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SOLUTION TO TIME-ENERGY COSTS OF QUANTUM CHANNELS

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We derive a formula for the time-energy costs of general quantum channels proposed in [Phys. Rev. A **88**, 012307 (2013)]. This formula allows us to numerically find the time-energy cost of any quantum channel using positive semidefinite programming. We also derive a lower bound to the time-energy cost for any channels and the exact the time-energy cost for a class of channels which includes the qudit depolarizing channels and projector channels as special cases.

Keywords: Time-energy cost, quantum channel, fidelity

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1 Introduction

A time-energy cost of a unitary matrix $U \in \mathrm{U}(r)$ is defined as [1]

$$\|U\|_{\max} = \max_{1\le j\le r} |\theta_j| \tag{1}$$

where U has eigenvalues $\exp(i\theta_j)$ for $j = 1, \ldots, r$. Here, we denote by $\mathrm{U}(r)$ the group of $r \times r$ unitary matrices, and we take the convention that $\theta_j \in (-\pi, \pi]$. This definition of time-energy cost was motivated [1, 2] from time-energy uncertainty relations [3, 4]. Essentially, this time-energy cost captures the idea that time and energy are a trade-off against each other and may be used as an indicator for the resource used by a quantum system. In particular,

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a closed quantum system with a time-independent Hamiltonian H evolves from the initial state $|\psi_{\rm i}\rangle$ to the final state $|\psi_{\rm f}\rangle$ according to the Schrödinger equation: $|\psi_{\rm f}\rangle = U|\psi_{\rm i}\rangle$ where $U = \exp(-iHt/\hbar)$ and t is the evolution time. The eigenvalues of the Hamiltonian H are the energies and thus the eigenvalues of $\log U$ correspond to the time-energy products, the absolute maximum of which is the time-energy cost $\|U\|_{\max}$ defined above. Note that to implement the same information processing task characterized by U, one may use a high energy H run for a short time or a low energy H run for a long time. The time-energy products in both cases are the same.

The definition for $\|U\|_{\max}$ in Eq. (1) is for unitary quantum channels. The time-energy cost has been extended to cover general quantum channels [2]. A quantum channel mapping n-dimensional density matrices to n-dimensional density matrices can be written as

$$\mathcal{K}(\rho) = \sum_{j=1}^{d} K_j \rho K_j^\dagger, \tag{2}$$

where $K_j \in \mathbb{C}^{n\times n}$ are the Kraus operators and $\sum_{j=1}^{d} K_j^\dagger K_j = I_n$. In this paper, we only consider finite dimensional systems. The time-energy cost for quantum channel $\mathcal{K}$ is defined as the time-energy cost of the most efficient unitary extension that implements $\mathcal{K}$ [2]:

$$\|\mathcal{K}\|_{\max} \equiv \min_{U} \|U\|_{\max} \tag{3}$$

$$\text{s.t. } \mathcal{K}(\rho) = \mathrm{Tr}_B[U_{BA}(|0\rangle_B\langle 0| \otimes \rho_A)U_{BA}^\dagger]\ \forall\rho,$$

where the channel $\mathcal{K}$ acts on quantum state ρ in system A and the unitary extension U_{BA} includes system B prepared in a standard state.

The time-energy cost has an interesting informational meaning. The cosine of this cost for a general quantum channel is exactly the worst-case entanglement fidelity of the channel [5], establishing a connection between the physical aspect (the time-energy cost) and the information aspect (the fidelity) of quantum channels. Fidelity is a popular quantity often used to characterize the performance of information processing tasks including quantum key distribution (as a security measure [6, 7]) and state discrimination (as the inconclusive probability [8, 9, 10]). Thus the study of the time-energy cost is important from a quantum information theoretical perspective. To be specific, the result of Ref. [5] shows that for any quantum channel $\mathcal{K}$, the worst-case entanglement fidelity $F_{\min}(\mathcal{K})$ of the channel is related to the time-energy cost by[b]

