Proof within the Western and the Eastern cultural traditions: Implications for mathematics education

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Presentation by SIU, Man Keung of the University of Hong Kong

There is something about mathematics that is universal, irrespective of race, culture or social context. For instance, no mathematician will accept the following "proof", offered as a "joke-proof" by Oscar Perron (1880-1975) but not without pedagogical purpose:

> **"Theorem":** 1 is the largest natural number.
> **"Proof":** Suppose N is the largest natural number, then N^2 cannot exceed *N*, so $N(N-1) = N^2 - N$ is not positive. This means that $N-1$ is not positive, or that N cannot exceed 1. But *N* is at least 1. Hence, $N = 1$. Q.E.D.

Likewise, well-known paradoxes on argumentation exist in both the Western and the Eastern world. The famous Liar Paradox, embodied in the terse but intriguing remark "I am a liar", is ascribed to the 4th century B.C.E. Greek philosopher Eubulides of Miletus. A similar flavour is conveyed in the famous shield-and-halberd story told by the Chinese philosopher Hon Fei Zi (Book 15, Section XXXVI, *Hon Fei Zi*, c.3rd Century B.C.E.):

> "My shields are so solid that nothing can penetrate them.
> My halberds are so sharp that they can penetrate anything."
> "How about using your halberds to pierce through your shields?"

In the Chinese language the term "*mao dun*", literally "halberd and shield", is used to mean "contradiction". Indeed, Hon Fei Zi used this story as an analogy to prove that the

Confucianist School was inadequate while the Legalist School was effective and hence superior[1]. His proof is by *reductio ad absurdum*.

In his book *A Mathematician's Apology*, English mathematician Godfrey Harold Hardy (1877-1947) said that "*reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons" (Hardy, 1940/1967, p. 94). Many people are led by this remark to see the technique of proof by contradiction as a Western practice, even to the extent that they wonder whether the technique is closely related to Greek, and hence Western, culture. I was once asked whether Chinese students would have inherent difficulty in learning proof by contradiction, because such argumentation was absent from traditional Chinese mathematics. My immediate response was that this learning difficulty shows up in a majority of students, Chinese or non-Chinese, and does not seem to be related to a student's cultural background. Nonetheless, this query urged me to look for examples of proof by contradiction in traditional Chinese thinking. Since then, I have gathered some examples, many of which are in a non-mathematical context. One mathematical presentation that approaches a proof by contradiction is Liu Hui's (c. 3rd century C.E.) argument in his commentary on Chapter 1 of *Jiu Zhang Suan Shu* (*Nine Chapters on the Mathematical Art*) explaining why the ancients were wrong in taking 3 to be the ratio of the perimeter of a circle to its diameter (Siu, 1993, p. 348). Still, I have

[1] The Confucianist School and the Legalist School were two streams of thought in ancient China, which would be too vast a subject to be explained, even in brief, here. If suffices to point out that the Legalist School maintained that good government was based on law and authority instead of on special ability and high virtue of the ruler who set an exemplar to influence the people. In particular, the story of shields and halberds was employed to stress that the two legendary leaders, Yao and Shun, whom the Confucianist School extolled as sage-kings, could not be both held in high regard.

not yet found a written proof in an ancient Chinese text that recognizably follows prominently and distinctly the Greek fashion of *reductio ad absurdum*.

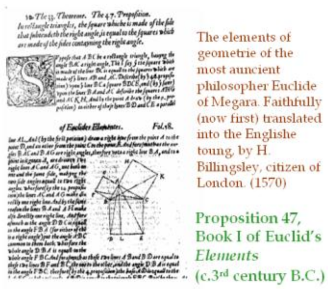
However, the notion of a proof is not so clear-cut when it comes to different cultures as well as different historical epochs. Mathematics practiced in different cultures and in different historical epochs may have its respective different styles and emphases. For the sake of learning and teaching it will be helpful to study such differences.

Unfortunately, many Western mathematicians have come to regard Eastern mathematical traditions as not ‘real’ mathematics. For example, take Hardy’s assessment:

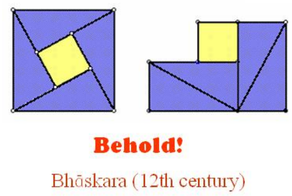
> The Greeks were the first mathematicians who are still ‘real’ to us to-day. Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. The Greeks first spoke a language which modern mathematicians can understand; as Littlewood said to me once, they are not clever schoolboys or ‘scholarship candidates’, but ‘Fellows of another college’. (Hardy, 1940/1967, pp. 80-81)

However, proper study of the different traditions leads one to disagree with Hardy’s assessment.

