On some properties of a class of multivariate Erlang mixtures with insurance applications

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Abstract

We discuss some properties of a class of multivariate mixed Erlang distributions with different scale parameters and describes various distributional properties related to applications in insurance risk theory. Some representations involving scale mixtures, generalized Esscher transformations, higher-order equilibrium distributions, and residual lifetime distributions are derived. These results allows for the study of stop-loss moments, premium calculation, and the risk allocation problem. Finally, some results concerning minimum and maximum variables are derived and applied to pricing joint life and last survivor policies.

Keywords: multivariate mixed Erlang, scale mixtures, generalized Esscher transformation, stop-loss moments, residual lifetime distribution, risk measures, capital allocation, joint and last survivor.

1 Introduction

In recent years there has been a great deal of study in the actuarial literature on multivariate models in a variety of insurance related contexts. Applications in ruin and surplus analysis, survival modelling, and insurance loss analysis have capitalized on the ready availability of computational resources. In particular, the aggregation of risks associated with dependent portfolios of insurance in group insurance or lines of business have received much attention. Various classes of distributions or copulas (e.g. Joe, 1997) have been employed both for loss and premium deficiency testing in aggregate for an entire line of business as well as for determination of regulatory risk capital requirements. Applications in other modelling situations such as competing risks involving simultaneous failure events, as in death to organ failure have also been studied.

In insurance applications, quantities of interest in connection with capital allocation to dependent risks include risk measures such as Tail Value-at-Risk (TVaR), which have been widely discussed in both the actuarial and finance areas. See Denault (2001), Denuit et al. (2005), Dhaene et al. (2008), Bargès et al. (2009), and Cossette et al. (2012) for a detailed discussion of the use and importance of dependency in insurance applications. Also see Artzner et al. (1999) and Goovaerts et al. (2010) for further details on risk measures.

For analysis of problems such as those described above, various multivariate distributions for the insurance losses have been utilized. Examples include multivariate normal distributions by Panjer (2002), multivariate Tweedie distributions by Furman and Landsman (2008, 2010), multivariate Pareto

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distributions by Chiragiev and Landsman (2007) as well Asimit et al. (2010). Multivariate gamma distributions are also of interest (e.g. Mathai and Moschopoulos (1991), Furman and Landsman (2005), and Kotz et al. (2000, Chapter 48)).

In recent years multivariate distributions with mixed Erlang marginals have been the subject of much research. For example, Cossette et al. (2013) utilized such a model under the Farlie-Gumble-Morgenstern (FGM) copula assumption. A systematic study of mathematical and statistical properties of multivariate mixed Erlang distributions with applications in insurance has been provided by Lee and Lin (2012). Such models share with multivariate phase-type distributions (MPH) (e.g. Assaf et al. (1984), Kulkarni (1989), and Cai and Li (2005a,b)) a semiparametric nature, as well as denseness in the set of all distributions on $[0,\infty)^n$. The use of multivariate mixed Erlang distributions is particularly convenient by virtue of the fact that a wide variety of attractive analytic properties enjoyed by univariate mixed Erlang distributions (e.g. Willmot and Lin (2011), Willmot and Woo (2007), and references therein) may be extended to a multivariate setting. The combination of analytic and computational tractability, together with the inherent flexibility of shape, bodes extremely well for the use of these models in a wide variety of insurance related modelling situations.

One of the features of the Lee and Lin (2012) model is the use a common scale parameter for all of the mixed Erlang components. This may be viewed to be a somewhat unnatural and/or inconvenient assumption in some insurance applications involving various lines of business. In particular, different scale parameters allow us to incorporate dependence structures across the company as well as to allow for distinct claims experience on different types of insurance.

In this paper we consider an extension of the model of Lee and Lin (2012) which involves different scale parameters. We first demonstrate that the use of different scale parameters is in principle only slightly more general than that involving the same scale parameter. That is, it may be assumed without loss of generality that the scale parameters are the same, but the support of the mixing weights must be assumed to be countable rather than simply finite. Indeed, this is a small and not inconvenient assumption which suggests that the model of Lee and Lin (2012) is actually extremely versatile, and perhaps more so than has previously been thought.

By virtue of the similarity of our model to that of Lee and Lin (2012), many of the analytic properties derived under the same scale parameter assumption extend without modification to the case with different scale parameters. As such, a second goal of the present paper is to study other useful analytic properties enjoyed by this very rich and important class of multivariate mixed Erlang distributions. These include analysis with scale mixtures which allows for incorporation of parameter uncertainty, generalized Esscher-type transforms, and higher-order equilibrium distributions. In addition, we study mean excess (or residual lifetime) distributions which are particularly important both for loss modelling in the presence of deductibles and for modelling future lifetimes, given attainment of fixed ages. These results allow for analysis of stop-loss moments, premium calculation, and risk allocation.

By judicious choice of the multivariate mixing weights, the present model is seen to include the joint distribution of arbitrary independent but not necessarily identically distributed mixed Erlang random variables. In this context, it is useful to study the distribution of the minimum and maximum of the jointly distributed random variables in the multivariate mixed Erlang model. We demonstrate that the minimum and maximum in our multivariate mixed Erlang model is again of mixed Erlang form, regardless of the nature of any dependency assumptions. This property, in conjunction with the multivariate mean excess results which we derive, allows for a direct application to joint-life and last-survivor insurance under a dependent mixed Erlang setting.

This paper is organized as follows: in the following section, we first introduce the multivariate mixed Erlang distribution with different scale parameters and derive some distributional properties which show that this class of distribution preserves its form under different types of transformations

such as a scale mixture, a generalized Esscher transformation, conditional distribution, and residual lifetime distribution. In Section 3, these results derived in the previous section have useful applications in an actuarial context including in particular calculation of risk capital allocation based on the risk contribution of the individual loss or business line involving TVaR and covariance. Finally, in Section 4, distributions of the minimum and the maximum variables are derived.

2 Distributional properties

We begin by introducing the form of multivariate mixed Erlang distribution. First, let $\tau_{j,\beta}(y)$ be Erang-j density given by

$$\tau_{j,\beta}(y) = \frac{\beta(\beta y)^{j-1}e^{-\beta y}}{(j-1)!}, \tag{1}$$

and its k-th moment is $\beta^{-k}(k+j-1)!/(j-1)!$. Then consider the random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$ having multivariate mixed Erlang with probability density function (pdf) (e.g. Lee and Lin (2012))

$$h(\mathbf{y}) = h(y_1, y_2, \ldots, y_k) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \varphi_{\mathbf{m}} \Big\{ \prod_{i=1}^{k} \tau_{m_i,\beta}(y_i) \Big\}, \tag{2}$$

where $\varphi_{\mathbf{m}}$ is a joint probability function (pf) defined as $\varphi_{\mathbf{m}} = \varphi_{m_1,\ldots,m_k}$ with $\sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \varphi_{\mathbf{m}} = 1$. The Laplace transform (LT) of (2) which can be obtained by the characteristic function in Lee and Lin (2012), is given by

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \varphi_{\mathbf{m}} \prod_{i=1}^{k} \left(\frac{\beta}{\beta + s_i} \right)^{m_i} = \Psi \left(\frac{\beta}{\beta + s_1}, \frac{\beta}{\beta + s_2}, \ldots, \frac{\beta}{\beta + s_k} \right), \tag{3}$$

where

$$\Psi(\mathbf{z}) = \Psi(z_1, z_2, \ldots, z_k) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \varphi_{\mathbf{m}} \Big(\prod_{i=1}^{k} z_i^{m_i} \Big). \tag{4}$$

Now we consider different rate parameters in (2), i.e. $\tau_{m_i,\beta_i}(y_i)$ instead of $\tau_{m_i,\beta}(y_i)$. In the following proposition, it can be shown that it is a still of the form (3). In this case, the marginal loss random variable (rv) Y_i is spread out differently.