$$F_{\min}(\mathcal{K}) = \cos \|\mathcal{K}\|_{\max}. \tag{4}$$

Here, the worst-case entanglement fidelity $F_{\min}(\mathcal{K})$ is defined as

$$F_{\min}(\mathcal{K}) \equiv \min_{|\Psi\rangle} F\big(|\Psi\rangle_{AC}\langle\Psi|, (\mathcal{K}_A \otimes I_C)(|\Psi\rangle_{AC}\langle\Psi|)\big), \tag{5}$$

where the channel acts on system A and the fidelity is taken between the channel input state (allowed to be entangled in systems A and C) and the corresponding output state. Here, $F(\rho,\rho') \equiv \mathrm{Tr}\sqrt{\rho^{1/2}\rho'\rho^{1/2}}$ is the fidelity between two mixed quantum states ρ and ρ' [11, 12].

[b] Note that Ref. [5] originally shows that $F_{\min}(\mathcal{K}) = \max(\cos\|\mathcal{K}\|_{\max}, 0)$. However, we should always consider taking the freedom of including an all-zero Kraus operator in the channel representation. In this case, $\cos\|\mathcal{K}\|_{\max}$ is never negative. See Theorem 1 and its proof.

This paper derives a formula for the time-energy cost $\|\mathcal{K}\|_{\max}$ defined in Eq. (3) and provides a numerical solution method via semidefinite programming. This in turn allows us to compute the the worst-case entanglement fidelity using Eq. (4). The difficulty in solving for $\|\mathcal{K}\|_{\max}$ stems from the freedom in the unitary extension. All the freedom we have for choosing different U without changing the channel consists of the following operations:

1. Change the last $(d+1)n - n$ columns of U.

2. Apply $V \otimes I_n$ to U on the left, where $V \in \mathrm{U}(d+1)$.

It turns out that one can apply an abstract mathematical result in unitary dilation theory [13] to solve the problem. One can then determine the optimal solution using semidefinite programming. Thus, we have a theoretical optimal solution that can be determined by numerical method. This is one of the best scenarios in solving an optimization problem if there is a closed form for the optimal solution of the given problem.

The organization of this paper is as follows. We solve problem (3) for $\|\mathcal{K}\|_{\max}$ in Sec. 2, and we derive a lower bound to the time-energy cost for any channels and compute the exact time-energy costs for special channels in Sec. 3. We formulate in Sec. 4 the problem of finding the time-energy cost as a semidefinite program (SDP) which can be solved numerically and efficiently. We give some mathematical remarks in Sec. 5 and conclude in Sec. 6

2 Main result

Theorem 1

$$\|\mathcal{K}\|_{\max} = \cos^{-1}\left[\max_{\mathbf{v}} \frac{1}{2}\lambda_{\min}\left(K_{\mathbf{v}} + K_{\mathbf{v}}^{\dagger}\right)\right] \tag{6}$$

where $\mathbf{v} \in \mathbb{C}^d$ has ℓ_2-norm $\|\mathbf{v}\| \leq 1$, $K_{\mathbf{v}} = \sum_{j=1}^{d} v_j K_j$, $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of its argument, and we take the convention that $\cos^{-1}$ returns an angle in the range $[0,\pi]$.

Proof: The most general form of U in Eq. (3) is

$$U = (V \otimes I_n) \underbrace{\begin{bmatrix} K_1 & * & * & \cdots & * \\ K_2 & * & * & \cdots & * \\ \vdots & & & & \vdots \\ K_d & * & * & \cdots & * \\ K_{d+1} & * & * & \cdots & * \end{bmatrix}}_{U'} \tag{7}$$

where $V \in \mathrm{U}(d+1)$ and only the first n columns of U' are fixed. Here, we append an all-zero Kraus operator $K_{d+1} = 0$ in order to make U the most general unitary implementing the channel $\mathcal{K}$. Certainly, both $\{K_1, \ldots, K_d\}$ and $\{K_1, \ldots, K_{d+1}\}$ are valid representations of $\mathcal{K}$. As we shall see, there is no need to add more than one extra all-zero operator.

