9: Approximated restricted optimal dividends for $1 \leq u_1 \leq 9$ and $1 \leq u_2 \leq 12$

In both Tables 4.8 and 4.9, the numbers in bold correspond to the positions where the values are identical to those in Tables 4.6 and 4.7. These cells are mainly where the initial surplus levels are of fairly balanced values. At many other positions, the restricted optimal barriers in Table 4.8 are much higher than the globally optimal barriers in Table 4.6. Moreover, since Table 4.9 is a result of constrained optimization, its values are no larger than those in Table 4.7. However, it is instructive to note that except when $u_1 = 1$ the dividend values in Table 4.9 are still comparable to those in Table 4.7. This suggests that applying the restricted optimal barriers can actually delay ruin (due to higher barriers) without sacrificing much dividends. Nonetheless, Table 4.8 shows the same phenomenon as in Table 4.6 that the restricted optimal barriers of the two lines are always of similar values. The intuitive reason is similar to that for the globally optimal barriers. When one turns to the problem of capital allocation based on the restricted optimal barriers, the results of the optimal allocation are identical to the case where the globally optimal barriers are used, at least up to $K = 10$. This is depicted in Figure 4.3.

INSERT FIGURE 4.3

Figure 4.3: Plot of approximated restricted optimal dividends against u_2 under $u_1 + u_2 = K$

4.4 A modified type of barrier strategy

In this section, we shall study a modified type of barrier strategy based on some observations from Table 4.7 regarding the dividend values under the globally optimal barriers. It has been always assumed that at time 0 the two lines of business fix their barrier levels that will not be changed later on. But if time 0 is a decision time to set the barriers, it would make sense to allow immediate dividends to be paid at time 0 so that the bivariate process moves to a better starting position from which the new globally optimal barriers are implemented.

The above idea can be illustrated with a concrete example as follows. Suppose that the bivariate risk process starts with initial surplus levels $(u_1, u_2) = (3, 8)$. From Tables 4.6 and 4.7, we know that the optimal joint dividend value is 14.460 under the optimal barriers $(7, 8)$. However, if line 2 pays an immediate dividend of 1, then the bivariate process moves to the new position $(3, 7)$ and the optimal barriers for the initial surplus levels $(3, 7)$ (which happen to be $(7, 8)$ also) can be applied. This will result in higher total joint dividends of $1 + 13.473 = 14.473$. But if line 2 continues paying an immediate dividend of 1, moving the bivariate process to $(3, 6)$, then the total joint dividends will be even higher at 14.498. The procedure continues, and no further improvement is possible upon reaching position $(3, 4)$ where total joint dividends of 14.534 can be enjoyed. To summarize, starting with $(u_1, u_2) = (3, 8)$, the overall strategy would be for line 2 to pay 4 at time 0, and then implement the barriers $(8, 9)$ so that the expected present value of future dividends is 10.534.

Following the above arguments, we have tabulated the optimal parameters of the modified barrier strategy in Table 4.10. In each cell, the upper pair is the target starting position while the lower pair represents the optimal barriers for the new starting position. Table 4.11 gives the dividend values accordingly.

u_1 \ u_2	1	2	3	4	5	6	7	8	9	10	11	12
1	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)	(1,1) (3,1)
2	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)	(2,1) (8,9)
3	(3,1) (8,9)	(3,2) (8,9)	(3,3) (8,9)	(3,4) (8,9)	(3,4) (8,9)	(3,4) (8,9)	(3,4) (8,9)	(3,4) (8,9)	(3,4) (8,9)	(3,4) (8,9)	(3,4) (8,9)	(3,4) (8,9)
4	(4,1) (8,10)	(4,2) (8,10)	(4,3) (8,10)	(4,4) (8,10)	(4,5) (8,9)	(4,6) (8,9)	(4,7) (8,9)	(4,8) (8,9)	(4,8) (8,9)	(4,8) (8,9)	(4,8) (8,9)	(4,8) (8,9)
5	(4,1) (8,10)	(5,2) (8,10)	(5,3) (8,10)	(5,4) (8,10)	(5,5) (8,10)	(5,6) (8,10)	(5,7) (8,10)	(5,8) (8,10)	(5,9) (8,10)	(5,10) (8,10)	(5,10) (8,10)	(5,10) (8,10)
6	(4,1) (8,10)	(5,2) (8,10)	(6,3) (8,10)	(6,4) (8,10)	(6,5) (8,10)	(6,6) (8,10)	(6,7) (8,10)	(6,8) (8,10)	(6,9) (8,10)	(6,10) (8,10)	(6,10) (8,10)	(6,10) (8,10)
7	(4,1) (8,10)	(5,2) (8,10)	(6,3) (8,10)	(6,4) (8,10)	(7,5) (8,10)	(7,6) (8,10)	(7,7) (8,10)	(7,8) (8,10)	(7,9) (8,10)	(7,10) (8,10)	(7,10) (8,10)	(7,10) (8,10)
8	(4,1) (8,10)	(5,2) (8,10)	(6,3) (8,10)	(6,4) (8,10)	(7,5) (8,10)	(7,6) (8,10)	(8,7) (8,10)	(8,8) (8,10)	(8,9) (8,10)	(8,10) (9,11)	(8,11) (9,11)	(8,11) (9,11)
9	(4,1) (8,10)	(5,2) (8,10)	(6,3) (8,10)	(6,4) (8,10)	(7,5) (8,10)	(7,6) (8,10)	(8,7) (8,10)	(8,8) (8,10)	(8,9) (8,10)	(9,10) (9,11)	(9,11) (9,11)	(9,11) (9,11)

