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Event-based Global Stabilization of Linear Systems via a Saturated Linear Controller

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SUMMARY

This paper investigates the problem of event-based linear control of systems subject to input saturation. First, for discrete-time systems with neutrally stable or double-integrator dynamics, novel event-triggered control algorithms with non-quadratic event-triggering conditions are proposed to achieve global stabilization. Compared with the quadratic event-triggering conditions, the non-quadratic ones can further reduce unnecessary control updates for the input-saturated systems. Furthermore, for continuous-time systems with neutrally stable or double-integrator dynamics, since that an inherent lower bound of the inter-event time does not exist for systems subject to input saturation, novel event-triggered control algorithms with an appropriately selected minimum inter-event time are proposed to achieve global stabilization. Finally, numerical examples are provided to illustrate the theoretical results. Copyright © 2015 John Wiley & Sons, Ltd.

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KEY WORDS: Event-based control, input saturation, global stabilization, non-quadratic event-triggering condition.

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1. INTRODUCTION

In practical control systems, actuators are always subject to saturation, making the issue of input saturation an important research problem for control science and engineering [1, 2]. When a system contains no strictly unstable modes, it can be globally stabilized in spite of the input saturation [3, 4, 5]. Furthermore, if every marginally unstable eigenvalue, except the ones corresponding to double-integrator dynamics, of the system is semi-simple, linear control law can be used for global stabilization [6, 7, 8]. In addition, linear controller works for semi-global stabilization as well if the initial value of the marginally stable system is known to be located within a bounded set [9, 10]. If the input-saturated system is strictly unstable, then only local stabilization can be achieved and research attention is focused on estimating the domain of attraction [11, 12, 13]. Recently, the input saturation issue is considered for network synchronization in [14, 15, 16, 17], where semi-global synchronization is achieved; and global input-saturated synchronization is considered in [18, 19]. In addition, the saturation problem is studied for singular Lipschitz systems in [20], where local stabilization is achieved.

Event-based sampling and control, which originate from the research on aperiodic sampling [21], has been extensively studied since the late 1990s [22]. This has led to the gradually establishing event-triggered control (ETC), which can prevent unnecessary samplings as well as information transmissions and require less control updates than the traditional periodic control method. The ETC theory is first systematically studied in [23] based on the Lyapunov stability theory. An event-triggering rule is guaranteed to be legitimate in the sense that the inter-event time is lower bounded such that accumulative events known as the Zeno behavior [24] do not exist. The event-trigger strategy is applied to sensor/actuator networks and generalized to a decentralized form in [25], where a minimum time τ is set *a priori* instead of being guaranteed by the local event-triggering functions to ensure the legitimacy. In addition, the distributed ETC is analyzed in [26] and the ETC for discrete-time network synchronization is addressed in [27]. The discrete-time ETC is first studied in [28]. And in [29], the periodic ETC is proposed for linear systems to combine the advantages of both ETC and the traditional sampled control.

In the past several years, the ETC strategy is applied to systems subject to actuator saturation to achieve local stabilization [30, 31, 32, 33]. Then, in this paper, the problem of event-based linear global stabilization of systems subject to input saturation is investigated. For discrete-time systems with neutrally stable or double-integrator dynamics, novel event-triggered control algorithms with non-quadratic event-triggering conditions are proposed. For continuous-time systems with neutrally stable or double-integrator dynamics, novel event-triggered control algorithms with an appropriately selected minimum inter-event time are proposed.

The contribution and significance of the results in this paper are three-fold: (i) both discrete-time and continuous-time event-based systems are tackled via a saturated linear controller; (ii) non-quadratic event-triggering conditions are proposed to generate less control updates than the quadratic conditions; (iii) a minimum inter-event time is appropriately selected in the continuous-time event-trigger strategy to prevent the Zeno behavior. It is pointed out for the first time in this paper that when global stabilization is considered, the event-triggering functions do not guarantee a lower bound of the inter-event time for continuous-time input-saturated systems. To solve this fatal problem, a minimum time τ is set *a priori* such that the event trigger is active only after the time interval τ during each updating process.

The remaining of the paper is organized as follows. In Section 2, the problem of event-based global stabilization is formulated. The main results for discrete-time systems are presented in Section 3. The main results for continuous-time systems are established in Section 4. Furthermore, numerical examples are provided in Section 5. Finally, conclusion is drawn in Section 6.

Nomenclature: Throughout this paper, $\mathbb{R}^p$ and $\mathbb{R}^{p\times q}$ represent the p-dimensional real vector space and the set of all $p \times q$ real matrices, respectively. For $x \in \mathbb{R}^p$, $\|x\|$ denotes its Euclidian norm; and $\|x\|_\infty \triangleq \max_i |x_i|$. For $X \in \mathbb{R}^{p\times p}$, $\rho(X)$ denotes its spectral radius; X is said to be Schur if $\rho(X) < 1$. The notation diag$\{\cdot\}$ denotes a diagonal or block diagonal matrix; 0 and I denote a zero matrix and an identity matrix, respectively, with compatible dimension. For $M \in \mathbb{R}^{p\times q}$, M^T

denotes its transpose and $\|M\| \triangleq \sqrt{\rho(M^TM)}$ denotes its spectral norm. A matrix pair (A, B) is stabilizable if there exists some matrix K such that $\rho(A-BK) < 1$. The saturation function with threshold of ϖ is defined as $\text{sat}_{\varpi}(u) \triangleq \text{sgn}(u)\min\{|u|, \varpi\}$.

2. PROBLEM STATEMENT

Consider the following linear system

$$x^+ = Ax(t) + B \cdot \sigma(u(t)), \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; $x^+ \triangleq \dot{x}(t)$ for continuous-time system and $x^+ \triangleq x(t+1)$ for discrete-time system; (A, B) is stabilizable; and $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is a saturation operator defined as $\sigma(u) \triangleq [\sigma_1(u_1), ..., \sigma_m(u_m)]^T$, with the saturation function $\sigma_i(u_i) \triangleq \text{sat}_{\varpi_i}(u_i) = \text{sgn}(u_i)\min\{|u_i|, \varpi_i\}$, where $\varpi_i > 0$ is an input-saturation threshold given *a priori* for u_i.

The problem of event-based linear stabilization is as follows: design an event-triggering condition to generate an event-triggered updating time sequence $\{t_0, t_1, ...\}$, and design a linear feedback law $u(t) = -Kx(t_k)$, which uses only the feedback information at the updating time t_k, $k = 0, 1, ...$, such that (1) is globally stabilized, that is, $\lim_{t\to+\infty} x(t) = 0$ for any initial condition $x(0) \in \mathbb{R}^n$. For the continuous-time system, the Zeno behavior has to be excluded, that is, a lower bound τ needs to be guaranteed for the inter-event time, $t_{k+1} - t_k \geq \tau > 0$, $\forall k \geq 0$. For the global stabilization of systems subject to input saturation, it will be shown that an inherent lower bound τ, similar to the one in [23] for unsaturated systems, does not exists. Therefore, in this paper, a novel continuous-time event-trigger strategy is designed in the way that a lower bound τ is appropriately selected and the event-triggering condition is checked only after $t = t_k + \tau$.

For any stabilizable linear systems, there exists a coordinate transformation such that (A, B) is in the form of $A = \text{diag}\{A_z, A_s\}$ and $B = [B_z^T, B_s^T]^T$, where (A_z, B_z) is controllable and A_s is stable. The linear controller can be designed as $K = [K_z, 0]$. Then, the closed-loop stability of the event-based system (1) is equivalent to that of the system modes corresponding to (A_z, B_z). In addition, by the results in [3, 4, 5, 6], a system containing strictly unstable open-loop modes cannot be globally stabilized via a saturated controller; and linear feedback laws can be used for global stabilization only if every eigenvalue, except the ones corresponding to double-integrator dynamics, of A_z is semi-simple. Therefore, without loss of generality, it is assumed in the sequel that $(A, B) = (A_z, B_z)$; and either A is neutrally stable with all eigenvalues being semi-simple, or the system dynamics are double integrators. The following lemma will be used for both discrete-time and continuous-time neutrally stable systems.

Lemma 1. Let $A \in \mathbb{R}^{n\times n}$ be a real matrix with all eigenvalues being semi-simple, that is, A is diagonalizable. Then, there exists a matrix P that transforms A to its real Jordan form:

$$PAP^{-1} = \mathrm{diag}\left\{C_{n_1}(a_1, b_1),\ C_{n_2}(a_2, b_2),\ \cdots,\ C_{n_p}(a_p, b_p),\ J_{n_q}(\lambda_q),\ \cdots,\ J_{n_r}(\lambda_r)\right\}, \qquad (2)$$

where $C_{n_i}(a_i, b_i) = \mathrm{diag}\left\{\left[\begin{smallmatrix} a_i & b_i \\ -b_i & a_i \end{smallmatrix}\right], \cdots, \left[\begin{smallmatrix} a_i & b_i \\ -b_i & a_i \end{smallmatrix}\right]\right\} \in \mathbb{R}^{2n_i\times 2n_i}$ and $J_{n_s}(\lambda_s) = \lambda_s I_{n_s}$, with $a_i \pm jb_i$ being a pair of imaginary eigenvalues of A and λ_s being a real eigenvalue of A. Furthermore, if all the eigenvalues are located on the unit circle, then $A^T P^T PA = P^T P$; if all the eigenvalues are located on the imaginary axis, then $A^T P^T P + P^T PA = 0$.