We first consider the freedom in U'. Let $d' = d+1$. We want to choose the last $d'n - n$ columns of U' so that its norm is the smallest. This is described as an optimization problem as follows:

$$\begin{aligned}\varphi \equiv\ & \min_{U'} \|U'\|_{\max} \\ & \text{s.t. } U'_{i1} = K_i \text{ for all } i = 1, \dots, d', \\ & \text{with } U' \in \mathrm{U}(d'n) \end{aligned} \tag{8}$$

where U'_{ij} denotes the (i,j) block of size $n \times n$.

By the result in Ref. [13], we know that there is a unitary matrix $\tilde{U} = (\tilde{U}_{rs})_{1 \le r,s \le 2} \in \mathrm{U}(2n)$ with eigenvalues $e^{\pm i\theta_j}$ for $j = 1, \dots, n$, such that $\tilde{U}_{11} = K_1$ and $\tilde{U}_{21} = \sqrt{I_n - K_1^\dagger K_1}$ where $\pi \ge \theta_1 \ge \cdots \ge \theta_n \ge 0$ and $\cos(\theta_1) = \lambda_{\min}(K_1 + K_1^\dagger)/2$. Note that there exists $W \in \mathrm{U}(d'n - n)$ such that $(I_n \oplus W)(\tilde{U} \oplus I_{d'n-2n})(I_n \oplus W)^\dagger$ satisfies the constraints in Eq. (8) and thus

$$\varphi \le \left\| \tilde{U} \right\|_{\max} = \cos^{-1}\left[\frac{1}{2}\lambda_{\min}\left(K_1 + K_1^\dagger\right)\right]. \tag{9}$$

Next, we lower bound φ. Consider U' satisfying the constraints in Eq. (8). By the interlacing inequalities (see, e.g., Ref. [14]), because $(K_1 + K_1^\dagger)/2$ is the principal submatrix of $(U' + U'^\dagger)/2$, the eigenvalues $a_1 \ge \cdots \ge a_{d'n}$ of $(U' + U'^\dagger)/2$ and the eigenvalues $b_1 \ge \cdots \ge b_n$ of $(K_1 + K_1^\dagger)/2$ satisfy

$$a_{d'n} \le b_n \le a_n,$$

and so

$$\cos^{-1}(a_{d'n}) \ge \cos^{-1}(b_n).$$

If U' has eigenvalues $\exp(i\theta_j)$, where $j = 1, \dots, d'n$ and $\theta_j \in (-\pi, \pi]$, then $a_{d'n} = \cos(\max_j |\theta_j|)$, giving

$$\max_j |\theta_j| \ge \cos^{-1}\left[\frac{1}{2}\lambda_{\min}\left(K_1 + K_1^\dagger\right)\right].$$

Thus, (8) is bounded by

$$\varphi \ge \cos^{-1}\left[\frac{1}{2}\lambda_{\min}\left(K_1 + K_1^\dagger\right)\right]. \tag{10}$$

Combining with Eq. (9) gives

$$\varphi = \cos^{-1}\left[\frac{1}{2}\lambda_{\min}\left(K_1 + K_1^\dagger\right)\right]. \tag{11}$$

Finally, we optimize V in Eq. (7) to obtain $\|\mathcal{K}\|_{\max}$. Note that φ which corresponds to the optimal solution of U' after adjusting the last $d'n - n$ columns depends only on the principal submatrix of U'. Thus,

$$\|\mathcal{K}\|_{\max} = \cos^{-1}\left[\max_{\mathbf{v}:\, \|\mathbf{v}\|=1} \frac{1}{2}\lambda_{\min}\left(K_{\mathbf{v}} + K_{\mathbf{v}}^\dagger\right)\right] \tag{12}$$

where $\mathbf{v} \in \mathbb{C}^{d+1}$ is the first row of V. Here, $K_{\mathbf{v}} = \sum_{j=1}^{d+1} v_j K_j$ represents the principal submatrix of U, where $\mathbf{v} = [v_1, \dots, v_{d+1}]$. Taking into account $K_{d+1} = 0$ gives the claim of the theorem. □

We remark that $\cos \|\mathcal{K}\|_{\max} \ge 0$.

3 Time-energy costs for special channels

In this section, we use Theorem 1 to compute the time-energy costs for a class of channels which includes the qudit depolarizing channels and projector channels as special cases.