Proposition 1 The multivariate mixed Erlang with (2) with different scale parameters can be re-expressed as

$$f(\mathbf{y}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} p_{\mathbf{m}} \Big\{ \prod_{i=1}^{k} \tau_{m_i,\beta}(y_i) \Big\}, \tag{5}$$

where

$$p_{\mathbf{m}} = \sum_{n_1=1}^{m_1} \cdots \sum_{n_k=1}^{m_k} \varphi_{\mathbf{n}} \prod_{i=1}^{k} \binom{m_i - 1}{n_i - 1} \left(\frac{\beta_i}{\beta} \right)^{n_i} \left(1 - \frac{\beta_i}{\beta} \right)^{m_i - n_i}, \tag{6}$$

and $\sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} p_{\mathbf{m}} = 1$ (see Willmot and Woo (2007) for univariate case).
Proof: Assuming that $\beta_i \leq \beta$ for the multivariate mixed Erlang with different scale parameters, we first have

$$\frac{\beta_i}{\beta_i + s_i} = Q_{\beta_i/\beta} \left(\frac{\beta}{\beta + s_i} \right),$$

where

$$Q_\phi(z) = \phi z/\{1-(1-\phi)z\} \tag{7}$$

(e.g. Willmot and Woo (2007)). Hence, for the more general LT $P(\mathbf{z}) = P(z_1, z_2, \ldots, z_k)$ than (3), we have for any $\beta \geq \sup_i \beta_i$,

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \varphi_{\mathbf{m}} \prod_{i=1}^{k} \left(\frac{\beta_i}{\beta_i+s_i}\right)^{m_i} = P\left(\frac{\beta_1}{\beta_1+s_1}, \frac{\beta_2}{\beta_2+s_2}, \ldots, \frac{\beta_k}{\beta_k+s_k}\right) = \Psi^*\left(\frac{\beta}{\beta+s_1}, \frac{\beta}{\beta+s_2}, \ldots, \frac{\beta}{\beta+s_k}\right), \tag{8}$$

where $\Psi^*(\mathbf{z}) = P\left\{Q_{\beta_1/\beta}(z_1), Q_{\beta_2/\beta}(z_2), \ldots Q_{\beta_k/\beta}(z_k)\right\}$. Thus, the result follows. □

Its tail distribution $\overline{F}(\mathbf{y}) = \Pr(Y_1 > y_1, \ldots, Y_k > y_k)$ is given by (e.g. Lee and Lin (2012)),

$$\overline{F}(\mathbf{y}) = e^{-\beta\sum_{i=1}^k y_i} \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \overline{P}_{\mathbf{m}} \left\{\prod_{i=1}^{k} \frac{(\beta y_i)^{m_i}}{m_i!}\right\} = \beta^{-k} \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \overline{P}_{\mathbf{m-1}} \left\{\prod_{i=1}^{k} \tau_{m_i,\beta}(y_i)\right\}, \tag{9}$$

where

$$\overline{P}_{\mathbf{m}} = \overline{P}_{m_1,\ldots,m_k} = \sum_{n_1=m_1+1}^{\infty} \cdots \sum_{n_k=m_k+1}^{\infty} p_{\mathbf{n}}, \tag{10}$$

and $\overline{P}_{\mathbf{m-1}} = \overline{P}_{m_1-1,\ldots,m_k-1}$. Also, the failure rate (or hazard rate or force of mortality) is given by

$$\lambda(\mathbf{y}) = \lambda(y_1, y_2, \ldots, y_k) = \frac{f(\mathbf{y})}{\overline{F}(\mathbf{y})} = \frac{\beta^k \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} p_{\mathbf{m+1}} \left\{\prod_{i=1}^k \frac{(\beta y_i)^{m_i}}{m_i!}\right\}}{\sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \overline{P}_{\mathbf{m}} \left\{\prod_{i=1}^k \frac{(\beta y_i)^{m_i}}{m_i!}\right\}},$$

where $p_{\mathbf{m+1}} = p_{m_1+1,\ldots,m_k+1}$. Thus, one finds $\mu(\mathbf{0}) = \beta^k p_{\mathbf{1}}$ where $\mathbf{0} = (0, \ldots, 0)$ and $p_{\mathbf{1}} = p_{1,\ldots,1}$.

Furthermore, a joint pdf proportional to a product of mixed Erlang pdfs can be written as

$$g(\mathbf{y}) \propto \prod_{i=1}^{k} \left\{\sum_{m_i=1}^{\infty} q_{m_i} \tau_{m_i,\beta}(y_i)\right\} = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} q_{\mathbf{m}} \left\{\prod_{i=1}^{k} \tau_{m_i,\beta}(y_i)\right\},$$

which is a form of the multivariate mixed Erlang in (2).

Throughout the paper, we consider the multivariate mixed Erlang distribution with different scale parameters but its joint pdf given in (5) is re-expressed in the form (2), which was considered in Lee and Lin (2012). Therefore, the results obtained therein also hold for this case. For example, the marginal distribution of Y_i obtained from the multivariate distribution with the LT in (8) is a mixed Erlang with a scale parameter β, and also the marginal distribution of $Y_1 + Y_2 + \cdots + Y_k$ is again a univariate mixed Erlang with parameter β. Also, direct applications for several measurements for bivariate dependency such as correlation coefficient, Spearman's rho and Kendall's tau are available.

2.1 Scale mixtures

Now, we consider the mixture of (8) over a scale parameter β, with the Laplace transform given by

$$\int_0^{\mu} \Psi^*\left(\frac{\beta}{\beta+s_1}, \frac{\beta}{\beta+s_2}, \ldots, \frac{\beta}{\beta+s_k}\right) dU(\beta) = P^*\left(\frac{\mu}{\mu+s_1}, \frac{\mu}{\mu+s_2}, \ldots, \frac{\mu}{\mu+s_k}\right),$$

where $P^*(z_1, z_2, \ldots, z_k) = \int_0^\mu \Psi^* \left\{Q_{\beta/\mu}(z_1), Q_{\beta/\mu}(z_2), \ldots, Q_{\beta/\mu}(z_k)\right\} dU(\beta)$ and $Q_\phi(z)$ is given by (7). This is still of the form in (4). To see this, we identify a joint probability function $p^*_{\mathbf{n}} = p^*_{n_1,n_2,\ldots,n_k}$ where $P^*(z_1, z_2, \ldots, z_k) = \sum_{n_1=1}^\infty \cdots \sum_{n_k=1}^\infty p^*_{\mathbf{n}}\big(\prod_{i=1}^k z_i^{n_i}\big)$. Note that from (5)

$$\Psi^* \left\{Q_\phi(z_1), Q_\phi(z_2), \ldots, Q_\phi(z_k)\right\} = \sum_{m_1=1}^\infty \cdots \sum_{m_k=1}^\infty p_{\mathbf{m}} \prod_{i=1}^k \left\{Q_\phi(z_i)\right\}^{m_i}, \tag{11}$$

and $\{Q_\phi(z)\}^{m_i} = z^{m_i}\left\{\frac{\phi}{1-(1-\phi)z}\right\}^{m_i} = \sum_{n_i=m_i}^\infty \binom{n_i-1}{n_i-m_i}\phi^{m_i}(1-\phi)^{n_i-m_i}z^{n_i}$, with the usual notational convention that $\binom{x}{0} = x^0 = 1$ for $x \in (-\infty, \infty)$. Thus,

$$\prod_{i=1}^k \{Q_\phi(z_i)\}^{m_i} = \sum_{n_1=m_1}^\infty \cdots \sum_{n_k=m_k}^\infty \left\{\prod_{i=1}^k \binom{n_i-1}{n_i-m_i}\right\}\phi^{\sum_{i=1}^k m_i}(1-\phi)^{\sum_{i=1}^k (n_i-m_i)}\left(\prod_{i=1}^k z_i^{n_i}\right).$$