A typical example of the cross-cultural difference in style and emphasis is the age-old result known in the Western world as Pythagoras’ Theorem. Compare the proof given in Proposition 47, Book I of Euclid’s *Elements* (c. 3rd century B.C.E.) (Figure 1) and that given by the Indian mathematician Bhaskara in the 12th century C.E. (Figure 2). The former is a deductive argument with justification provided at every step. The latter is a visually clear dissect-and-reassemble procedure, so clear that Bhaskara found it adequate to simply qualify the argument by a single word, "Behold!"



Euclid's proof of Pythagoras' Theorem
Fig. 1



Bhaskara's proof of Pythagoras' Theorem
Fig. 2

In her plenary address (this volume), Judith Grabiner noted how the notion of proof permeates other human endeavour in the Western world. Indeed, one finds the following passage in Book1.10 in *Institutio Oratoria* by Marcus Fabius Quintilianus (1st century C.E.):

> Geometry [Mathematics] is divided into two parts, one dealing with Number, the other with Form. Knowledge of numbers is essential not only to the orator, but to anyone who has had even a basic education. (…) In the first place, order is a necessary element in geometry; is it not also in eloquence? Geometry proves subsequent propositions from preceding ones, the uncertain from the certain: do we not do the same in speaking? Again: does not the solution of the problems rest almost wholly on Syllogisms? (…) Finally, the most powerful proofs are commonly called "linear demonstrations". And what is the aim of oratory if not proof? Geometry also uses reasoning to detect falsehoods which appear like truths. (…) So, if (as the next book will prove) an orator has to speak on all subjects, he cannot be an orator without geometry [mathematics]. (Quintilian, 2001, pp. 231, 233, 237)

Stephen Toulmin, in examining "how far logic *can* hope to be a formal science, and yet retain the possibility of being applied in the critical assessment of actual arguments"

(Toulmin 1958, p.3), opines that one source from which the notion of proof arose is argument on legal matters. He propounds a need for a rapprochement between logic and epistemology, for a re-introduction of historical, empirical and even anthropological considerations into the subject which philosophers have prided themselves on purifying:

> The patterns of argument in geometrical optics, for instance (…) are distinct from the patterns to be found in other fields: e.g. in a piece of historical speculation, a proof in the infinitesimal calculus, or the case for the plaintiff in a civil suit alleging negligence. Broad similarities there may be between arguments in different fields, (…) it is our business, however, not to insist on finding such resemblances at all costs but to keep an eye open quite as much for possible differences. (Toulmin, 1958, p. 256)

This year (2009) is the 200th anniversary of the birth of the great English naturalist Charles Darwin (1809-1882) and the 150th anniversary of the publication of *On the Origin of Species* (1859). Not many may have noted what Darwin once said in his autobiography about mathematics:

> I attempted mathematics, and even went during the summer of 1828 with a private tutor (a very dull man) to Barmouth, but I got on very slowly. This work is repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. (Darwin, 1887, Chapter II, Volume I, p. 46)

This kind of *extra sense* shows up in another important historical figure, the American polymath Benjamin Franklin (1706-1790). He thought in a precise, rational way even about seemingly non-mathematical issues and used mathematical argument for a social debate (Pasles, 2008, Chapter 1, Chapter 4).

The same use of mathematical argument in other contexts also happens in the Eastern world. For example, the Indian-British scholar and recipient of the 1998 Nobel Prize in Economics, Amartya Sen, presents an interesting discussion of the case in India in his book *The Argumentative Indian: Writings on Indian Culture, History and Identity* (2005).

Next, I draw your attention to two styles in doing mathematics, using terms borrowed from Peter Henrici (Henrici, 1974), who labels the two styles as “dialectic” and “algorithmic”. Broadly speaking, dialectic mathematics is a rigorously logical science, in which “statements are either true or false and objects with specified properties either do or do not exist.” (Henrici, 1974, p.80) On the other hand, algorithmic mathematics is a tool for solving problems, in which “we are concerned not only with the existence of a mathematical object but also with the credentials of its existence” (Henrici, 1974, p. 80). In a lecture (July, 2002), I attempted to synthesize the two aspects from a pedagogical viewpoint with examples from historical mathematical developments in Western and Eastern cultures. In this 19th ICMI Study Conference, I reiterated this theme, focusing on proof, and discussed how the two aspects complement and supplement each other in proof activity (Siu, 2009b). A procedural (algorithmic) approach helps to prepare more solid ground on which to build up conceptual understanding; conversely, better conceptual (dialectical) understanding enables one to handle algorithms with more facility, or even to devise improved or new algorithms. Like *yin* and *yang* in Chinese philosophy, these two aspects complement and supplement each other, each containing some part of the other.