Lemma 1 Any channel $\mathcal{K}$ can be described by an equivalent form with the Kraus operators $\{K_j \in \mathbb{C}^{n\times n} : j = 1, \dots, d\}$ satisfying

$$\mathrm{Tr}(K_j) = 0, \;\; j = 2, \dots, d.$$

Proof: Two sets of Kraus operators $\{K_1, \dots, K_d\}$ and $\{\tilde{K}_1, \dots, \tilde{K}_d\}$ describe the same quantum channel if and only if

$$K_i = \sum_{j=1}^{d} w_{ij} \tilde{K}_j, \text{ for } i = 1, \dots, d \tag{13}$$

and for some unitary matrix $W \equiv [w_{ij}]$ of dimension d (see, e.g., Theorem 8.2 of Ref. [15]). By taking the trace of Eq. (13), we see that there must exist W that can bring $d-1$ terms to zero. In particular, we have

$$K_1 = \left(\sum_{j=1}^{d} |\mathrm{Tr}(\tilde{K}_j)|^2 \right)^{-\frac{1}{2}} \sum_{j=1}^{d} \mathrm{Tr}^\dagger(\tilde{K}_j)\tilde{K}_j. \tag{14}$$

□

(If $d = 1$, we can pad the channel with $K_2 = 0$ to make Lemma 1 automatically hold.)

Lemma 2 For any channel $\mathcal{K}$ that can be described by Kraus operators $\{K_j \in \mathbb{C}^{n\times n} : j = 1, \dots, d\}$ of the form

$$\mathrm{Tr}(K_j) = 0, \;\; j = 2, \dots, d,$$

we have

$$\cos^{-1}\left[\frac{1}{n}\left|\mathrm{Tr}\left(K_1\right)\right|\right] \le \|\mathcal{K}\|_{\max}. \tag{15}$$

Proof: We consider the middle term of Eq. (6):

$$\begin{aligned}
\frac{1}{2}\lambda_{\min}\left(K_{\mathbf{v}} + K_{\mathbf{v}}^\dagger\right) &\le \frac{1}{2n}\sum_{i=1}^{n}\lambda_i\left(K_{\mathbf{v}} + K_{\mathbf{v}}^\dagger\right) \\
&= \frac{1}{2n}\mathrm{Tr}\left(K_{\mathbf{v}} + K_{\mathbf{v}}^\dagger\right) \\
&= \frac{1}{n}\,\mathrm{Re}\left[\mathrm{Tr}\left(K_{\mathbf{v}}\right)\right] \\
&= \frac{1}{n}\,\mathrm{Re}\left[v_1 \mathrm{Tr}\left(K_1\right)\right]
\end{aligned}$$

where the first line is because the minimum is no greater than the average and λ_i denotes the ith eigenvalue. Maximizing over $\mathbf{v}$ gives the claim. □

Theorem 2 (Time-energy lower bound) For any channel $\mathcal{K}$ described by Kraus operators $\{K_j \in \mathbb{C}^{n\times n} : j = 1, \ldots, d\}$, we have

$$\cos^{-1}\left[\frac{1}{n}\sqrt{\sum_{j=1}^{d} |\mathrm{Tr}\,(K_j)|^2}\right] \le \|\mathcal{K}\|_{\max}. \tag{16}$$

Proof: This follows from Lemma 1 and Lemma 2. □

Theorem 3 (Time-energy for special channels) For any channel $\mathcal{K}$ that can be described by Kraus operators $\{K_j \in \mathbb{C}^{n\times n} : j = 1, \ldots, d\}$ of the form

$$\begin{aligned} K_1 &= \alpha I \text{ where } \alpha \in \mathbb{C} \\ \mathrm{Tr}(K_j) &= 0,\ \ j = 2, \ldots, d, \end{aligned} \tag{17}$$

its time-energy cost is

$$\|\mathcal{K}\|_{\max} = \cos^{-1}|\alpha|. \tag{18}$$

Proof: From Eq. (15), we have $\cos^{-1}|\alpha| \le \|\mathcal{K}\|_{\max}$.