Proof

Applying [34, Theorem 3.4.5] and noting that A is diagonalizable, it is straightforward to verify that the real Jordan form of A is in the form of (2). If all the eigenvalues are located on the unit circle, one has $\left[\begin{smallmatrix} a_i & b_i \\ -b_i & a_i \end{smallmatrix}\right]^T \left[\begin{smallmatrix} a_i & b_i \\ -b_i & a_i \end{smallmatrix}\right] = I$ and $(J_{n_s}(\lambda_s))^T J_{n_s}(\lambda_s) = I$ since $a_i^2 + b_i^2 = \lambda_s^2 = 1$. Then, it is straightforward that $(PAP^{-1})^T PAP^{-1} = I$ and $A^T P^T PA = P^T P$. If all the eigenvalues are located on the imaginary axis, one has $\left[\begin{smallmatrix} a_i & b_i \\ -b_i & a_i \end{smallmatrix}\right]^T + \left[\begin{smallmatrix} a_i & b_i \\ -b_i & a_i \end{smallmatrix}\right] = 0$ and $(J_{n_s}(\lambda_s))^T + J_{n_s}(\lambda_s) = 0$ since $a_i = \lambda_s = 0$. Then, it is immediate that $(PAP^{-1})^T + PAP^{-1} = 0$ and $A^T P^T P + P^T PA = 0$. □

3. DISCRETE-TIME SYSTEMS

In this section, system (1) is treated as a discrete-time system subject to input saturation. The discrete-time event-trigger strategy is described in Section 3.1. The discrete-time neutrally stable dynamics are considered in Section 3.2. The discrete-time double-integrator system is dealt with in Section 3.3.

3.1. Event-trigger Strategy

In this subsection, an event-triggered mechanism is described to generate the updating time sequence $\{t_k\}$ for discrete-time systems.

Algorithm 1. Event-based updating:

Step 1. The initial time is set as the first event time: $t_0 \triangleq 0$. At the beginning of each updating process, $t = t_k$, $k \geq 0$, the feedback control input u is updated. An event-triggering function $f(t)$ satisfying that $f(t_k) \leq 0$ will be designed later.

Step 2. For $t \geq t_k + 1$, the next updating event is triggered at instant $t_{k+1} \geq t_k + 1$ when $f(t_{k+1}) > 0$ and $f(t) \leq 0$ for all $t \in [t_k, t_{k+1})$. If no such an event occurs, denote $t_{k+1} \triangleq +\infty$. The feedback control will be designed in the linear form of $u(t) = -Kx(t_k),\ t \in [t_k, t_{k+1})$.

Step 3. When a finite t_{k+1} is triggered, a new updating cycle will begin; then, go to Step 1 and redefine the event-triggering function $f(t)$ such that $f(t_{k+1}) \leq 0$. Thus, $f(t) \leq 0$ holds all the time.

Remark 1. In event-triggered control [23], the event-triggering function $f(t)$ is dependent on the state error $(x(t) - x(t_k))$ or the control error $(\tilde{u}(t) - u)$, where $\tilde{u}(t)$ is the desirable control defined by $x(t)$ and u is the true control defined by $x(t_k)$. When the state error $(x(t) - x(t_k))$ is small, one has $f(t) \leq 0$, and feedback updating is unnecessary. When $(x(t) - x(t_k))$ becomes large enough at $t = t_{k+1}$ such that $f(t_{k+1}) > 0$, an updating event is triggered. After an event is triggered at t_{k+1}, the state error is updated as $(x(t) - x(t_{k+1}))$ and the control error is updated using the information of $x(t_{k+1})$. Since $x(t_{k+1}) - x(t_{k+1}) = 0$, the event-triggering function $f(t)$ can be redefined using the updated feedback $x(t_{k+1})$ such that $f(t_{k+1}) \leq 0$.

3.2. Neutrally Stable Dynamics

Assumption 1

The system matrix A is neutrally stable with all eigenvalues located on the unit circle and being semi-simple; and the pair (A, B) is controllable.

3.2.1. Control Protocol

The design of the control protocol is performed as follows.

Algorithm 2. Event-triggered Control Protocol:

Step 1. Find a similarity transformation matrix P such that PAP^{-1} is in the real Jordan form [34]. By Lemma 1 and Assumption 1, one has

$$A^T P^T P A = P^T P. \tag{3}$$

Step 2. Choose two event-trigger parameters $\rho_d \in (0, 1)$ and $\tilde{\rho}_d \in (0, 1)$. Then, set the control gain parameter μ such that $0 < \mu < 2(1 - \rho_d)/\|PB\|^2$, that is,

$$\frac{2(1-\rho_d)}{\mu} > \rho(B^T P^T P B). \tag{4}$$

Step 3. Design the linear controller matrix as

$$K \triangleq \mu B^T P^T P A. \tag{5}$$

Step 4. The event-triggering function $f(t)$ in Algorithm 1 is designed as

$$\begin{aligned} f(t) &\triangleq \max\{f_1(t), f_2(t)\}, \\ f_1(t) &\triangleq \frac{1}{\mu}\sigma(u)^T K x(t) + \frac{1}{2}\sigma(u)^T B^T P^T P B \sigma(u) + \frac{\rho_d}{\mu}\|\sigma(u)\|^2, \\ f_2(t) &\triangleq \|\sigma(-Kx(t)) - \sigma(u)\| - \tilde{\rho}_d \|\sigma(-Kx(t))\|. \end{aligned} \tag{6}$$

Step 5. The event-based control input is designed as

$$u(t) = -Kx(t_k),\ t \in [t_k, t_{k+1}), \tag{7}$$

where the updating times $\{t_k\}$ are generated by Algorithm 1.

Remark 2. (i) If A is already in the real Jordan form, one has $P^TP = I$, (3) reduces to $A^TA = I$, and (5) reduces to $K = \mu B^TA$.

(ii) The event-triggering function $f(t)$ in (6) is non-quadratic with respect to $x(t)$. Thus, the event-triggering condition $f(t) \geq 0$ is different from the quadratic ones in [29, 31, 32, 33]. The advantage of the non-quadratic conditions over the quadratic conditions is shown in Example 1 in Section 5, where non-quadratic condition triggers less feedback updates.

(iii) When the event-trigger parameter $\tilde{\rho}_d$ is set as zero, one has $f(t) \geq 0$ for $t = t_k + 1$. Thus, following Algorithm 1, $t_{k+1} = t_k + 1$, and system (1) becomes a traditional input-saturated system with no effect of event-triggering conditions.

3.2.2. *Event-based Global Stabilization*

Theorem 1

Consider the linear discrete-time input-saturated system (1). Let Assumption 1 hold. Then, Algorithms 1 and 2 can achieve global stabilization of system (1), that is, $\lim_{t\to+\infty} x(t) = 0$ for any initial condition $x(0) \in \mathbb{R}^n$.

Proof

For $t \geq t_k$, $k \geq 0$, define the *desired control* as $\tilde{u}(t) = -Kx(t)$, and the *error variable* as

$$e(t) \triangleq \sigma(\tilde{u}(t)) - \sigma(u), \tag{8}$$

such that $\sigma(\tilde{u}) = e + \sigma(u)$. At $t = t_k$, $e = 0$, one has $f_2(t_k) = -\tilde{\rho}_d\|\sigma(u)\| \leq 0$, and by (4),

$$\begin{aligned} f_1(t_k) =& -\frac{1}{\mu}\sigma(u)^Tu + \frac{1}{2}\sigma(u)^TB^TP^TPB\sigma(u) + \frac{\rho_d}{\mu}\|\sigma(u)\|^2 \\ \leq& -\left(\frac{(1-\rho_d)}{\mu} - \frac{1}{2}\rho(B^TP^TPB)\right)u^T\sigma(u) \leq 0. \end{aligned}$$

Thus, $f(t_k) \leq 0$, and Algorithm 1 is feasible.

Now, consider the following quadratic Lyapunov function candidate:

$$V(x(t)) \triangleq \frac{1}{2}x(t)^TP^TPx(t), \tag{9}$$

where $P^TP \succ 0$ is given in (3). Denote $\tilde{x}(t+1) \triangleq Ax(t) + B\sigma(u)$. For $t \in [t_k, t_{k+1})$, one has $x(t+1) = \tilde{x}(t+1)$, $V(x(t+1)) = V(\tilde{x}(t+1))$, and the variation of V along the discrete-time trajectories of x within the time interval $[t_k, t_{k+1})$, that is, $\Delta V(t) \triangleq V(x(t+1)) - V(x(t))$, can be evaluated as follows:

$$\begin{aligned}\Delta V(t) =& \frac{1}{2}(Ax(t) + B\cdot\sigma(u))^T P^T P(Ax(t) + B\cdot\sigma(u)) - \frac{1}{2}x(t)^T P^T Px(t)\\ =& \sigma(u)^T B^T P^T PAx(t) + \frac{1}{2}\sigma(u)^T B^T P^T PB\sigma(u)\\ =& f_1(t) - \frac{\rho_d}{\mu}\|\sigma(u)\|^2 \leq -\frac{\rho_d}{\mu}\|\sigma(u)\|^2 \leq 0. \end{aligned}\tag{10}$$