Several main issues in mathematics education are rooted in understanding these two complementary aspects, “dialectic mathematics” and “algorithmic mathematics”. Those issues include: (1) procedural versus conceptual knowledge; (2) process versus object in learning theory; (3) computer versus computerless learning environments; (4) “symbolic” versus “geometric” emphasis in learning and teaching; and (5) “Eastern” versus “Western” learners/teachers. In a seminal paper, Anna Sfard explicates this duality and develops it into a deeper model of concept formation through interplay of the “operational” and “structural” phases (Sfard, 1991).

Tradition holds that Western mathematics, developed from that of the ancient Greeks, is dialectic, while Eastern mathematics, developed from that of the ancient Egyptians, Babylonians, Chinese and Indians, is algorithmic. Even if it holds an element of truth as a broad statement, under more refined examination this thesis is an over-simplification. Karine Chemla has explained this point in detail (Chemla, 1996). In this respect, the other two speakers in this plenary panel attend primarily to Chinese mathematical classics. For my part, I will discuss the issue with examples from Euclid’s *Elements*.

Saul Stahl has summarized the ancient Greek’s contribution to mathematics:

> Geometry in the sense of mensuration of figures was spontaneously developed by many cultures and dates to several millennia B.C. The science of geometry as we know it, namely, a collection of abstract statements regarding ideal figures, the verification of whose validity requires only pure reason, was created by the Greeks. (Stahl 1993, p. 1)

A systematic and organized presentation of this body of knowledge is found in Euclid's *Elements*.

Throughout history, many famed Western scholars have recounted the benefit they received from learning geometry through reading Euclid's *Elements* or some variation thereof. For example, Bertrand Russell (1872-1970) wrote in his autobiography:

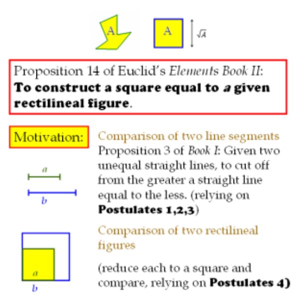
> At the age of eleven, I began Euclid, with my brother as tutor. This was one of the great events of my life, as dazzling as first love. (…) I had been told that Euclid proved things, and was much disappointed that he started with axioms. At first, I refused to accept them unless my brother could offer me some reason for doing so, but he said, 'If you don't accept them, we cannot go on', and as I wished to go on, I reluctantly admitted them *pro temp*. (Russell, 1967, p. 36)

Another example, Albert Einstein (1879-1955), wrote in his autobiography:,

> At the age of twelve I experienced a second wonder of a totally different nature: in a little book dealing with Euclidean plane geometry, which came into my hands at the beginning of a school year. (…) The lucidity and certainty made an indescribable impression upon me. (…) it is marvelous enough that man is capable at all to reach such a degree of certainty and purity in pure thinking as the Greeks showed us for the first time to be possible in geometry. (Schilepp, 1949, pp. 9, 11)

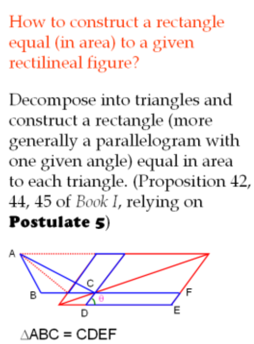
That axiomatic and logical aspect of Euclid's *Elements* has long been stressed. However, reasoning put forth by S.D. Agashe (1989) leads one to look at an alternative feature of the *Elements*; namely, right from the start metric geometry plays a key role, not just in the exposition but even in the motivation of the book's design. In addition, there is a procedural flavour to the reasoning.

For example, Proposition 14 of *Elements*, *Book II* proposes, “To construct a square equal to a given rectilineal figure.” The problem of interest is to compare two polygons. To achieve the one-dimensional analogue, comparing two straight line segments, is easy; one simply overlays one segment on the other and checks whether one segment lies completely inside the other or whether the two are equal. This is in fact what Proposition 3 of *Book I* attempts: “Given two unequal straight lines, to cut off from the greater a straight line equal to the less.” To justify the result, one relies on Postulates 1, 2 and 3. The two-dimensional problem is not so straightforward, except for the special case when both polygons are squares; in this case, one can compare their areas through a comparison of their sides, by placing the smaller square at the lower left corner of the larger square. Incidentally, here one needs to invoke Postulate 4. What Proposition 14 of *Book II* sets out to do is to reduce the comparison of two polygons to that of two squares (Figure 3).