Table 4.10: Optimal parameters of the modified barrier strategy

u_1 \ u_2	1	2	3	4	5	6	7	8	9	10	11	12
1	4.309	5.309	6.309	7.309	8.309	9.309	10.309	11.309	12.309	13.309	14.309	15.309
2	5.839	6.839	7.839	8.839	9.839	10.839	11.839	12.839	13.839	14.839	15.839	16.839
3	7.250	8.400	9.498	10.534	11.534	12.534	13.534	14.534	15.534	16.534	17.534	18.534
4	8.371	9.701	10.959	12.133	13.241	14.295	15.315	16.318	17.318	18.318	19.318	20.318
5	9.371	10.766	12.156	13.445	14.641	15.763	16.831	17.863	18.874	19.875	20.875	21.875
6	10.371	11.766	13.164	14.539	15.807	16.986	18.098	19.160	20.189	21.198	22.198	23.198
7	11.371	12.766	14.164	15.539	16.831	18.051	19.193	20.276	21.318	22.332	23.332	24.332
8	12.371	13.766	15.164	16.539	17.831	19.051	20.206	21.300	22.348	23.374	24.383	25.383
9	13.371	14.766	16.164	17.539	18.831	20.051	21.206	22.300	23.348	24.379	25.390	26.390

Table 4.11: Approximated optimal dividends under modified barrier strategy

First, the cells in Tables 4.9-10 with white background indicate positions (i) where there does not exist any modified barrier strategy that can beat the (globally) optimal barrier strategy in Tables 4.6-7; (ii) that are not the target starting positions for other starting initial surplus levels under consideration; and (iii) that do not involve any immediate dividends at time 0. Second, a (partial) column or row with black numbers and the same grey background in Tables 4.9-10 represents positions that all collapse to the uppermost or leftmost cell within that (partial) column or row. For example, as long as $u_1 = 3$ and $4 \leq u_2 \leq 12$, line 2 should pay a dividend of $u_2 - 4$ at time 0 and then the two lines should implement $(8, 9)$ as the barriers. As another example, within the group where $5 \leq u_1 \leq 9$ and $u_2 = 2$, line 1 immediately pays out $u_1 - 5$ and then the two lines apply the barriers $(8, 10)$. Except for the target positions, all these cells with grey background in Table 4.10 have strictly higher dividend values than the corresponding ones under the (globally) optimal barrier strategy given in Table 4.7. Finally, each remaining cell with the darkest background and white number actually has the same modified barrier strategy as the (globally) optimal barrier strategy, but the interpretation is slightly more complicated. For example, when $u_1 = 5$ and $10 \leq u_2 \leq 12$, Table 4.6 indicates (globally) optimal barriers of $(b_1^*, b_2^*) = (8, 10)$. Since $u_2 \geq b_2^*$, this essentially means that line 2 should pay an immediate dividend of $u_2 - 10$ (moving the bivariate process to $(5, 10)$) and continue to apply the barriers $(8, 10)$. Again from Table 4.6, it is known that the (globally) optimal barriers corresponding to the initial surplus levels $(u_1, u_2) = (5, 10)$ are also $(8, 10)$. Therefore, the above description is indeed identical to the modified strategy given in Table 4.10, with $(5, 10)$ being the target starting position and $(8, 10)$ the new barriers.