Therefore, one has $V(x(t+1)) \leq V(x(t))$ for all $t \in [t_k, t_{k+1})$, and "=" holds if and only if $Kx(t_k) = 0$. Thus, $V(x(t))$ is non-increasing and non-negative, and $\lim_{t\to+\infty} V(x(t))$ exists, which implies that $\lim_{t\to+\infty} \Delta V(t) = 0$. By (10), one has $u(t_k) = 0$ and $t_{k+1} = +\infty$ for some $k \geq 0$, or $\lim_{t_k\to+\infty} u(t_k) = 0$. If $t_{k+1} = +\infty$, by (10), one has $u(t_k) = 0$, and $x(t) = A^{t-t_k}x(t_k)$, $f_2(t) = (1-\tilde{\rho}_d)\|\sigma(-Kx(t))\| \leq 0$, $B^T P^T PA^{t-t_k+1}x(t_k) = 0$, $\forall\ t \geq t_k$. By (3), $B^T P^T PA^{-s}x(t_k+n) = B^T(A^T)^s P^T Px(t_k+n) = 0$, $s = 0, 1, ..., n-1$. Since (A, B) is controllable, one has that $\left(B^T P^T, (P^T)^{-1}A^T P^T\right)$ is observable, which implies that $Px(t_k+n) = 0$ and $x(t_k) = 0$. If $\lim_{t_k\to+\infty} u(t_k) = 0$, $\lim_{t\to+\infty} u(t) = 0$, then $f_2(t) \leq 0$ implies $\lim_{t\to+\infty} Kx(t) = 0$, and $\lim_{t\to+\infty} Kx(t+s) = \lim_{t\to+\infty} KA^s x(t) = 0$, $s = 0, 1, ..., n$. Consequently, one has $\lim_{t\to+\infty} x(t) = 0$. This completes the proof of Theorem 1. □

Remark 3. (i) A key feature of the event-triggered control [23] is that the feedback updating is performed only when the error is large enough and the desired control deviates too much from the true control. In the above Lyapunov analysis, $V(x(t+1))$ is the Lypunov function of the closed-loop system having applied the true control given in (7); the desired control $\tilde{u}(t) = -Kx(t)$ is only used for defining the error variable $e(t)$ in (8), but not for the Lyapunov analysis of the closed-loop system. After an event is triggered at $t = t_{k+1} \geq t_k + 1$, the feedback information of $x(t_{k+1})$ can be used by the controller and the event trigger, that is, $u(t_k)$ is substituted by $u(t_{k+1})$, and the state error is updated as $(x(t) - x(t_{k+1}))$. Furthermore, the Lyapunov analysis is updated in a new cycle. As shown in the proof of Theorem 1, at the time instant t_k, one has $V(x(t_k+1)) \leq V(x(t_k))$. Then, at the beginning of a new cycle, one similarly has $V(x(t_{k+1}+1)) \leq V(x(t_{k+1}))$.

(ii) By adopting the non-quadratic event-triggering function $f(t)$ in (6), with $f_1(t)$ based on the variation of the Lyapunov function and $f_2(t)$ based on the error $e(t)$ defined in (8), the number of control updates can be significantly reduced for system (1). The effectiveness of the non-quadratic event-triggering condition is illustrated in Example 1 in Section 5.

3.3. *Double-integrator Dynamics*

Consider the ZOH-discretized double integrators

$$x_1(t+1) = x_1(t) + hx_2(t) + \tfrac{1}{2}h^2\sigma(u),\, x_2(t+1) = x_2(t) + h\sigma(u),\, u = -k_1x_1(t_k) - k_2x_2(t_k),$$

where $x_1, x_2, u \in \mathbb{R}^{\frac{n}{2}}$ and h is the sampling period, that is, the system matrices are

$$A = \begin{bmatrix} I & hI \\ 0 & I \end{bmatrix}, \quad B = \begin{bmatrix} \frac{h^2}{2}I \\ hI \end{bmatrix}, \quad K = \begin{bmatrix} k_1I, k_2I \end{bmatrix}. \tag{11}$$

It is straightforward to verify that $(A - BK)$ is Schur if and only if $0 < hk_1/2 < k_2 < 2/h$. To further guarantee the closed-loop stability of the event-based system (1), we assume that

$$0 < \frac{3h}{2}k_1 \le k_2 < \frac{3}{2h}. \tag{12}$$

Denoting $a \triangleq \frac{h^2}{2}k_1$, $b \triangleq hk_2$, and $c \triangleq a - b + 1$, one has that condition (12) implies $a < (1-c^2)/2$, that is, $(\frac{h^2}{2}k_1 - hk_2 + 1)^2 + h^2k_1 < 1$.

3.3.1. *Control Protocol*

The design of the control protocol is performed as follows.

Algorithm 3. Event-triggered Control Protocol:

Step 1. For any k_1, k_2 satisfying (12), make the coordinate transformation $y = Tx$:

$$y_1 = -Kx = -k_1x_1 - k_2x_2,\ y_2 = -k_1x_2. \tag{13}$$

Denote $c_1 \triangleq -k_1$, $c_2 \triangleq -\frac{h}{2}k_1 - k_2$, and

$$\rho_0 \triangleq 1 + c_1h^2 - (c_2h - c_1h^2 + 1)^2 > 0. \tag{14}$$

Step 2. Consider the following Lyapunov function candidate:

$$V(y) \triangleq 2y_1^T\sigma(y_1) - \|\sigma(y_1)\|^2 - 2hy_2^T\sigma(y_1) + \frac{1}{k_1}\|y_2\|^2, \tag{15}$$

which is positive definite since $k_1 < 1/h^2$. For $t \geq t_k$, define

$$\tilde{y}(t+1) = \begin{bmatrix} \tilde{y}_1(t+1) \\ \tilde{y}_2(t+1) \end{bmatrix} \triangleq \begin{bmatrix} y_1(t) + hy_2(t) + hc_2\sigma(u) \\ y_2(t) + hc_1\sigma(u) \end{bmatrix}; \tag{16}$$

and define the *error variable* as

$$e(t) \triangleq \sigma(y_1(t)) - \sigma(u). \tag{17}$$

Step 3. Choose two event-trigger parameters $\rho_1 \in (0, \rho_0)$ and $\rho_2 \in (0, 1)$, where ρ_0 is defined in (14).

Step 4. Using (16) and (17), the event-triggering function $f(t)$ in Algorithm 1 is designed as

$$\begin{aligned} f(t) &\triangleq \max\{f_1(t), f_2(t)\}, \\ f_1(t) &\triangleq V(\tilde{y}(t+1)) - V(y(t)) + \rho_1\|\sigma(u)\|^2, \\ f_2(t) &\triangleq \|e(t)\| - \rho_2\|\sigma(y_1(t))\|. \end{aligned} \tag{18}$$

Step 5. The event-based control input is designed as

$$u(t) = -Kx(t_k), \ t \in [t_k, t_{k+1}), \tag{19}$$

where the updating times $\{t_k\}$ are generated by Algorithm 1 and K is given in (11).

Remark 4. The Lyapunov function (15) is inspired by the proof of [7, Theorem 2]. It can be easily verified that $0 < hk_1/2 < k_2 < 2/h$ if and only if $\frac{h}{2}c_1 - \frac{2}{h} < c_2 < hc_1 < 0$, which is consistent with [7, Condition (3)]. By adopting the non-quadratic event-triggering function $f(t)$ in (18), with $f_1(t)$ based on the Lyapunov function and $f_2(t)$ based on the error $e(t)$ defined in (17), the number of control updates can be significantly reduced.

3.3.2. Event-based Global Stabilization

Theorem 2

Consider the discrete-time input-saturated double-integrator system (1) with (A, B) given in (11). Algorithms 1 and 3 can achieve global stabilization of system (1), that is, $\lim_{t\to+\infty} x(t) = 0$ for any initial condition $x(0) \in \mathbb{R}^n$.

Proof

The dynamics of y are $y_1^+ = y_1 + hy_2 + hc_2\sigma(u)$, $y_2^+ = y_2 + hc_1\sigma(u)$. At $t = t_k$, $e = 0$, one has $f_2(t_k) = -\rho_2\|\sigma(u)\| \leq 0$; denoting $\tilde{y}_1 \triangleq \tilde{y}_1(t_k+1) = [\tilde{y}_{11}, ..., \tilde{y}_{1m}]^T$ and $y_1(t_k) = u = [u_1, ..., u_m]^T$ with $m = n/2$, by (15), one has

$$
\begin{aligned}
f_1(t_k) =& 2u^T\sigma(\tilde{y}_1) + 2h(c_2 - hc_1)\sigma(u)^T\sigma(\tilde{y}_1) - \|\sigma(\tilde{y}_1)\|^2 - 2u^T\sigma(u) + (1 - c_1h^2 + \rho_1)\|\sigma(u)\|^2 \\
\triangleq& \sum_{i=1}^{m} f_{1i} = \sum_{i=1}^{m} \big(2u_i\sigma_i(\tilde{y}_{1i}) + 2h(c_2 - hc_1)\sigma_i(u_i)\sigma_i(\tilde{y}_{1i}) - \|\sigma_i(\tilde{y}_{1i})\|^2 \\
& - 2u_i\sigma_i(u_i) + (1 - c_1h^2 + \rho_1)\|\sigma_i(u_i)\|^2\big) .
\end{aligned}
$$

Similar to the proof of [7, Theorem 2], one has that $f_{1i} = -2(u_i - \varpi_i)(\varpi_i - \sigma_i(\tilde{y}_{1i})) - (\sigma_i(\tilde{y}_{1i}) - (c_2h - c_1h^2 + 1)\varpi_i)^2 - (\rho_0 - \rho_1)\varpi_i^2$ if $u_i \geq \varpi_i$; $f_{1i} = 2(u_i + \varpi_i)(\varpi_i + \sigma_i(\tilde{y}_{1i})) - (\sigma_i(\tilde{y}_{1i}) + (c_2h - c_1h^2 + 1)\varpi_i)^2 - (\rho_0 - \rho_1)\varpi_i^2$ if $u_i \leq -\varpi_i$; and $f_{1i} = -(\sigma_i(\tilde{y}_{1i}) - (c_2h - c_1h^2 + 1)u_i)^2 - (\rho_0 - \rho_1)u_i^2$ if $|u_i| < \varpi_i$. Thus, $f_1(t_k) \leq -(\rho_0 - \rho_1)\|\sigma(u)\|^2 \leq 0$, $f(t_k) \leq 0$, and Algorithm 1 is feasible.