On the other hand, by choosing a particular $\mathbf{v}$,

$$\begin{aligned} &\max_{\mathbf{v}} \frac{1}{2}\lambda_{\min}\left(K_{\mathbf{v}} + K_{\mathbf{v}}^{\dagger}\right) \\ \ge &\max_{\theta_1} \frac{1}{2}\lambda_{\min}\left(e^{i\theta_1}K_1 + e^{-i\theta_1}K_1^{\dagger}\right) \\ = &|\alpha|. \end{aligned}$$

Therefore, $\|\mathcal{K}\|_{\max} \le \cos^{-1}|\alpha|$ and the claim is proved. □

Note that this theorem is slightly more general than Eq. (52) of Ref. [2] in which α is real and positive. As noted in Ref. [2], channels satisfying Eq. (17) include the qudit depolarizing channels. In the following, we show that projector channels also satisfy Eq. (17).

In general, given a channel, we can find an equivalent form according to Lemma 1 and compute the new K_1 using Eq. (14). If this new K_1 satisfies Eq. (17), then the time-energy cost of the channel is immediately given by Theorem 3. Otherwise, we can lower bound it using Theorem 2.

Theorem 4 (Projector channels) For any channel $\mathcal{K}$ that can be described by Kraus operators $\{K_j \in \mathbb{C}^{n\times n} : j = 1, \ldots, d\}$ of the form $K_j = s_j P_j$ with $P_j = P_j^2 = P_j^{\dagger}$ being a projector of rank r and $s_j \in \mathbb{C}$, we have

$$\|\mathcal{K}\|_{\max} = \cos^{-1}\left(\sqrt{\frac{r}{n}}\right). \tag{19}$$

Proof: Note that $\mathrm{Tr}(K_j) = s_j r$ for all j. Using Lemma 1 and Eq. (14), an equivalent description of $\mathcal{K}$ satisfies

$$\begin{aligned} K_1' &= \frac{1}{\sqrt{\sum_{i=1}^{d} |s_i|^2}} I, \\ \mathrm{Tr}(K_j') &= 0,\ \ j = 2, \ldots, d. \end{aligned}$$

Next, note that the trace-preserving constraint of quantum channels implies that $I_n = \sum_{j=1}^{d} K_j^{\dagger} K_j = \sum_{j=1}^{d} |s_j|^2 P_j$ and taking the trace of it gives $n/r = \sum_{j=1}^{d} |s_j|^2$. Then by Theorem 3, the claim is proved. □

4 Efficient numerical solution using semidefinite programming

Our main result (6) in Theorem 1 can be formulated as an SDP. We can write $K_j = A_j + iB_j$, where $A_j, B_j \in \mathbb{C}^{n\times n}$ are Hermitian, and also write $v_j = a_j - ib_j$ with $a_j, b_j \in \mathbb{R}$ for $j = 1, \ldots, d$. Then the problem is equivalent to

$$\begin{aligned} \max \quad & \lambda_{\min}\left(\sum_{i=1}^{d}(a_j A_j + b_j B_j)\right) \\ \text{s.t.} \quad & \sum_{j=1}^{d}(a_j^2 + b_j^2) \le 1 \end{aligned} \tag{20}$$

where the maximization is over $a_1, b_1, \ldots, a_d, b_d \in \mathbb{R}$. We show that this problem can be cast as a complex SDP which has the following form:

$$\begin{aligned} \min \quad & g^T x \\ \text{s.t.} \quad & x_1 G_1 + \cdots + x_m G_m + H \succeq 0 \end{aligned} \tag{21}$$

where the minimization is over $x \in \mathbb{R}^m$. Here, $g \in \mathbb{R}^m$, and $G_1, \ldots, G_m, H$ are complex Hermitian matrices. Note that a complex SDP can always be cast as a real SDP in which $G_1, \ldots, G_m, H$ are real symmetric matrices.