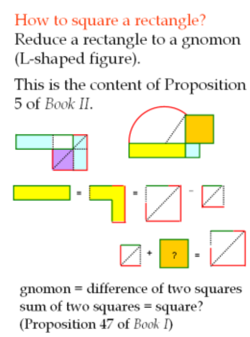


Proposition 14 of *Elements Book II*
Fig. 3

The proof of Proposition 14 of *Book II* can be divided into two steps: (1) construct a rectangle equal (in area) to a given polygon (Figure 4); (2) construct a square equal (in area) to a given rectangle (Figure 5). Note that (1) is already explained through Propositions 42, 44 and 45 of *Book I*, by triangulating the given polygon then converting each triangle into a rectangle of equal area. Incidentally, one has to rely on the famous (notorious?) Postulate 5 on (non-)parallelism to prove those results. To achieve the solution in (2), one makes the preliminary step of converting the given rectangle into an L-shaped gnomon of equal area. This is illustrated in Proposition 5 of *Book II*, “If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segment of the whole together with the square on the straight line between the points of section is equal to the square on the half.”



Construction of a rectangle equal (in area) to a given polygon
Fig. 4



Squaring a rectangle
Fig. 5

Proposition 5 of *Book II* asserts that a certain rectangle is equal (in area) to a certain gnomon which is a square (c^2) minus another square (b^2). To finalize step (2), one must construct a square (a^2) equal to the difference between two squares ($c^2 - b^2$); or equivalently, the square (c^2) is a sum of the two squares ($a^2 + b^2$). This leads naturally to Pythagoras’ Theorem, Proposition 47 of *Book I*, which epitomizes the interdependence between shape and number, between geometry and algebra. (For an enlightening exposition of Pythagoras’ Theorem in Clairaut’s *Eléments de géométrie* [1741,1753], see (Siu 2009a, pp.106-107.)) In studying this problem to compare two polygons we see how algorithmic mathematics blends in with dialectic mathematics in Book I and Book II of *Elements*.

However, despite such evidence of parallels between the Western and Eastern mathematical traditions, some teachers hesitate to integrate history of mathematics with the learning and teaching of mathematics in the classroom. They cite their concern that students lack enough knowledge on culture in general to appreciate history of mathematics in particular. This is probably true, but one can look at the problem from the reverse, seeing the integration of history of mathematics into the day-to-day mathematics classes as an opportunity to let students know more about other cultures in general and other mathematical traditions in particular. They can thus come into contact with other variations in the development of proof and proving. Proof is such an important ingredient in a proper education in mathematics that we can ill afford to miss such an opportunity.

Earlier, I suggested (Siu, 2008) four examples that might be used in such teaching. The first examines how the exploratory, venturesome spirit of the ‘era of exploration’ in the 15th and 16th centuries C.E. influenced the development of mathematical practice in Europe. It resulted in a broad change of mentality in mathematical pursuit, not just affecting its presentation but, more important, bringing in an exploratory spirit. The second example deals with a similar happening in the Orient, though with more emphasis on the aspect of argumentation. It describes the influence of the intellectual milieu in the period of the Three Kingdoms and the Wei-Jin Dynasties from the 3rd to the 6th centuries C.E. in China on mathematical practice as exemplified in the work of Liu Hui. The third example, the influence of Daoism on mathematics in ancient China, particularly astronomical measurement and surveying from a distance, examines the role religious, philosophical (or even mystical) teachings may play in mathematical pursuit. The fourth example, the influence of Euclid's *Elements* in Western culture compared to that in China after the first Chinese translation by the Ming Dynasty scholar-minister Xu Guang Qi (1562-1633) and the Italian Jesuit Matteo Ricci (1552-1610) in 1607 points out a kind of 'reverse' influence; namely, how the mathematical thinking may stimulate thinking in other areas of human endeavour. As a 'bonus', these examples sometimes suggest ways to enhance understanding of specific topics in the classroom.

Finally, one benefit of learning proof and proving is important but seldom emphasized in Western education, namely, its value in character building. This point had been emphasized in the Eastern world rather early, perhaps as a result of the influence of the Confucian philosophical heritage.

In an essay on the Chinese translation of the *Elements*, the co-translator Xu wrote:

> The benefit derived from studying this book [the *Elements*] is many. It can dispel shallowness of those who learn the theory and make them think deep. It can supply facility for those who learn the method and make them think elegantly. Hence everyone in this world should study the book. (…) Five categories of personality will not learn from this book: those who are impetuous, those who are thoughtless, those who are complacent, those who are envious, those who are arrogant. Thus to learn from this book one not only

strengthens one's intellectual capacity but also builds a moral base. (cited in (Siu, 2009a, p. 110))

Such emphasis on proof for a moral reason still sometimes echoes in modern times. As the late Russian mathematics educator Igor Fedorovich Sharygin (1937-2004) once put it, “Learning mathematics builds up our virtues, sharpens our sense of justice and our dignity, and strengthens our innate honesty and our principles. The life of mathematical society is based on the idea of proof, one of the most highly moral ideas in the world.” (cited in (Siu, 2009a, p. 110))

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