For any $t \in [t_k, t_{k+1})$, one has $V(y(t+1)) = V(\tilde{y}(t+1))$, and the variation $\Delta V(t) \triangleq V(y(t+1)) - V(y(t))$ along the trajectories of y within the time interval $[t_k, t_{k+1})$ satisfies

$$
\Delta V(t) = f_1(t) - \rho_1\|\sigma(u)\|^2 \leq -\rho_1\|\sigma(u)\|^2 \leq 0. \tag{20}
$$

Therefore, one has $V(y(t+1)) \leq V(y(t))$ for all $t \in [t_k, t_{k+1})$, and "=" holds if and only if $y_1(t_k) = 0$ and $y_1(t_k+1) = 0$, which implies that $x(t_k) = 0$. Thus, $V(y(t))$ is non-negative and strictly decreasing until $y(t_k) = x(t_k) = 0$ for some possible t_k. Therefore, $\lim_{t\to+\infty} V(y(t))$ exists, which implies that $\lim_{t\to+\infty} \Delta V(t) = 0$. By (20), one has $u(t_k) = 0$ and $t_{k+1} = +\infty$ for some $k \geq 0$, or $\lim_{t_k\to+\infty} u(t_k) = 0$. If $t_{k+1} = +\infty$, by (20), one has $x(t_k) = 0$. If $\lim_{t_k\to+\infty} u(t_k) = 0$, $\lim_{t\to+\infty} u(t) = 0$, then $f_2(t) \leq 0$ implies $\lim_{t\to+\infty} y_1(t) = 0$, and $\lim_{t\to+\infty} y_1(t+1) = \lim_{t\to+\infty} hy_2(t) = 0$. Consequently, one has $\lim_{t\to+\infty} y(t) = 0$ and $\lim_{t\to+\infty} x(t) = 0$. This completes the proof of Theorem 2. □

4. CONTINUOUS-TIME SYSTEMS

In this section, system (1) is treated as a continuous-time system subject to input saturation. The continuous-time event-trigger strategy is proposed in Section 4.1. The continuous-time neutrally stable dynamics are considered in Section 4.2. The continuous-time double-integrator system is dealt with in Section 4.3.

4.1. Event-trigger Strategy

In this subsection, an event-triggered mechanism is described to generate the updating time sequence $\{t_k\}$ for continuous-time systems.

Algorithm 4. Event-based updating:

Step 1. The initial time is set as the first event time: $t_0 \triangleq 0$. At the beginning of each updating process, $t = t_k$, $k \geq 0$, the feedback control input u is updated. A minimum inter-event time $\tau > 0$, which will be designed later, is set *a priori*. During the time interval $[t_k, t_k + \tau)$, the event trigger is inactive.

Step 2. At the time instant $t = t_k + \tau$, the event trigger is activated. An event-triggering function $f(t)$ will be designed later. Either if $f(t_k + \tau) > 0$, or if $f(t_k + \tau) = 0$ while $\|x(t_k + \tau)\| \neq 0$, the next event time is triggered as $t_{k+1} = t_k + \tau$; if $f(t_k + \tau) < 0$, for $t > t_k + \tau$, the next updating event is triggered at instant $t_{k+1} > t_k + \tau$ when $f(t_{k+1}) = 0$ and $f(t) < 0$ for all $t \in [t_k + \tau, t_{k+1})$; if $f(t_k + \tau) = \|x(t_k + \tau)\| = 0$, stabilization has been achieved in finite time; if $f(t) < 0$ for all $t \in [t_k + \tau, +\infty)$, denote $t_{k+1} \triangleq +\infty$. The feedback control will be designed in the linear form of $u(t) = -Kx(t_k),\ t \in [t_k, t_{k+1})$.

Step 3. When a finite t_{k+1} is triggered, a new updating cycle will begin; then, go to Step 1.

Remark 5. A key feature of Algorithm 4 is that a minimum inter-event time τ is appropriately selected to prevent the Zeno behavior. It is pointed out for the first time in this paper that when global stabilization is considered, the event-triggering functions do not guarantee a lower bound of the inter-event time for continuous-time input-saturated systems, see Examples 2 and 4 in Section 5.

Denote $\mathbb{T} \triangleq \bigcup_{k\geq 0}(t_k+\tau, t_{k+1}) = [0,\infty) \setminus \bigcup_{k\geq 0}[t_k, t_k+\tau]$. The closed-loop stability of the event-based continuous-time system (1) will be established via a Lyapunov function V satisfying that V is non-increasing on the time sequence $\{t_k\}$ and non-increasing on $\mathbb{T}$ as well.

4.2. Neutrally Stable Dynamics

Assumption 2

The system matrix A is neutrally stable with all eigenvalues located on the imaginary axis and being semi-simple; and the pair (A, B) is controllable.

4.2.1. Control Protocol

The design of the control protocol is performed as follows.

Algorithm 5. Event-triggered Control Protocol:

Step 1. Find a similarity transformation matrix P such that PAP^{-1} is in the real Jordan form [34]. By Lemma 1 and Assumption 2, one has

$$A^T P^T P + P^T P A = 0. \tag{21}$$

Denote $A_d \triangleq e^{A\tau} = I + \Psi(\tau)A$ and $B_d \triangleq \int_0^\tau e^{A(\tau-s)}ds B = \Psi(\tau)B$, where τ is the minimum inter-event time for Algorithm 4 and $\Psi(\tau) \triangleq \sum_{i=0}^{+\infty} \frac{1}{(i+1)!}\tau^{i+1}A^i$. One has $A_d^T P^T P A_d = P^T P$. Then, the minimum inter-event time τ is chosen such that (A_d, B_d) is controllable.

Step 2. Set the control gain parameter μ such that $0 < \mu < 2\tau/\|PB_d\|^2$, that is,

$$\frac{2\tau}{\mu} > \rho(B_d^T P^T P B_d). \tag{22}$$

Step 3. Design the linear controller matrix as

$$K \triangleq \frac{\mu}{\tau} B_d^T P^T P A_d = \mu B^T \Phi(\tau) P^T P, \tag{23}$$

where $\Phi(\tau) \triangleq \sum_{i=0}^{+\infty} \frac{1}{(i+1)!}(-\tau)^i A^i$ satisfying $\lim_{\tau\to 0} \Phi(\tau) = I$.

Step 4. Choose two event-trigger parameters $\rho_c \in (0,1)$ and $\tilde{\rho}_c \in (0,1)$. The event-triggering function $f(t)$ in Algorithm 4 is designed as

$$\begin{aligned} f(t) &\triangleq \max\{f_1(t), f_2(t)\}, \\ f_1(t) &\triangleq \sigma(u)^T B^T P^T P x(t) + \frac{\rho_c}{\mu}\|\sigma(u)\|^2, \\ f_2(t) &\triangleq \|\sigma(-Kx(t)) - \sigma(u)\| - \tilde{\rho}_c\|\sigma(-Kx(t))\|. \end{aligned} \tag{24}$$

Step 5. The event-based control input is designed as

$$u(t) = -Kx(t_k),\ t \in [t_k, t_{k+1}), \tag{25}$$

where the updating times $\{t_k\}$ are generated by Algorithm 4.

Remark 6. (i) If A is already in the real Jordan form, one has $P^T P = I$, (21) reduces to $A^T + A = 0$, and (23) reduces to $K = \frac{\mu}{\tau} B_d^T A_d = \mu B^T \Phi(\tau)$.

(ii) When setting τ such that (A_d, B_d) is controllable, for single-integrator dynamics, that is, $A = 0$ and $B = I$, one has $A_d = I$, $B_d = \tau I$, and (A_d, B_d) is controllable for any $\tau > 0$. For planar dynamics [8], that is, $A = \left[\begin{smallmatrix} 0 & \omega \\ -\omega & 0 \end{smallmatrix}\right]$ and $B = [b_1\ b_2]^T$ with $\omega > 0$ and $b_1^2 + b_2^2 \neq 0$, using the facts that $\cos x = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ and $\sin x = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$, one obtains

$$\begin{aligned} A_d &= \begin{bmatrix} \cos(\omega\tau) & \sin(\omega\tau) \\ -\sin(\omega\tau) & \cos(\omega\tau) \end{bmatrix}, \quad B_d = (A_d - I)A^{-1}B, \\ B_d &= \frac{1}{\omega}\left[b_1 \sin(\omega\tau) + b_2(1 - \cos(\omega\tau)), -b_1(1 - \cos(\omega\tau)) + b_2 \sin(\omega\tau)\right]^T. \end{aligned} \tag{26}$$

Since $\omega > 0$ and $b_1^2 + b_2^2 \neq 0$, it can be easily verified that (A_d, B_d) is controllable if and only if $\sin(\omega\tau) \neq 0$. Thus, one can choose any $\tau \in (0, +\infty) \setminus \{k\pi/\omega | k = 1, 2, ...\}$ such that (A_d, B_d) is controllable.