In turn, (11) can be rewritten as

$$\begin{aligned}
&\Psi^* \left\{Q_\phi(z_1), Q_\phi(z_2), \ldots, Q_\phi(z_k)\right\} \\
&= \sum_{m_1=1}^\infty \cdots \sum_{m_k=1}^\infty \sum_{n_1=m_1}^\infty \cdots \sum_{n_k=m_k}^\infty p_{\mathbf{m}}\left\{\prod_{i=1}^k \binom{n_i-1}{n_i-m_i}\right\}\phi^{\sum_{i=1}^k m_i}(1-\phi)^{\sum_{i=1}^k (n_i-m_i)}\left(\prod_{i=1}^k z_i^{n_i}\right) \\
&= \sum_{n_1=1}^\infty \cdots \sum_{n_k=1}^\infty \left(\prod_{i=1}^k z_i^{n_i}\right) \sum_{m_1=1}^{n_1} \cdots \sum_{m_k=1}^{n_k} p_{\mathbf{m}}\left\{\prod_{i=1}^k \binom{n_i-1}{n_i-m_i}\right\}\phi^{\sum_{i=1}^k m_i}(1-\phi)^{\sum_{i=1}^k (n_i-m_i)}.
\end{aligned}$$

Finally, replacing ϕ by β/μ and integrating over β followed by equating coefficients of $z_1^{n_1} z_2^{n_1} \cdots z_k^{n_k}$ results in

$$p^*_{\mathbf{n}} = \sum_{m_1=1}^{n_1} \cdots \sum_{m_k=1}^{n_k} p_{\mathbf{m}}\left\{\prod_{i=1}^k \binom{n_i-1}{n_i-m_i}\right\} \int_0^\mu \left(\frac{\beta}{\mu}\right)^{\sum_{i=1}^k m_i}\left(1-\frac{\beta}{\mu}\right)^{\sum_{i=1}^k (n_i-m_i)} dU(\beta), \tag{12}$$

and the integral is easy to evaluate if $U(\cdot)$ is a rescaled beta distribution.

Example 1 With a beta mixing distribution $U'(\beta) = \frac{\Gamma(a+b)}{\mu\Gamma(a)\Gamma(b)}(\frac{\beta}{\mu})^{a-1}(1-\frac{\beta}{\mu})^{b-1}$ for $a, b > 0$ and $0 < \beta < \mu$, the integral in (12) becomes a mixed binomial with a beta mixing distribution obtained as

$$\int_0^\mu \left(\frac{\beta}{\mu}\right)^{\sum_{i=1}^k m_i}\left(1-\frac{\beta}{\mu}\right)^{\sum_{i=1}^k (n_i-m_i)} dU(\beta) = \frac{\Gamma(a+b)\Gamma(a+\sum_{i=1}^k m_i)\Gamma(b+\sum_{i=1}^k n_i-\sum_{i=1}^k m_i)}{\Gamma(a)\Gamma(b)\Gamma(a+b+\sum_{i=1}^k n_i)}.$$

Therefore (12) becomes

$$p^*_{\mathbf{n}} = \sum_{m_1=1}^{n_1} \cdots \sum_{m_k=1}^{n_k} p_{\mathbf{m}}\left\{\prod_{i=1}^k \binom{n_i-1}{n_i-m_i}\right\} \frac{\Gamma(a+b)\Gamma(a+\sum_{i=1}^k m_i)\Gamma(b+\sum_{i=1}^k n_i-\sum_{i=1}^k m_i)}{\Gamma(a)\Gamma(b)\Gamma(a+b+\sum_{i=1}^k n_i)}.$$

□

2.2 Generalized Esscher transformation

Let us consider a generalized Esscher transform of (5), which we define as

$$f_*(y_1, y_2, \ldots, y_k) = \frac{\prod_{i=1}^{k} y_i^{n_i} e^{-\mu y_i} f(y_1, y_2, \ldots, y_k)}{\int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^{k} t_i^{n_i} e^{-\mu t_i} \right) f(t_1, t_2, \ldots, t_k) dt_1 \cdots dt_k}, \tag{13}$$

where $n_i = 0, 1, 2, \ldots$ and $\mu > -\beta$. If $n_i = 0$ for $i = 1, 2, \ldots, k$, (13) reduces to the Esscher transform of the multivariate mixed Erlang in (5). This result is relevant for premium calculation, as is discussed in a later section.

It can be shown that (13) may be expressed in the form (5) as

$$f_*(y_1, y_2, \ldots, y_k) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} q^*_{\mathbf{m}} \left\{ \prod_{i=1}^{k} \tau_{m_i+n_i, \mu+\beta}(y_i) \right\}, \tag{14}$$

where the Erlang density $\tau(y)$ is given by (1), and the mixing weights are given by

$$q^*_{\mathbf{m}} = q_{m_1,\ldots,m_k} = \frac{p_{\mathbf{m}} \prod_{i=1}^{k} \left(\frac{\beta}{\mu+\beta} \right)^{m_i} \binom{m_i+n_i-1}{n_i}}{\sum_{r_1=1}^{\infty} \cdots \sum_{r_k=1}^{\infty} p_{\mathbf{r}} \prod_{i=1}^{k} \left(\frac{\beta}{\mu+\beta} \right)^{r_i} \binom{r_i+n_i-1}{n_i}},$$

which is proportional to $p_{\mathbf{m}}$ and the Pascal distribution (a special case of negative binomial with an integer-valued shape parameter).

In particular, when $n_i = 0$ for $i = 1, 2, \ldots, k$, the above result simplifies to the Esscher transform of (5), which is the multivariate mixed Erlang with the scale parameter shifted by μ with the joint pdf given by

$$f^E(y_1, y_2, \ldots, y_k) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} q^E_{\mathbf{m}} \left\{ \prod_{i=1}^{k} \tau_{m_i, \mu+\beta}(y_i) \right\}, \tag{15}$$

where

$$q^E_{\mathbf{m}} = \frac{p_{\mathbf{m}} \prod_{i=1}^{k} \left(\frac{\beta}{\mu+\beta} \right)^{m_i}}{\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} p_{\mathbf{n}} \prod_{i=1}^{k} \left(\frac{\beta}{\mu+\beta} \right)^{n_i}}. \tag{16}$$

From (9), its tail distribution is $\overline{F}^E(y_1, y_2, \ldots, y_k) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \overline{Q}^E_{\mathbf{m}-\mathbf{1}} \{ \prod_{i=1}^{k} \frac{\tau_{m_i,\mu+\beta}(y_i)}{\mu+\beta} \}$, where $\overline{Q}^E_{\mathbf{m}-\mathbf{1}} = \overline{Q}^E_{m_1-1,\ldots,m_k-1} = \sum_{n_1=m_1}^{\infty} \cdots \sum_{n_k=m_k}^{\infty} q^E_{\mathbf{n}}$.

2.3 Conditional distributions and moments

Assume that n variables X_i for $i = 1, 2, \ldots, n$ are jointly distributed with pdf (5). Let the conditional pdf of k random variables given the other $n-k$ random variable be given by $g_{n-k}(x_1, x_2, \ldots, x_k | x_{k+1}, \ldots, x_n)$ $= f(x_1, x_2, \ldots, x_n) / \{ \int_0^\infty \ldots \int_0^\infty f(x_1, x_2, \ldots, x_n) dx_1 \ldots dx_k \}$, and the $(n-k)$-variate marginal in the

denominator is a $(n-k)$-variate mixed Erlang (Lee and Lin (2012)), i.e.

$$g_{n-k}(x_1,x_2,\ldots,x_k|x_{k+1},\ldots,x_n)=\frac{\sum\limits_{m_1=1}^{\infty}\cdots\sum\limits_{m_n=1}^{\infty}p_{\mathbf{m}}\left\{\prod\limits_{i=1}^{n}\tau_{m_i,\beta}(x_i)\right\}}{\sum\limits_{m_1=1}^{\infty}\cdots\sum\limits_{m_n=1}^{\infty}p_{\mathbf{m}}\left\{\prod\limits_{i=k+1}^{n}\tau_{m_i,\beta}(x_i)\right\}}.$$