Note that we can rewrite the objective function as follows:

$$\begin{aligned} \min \quad & -\lambda \\ \text{s.t.} \quad & \sum_{j=1}^{d}(a_j^2 + b_j^2) \le 1 \\ & \sum_{i=1}^{d}(a_j A_j + b_j B_j) \succeq \lambda I \end{aligned} \tag{22}$$

where the maximization is over $a_1, b_1, \ldots, a_d, b_d, \lambda \in \mathbb{R}$. Next, we convert this inequality constraint to a positive semidefinite constraint. Let $c = \sqrt{\sum_{j=1}^{d}(a_j^2 + b_j^2)}$. Consider the matrix

$$C = \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix}$$

which has eigenvalues $1 \pm c$. Thus, the constraint $c \le 1$ is equivalent to the constraint $C \succeq 0$. Note that $C \oplus I_{2d-1}$ is unitarily similar to

$$a_1 F_1 + \ldots a_d F_d + b_1 F_{d+1} + \cdots + b_d F_{2d} + I_{2d+1}$$

where $F_j = E_{j,2d+1} + E_{2d+1,j}$ and $E_{i,j}$ is an $(2d+1) \times (2d+1)$ matrix with one at the (i, j) position. Then, the problem becomes

$$\begin{aligned} \min \quad & -\lambda \\ \text{s.t.} \quad & a_1 F_1 + \ldots a_d F_d + b_1 F_{d+1} + \cdots + b_d F_{2d} + I_{2d+1} \succeq 0 \\ & \sum_{i=1}^{d}(a_j A_j + b_j B_j) - \lambda I \succeq 0 \end{aligned} \tag{23}$$

where the maximization is over $a_1, b_1, \ldots, a_d, b_d, \lambda \in \mathbb{R}$. This is in the SDP form (21). Thus, one can apply standard positive semidefinite programming to determine the time-energy cost of a general quantum channel given in Eq. (6).

5 Mathematical remarks

- We may replace K_1 by $e^{i\theta_1}K_1$ without affecting the quantum channel. Thus, we can select $\theta_1 \in [0, 2\pi)$ to maximize the smallest eigenvalue of $e^{i\theta_1}K_1 + e^{-i\theta_1}K_1^\dagger$. To this end, we can use the numerical range of K_1 defined as

$$W(K_1) = \{\langle x|K_1|x\rangle : |x\rangle \in \mathbb{C}^n, \langle x|x\rangle = 1\}.$$

  This is a compact convex set in $\mathbb{C}$, and can be obtained as the intersection of the half spaces

$$Q_{\theta_1} = \big\{\mu \in \mathbb{C} : e^{i\theta_1}\mu + e^{-i\theta_1}\bar{\mu} \geq \lambda_{\min}(e^{i\theta_1}K_1 + e^{-i\theta_1}K_1^\dagger)\big\}, \quad \theta_1 \in [0, 2\pi).$$

  So, maximizing the smallest eigenvalue of $e^{i\theta_1}K_1 + e^{-i\theta_1}K_1^\dagger$ corresponds to finding the half space Q_{θ_1} whose intersection with the unit disk has the smallest area.

- A heuristic approach to upper bound Eq. (6) is as follows. We separately consider $v_jK_j, j = 1, \ldots, d$ and let $v_j = c_j \exp(i\theta_j)$ where $c_j \in \mathbb{R}_+$. Choose $\theta_j \in [0, 2\pi)$ to maximize the smallest eigenvalue σ_j of $e^{i\theta_j}K_j + e^{-i\theta_j}K_j^\dagger$. This is equivalent to rotating the numerical range $W(K_j)$ so that the left support line is as close to the right side as possible. Then choose a nonnegative unit vector $(c_1, \ldots, c_d)$ to maximize $\sum_{j=1}^d c_j\sigma_j$. If $K_{\mathbf{v}} = \sum_{j=1}^d c_j \exp(i\theta_j)K_j$, then $\lambda_{\min}\left(K_{\mathbf{v}} + K_{\mathbf{v}}^\dagger\right) \geq \sum_{j=1}^d c_j\sigma_j$. Thus, $\|\mathcal{K}\|_{\max} \leq \cos^{-1}(\sum_{j=1}^d c_j\sigma_j/2)$.

6 Conclusions

The physical meaning of the time-energy cost is its relation with the channel fidelity [5]. In this paper, we show that the time-energy cost of any general quantum channel is given by Eq. (6). It has closed formulas for special channels. For general channels, the problem of finding the time-energy cost can be formulated as an SDP which can be solved efficiently on computers.

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