It is instructive to note that the dividend values in Table 4.11 are no less than those in Table 4.7 under the (globally) optimal barriers. More importantly, in some cases where the (globally) optimal barriers are low in Table 4.6, application of our proposed modified barrier strategy can lead to later ruin time as well. For example, when $u_1 = 2$ and $4 \leq u_2 \leq 12$, under both strategies in Tables 4.6 and 4.10 the bivariate process essentially starts at the initial surplus levels $(2, 1)$ after payment of an immediate dividend at time 0. But the higher barriers of $(8, 9)$ applied under the modified strategy (compared to the barriers $(3, 1)$ in Table 4.6) mean that the bivariate process can now survive longer. Therefore, our proposed modified strategy could have the advantage of increased joint dividends and delayed ruin time in comparison with the standard barrier strategy. Finally, Figure 4.4 shows that our modified barrier strategy leads to the same optimal capital allocation as in Figures 4.2 and 4.3 for at least up to $K = 10$.

INSERT FIGURE 4.4

Figure 4.4: Plot of approximated optimal dividends against u_2 under $u_1 + u_2 = K$ with the modified barrier strategy

5 Concluding Remarks

In this paper, a discretization procedure is developed to approximate a continuous-time bivariate risk process. Applications to related optimal problems in reinsurance, capital allocation and dividends are illustrated with numerical examples. A modified dividend barrier strategy which can lead to increased dividends and longer survival time is proposed.

There are various directions for future research. First, with the barrier levels (b_1, b_2) in the continuous-time model along with the scaling factors (β_1, β_2), one needs to solve a system of $(\beta_1 b_1 + 1)(\beta_2 b_2 + 1)$ linear equations in the discrete model according to the approximation procedures outlined at the beginning of Section 2.4. In cases where (b_1, b_2) and (β_1, β_2) are both large, the computer can actually run out of memory. (This explains the choices of low barriers in Section 3 and low scaling factors in Section 4.) More efficient computational methods should be explored. Second, in principle our procedures can be extended from bivariate to multivariate processes. But the calculations will be far more tedious, and again better algorithms will be needed. Third, the present model may be modified so that capital transfer between lines (e.g. Hult and Lindskog (2006)) is possible when one business line is in danger while the other has abundant capital. Finally, one may also attempt to obtain explicit expressions under the simplest model assumptions such as exponential claims with common shocks only or under proportional reinsurance. We leave these as open questions.

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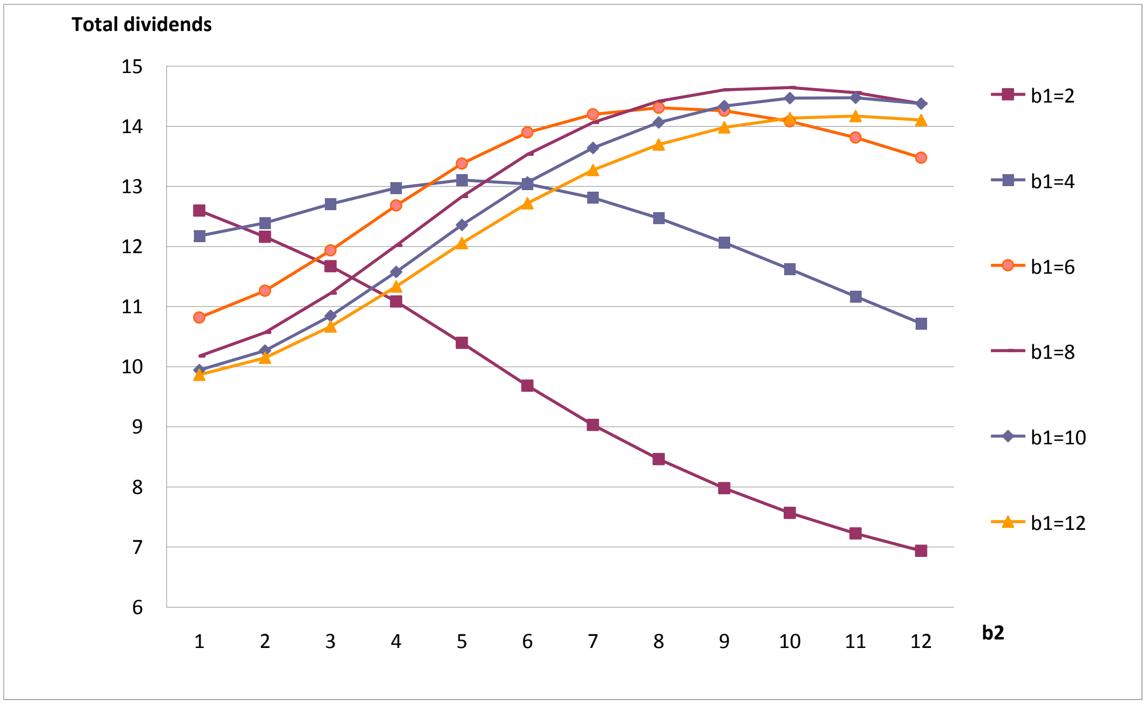