Hence, with $\mathbf{x_{-k}}=(x_{k+1},x_{k+2},\ldots,x_n)$, it can still be expressed in the form (5) as

$$g_{n-k}(x_1,x_2,\ldots,x_k|x_{k+1},\ldots,x_n)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}p_{\mathbf{m_k}}(\mathbf{x_{-k}})\left\{\prod_{i=1}^{k}\tau_{m_i,\beta}(x_i)\right\},\tag{17}$$

where the mixing probability function is

$$p_{\mathbf{m_k}}(\mathbf{x_{-k}})=p_{m_1,\ldots,m_k}(x_{k+1},\ldots,x_n)=\frac{\sum\limits_{m_{k+1}=1}^{\infty}\cdots\sum\limits_{m_n=1}^{\infty}p_{\mathbf{m}}\left\{\prod\limits_{i=k+1}^{n}\tau_{m_i,\beta}(x_i)\right\}}{\sum\limits_{r_1=1}^{\infty}\cdots\sum\limits_{r_n=1}^{\infty}p_{\mathbf{r}}\left\{\prod\limits_{i=k+1}^{n}\tau_{r_i,\beta}(x_i)\right\}}.$$

Then, the r-th conditional moment of $(X_1,X_2,\ldots,X_k)$ given the $n-k$ variables is simply

$$E[(X_1X_2\cdots X_k)^r|X_{k+1}=x_{k+1},\ldots,X_n=x_n]=\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}p_{\mathbf{m_k}}(\mathbf{x_{-k}})\left\{\prod_{i=1}^{k}\beta^{-r}(m_i+r-1)^{(r)}\right\},$$

where the notation $(m_i{+}r{-}1)^{(r)}=(m_i{+}r{-}1)\cdots m_i$ is used for the r-th descending factorial of $(m_i{+}r{-}1)$.

For the bivariate case having the joint pdf given in (5), the conditional pdf of $X_1|X_2$ (i.e. when $n=2$ and $k=1$) is from (17) $g_1(x_1|x_2)=\sum_{m_1=1}^{\infty}p_{m_1}(x_2)\tau_{m_1,\beta}(x_1)$, where $p_{m_1}(x_2)=\sum_{m_2=1}^{\infty}p_{m_1,m_2}\tau_{m_2,\beta}(x_2)$ $\left\{\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}p_{n_1,n_2}\tau_{n_2,\beta}(x_2)\right\}^{-1}$, and the conditional r-th moment of Y_1 is obtained as $E[X_1^r|X_2=x_2]=\beta^{-r}\sum\limits_{m_1=1}^{\infty}p_{m_1}(x_2)(m_1+r-1)^{(r)}$.

2.4 The higher-order equilibrium distribution

In connection with the analysis of the multivariate stop-loss moments, it is useful to study the r-th equilibrium distribution (e.g. Willmot et al. (2005)) of the aggregate loss $S_N=\sum_{i=1}^{N}X_i$ where the individual losses $(X_1,\ldots,X_k)$ follow the multivariate mixed Erlang distribution with the joint LT in (8) and N is a discrete rv having pf $p_n=\Pr(N=n)$. Since the mixed Erlang distribution may be viewed as a compound random sum of exponential rvs, the aggregate loss can be rewritten as $S_N=\sum_{i=1}^{M_1+\cdots+M_N}E_i$ where E_i is an exponential rv with mean $1/\beta$, and M_is are discrete rvs with the joint pf $p_{\mathbf{m}}$ given by (6).

From Theorem 5.1 in Lee and Lin (2012), when the joint pdf of the X_is is assumed to be given by (2), it follows that the distribution of S_k (i.e. the sum of X_is) is a univariate Erlang mixture with mixing weights given by $\sum_{m_1+\cdots+m_k=i}\varphi_{\mathbf{m}}$ for $i=1,2,\ldots$. Instead, we consider the distribution of S_N (i.e. the compound sum of X_is) which is a univariate Erlang mixture with mixing weights given by $\sum_{m_1+\cdot+m_N=i}p_{\mathbf{m}}$ for $i=1,2,\ldots$, then it is convenient mathematically to utilize the Laplace-Stieltjes transform (LST), given as

$$\widetilde{g}(s)=\sum_{n=1}^{\infty}q_n\left(\frac{\beta}{\beta+s}\right)^n,\tag{18}$$

where the mixing weights are given by

$$q_n = \sum_{i=1}^{\infty} p_{0,i} \Pr(M_1 + \cdots + M_i = n) = \sum_{i=1}^{\infty} p_{0,i} \Big\{ \sum_{m_1+\cdots+m_i=n} p_{\mathbf{m}} \Big\}, \qquad n = 1, 2, \ldots, \tag{19}$$

where $p_{0,i} = p_i/(1-p_0)$. Then, the pdf of S_N is the mixed Erlang pdf

$$g(x) = \sum_{n=1}^{\infty} q_n \tau_{n,\beta}(y), \tag{20}$$

where q_n and $\tau_{n,\beta}(y)$ are given by (19) and (1) respectively.

Assume that the tail distribution of S_N is $\overline{G}(y) = 1 - G(y)$ where $G(y)$ is the distribution function (df) of S_N, and define the equilibrium distribution function (df) for S_N as $G_1(y) = \int_0^y \overline{G}(t)dt/E[S_N]$, then from (9), its pdf $g_1(y) = G_1'(y)$ is $g_1(y) = \overline{G}(y)/\{\int_0^\infty \overline{G}(t)dt\} = \sum_{j=1}^{\infty} q_j^* \tau_{j,\beta}(y)$, where $q_j^* = \overline{Q}_{j-1}/\{\sum_{n=1}^{\infty} n q_n\}$ with $\overline{Q}_j = \sum_{k=j+1}^{\infty} q_k$. Then, using the result in Example 2.1 of Willmot et al. (2005), the r-th equilibrium df of S_N recursively defined as $G_r(y) = \int_0^y \overline{G}_{r-1}(t)dt/\int_0^\infty \overline{G}_{r-1}(t)dt$ can be obtained as

$$\overline{G}_r(y) = e^{-\beta y} \sum_{n=0}^{\infty} \overline{Q}_{r,n} \frac{(\beta y)^n}{n!} = \beta^{-1} \sum_{n=1}^{\infty} \overline{Q}_{r,n-1} \tau_{n,\beta}(y), \qquad r = 1, 2, \ldots, \tag{21}$$

where $\overline{Q}_{r,n} = \sum_{j=0}^{\infty} \binom{j+r}{r} q_{n+j+1} \big/ \big\{ \sum_{j=0}^{\infty} \binom{j+r}{r} q_{j+1} \big\}$. Also, its tail is easily obtained as

$$\int_x^\infty \overline{G}_r(y)dy = \beta^{-1} e^{-\beta x} \sum_{j=0}^{\infty} \overline{Q}_{r,j}^* \frac{(\beta x)^j}{j!}, \tag{22}$$

where $\overline{Q}_{r,j}^* = \sum_{n=j+1}^{\infty} \overline{Q}_{r,n-1} = \sum_{n=j}^{\infty} \overline{Q}_{r,n}$. As discussed in Willmot and Lin (2011), asymptotic results (e.g. Embrechts et al. (1985), Grandell (1997), Willmot (1989b)) may be used to yield asymptotic estimates for (21) and (22).

Furthermore, the multivariate equilibrium or integrated tail distribution of $(Y_1, Y_2, \ldots, Y_k)$ with a tail df given in (9) is

$$f_e(\mathbf{y}) = \frac{\overline{F}(\mathbf{y})}{\int_0^\infty \cdots \int_0^\infty \overline{F}(\mathbf{y})dy_1 \cdots dy_k} = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} q_{\mathbf{m}}^e \Big\{ \prod_{i=1}^{k} \tau_{m_i,\beta}(y_i) \Big\}, \tag{23}$$

where

$$q_{\mathbf{m}}^e = \frac{\overline{P}_{\mathbf{m-1}}}{\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \overline{P}_{\mathbf{n-1}}}, \tag{24}$$

again the multivariate mixed Erlang distribution but with different mixing weights, and then its tail $\overline{F}_e(\mathbf{y}) = \int_{y_1}^\infty \cdots \int_{y_k}^\infty f_e(\mathbf{x})dx_1 \cdots dx_k$ is given by

$$\overline{F}_e(\mathbf{y}) = e^{-\beta \sum_{i=1}^k y_i} \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \overline{Q}_{\mathbf{m}}^e \Big\{ \prod_{i=1}^{k} \frac{(\beta y_i)_i^m}{m_i!} \Big\} = \beta^{-k} \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \overline{Q}_{\mathbf{m-1}}^e \Big\{ \prod_{i=1}^{k} \tau_{m_i,\beta}(y_i) \Big\}, \tag{25}$$

where $\overline{Q}_{\mathbf{m}}^e = \sum_{n_1=m_1+1}^{\infty} \cdots \sum_{n_k=m_k+1}^{\infty} q_{\mathbf{n}}^e$.