4.2.2. Event-based Global Stabilization

Theorem 3

Consider the linear continuous-time input-saturated system (1) satisfying Assumption 2. Algorithms 4 and 5 can achieve global stabilization of system (1), that is, $\lim_{t\to+\infty} x(t) = 0$ for any initial condition $x(0) \in \mathbb{R}^n$.

Proof

For $t \geq t_k$, $k \geq 0$, denote $\tilde{u}(t) = -Kx(t)$, and define the *error variable* as

$$e(t) \triangleq \sigma(\tilde{u}(t)) - \sigma(u). \tag{27}$$

Consider the following quadratic Lyapunov function candidate:

$$V(x(t)) \triangleq \frac{1}{2} x(t)^T P^T P x(t), \tag{28}$$

where $P^T P \succ 0$ is given in (21). Noting that $x(t_k + \tau) = A_d x(t_k) + B_d \sigma(u)$, similar to the proof of Theorem 1, one has that

$$\begin{aligned} V\left(x(t_k+\tau)\right) - V\left(x(t_k)\right) &= \sigma(u)^T B_d^T P^T P A_d x(t_k) + \frac{1}{2}\sigma(u)^T B_d^T P^T P B_d \sigma(u) \\ &\leq -\left(\frac{\tau}{\mu} - \frac{1}{2}\rho(B_d^T P^T P B_d)\right) u^T \sigma(u) \leq 0. \end{aligned} \tag{29}$$

Therefore, if $t_{k+1} = t_k + \tau$, $V\left(x(t_{k+1})\right) \leq V\left(x(t_k)\right)$, where "=" holds if and only if $Kx(t_k) = 0$. If $t_k + \tau < t_{k+1} < +\infty$, for $t \in [t_k + \tau, t_{k+1})$, by Algorithm 4 and (24), one has $u \neq 0$ and $\dot{V} = \sigma(u)^T B^T P^T P x(t) = f_1(t) - \frac{\rho_c}{\mu}\|\sigma(u)\|^2 < -\frac{\rho_c}{\mu}\|\sigma(u)\|^2 < 0$, which implies that $V\left(x(t_{k+1})\right) < V\left(x(t_k)\right)$. If $f(t_k + \tau) = \|x(t_k + \tau)\| = 0$, one has $u(t_k) = 0$, $x(t) = 0$, $V(x(t)) = 0$, $\forall t \geq t_k + \tau$. If $f(t) < 0$ so that $f_2(t) < 0$ for all $t \in [t_k + \tau, +\infty)$, one has $u(t_k) \neq 0$, $\dot{V} < -\frac{\rho_c}{\mu}\|\sigma(u)\|^2 < 0$, V is decreasing with at least a linear decay rate, which is impossible by the positive definiteness of V.

Thus, $V(x(t))$ is non-negative and non-increasing on both $\{t_k\}$ and $\mathbb{T} = \bigcup_{k \geq 0}(t_k + \tau, t_{k+1})$ so that $\lim_{t\to+\infty} V(x(t))$ exists, which implies that $\lim_{t\to+\infty} u(t) = 0$, $x(t)$ is bounded, and $\lim_{t\to+\infty} V'_+(t) = 0$. Here, $V'_+(t)$ denotes the right-derivative of $V(x(t))$. Then, similar to the proof of Theorem 1, one obtains $\lim_{t\to+\infty} x(t) = 0$. This completes the proof of Theorem 3. □

Remark 7. The continuous-time event-trigger strategy is based on the discrete-time results in Section 3.2. The necessity of setting a non-inherent minimum inter-event time τ is illustrated in Example 2 in Section 5. The non-quadratic event-triggering function $f(t)$ in (24), with $f_1(t)$ based on the Lyapunov function and $f_2(t)$ based on $e(t)$ in (27), can significantly reduce the number of control updates.

The advantage of $e(t)$ in (27) over the one used for the quadratic event-triggering condition, that is, $e_q(t) \triangleq Kx(t) - Kx(t_k)$ for the quadratic function $f_q(t) \triangleq (Kx(t) - Kx(t_k))^T(Kx(t) - Kx(t_k)) - \tilde{\rho}_c^2 x(t)^T K^T Kx(t)$, is demonstrated in Proposition 1 below.

Proposition 1

For single-input system (1) with $m = 1$, when $t \geq t_k$, if t_{k+1} is triggered by the quadratic condition $|Kx(t_{k+1}) - Kx(t_k)| = \tilde{\rho}_c|Kx(t_{k+1})|$ and $|Kx(t) - Kx(t_k)| < \tilde{\rho}_c|Kx(t)|$ for all $t \in [t_k, t_{k+1})$, then $|\sigma(Kx(t_{k+1})) - \sigma(Kx(t_k))| \leq \tilde{\rho}_c|\sigma(Kx(t_{k+1}))|$. Thus, the quadratic event-triggering condition leads to more frequent control updates.

Proof

First, for any $\varpi > 0$ and $a, b \in \mathbb{R}$, $a \neq 0$, it will be shown that

$$\frac{|a-b|}{|a|} < \frac{|\text{sat}_\varpi(a) - \text{sat}_\varpi(b)|}{|\text{sat}_\varpi(a)|} \quad \Rightarrow \quad \frac{|a-b|}{|a|} \geq 1 > \tilde{\rho}_c. \tag{30}$$

On one hand, if $ab \leq 0$, then $|a - b| \geq |a|$. On the other hand, if $a, b \in [-\varpi, \varpi]$ or $\text{sat}_\varpi(a) = \text{sat}_\varpi(b)$, then $|a - b| \cdot |\text{sat}_\varpi(a)| \geq |a| \cdot |\text{sat}_\varpi(a) - \text{sat}_\varpi(b)|$. Thus, only the following three cases need to be discussed: (i) $|b| > \varpi$, $|a| < \varpi$; (ii) $0 < b < \varpi$, $a > \varpi$; and (iii) $-\varpi < b < 0$, $a < -\varpi$. In case (i), $|b - a| \geq |\text{sat}_\varpi(b) - a| \geq \varpi - |a|$; in case (ii), $-b/a > -b/\varpi$; in case (iii), $-b/a > b/\varpi$. Consequently, in all three cases, one obtains $|a - b| \cdot |\text{sat}_\varpi(a)| \geq |a| \cdot |\text{sat}_\varpi(a) - \text{sat}_\varpi(b)|$. Therefore, (30) holds.

Suppose that $|\sigma(Kx(t_{k+1})) - \sigma(Kx(t_k))| > \tilde{\rho}_c|\sigma(Kx(t_{k+1}))|$, which implies $Kx(t_{k+1}) \neq 0$. By (30), one has that either

$$\frac{|Kx(t_{k+1}) - Kx(t_k)|}{|Kx(t_{k+1})|} \geq \frac{|\sigma(Kx(t_{k+1})) - \sigma(Kx(t_k))|}{|\sigma(Kx(t_{k+1}))|} > \tilde{\rho}_c$$

or

$$\frac{|\sigma(Kx(t_{k+1})) - \sigma(Kx(t_k))|}{|\sigma(Kx(t_{k+1}))|} > \frac{|Kx(t_{k+1}) - Kx(t_k)|}{|Kx(t_{k+1})|} \geq 1 > \tilde{\rho}_c,$$

both of which lead to a contradiction. □

4.3. Double-integrator Dynamics

Consider the double integrators

$$\dot{x}_1(t) = x_2(t), \dot{x}_2(t) = \sigma(u), u = -k_1 x_1(t_k) - k_2 x_2(t_k),$$

where $x_1, x_2, u \in \mathbb{R}^{\frac{n}{2}}$, that is, the system matrices are

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad K = \begin{bmatrix} k_1 I, k_2 I \end{bmatrix}. \tag{31}$$

By Routh-Hurwitz criteria, $(A - BK)$ is stable if and only if $k_1 > 0$ and $k_2 > 0$.

4.3.1. Control Protocol

The design of the control protocol is performed as follows.

Algorithm 6. Event-triggered Control Protocol:

Step 1. For any $k_1 > 0$ and $k_2 > 0$, make the coordinate transformation $y = Tx$:

$$y_1 = -Kx = -k_1 x_1 - k_2 x_2, \; y_2 = -k_1 x_2. \tag{32}$$

Let the minimum inter-event time τ chosen in Algorithm 4 satisfy that

$$\tau < \min\left\{ \frac{3}{2k_2}, \frac{2k_2}{3k_1} \right\}. \tag{33}$$

Step 2. For $t \geq t_k + \tau$, define the *error variable* as

$$e(t) \triangleq \sigma(y_1(t)) - \sigma(u); \tag{34}$$

denote $y_1(t) = [y_{11}, ..., y_{1m}]^T$, $\hat{y}_1(t) = [\hat{y}_{11}, ..., \hat{y}_{1m}]^T \triangleq y_2(t) - k_2\sigma(u)$ with $m = n/2$, and reset $\hat{y}_{1i} = 0$ if $|y_{1i}| > \varpi_i$. Then, $\frac{d}{dt}\sigma(y_1(t)) = \hat{y}_1(t)$.