Figure 4.1

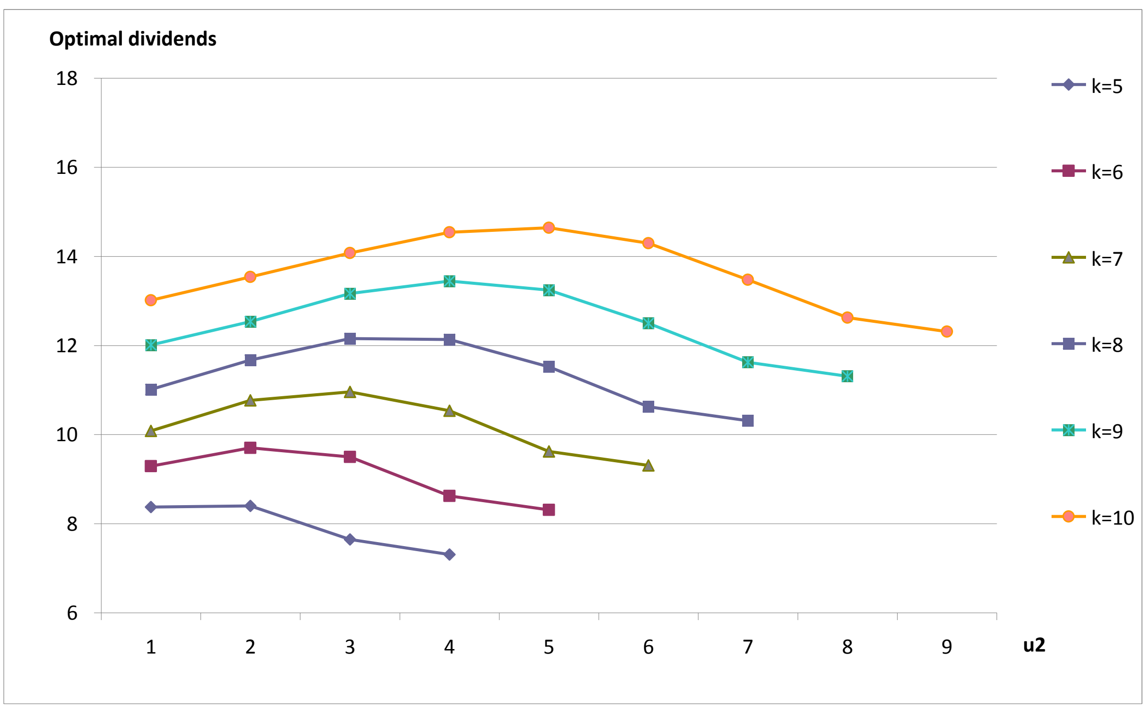

Figure 4.2