2.5 The residual lifetime distribution

Assume that there are "deductibles" given by $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ for the individual losses $\mathbf{X} = (X_1, X_2, \ldots, X_k)$ with a joint pdf given in (5). We are interested in the multivariate excess loss (or residual life time), i.e. $\mathbf{Y_x} = \mathbf{X} - \mathbf{x}|\mathbf{X} > \mathbf{x}$. Its pdf is given by (e.g. Theorem 5.3 in Lee and Lin (2012)),

$$f_{\mathbf{x}}(\mathbf{y}) = \frac{f(\mathbf{y}+\mathbf{x})}{\overline{F}(\mathbf{x})}, \qquad \text{where } \mathbf{x} = (x_1, \ldots, x_k).$$

Using $\tau_{j,\beta}(x+y) = \beta^{-1}\sum_{i=1}^{j}\tau_{i,\beta}(x)\tau_{j-i+1,\beta}(y)$ and (5), it follows that

$$f_{\mathbf{x}}(\mathbf{y}) = \frac{\beta^{-k}}{\overline{F}(\mathbf{x})}\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty} p_{\mathbf{m}}\Big\{\prod_{i=1}^{k}\sum_{n_i=1}^{m_i}\tau_{n_i,\beta}(y_i)\tau_{m_i-n_i+1,\beta}(x_i)\Big\} = \sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty} p_{\mathbf{n}}(\mathbf{x})\Big\{\prod_{i=1}^{k}\tau_{n_i,\beta}(y_i)\Big\}, \tag{26}$$

where

$$p_{\mathbf{n}}(\mathbf{x}) = \frac{\beta^{-k}}{\overline{F}(\mathbf{x})}\sum_{m_1=n_1}^{\infty}\cdots\sum_{m_k=n_k}^{\infty} p_{\mathbf{m}}\Big\{\prod_{i=1}^{k}\tau_{m_i-n_i+1,\beta}(x_i)\Big\} = \frac{\beta^{-k}}{\overline{F}(\mathbf{x})}\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty} p_{\mathbf{m+n-1}}\Big\{\prod_{i=1}^{k}\tau_{m_i,\beta}(x_i)\Big\}, \tag{27}$$

and $\overline{F}(\cdot)$ is given by (9), i.e. again a multivariate mixed Erlang pdf as in (2) but with different mixing weights.

In addition, the mean residual lifetime $E[\mathbf{Y_x}]$ denoted as $r(\mathbf{x})$ can be obtained by calculating $\int_0^\infty\cdots\int_0^\infty(\prod_{i=1}^{k}y_i)f_{\mathbf{x}}(\mathbf{y})dy_1\cdots dy_k$, i.e.

$$r(\mathbf{x}) = \frac{\int_0^\infty\cdots\int_0^\infty(y_1\cdots y_k)f(y_1+x_1,\cdots,y_k+x_k)dy_1\cdots dy_k}{\overline{F}(\mathbf{x})} = \frac{\int_{x_1}^\infty\cdots\int_{x_k}^\infty\overline{F}(\mathbf{y})dy_1\cdots dy_k}{\overline{F}(\mathbf{x})}, \tag{28}$$

the last equality resulting from the integration by parts on the previous k integrals on $y_1, \ldots, y_k$. From (26), one finds

$$r(\mathbf{x}) = \beta^{-k}\sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty} p_{\mathbf{n}}(\mathbf{x})\Big\{\prod_{i=1}^{k}n_i\Big\}, \tag{29}$$

where $p_{\mathbf{n}}(\mathbf{x})$ is given by (27), and it implies that when $x_1 = x_2 = \cdots = x_k = 0$,

$$r(\mathbf{0}) = E[\mathbf{Y}] = \beta^{-k}\sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty} p_{\mathbf{n}}\Big\{\prod_{i=1}^{k}n_i\Big\} = \beta^{-k}\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\overline{P}_{\mathbf{m-1}},$$

due to $p_{\mathbf{n}}(\mathbf{0}) = p_{\mathbf{n}}$ from (27). Alternatively, we first find the failure rate of the multivariate equilibrium distribution using (23) and (25),

$$\lambda_e(\mathbf{x}) = \frac{f_e(\mathbf{x})}{\overline{F}_e(\mathbf{x})} = \frac{\overline{F}(\mathbf{x})}{\int_{x_1}^\infty\cdots\int_{x_k}^\infty\overline{F}(\mathbf{y})dy_1\cdots dy_k} = \frac{1}{r(\mathbf{x})},$$

which is the same as the reciprocal of the mean residual lifetime in (28) as in the univariate case which

is well known. Hence, from (9) with (23) and (25) one finds

$$r(\mathbf{x}) = \frac{\beta^{-k} \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \overline{Q}^e_{\mathbf{m-1}} \left\{ \prod_{i=1}^{k} \tau_{m_i,\beta}(x_i) \right\}}{\overline{F}(\mathbf{x}) / \{\beta^{-k} \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \overline{P}_{\mathbf{m-1}}\}}$$
$$= \frac{\beta^{-2k}}{\overline{F}(\mathbf{x})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \sum_{n_1=m_1}^{\infty} \cdots \sum_{n_k=m_k}^{\infty} \overline{P}_{\mathbf{n-1}} \left\{ \prod_{i=1}^{k} \tau_{m_i,\beta}(x_i) \right\},$$

where $\overline{Q}^e_{\mathbf{m-1}} = \sum_{n_1=m_1}^{\infty} \cdots \sum_{n_k=m_k}^{\infty} q^e_{\mathbf{n}}$ and $q^e_{\mathbf{n}}$ is given by (24). Because,

$$\sum_{l_1=m_1}^{\infty} \cdots \sum_{l_k=m_k}^{\infty} \overline{P}_{\mathbf{l-1}} = \sum_{l_1=1}^{\infty} \cdots \sum_{l_k=1}^{\infty} \sum_{n_1=l_1}^{\infty} \cdots \sum_{n_k=l_k}^{\infty} p_{\mathbf{n+m-1}} = \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} p_{\mathbf{n+m-1}} \left\{ \prod_{i=1}^{k} n_i \right\},$$

it gives the same expression as (29) with (27).

3 Applications

3.1 Stop-loss moments

In the previous section, the pdf of the aggregate loss $S_N = \sum_{i=1}^{N} X_i$ where $(X_1, X_2, \ldots, X_k)$ has the multivariate mixed Erlang pdf given in (5) was obtained as (20) and its LST is given in (18). Hence, using (e.g. Willmot (2007)) $g(x+y) = \beta^{-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_{j+k+1} \tau_{j+1,\beta}(x) \tau_{k+1,\beta}(y)$, where q_n is given by (19), the stop loss moment of any positive order can be obtained as (e.g. Willmot and Woo (2007, p.106), Willmot and Lin (2011, p.8)),

$$E\{(S_N - x)^r_+\} = \int_x^{\infty} (y-x)^r dG(y) = e^{-\beta x} \sum_{j=0}^{\infty} a_{j,r} \frac{(\beta x)^j}{j!}, \tag{30}$$

where $x_+ = \max(x, 0)$ and $a_{j,r} = \beta^{-r} \sum_{k=1}^{\infty} q_{j+k} \frac{\Gamma(r+k)}{(k-1)!}$. Certainly, when $r = 0$ the tail distribution of S can be obtained, and when $r = 1$, (30) results in the stop-loss premium with $a_{j,1} = \beta^{-1} \sum_{k=1}^{\infty} k q_{j+k} = \beta^{-1} \sum_{i=j}^{\infty} \overline{q}_i$ where $\overline{q}_n = \sum_{m=n+1}^{\infty} q_m$. In particular, when $N = k$ (i.e. $p_{0,k} = 1$ for $i = k$ in (19) and then $q_n = \sum_{m_1+\cdots+m_i=n} p_{\mathbf{m}}$), it reduces to the stop-loss premium result by Lee and Lin (2012, p.167) in the presence of a deductible level d as $E\{(S_k - d)_+\} = \beta^{-1} e^{-\beta d} \sum_{j=0}^{\infty} \left(\sum_{i=j}^{\infty} \overline{q}_i \right) \frac{(\beta d)^j}{j!}$.