Step 3. Choose two event-trigger parameters $\rho_1 \in (0, 1)$ and $\rho_2 \in (0, 1)$.

Step 4. The event-triggering function $f(t)$ in Algorithm 4 is designed as

$$f(t) \triangleq \max\{f_1(t), f_2(t)\}, \tag{35}$$

$$\begin{aligned} f_1(t) \triangleq & 2\left(y_2(t) - k_2\sigma(u)\right)^T \sigma(y_1(t)) + 2y_1(t)^T \hat{y}_1(t) - 2\sigma(y_1(t))^T \hat{y}_1(t) \\ & + 2\tau k_1 \sigma(u)^T \sigma(y_1(t)) - 2\tau y_2(t)^T \hat{y}_1(t) - 2\sigma(u)^T y_2(t) + \rho_1(k_2 - k_1\tau)\|\sigma(u)\|^2, \\ f_2(t) \triangleq & \|e(t)\| - \rho_2\|\sigma(y_1(t))\|. \end{aligned}$$

Step 5. The event-based control input is designed as

$$u(t) = -Kx(t_k),\ t \in [t_k, t_{k+1}), \tag{36}$$

where $\{t_k\}$ are generated by Algorithm 4 and K is given in (31).

4.3.2. *Event-based Global Stabilization*

Theorem 4

Consider the continuous-time input-saturated double-integrator system (1) with (A, B) given in (31). Algorithms 4 and 6 can achieve global stabilization of system (1), that is, $\lim_{t\to+\infty} x(t) = 0$ for any initial condition $x(0) \in \mathbb{R}^n$.

Proof

By (33), one has $0 < \frac{3\tau}{2}k_1 \leq k_2 < \frac{3}{2\tau}$ and $\frac{1}{k_1} > \tau^2$. Similar to Algorithm 3, denote $c_1 \triangleq -k_1$, $c_2 \triangleq -\frac{\tau}{2}k_1 - k_2$, and $\rho_0 \triangleq 1 + c_1\tau^2 - (c_2\tau - c_1\tau^2 + 1)^2 > 0$. Consider the following Lyapunov function candidate:

$$V(y) \triangleq 2y_1^T\sigma(y_1) - \|\sigma(y_1)\|^2 - 2\tau y_2^T\sigma(y_1) + \frac{1}{k_1}\|y_2\|^2, \tag{37}$$

which is positive definite and satisfies $\frac{d}{dt}V(y(t)) = f_1 - \rho_1(k_2 - k_1\tau)\|\sigma(u)\|^2$ with $k_2 - k_1\tau > 0$. The dynamics of y are $\dot{y}_1 = y_2 - k_2\sigma(u)$, $\dot{y}_2 = -k_1\sigma(u)$. One has $y_2(t_k + \tau) = y_2(t_k) + \tau c_1\sigma(u)$ and $y_1(t_k + \tau) = y_1(t_k) + \tau y_2(t_k) + \tau c_2\sigma(u)$. Similar to the proof of Theorem 2, one obtains

$$V\left(y(t_k + \tau)\right) - V\left(y(t_k)\right) \leq -\rho_0\|\sigma(u)\|^2 \leq 0. \tag{38}$$

Therefore, if $t_{k+1} = t_k + \tau$, $V\left(y(t_{k+1})\right) \leq V\left(y(t_k)\right)$, where "=" holds if and only if $x(t_k) = 0$. If $t_k + \tau < t_{k+1} < +\infty$, for $t \in [t_k + \tau, t_{k+1})$, by Algorithm 4 and (35), one has $u \neq 0$ and $\dot{V} = f_1 - \rho_1(k_2 - k_1\tau)\|\sigma(u)\|^2 < -\rho_1(k_2 - k_1\tau)\|\sigma(u)\|^2 < 0$, which implies that $V\left(y(t_{k+1})\right) < V\left(y(t_k)\right)$. If $f(t_k + \tau) = \|x(t_k + \tau)\| = 0$, one has $u(t_k) = 0$, $x(t) = 0$, $V(y(t)) = 0$, $\forall t \geq t_k + \tau$. If $f(t) < 0$ so that $f_2(t) < 0$ for all $t \in [t_k + \tau, +\infty)$, one has $u(t_k) \neq 0$, $\dot{V} < -\rho_1(k_2 - k_1\tau)\|\sigma(u)\|^2 < 0$, V is decreasing with at least a linear decay rate, which is impossible by the positive definiteness of V.

Thus, $V(y(t))$ is non-negative and non-increasing on both $\{t_k\}$ and $\mathbb{T} = \bigcup_{k\geq 0}(t_k + \tau, t_{k+1})$ so that $\lim_{t\to+\infty} V(y(t))$ exists, which implies that $\lim_{t\to+\infty} u(t) = 0$, $\lim_{t\to+\infty} y_1(t) = 0$,

$\lim_{t\to+\infty} V(y(t)) = \lim_{t\to+\infty} \|y_2(t)\|^2/k_1$. If an infinite sequence $\{t_k\}$ is not generated by the event-trigger, one has $f(t_k+\tau) = \|x(t_k+\tau)\| = 0$ for some $k \geq 0$, then $u(t_k) = 0$, $x(t) = 0$, $\forall t \geq t_k + \tau$. If $t_k < \infty$, $\forall k \geq 0$, one has $\lim_{k\to+\infty} y_1(t_k+\tau) = \lim_{k\to+\infty} \tau y_2(t_k) = 0$. Consequently, one obtains $\lim_{t\to+\infty} y_2(t) = 0$ and $\lim_{t\to+\infty} x(t) = 0$. This completes the proof of Theorem 4. □

Remark 8. The continuous-time event-trigger strategy is based on the discrete-time results in Section 3.3. The non-quadratic event-triggering function $f(t)$ in (35), with $f_1(t)$ based on the Lyapunov function and $f_2(t)$ based on $e(t)$ in (34), can significantly reduce the number of control updates. The necessity of setting a non-inherent minimum inter-event time τ is illustrated in Example 4 in Section 5.

5. NUMERICAL EXAMPLES

In this section, numerical examples are provided to illustrate the theoretical results.

Example 1. Consider the discrete-time system (1) with

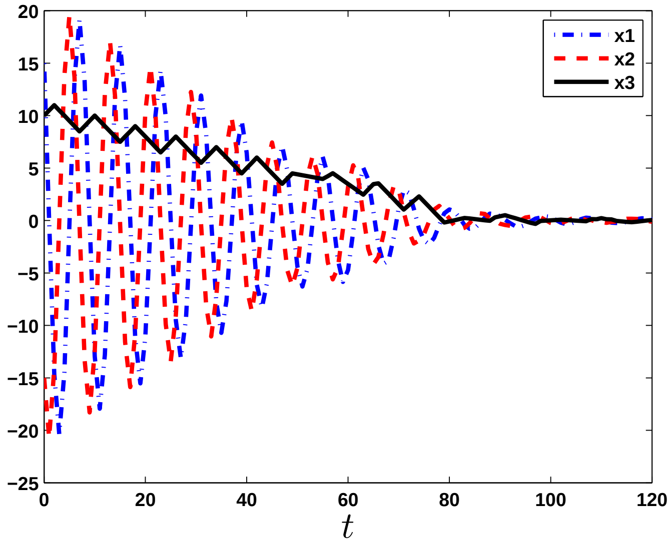
$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and input saturation threshold } \varpi = 0.5.$$

In Algorithm 2, we take $P = I$, $\rho_d = 0.01$, and $\tilde{\rho}_d = 0.95$. The control gain is set as $\mu = 0.9405$ such that $2(1-\rho_d)/\mu = 2.1053 > \rho(B^T P^T P B) = 2$. Then, the linear controller matrix is obtained as $K = [-0.6650, 0.6650, 0.9405]$. Following Algorithm 1 and choosing the initial state as $x(0) = [15, -15, 10]^T$, there are 38 event-triggered feedback updates in 120 discrete-time steps. The simulation result is shown in Fig. 1(a).

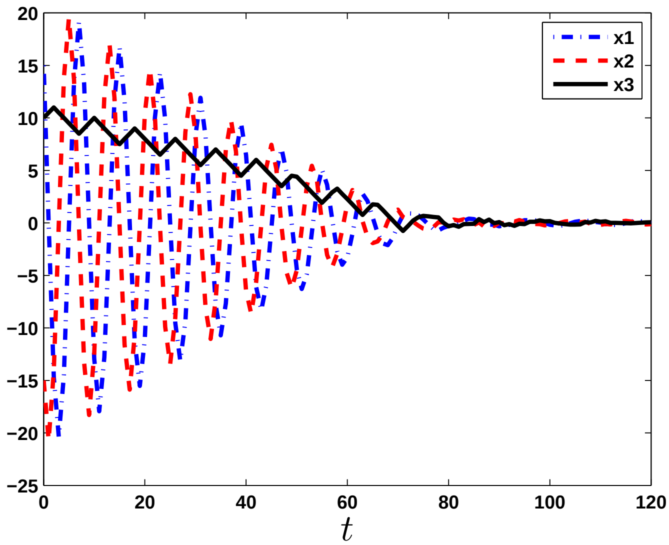
If the non-quadratic function $f_2(t)$ in (6) is substituted by the following quadratic function

$$f_q(t) \triangleq (Kx(t) - Kx(t_k))^T (Kx(t) - Kx(t_k)) - \tilde{\rho}_d^2 x(t)^T K^T K x(t),$$

the number of feedback updates in 120 steps increases to 61 while the control performance is only slightly improved, see Fig. 1(b).