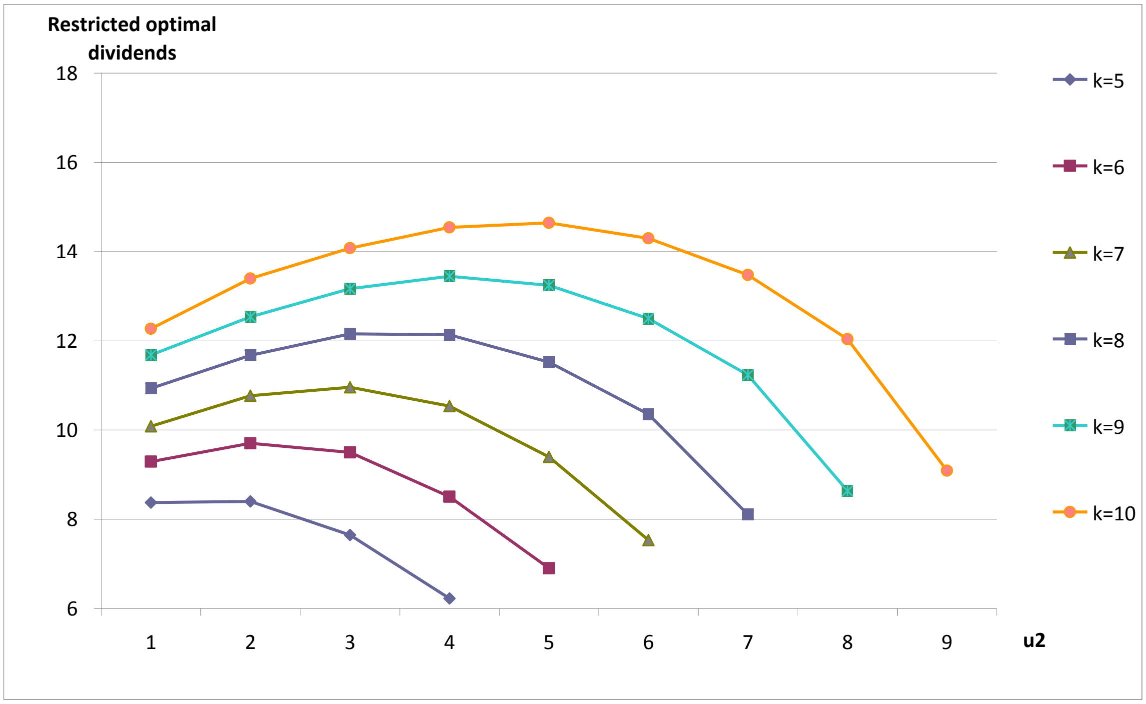

Figure 4.3

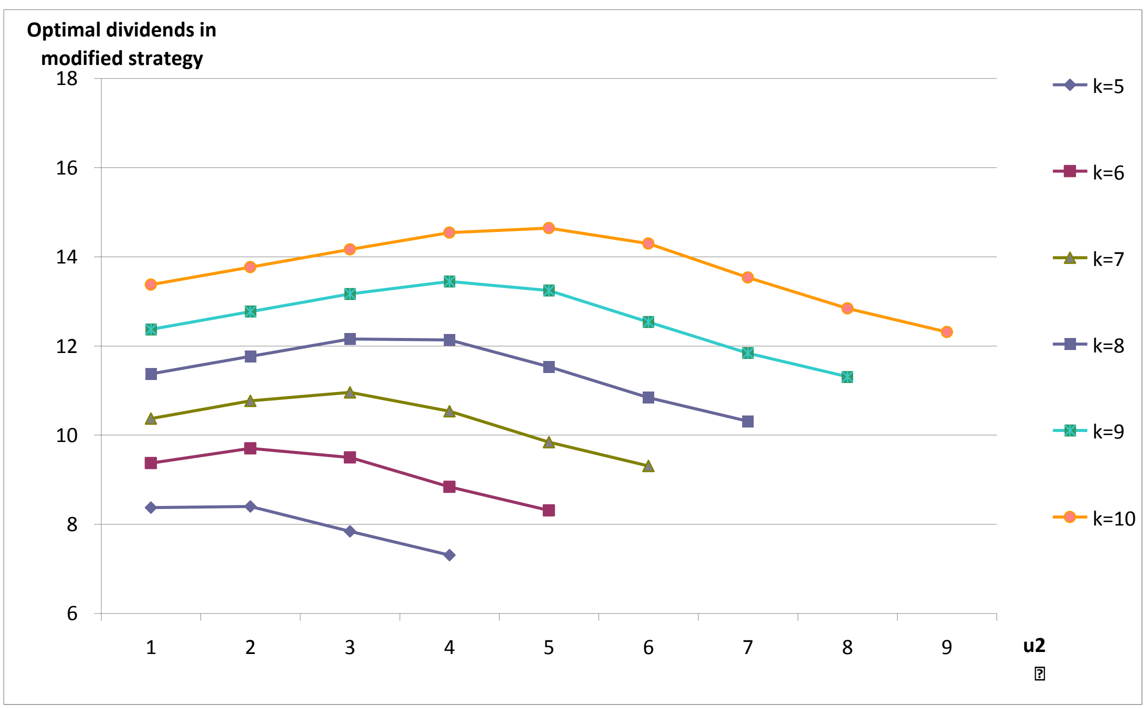

Figure 4.4