3.2 Premium calculations

In this section, we calculate several types of premiums previously considered in the literature when the multivariate loss variables have a joint pdf with the form (5) (e.g. see Kijima (2006) for multivariate extension of equilibrium pricing transform).

First, consider the Esscher premium defined as $E[Xe^{\mu X}]/E[e^{\mu X}]$ in the univariate case (e.g. Bühlmann (1980), Gerber (1980a)). For multivariate mixed Erlang losses, this premium can be easily obtained from (15) by

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} q^E_{\mathbf{m}} \left\{ \prod_{i=1}^{k} \frac{m_i}{\mu + \beta} \right\},$$

where $q^E_{\mathbf{m}}$ is given by (16).

Second, the joint size-biased distribution of order n_i for each rv Y_i is defined as (13) with $\mu = 0$, e.g. see Patil and Ord (1976) for the univariate case. Then from (14) its pdf is still a multivariate mixed Erlang with the same scale parameter β, but with new shape parameters shifted by n_i and new mixing weights. The weighted premium defined as $E[X^{1+c}]/E[X^c]$ where $c \in (0,1]$ for the univariate loss, is also calculated in this case assuming $n_i \in (0,1]$ for $i = 1,2,\ldots,k$ instead of c. It is a simply the expectation of the multivariate random vectors with (14) when $\mu = 0$ (i.e. multivariate mixed gamma), that is,

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} q^*_{\mathbf{m}} \left\{ \prod_{i=1}^{k} \frac{m_i + n_i}{\beta} \right\}.$$

Similar to what Furman and Zitikis (2008) pointed out, this premium converges to the net premium as $n_i \to 0$ for all i, and to the modified variance premium (e.g. Heilmann (1989)) as $n_i \to 1$ for all i.

Lastly, as first studied in Furman an Landsman (2006), here we calculate the tail variance premium for multivariate mixed Erlang losses. In the univariate case, this premium is defined with two risk measures including the conditional tail expectation (CTE) and conditional tail variance (TV) as $\mathrm{CTE}_p[X] + \alpha \mathrm{TV}_p[X]$ where $\alpha \geq 0$ and $p \in (0,1)$. In this case, from (5) one finds

$$\prod_{i=1}^{k} y_i^n f(y_1,\ldots,y_k) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} p_{\mathbf{m}} \left\{ \prod_{i=1}^{k} \beta^{-n} \frac{(n+m_i-1)!}{(m_i-1)!} \tau_{m_i+n,\beta}(y_i) \right\},$$

and calculate $q = \Pr(X_1 > F^{-1}_{X_1}(p), \ldots, X_k > F^{-1}_{X_k}(p))$ with the quantile $F^{-1}_{X_i}(p)$ for each X_i where F_{X_i} is a marginal cumulative distribution function (cdf) for X_i for $i = 1,2,\ldots,k$. Then, the first term of the formula for the tail variance premium in this multivariate case is given by

$$q^{-1} \left\{ \int_{F^{-1}_{X_1}(p)}^{\infty} \cdots \int_{F^{-1}_{X_k}(p)}^{\infty} h_1(x_1,\ldots,x_k) dx_1 \cdots dx_k \right\},$$

and similarly the second term is also found using h_n when $n = 2$. Since its evaluation is straightforward using the tail of the Erlang distribution in (1), it is omitted here.

3.3 Risk capital allocation

In this section, we study the capital allocation problem based on the risk contribution of the individual loss or business line X_j to the total risk $S = \sum_{i=1}^{k} X_i$ where the joint LT of these losses (or line of business) is given by (8). Here we use two different rules (based on TVaR and covariance) as in Cossette et al. (2013). They studied capital allocation problems when the multivariate distribution of X_is are defined via the FGM copula with mixed Erlang marginals and the total risk is a sum of these X_is.

3.3.1 Tail Value-at-Risk (TVaR)

To begin, let us define the Value-at-Risk (VaR) for X at level $t \in (0,1)$ by $\mathrm{VaR}_X(t) = \inf\{x : F_X(x) \geq t\}$, and the TVaR at level t for the continuous distribution as

$$\mathrm{TVaR}_X(t) = E[X | X > \mathrm{VaR}_X(t)] = \frac{1}{1-t} \int_t^1 \mathrm{VaR}_X(u) du.$$

Let v_t be the $\text{VaR}_S(t)$ of the total loss S at level t, then from the tail df of S, it is the solution of the equation (e.g. Corollary 5.1 of Lee and Lin (2011)),

$$1-t = e^{-\beta v_t}\sum_{n=0}^{\infty}\overline{Q}_n\frac{(\beta v_t)^n}{n!} = \beta^{-1}\sum_{n=1}^{\infty}\overline{Q}_{n-1}\tau_{n,\beta}(v_t), \tag{31}$$

where $\overline{Q}_n = \sum_{j=n+1}^{\infty} q_j$ with q_n given by (19) when $p_{0,i}=1$ for $i=k$ and $p_{0,i}=0$ for $i\neq k$, and $\tau_{n,\beta}(y)$ is given by (1). Then, its TVaR at level t is obtained as $\text{TVaR}_S(t) = v_t + E[S-v_t|S>v_t] = v_t + r(v_t)$ where $r(v_t)$ is the mean residual lifetime of $S_{v_t} = S-v_t|S>v_t$. Then using (23)-(25) with $\overline{Q}_n$ instead of $\overline{P}_n$, it follows

$$r(v_t) = \frac{\sum_{m=0}^{\infty}\overline{Q}^e_m\frac{(\beta v_t)^m}{m!}}{\beta\sum_{n=0}^{\infty}q^e_{n+1}\frac{(\beta v_t)^n}{n!}} = \frac{\beta^{-1}}{1-t}\sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty}\overline{Q}_n\right)\frac{(\beta v_t)^m}{m!},$$

since $q^e_{n+1} = \overline{Q}_n/\sum_{n=1}^{\infty}\overline{Q}_{n-1} = \overline{Q}_n/\sum_{j=1}^{\infty} jq_j$ in this case, and the denominator is simplified using (31).

Given the total risk capital at level t, let us define the risk capital $C_i(t)$ allocated for the individual loss i or business line i based on its TVaR, as $C_i(t) = \text{TVaR}_{X_i|S}(t) = E[X_i|S>v_t]$. Then a sum of all individual risks equal to the total risk, i.e.

$$\text{TVaR}_S(t) = \sum_{i=1}^{k} C_i(t), \tag{32}$$

and the amount of capital requirement for risk i for $i=1,2,\ldots,k$, given (32) is obtained as

$$C_i(t) = E[X_i|S>v_t] = \frac{1}{1-t}\int_{v_t}^{\infty} E[X_i|S=y]\,g(y)dy = \frac{1}{1-t}\int_{v_t}^{\infty}\int_0^y x f_{X_i,S}(x,y)dxdy, \tag{33}$$

where $g(y)$ is a pdf of S given by (20) when $N=k$. Then, the expression for $C_i(t)$ can be found in the following theorem.