(a) Non-quadratic condition triggers 38 updates



(b) Quadratic condition triggers 61 updates

Figure 1. Event-based stabilization of neutrally stable systems via a saturated linear controller with $x(0) = [15, -15, 10]^T$ and $\varpi = 0.5$: (a) the event-triggering function $f(t)$ is based on the non-quadratic function $f_2(t)$; (b) the event-triggering function $f(t)$ is based on the quadratic function $f_q(t)$.

Example 2. Consider the continuous-time system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; u = -Kx(t_k), \text{ and } \sigma(u) = \text{sat}_{\varpi}(u), \; \varpi > 0.$$

It is straightforward that $K = [k_1 \;\; k_2]$ needs to satisfy the condition $k_1 > -1, k_2 > 0$ so that $(A - BK)$ is stable. If $u(t) = -Kx(t)$ and the control updating is not event-triggered, the closed-loop system is stable with the following Lyapunov function [6, 8]:

$$V(x) \quad = \quad \begin{cases} \frac{1}{2}\|x\|^2 + \frac{1}{k_1}\int_0^{-k_1 x_1} \sigma(\nu) d\nu, & \text{if} \quad k_1 \neq 0; \\ \frac{1}{2}\|x\|^2, & \text{if} \quad k_1 = 0. \end{cases}$$

However, if event-trigger strategy is applied to the global stabilization, it will be shown that the property $\dot{V} \le 0$ cannot be established. Thus, an inherent lower bound τ for the inter-event time, which guarantees that V is non-increasing on $[t_k, t_k + \tau]$, does not exist. For an arbitrarily small constant $\tilde{\tau} > 0$, let $u(t) = -Kx(0)$ for $t \in [0, \tilde{\tau}]$. Then, the trajectory of $x(t)$ can be solved as $x_1(t) = r_0 \sin(t + \phi_0) + \sigma(u)$, $x_2(t) = r_0 \cos(t + \phi_0)$, where r_0 and ϕ_0 are the initial condition parameters to be determined.

If $k_1 = 0$, one has $\dot{V}(\tilde{\tau}) = x_2(\tilde{\tau})\sigma(u) = r_0 \cos(\tilde{\tau} + \phi_0)\sigma(u)$. Letting $\phi_0 = (\pi - \tilde{\tau})/2$ and $r_0 = \varpi / \left(2k_2 \sin(\frac{1}{2}\tilde{\tau})\right)$, one obtains that $x(0) = [(\varpi \cot(\tilde{\tau}/2)/(2k_2)) - (\varpi/2), \; \varpi/(2k_2)]^T$, $u = -\varpi/2$, and $\dot{V}(\tilde{\tau}) = \varpi^2/(4k_2) > 0$ for any $\tilde{\tau} \in (0, \pi)$. When $k_1 \neq 0$, one has $\dot{V}(\tilde{\tau}) = x_2(\tilde{\tau})(\sigma(u) + k_1 x_1(\tilde{\tau}))$. If $k_1 > 0$, letting $\phi_0 = \pi/2$ and $r_0 = \left(\frac{1}{2} + \frac{1}{2k_1}\right)\varpi$, one obtains that $x(0) = [\varpi/(2k_1), \; 0]^T$, $u = -\varpi/2$, and

$$\dot{V}(\tilde{\tau}) = r_0 \sin(\tilde{\tau}) \left(\tfrac{\varpi}{2} - k_1(r_0 \cos(\tilde{\tau}) + u)\right) > r_0 \sin(\tilde{\tau}) \left(\tfrac{\varpi}{2} - k_1(r_0 + u)\right) = 0$$

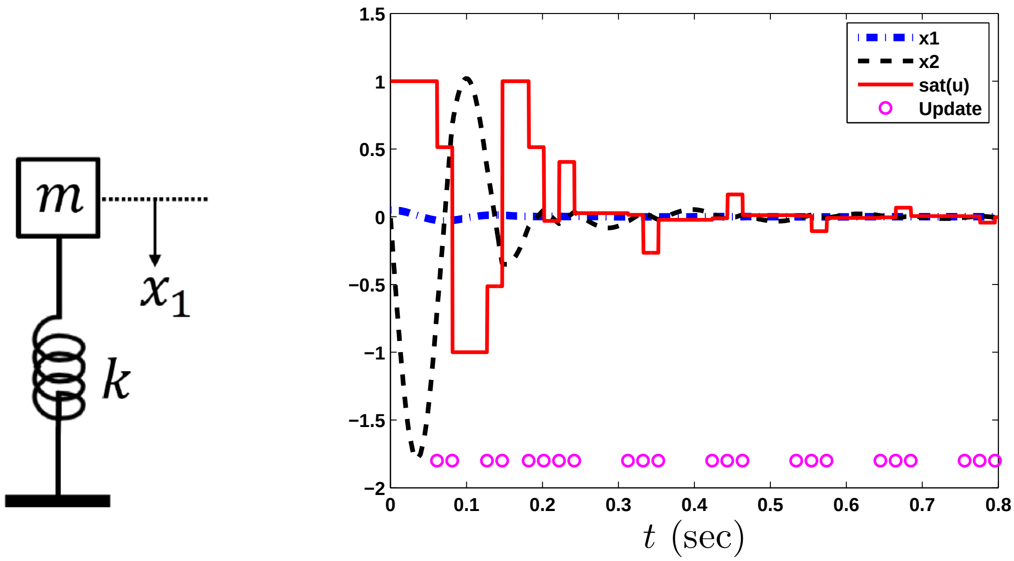
for any $\tilde{\tau} \in (0, \pi)$. If $-1 < k_1 < 0$, for $\tilde{\tau} \in (0, 2\arctan(-k_1/k_2))$, letting $\phi_0 = (\pi - \tilde{\tau})/2$ and $r_0 = (1 + k_1)\varpi/(-2k_1 \cos(\tilde{\tau}/2) - 2k_2 \sin(\tilde{\tau}/2)) > 0$, one obtains that $x(0) = [r_0 \cos(\tilde{\tau}/2) + \varpi/2, \; r_0 \sin(\tilde{\tau}/2)]^T$, $u = \varpi/2$, and

$$\dot{V}(\tilde{\tau}) = r_0 \sin(\tilde{\tau}/2) \left(-\tfrac{\varpi}{2} - k_1(r_0 \cos(\tilde{\tau}/2) + u)\right) = k_2(r_0 \sin(\tilde{\tau}/2))^2 > 0.$$

Consequently, for any $K = [k_1 \;\; k_2]$ satisfying $k_1 > -1$ and $k_2 > 0$, and any sufficiently small $\tilde{\tau} > 0$, there always exists some initial value $x(0)$ such that $V(x(t))$ is strictly increasing at $t = \tilde{\tau}$.

Therefore, a lower bound $\tau > 0$ cannot be inherently guaranteed for the inter-event time and has to be set *a priori* as in Algorithm 4.

Example 3. Consider a spring-mass oscillator system, as shown in Fig. 2(a), where the mass of the body is $m = 0.05$ kg and the stiffness of the spring is $k = 100$ N/m. The control input u, which is subject to the magnitude constraint $|u| \leq F_{\max} = 1$ N, is the force exerted on the mass.



(a) A spring-mass oscillator

(b) Event-based system response

Figure 2. Event-based stabilization of a spring-mass oscillator system via a saturated linear controller with $x(0) = [0.05, -0.01]^T$, $\varpi = 1$, and $K = [-89.7683,\ 4.1851]$.

Denote the displacement of the mass by x_1 m ($x_1 = 0$ at equilibrium), and the velocity by x_2 m/s. Then, the motion can be described by the continuous-time system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2000 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix},$$

and $\sigma(u) = \mathrm{sat}_{\varpi}(u),\ \varpi = 1$. Denote $P = \left[\begin{smallmatrix} \omega & 1 \\ -\omega & 1 \end{smallmatrix}\right]$ with $\omega = \sqrt{2000}$, such that $P^{-1} = \frac{1}{2\omega} \cdot \left[\begin{smallmatrix} 1 & -1 \\ \omega & \omega \end{smallmatrix}\right]$, and $\tilde{A} \triangleq PAP^{-1} = \left[\begin{smallmatrix} 0 & \omega \\ -\omega & 0 \end{smallmatrix}\right]$. Similar to (26), one has

$$\begin{aligned} \tilde{A}_d &\triangleq e^{\tilde{A}\tau} = \begin{bmatrix} \cos(\omega\tau) & \sin(\omega\tau) \\ -\sin(\omega\tau) & \cos(\omega\tau) \end{bmatrix}, \quad A_d = P^{-1}\tilde{A}_d P = \begin{bmatrix} 0.6260 & 0.0174 \\ -34.8760 & 0.6260 \end{bmatrix}, \\ B_d &= (A_d - I)A^{-1}B = P^{-1}(\tilde{A}_d - I)\tilde{A}^{-1}PB = [0.0037,\ 0.3488]^T. \end{aligned}$$

The minimum inter-event time is set *a priori* as $\tau = 0.02$ second satisfying that $\sin(\omega\tau) = 0.7799 \neq 0$ and (A_d, B_d) is controllable. Following Algorithm 5, the control gain is selected as $\mu = 0.12$ such that $2\tau/\mu = 0.3333 > \|PB_d\|^2 = 0.2992$. Thus, the controller formulated in (23) is obtained as $K = [-89.7683,\ 4.1851]$. Setting $\rho_c = 0.05$, $\tilde{\rho}_c = 0.95$, and applying Algorithms 4 and 5, the event-based system response is shown in Fig. 2(b). In 0.8 second, there are 24 feedback control updates, including the one at the initial time $t = 0$.

Example 4. Consider the continuous-time system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ u = -Kx(t_k), \text{ and } \sigma(u) = \mathrm{sat}_{\varpi}(u),\ \varpi > 0.$$

For any $K = [k_1\ k_2]$ satisfying $k_1 > 0$ and $k_2 > 0$, if $u(t) = -Kx(t)$ and the control updating is not event-triggered, the closed-loop stability can be shown via the following Lure-Posnikov Lyapunov function [6]:

$$V_2(x) = k_1 x_2^2 + 2\int_0^{-k_1x_1 - k_2x_2} \sigma(\nu)d\nu.$$

If event-trigger strategy is applied to the global stabilization, it will be shown that the property $\dot{V}_2 \leq 0$ cannot be established either. Thus, an inherent lower bound τ for the inter-event time does not exist.

When a lower bound τ is not set *a priori* in Algorithm 4, by removing the term $(-2\tau y_2^T\sigma(y_1))$ in (37), one can easily verify that $V_2(x) = V(y) = 2y_1^T\sigma(y_1) - \|\sigma(y_1)\|^2 + \frac{1}{k_1}\|y_2\|^2$. For any constant $\tilde{\tau} > 0$, performing $u(t) = -Kx(0)$ on $[0, \tilde{\tau}]$, the trajectory of $x(t)$ can be solved as $x_1(t) = x_1(0) + x_2(0)t + \frac{1}{2}\sigma(u)t^2$, $x_2(t) = x_2(0) + \sigma(u)t$. Then, the initial value is chosen to be

$x(0) = [-k_2 r_0,\ k_1 r_0 - (\varpi/(2k_2))]^T$ with $r_0 > 0$ to be determined. One has $u = \varpi/2$, and

$$\begin{aligned}\dot{V}_2(\tilde{\tau}) =& k_1 x_2(\tilde{\tau})\varpi + 2\sigma(Kx(\tilde{\tau}))K(Ax(\tilde{\tau}) + Bu)\\ =& k_1 x_2(\tilde{\tau})\varpi + 2\sigma(Kx(\tilde{\tau}))(k_1 x_2(\tilde{\tau}) + k_2\varpi/2).\end{aligned}$$

Letting r_0 be sufficiently large such that $x_2(\tilde{\tau}) = x_2(0) + \tilde{\tau}\varpi/2 = k_1 r_0 - (\varpi/(2k_2)) + \tilde{\tau}\varpi/2 > 0$ and $Kx(\tilde{\tau}) = k_1^2 r_0 \tilde{\tau} + (k_1 k_2 \varpi \tilde{\tau}^2 + 2k_2^2 \tilde{\tau}\varpi - 2k_1\varpi\tilde{\tau} - 2k_1 k_2 \varpi)/(4k_2) \geq \varpi$, one obtains that $\dot{V}_2(\tilde{\tau}) = 3k_1\varpi x_2(\tilde{\tau}) + \varpi^2 k_2 > 0$. Consequently, for any $K = [k_1\ k_2]$ satisfying $k_1, k_2 > 0$ and any $\tilde{\tau} > 0$, there always exists some $x(0) \in \mathbb{R}^2$ such that $V_2(x(t))$ is strictly increasing at $t = \tilde{\tau}$.

Example 5. Consider the double-integrator dynamics in Example 4 with $\varpi = 1$, $x(0) = [10,\ 10]^T$, and $K = [1,\ 2]$, which are the same as [6, Example 4.4]. When there is no event trigger, the system response is shown in Fig. 3(a). If event-trigger strategy is adopted, in Algorithm 6, we take $\tau = 0.4 < 0.75 = 3/(2k_2)$, $\rho_1 = 0.05$, and $\rho_2 = 0.95$. Following Algorithm 4, the system response is shown in Fig. 3(b). There are 34 feedback control updates in 70 seconds; and only 14 updates within the first 50 seconds. After the initial time, the next event time t_1 is triggered at $t_1 = 19.0893$ second (simulation time unit). The accumulated number of updates is shown in Fig. 4(a); and the number of updates in each second is shown in Fig. 4(b).

6. CONCLUSION

In this paper, the event-based global stabilization of linear systems subject to input saturation have been studied. For discrete-time neutrally stable and double-integrator systems, novel event-trigger strategies based on the non-quadratic event-triggering conditions have been proposed, which can lead to less control updates than the ones based on the quadratic event-triggering conditions. For continuous-time neutrally stable and double-integrator systems, novel event-trigger strategies with an appropriately selected minimum inter-event time have been proposed to avoid the problem that an inherent lower bound of the inter-event time does not exist for systems subject to input saturation. Future studies include extension of the state-feedback results to output-feedback systems.

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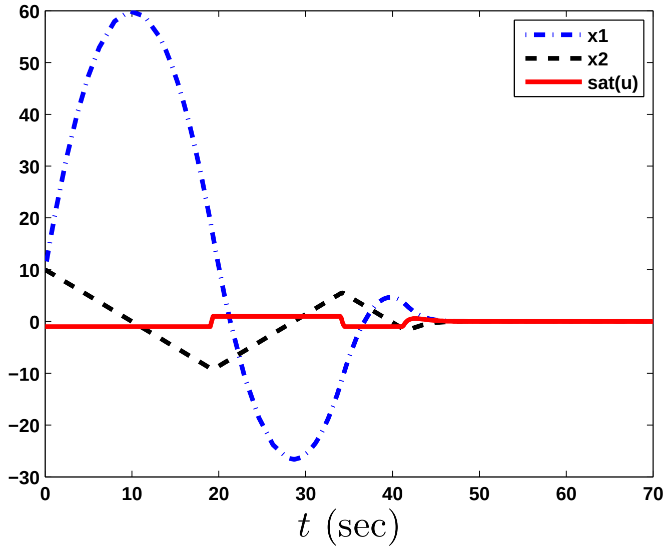
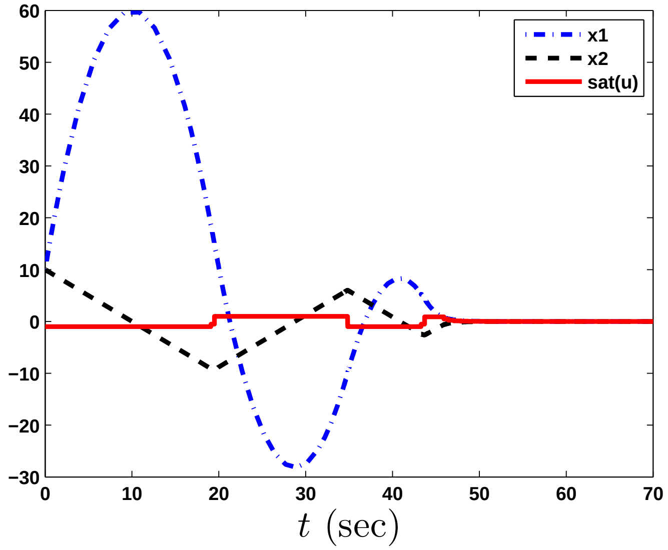


(a) System response without event trigger



(b) Event-based system response

Figure 3. Stabilization of double integrators via a saturated linear controller with $x(0) = [10, 10]^T$ and $\varpi = 1$.

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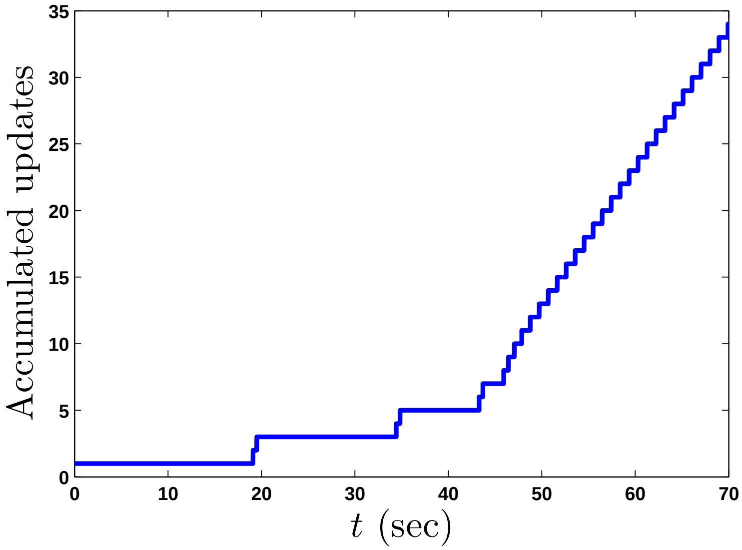
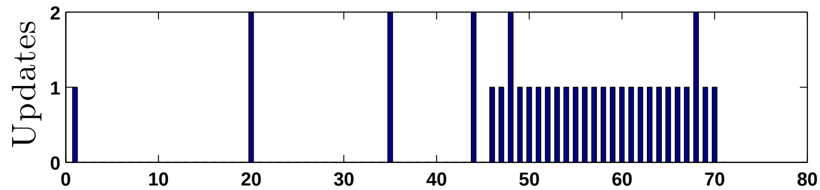


(a) The accumulated number of updates



(b) Updates in each second

Figure 4. The number of event-triggered feedback control updates.

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