Theorem 1 Assuming that the individual losses $(X_1,\ldots,X_k)$ follow the multivariate mixed Erlang distribution with the joint pdf in (5), the risk capital required for loss X_i for $i=1,2,\ldots,k$ in (33) can be found as

$$C_i(t) = \frac{1}{1-t}\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}a_{\mathbf{m}}\int_{v_t}^{\infty}\tau_{m+1,\beta}(y)dy = \frac{1}{1-t}\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}a_{\mathbf{m}}\left\{\sum_{n=0}^{m}\frac{e^{-\beta v_t}(\beta v_t)^n}{n!}\right\},$$

where $m=\sum_{j=1}^{k} m_j$ and

$$a_{\mathbf{m}} = p_{\mathbf{m}}\sum_{j=0}^{m_{-i}-1}\frac{\beta^{-1}(-1)^{m_{-i}-1-j}m^{(m_{-i}+1)}}{(m_{-i}-j-1)!j!(m-j)}. \tag{34}$$

Proof: To obtain $C_i(t)$, it is necessary to evaluate

$$\int_0^y x f_{X_i,S}(x,y)dx = \int_0^y x f_{X_i,S-X_i}(x,y-x)dx. \tag{35}$$

Let $S_{-i} = S - X_i = X_1 + \cdots + X_{i-1} + X_{i+1} + \cdots + X_k$, and since X_i and S_{-i} are dependent, the joint pdf of (X_i, S_{-i}) is obtained as

$$f_{X_i,S_{-i}}(x, y-x)$$
$$= \int_0^{y-x} \int_0^{y-x-x_1} \cdots \int_0^{y-x-\sum_{j=1,j\neq i}^{k-1} x_j} f(x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,y-x-\sum_{j=1,j\neq i}^{k-1} x_j)dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_{k-1},$$

and note that from (5) the above integral involves a convolution of independent $(k-1)$ Erlang rvs with each shape parameter m_j for $j = 1, 2, \ldots, k$ and $j \neq i$, and a common scale parameter β, i.e. univariate Erlang with a shape parameter $m_{-i} = m_1 + \cdots + m_{i-1} + m_{i+1} + \cdots + m_k$. Thus, it follows that

$$\begin{aligned} f_{X_i,S_{-i}}(x, y-x) &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} p_{\mathbf{m}} \tau_{m_i,\beta}(x)\tau_{m_{-i},\beta}(y-x) \\ &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} p_{\mathbf{m}} \left\{ \frac{e^{-\beta x} x^{m_i-1}\beta^{m_i}}{(m_i-1)!} \right\} \left\{ \frac{e^{-\beta(y-x)}(y-x)^{m_{-i}-1}\beta^{m_{-i}}}{(m_{-i}-1)!} \right\} \\ &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} p_{\mathbf{m}} \left\{ \frac{e^{-\beta y} x^{m_i-1}\beta^{m}}{(m_i-1)!(m_{-i}-1)!} \right\} \left\{ \sum_{j=0}^{m_{-i}-1} \binom{m_{-i}-1}{j} y^j (-x)^{m_{-i}-1-j} \right\}, \quad 0 < x < y, \end{aligned} \tag{36}$$

where $m = m_i + m_{-i} = \sum_{j=1}^{k} m_j$. Substitution of (36) into (35) results in

$$\begin{aligned} \int_0^y x f_{X_i,S}(x,y)dx &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} p_{\mathbf{m}} \left\{ \sum_{j=0}^{m_{-i}-1} \frac{e^{-\beta y}\beta^m (-1)^{m_{-i}-1-j}}{(m_i-1)!(m_{-i}-1)!} \binom{m_{-i}-1}{j} y^j \left(\int_0^y x^{m-j-1} dx \right) \right\} \\ &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} p_{\mathbf{m}} \left\{ \sum_{j=0}^{m_{-i}-1} \frac{\beta^{-1}(-1)^{m_{-i}-1-j} m!}{(m_i-1)!(m_{-i}-j-1)!j!(m-j)} \right\} \tau_{m+1,\beta}(y). \end{aligned} \tag{37}$$

In turn, the result follows. □

3.3.2 Covariance

According to Hesselager and Andersson (2002), the covariance principle can also be used for the capital allocation problem in a risk sharing scheme for a multi-line of an insurance company. If we apply the covariance-based rule to calculate the capital allocation amount $C_i(t)$ given (32), for each individual i, we can obtain

$$C_i(t) = E[X_i] + \frac{\mathrm{Cov}(X_i, S)}{\mathrm{Var}(S)}(\mathrm{TVaR}_S(t) - E[S]), \qquad i = 1, 2, \ldots, n.$$

Since $E[X_i]$ and $\mathrm{VaR}(S)$ are immediately available using the marginal distribution of X_i and the distribution of S provided previously, and $\mathrm{TVaR}_S(t)$ is given in previous section, we conclude this section with the expression for $E[X_iS]$ required for covariance calculation. From (36), we have $f_{X_i,S}(x,y) = f_{X_i,S_{-i}}(x, y-x)$ for $x < y$, and then using (37), one finds

$$E[X_iS] = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} a_{\mathbf{m}} \int_0^{\infty} y\tau_{m+1,\beta}(y)dy = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} a_{\mathbf{m}} \left(\frac{m+1}{\beta} \right),$$

where $a_{\mathbf{m}}$ is given by (34) and $m = \sum_{i=1}^{k} m_i$.

4 Minimum and maximum in multiple lives

In this section, we derive distributions of the minimum and the maximum variables where the joint distribution of k variables is assumed to be (5). Mixed Erlang models for human lifetimes are potentially very useful because these distribution can accurately approximate any positive distributions (e.g. Lee and Lin (2012)), even those as complicated as human survival distributions. Utilizing the distribution of the minimum and maximum random variables, we study the pricing of the joint and the last survivor statuses of several lifetimes. Some classes of multivariate distribution have nice properties which provide well-defined forms of joint distributions of the minimum and the maximum (even order statistics) such as a MPH distribution (e.g. Cai and Li (2005b)), a multivariate normal distribution (e.g. Gupta and Gupta (2001)), and a generalized exponential distributions (e.g. Franco and Vivo (2009)).

First, let $p_{\mathbf{m}}$ in (5) be a joint probability mass function (pmf) of $(N_1, N_2, \ldots, N_k)$, i.e.

$$p_{\mathbf{m}} = \Pr(N_1 = m_1, N_2 = m_2, \ldots, N_k = m_k) = \Pr\big(\cap_{l=1}^{k} \{N_l = m_l\}\big),$$

and let its tail df in (10) be,

$$\overline{P}_{\mathbf{m}} = \Pr(N_1 \geq m_1 + 1, N_2 \geq m_2 + 1, \ldots, N_k \geq m_k + 1) = \Pr\big(\cap_{l=1}^{k} \{N_l \geq m_l + 1\}\big).$$

Note that for $j = 1, 2, \ldots$, one has

$$-\frac{d}{dy}\ \tau_{j,\beta}(y) = \beta\{\tau_{j,\beta}(y) - \tau_{j-1,\beta}(y)\},$$

where we define $\tau_{0,\beta}(y) = 0$. Then, the tail df of the minimum $Y_{(1)} = \min(Y_1, \ldots, Y_k)$ is obtained as $\Pr(Y_{(1)} > y) = \overline{F}(y, \ldots, y)$, and from (9) and the product rule the pdf of $Y_{(1)}$ denoted by $f_{(1)}(y)$ is

$$-\frac{d}{dy}\overline{F}(y, \ldots, y) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \Pr\big(\cap_{l=1}^{k} \{N_l \geq m_l\}\big) \sum_{j=1}^{k} \{\tau_{m_j,\beta}(y) - \tau_{m_j-1,\beta}(y)\} \prod_{i=1, i\neq j}^{k} \frac{\tau_{m_i,\beta}(y)}{\beta}. \quad (38)$$

Because

$$\begin{aligned}\sum_{m_j=1}^{\infty} \Pr\big(\cap_{l=1}^{k} \{N_l \geq m_l\}\big)\tau_{m_j-1,\beta}(y) &= \sum_{m_j=0}^{\infty} \Pr\big(N_j \geq m_j + 1,\ \cap_{l=1, l\neq j}^{k} \{N_l \geq m_l\}\big)\tau_{m_j,\beta}(y) \\ &= \sum_{m_j=1}^{\infty} \Pr\big(N_j \geq m_j + 1,\ \cap_{l=1, l\neq j}^{k} \{N_l \geq m_l\}\big)\tau_{m_j,\beta}(y),\end{aligned}$$

(due to $\tau_{0,\beta}(y) = 0$ in the last line), it follows that

$$\sum_{m_j=1}^{\infty} \Pr\big(\cap_{l=1}^{k} \{N_l \geq m_l\}\big)\{\tau_{m_j,\beta}(y) - \tau_{m_j-1,\beta}(y)\} = \sum_{m_j=1}^{\infty} \overline{q}_{j,\mathbf{m}}\tau_{m_j,\beta}(y),$$

where $\overline{q}_{j,\mathbf{m}} = \Pr\big(N_j = m_j,\ \cap_{l=1, l\neq j}^{k} \{N_l \geq m_l\}\big)$. Thus (38) may be expressed as

$$-\frac{d}{dy}\overline{F}(y, \ldots, y) = \sum_{j=1}^{k} \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \overline{q}_{j,\mathbf{m}} \frac{\prod_{i=1}^{k} \tau_{m_i,\beta}(y)}{\beta^{k-1}}. \quad (39)$$

Also, note that in combinatorial notation $\binom{n}{n_1,n_2,\ldots,n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$, it follows that

$$\frac{\prod_{i=1}^{k}\tau_{m_i,\beta}(y)}{\beta^{k-1}} = \frac{\binom{\sum_{i=1}^{k} m_i - k}{m_1-1,m_2-1,\ldots,m_k-1}}{k^{\sum_{i=1}^{k} m_i - k+1}} \left\{\tau_{\sum_{i=1}^{k} m_i - k+1,k\beta}(y)\right\},$$

and so (39) can be expressed in mixed Erlang form as

$$-\frac{d}{dy}\overline{F}(y,\ldots,y) = \sum_{j=1}^{k}\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\overline{q}_{j,\mathbf{m}}\frac{\binom{\sum_{i=1}^{k} m_i - k}{m_1-1,m_2-1,\ldots,m_k-1}}{k^{\sum_{i=1}^{k} m_i - k+1}}\left\{\tau_{\sum_{i=1}^{k} m_i - k+1,k\beta}(y)\right\}. \tag{40}$$

Finally, from (40)

$$f_{(1)}(y) = \sum_{n=1}^{\infty} w_n \tau_{n,k\beta}(y),$$

where

$$w_n = k^{-n}\sum_{j=1}^{k}\sum_{A_{n,k}}\overline{q}_{j,\mathbf{m}}\binom{n-1}{m_1-1,m_2-1,\ldots,m_k-1}$$

and

$$A_{n,k} = \left\{(m_1,m_2,\ldots,m_k)\middle|\sum_{i=1}^{k} m_i = n+k-1,\ m_i \geq 1 \text{ for all } i\right\}. \tag{41}$$

Next, to find the pdf of the maximum $Y_{(k)} = \max(Y_1,\ldots,Y_k)$, recall that

$$\int_0^y \tau_{n,\beta}(x)dx = 1 - \int_y^{\infty}\tau_{n,\beta}(x)dx = 1 - e^{-\beta y}\sum_{m=0}^{n-1}\frac{(\beta y)^m}{m!} = e^{-\beta y}\sum_{m=n}^{\infty}\frac{(\beta y)^m}{m!},$$

and thus

$$\int_0^y \tau_{n,\beta}(x)dx = \sum_{m=n}^{\infty}\frac{\tau_{m+1,\beta}(y)}{\beta} = \sum_{m=n+1}^{\infty}\frac{\tau_{m,\beta}(y)}{\beta}.$$

Then, from (5), using the above result yields the joint df of $(Y_1, Y_2, \cdots, Y_k)$ as

$$F(y_1,y_2,\ldots,y_k) = \sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}p_{\mathbf{n}}\left\{\prod_{i=1}^{k}\int_0^{y_i}\tau_{n_i,\beta}(x_i)dx_i\right\} = \sum_{m_1=2}^{\infty}\cdots\sum_{m_k=2}^{\infty}\sum_{n_1=1}^{m_1-1}\cdots\sum_{n_k=1}^{m_k-1}p_{\mathbf{n}}\prod_{i=1}^{k}\frac{\tau_{m_i,\beta}(y_i)}{\beta},$$

i.e.

$$F(y_1,y_2,\ldots,y_k) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\Pr\left(\cap_{l=1}^{k}\{N_l \leq m_l\}\right)\prod_{i=1}^{k}\frac{\tau_{m_i+1,\beta}(y_i)}{\beta}.$$

Therefore, the pdf of the maximum $Y_{(k)}$ is, as in (38),

$$\frac{d}{dy}F(y,\ldots,y) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\Pr\left(\cap_{l=1}^{k}\{N_l \leq m_l\}\right)\sum_{j=1}^{k}\{\tau_{m_j,\beta}(y) - \tau_{m_j+1,\beta}(y)\}\prod_{i=1,i\neq j}^{k}\frac{\tau_{m_i+1,\beta}(y)}{\beta}. \tag{42}$$

As

$$\sum_{m_j=1}^{\infty}\Pr\left(\cap_{l=1}^{k}\{N_l \leq m_l\}\right)\tau_{m_j,\beta}(y) = \sum_{m_j=0}^{\infty}\Pr\left(N_j \leq m_j+1,\ \cap_{l=1,l\neq j}^{k}\{N_l \leq m_l\}\right)\tau_{m_j+1,\beta}(y)$$

it follows (noting that $N_j \geq 1$) that

$$\sum_{m_j=1}^{\infty} \Pr\left(\cap_{l=1}^{k} \{N_l \leq m_l\}\right)\{\tau_{m_j,\beta}(y) - \tau_{m_j+1,\beta}(y)\} = \sum_{m_j=0}^{\infty} q_{j,\mathbf{m}}\tau_{m_j+1,\beta}(y),$$

where $q_{j,\mathbf{m}} = \Pr\left(N_j = m_j + 1,\ \cap_{l=1,l\neq j}^{k} \{N_l \leq m_l\}\right)$. Thus, because $N_j \geq 1$ for all j, (42) may be expressed as in (39), namely,

$$\frac{d}{dy}F(y,\ldots,y) = \sum_{j=1}^{k}\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty} q_{j,\mathbf{m}}\frac{\prod_{i=1}^{k}\tau_{m_i+1,\beta}(y)}{\beta^{k-1}} = \sum_{j=1}^{k}\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty} q_{j,\mathbf{m}-\mathbf{1}}\frac{\prod_{i=1}^{k}\tau_{m_i,\beta}(y)}{\beta^{k-1}},$$

and the rest is the same as the minimum case. Essentially, the pdf of the maximum is given by

$$f_{(k)}(y) = \sum_{n=1}^{\infty} w_n^* \tau_{n,k\beta}(y),$$

where

$$w_n^* = k^{-n}\sum_{j=1}^{k}\sum_{A_{n,k}} q_{j,\mathbf{m}-\mathbf{1}}\binom{n-1}{m_1-1, m_2-1, \ldots, m_k-1}$$

and, as in (41), $A_{n,k} = \left\{(m_1, m_2, \ldots, m_k)\middle|\, \sum_{i=1}^{k} m_i = n+k-1,\ m_i \geq 1 \text{ for all } i\right\}$.

Lastly, we remark that the joint life and last survivor statuses are easily dealt with using the residual lifetime result obtained from (26), which is still of the form in (5) but with new mixing weights. Also, as mentioned, the use of the lifetime model can be justified by the denseness of the multivariate mixed Erlang as shown by Lee an Lin (2012).

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