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Positive State-bounding Observer for Interval Positive Systems under L_1 Performance

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Abstract: In this paper, the positive observer problem is investigated for interval positive systems under the L_1-induced performance. To estimate the state of positive systems, a pair of state-bounding positive observers is designed. A novel characterization is first proposed under which the augmented system is stable and satisfies the L_1-induced performance. Necessary and sufficient conditions are then presented to design the observers. The results obtained in this paper are expressed in terms of linear programming problems, and can be easily solved by standard software. In the end, we present a numerical example to show the effectiveness of the derived design procedures.

Key Words: Interval systems, Linear Lyapunov functions, Linear programming, L_1-induced performance, Positive observer, Positive systems

1 Introduction

Positive systems exist in different fields such as engineering, physical and social sciences [1], [2]. The inputs, states, and outputs of such systems take non-negative values at all times since they usually denote the concentrations or amounts of material in application fields. In other words, such systems involve quantities which are naturally nonnegative. Different from general systems, positive systems are defined on cones rather than linear spaces. Due to the positivity of the variables, new problems appear and previous methods developed for general systems cannot be used for positive systems.

In recent years, the property characterization, behavioral analysis and stabilization of positive systems have been investigated by many researchers. After the unifying approach of system theory has been proposed for positive systems in [3], many contributions can be found in [4], [5], [6] and [7]. For example, the positive realization problem has been investigated to a great extent in [8], [9]. Controllability and reachability for positive systems have been developed in [10], [11]. The controller synthesis problem for positive systems has been addressed through the linear matrix inequality (LMI) approach and the linear programming approach in [12], [13], respectively. Some fundamental problems for compartmental dynamic systems has been studied in [14]. Positivity preserving filtering problem for positive systems has been solved in [15]. In [16], the linear time-invariant exponentially stable systems have been transformed into cooperative systems. Moreover, the exponentially stable observers have been established. The analysis and synthesis of 2-D positive systems has been studied in [17]. For positive systems with delays, some basic problems has been thoroughly addressed in [18], [19], [20], [21], [22]. In addition, the model reduction problem for positive systems has been tackled in [23], [24].

It is noted that previous approach developed for general systems cannot be directly used for positive systems, since the estimated state cannot be guaranteed to be always nonnegative. Due to the positivity of the system state, a valid estimate is required to be nonnegative. Recently, many results which can guarantee the positivity of the observer have been reported in [25], [26]. Unfortunately, these approaches lead to some constraint on the structure of positive observers and are not applicable to positive systems with parameter uncertainties. In addition, we note that most of the previous results about the positive systems are obtained with the quadratic Lyapunov function. Correspondingly, many results are treated under the linear matrix inequality (LMI) framework [27]. In recent years, some researchers are devoted to investigating positive systems with the linear Lyapunov function [28], [29], [30], [31], [32]. Due to the positivity of the state of positive systems, a linear Lyapunov function can be chosen as a valid candidate. This forms the motivation for using linear Lyapunov functions. By using a linear Lyapunov function, a novel method can be derived to address the observer design problem for positive systems. Moreover, we note that the system state can only be estimated in an asymptotic way with conventional observers. Consequently, it is meaningful to design new observers which can give the information of the transient state of positive systems. This motivates our research.

In this paper, we study the problem of L_1-performance based observer design for interval positive systems with their positivity preserved in the observer. More specifically, a pair of L_1-performance based positive observers is first proposed. Then, we establish necessary and sufficient conditions to design the positive observers. It should be mentioned that the results obtained in this paper are expressed in terms of linear programming problems.

The rest of this paper is organized as follows. In Section 2, some notations and preliminaries are introduced. The problem of state-bounding positivity preserving observers is formulated in Section 3. In Section 4, the positive state-bounding observer is designed for positive systems. To show the application of the theoretical results, an example is given in Section 5. Finally, the results are summarized in Section 6.

2 Preliminaries

In this section, we introduce notations and preliminaries about positive systems.

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Let $\mathbb{R}$ be the set of real numbers; $\mathbb{R}^n$ denotes the n-column real vectors; $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimension $n \times m$. For a matrix $A \in \mathbb{R}^{m\times n}$, $[A]_{ij}$ denotes the element located at the ith row and the jth column; $[A]_{r,i}$, and $[A]_{c,j}$ denote the ith row, and the jth column, respectively. $A \geq\geq 0$ (respectively, $A >> 0$) means that for all i and j, $[A]_{ij} \geq 0$ (respectively, $[A]_{ij} > 0$). A matrix A is called Metzler, if all its off-diagonal elements are nonnegative, i.e., $\forall (i,j), i \neq j$, $[A]_{ij} \geq 0$. The notation $A \geq\geq B$ (respectively, $A >> B$) means that the matrix $A - B \geq\geq 0$ (respectively, $A - B >> 0$). For matrices $A, \underline{A}, \overline{A} \in \mathbb{R}^{n\times m}$, the notation $A \in [\underline{A}, \overline{A}]$ means that $\underline{A} \leq\leq A \leq\leq \overline{A}$. Let $\bar{\mathbb{R}}^n_+$ denote the nonnegative orthants of $\mathbb{R}^n$; that is, if $x \in \mathbb{R}^n$, then $x \in \bar{\mathbb{R}}^n_+$ is equivalent to $x \geq\geq 0$. The superscript "T" denotes matrix transpose. $\|\cdot\|$ represents the Euclidean norm for vectors. The 1-norm of a vector $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is defined as $\|x(t)\|_1 \triangleq \sum_{i=1}^{n} |x_i(t)|$. The L_1-norm of a Lebesgue integrable function x is defined as $\|x\|_{L_1} \triangleq \int_0^\infty \|x(t)\|_1 dt$. The space of all vector-valued functions defined on $\bar{\mathbb{R}}^n_+$ with finite L_1 norm is denoted by $L_1(\bar{\mathbb{R}}^n_+)$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations. Vector $\mathbf{1} = [1, 1, \ldots, 1]^T$.

Consider a continuous-time linear system:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bw(t), \\ y(t) &= Cx(t) + Dw(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n, w(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the system state, input and output, respectively; A, B, C and D are system matrices with compatible dimensions.

In this paper, the system matrices A, B, C and D belong to the following interval uncertainty domain:

$$A \in [\underline{A}, \overline{A}],\ B \in [\underline{B}, \overline{B}],\ C \in [\underline{C}, \overline{C}],\ D \in [\underline{D}, \overline{D}]. \tag{2}$$

Some definitions are introduced in the following.

Definition 1 *System (1) is said to be a continuous-time positive linear system if for all $x(0) \geq\geq 0$ and $w(t) \geq\geq 0$, we have $x(t) \geq\geq 0$ and $y(t) \geq\geq 0$ for $t > 0$.*

Definition 2 *System (1) is said to be positive and robustly stable if it is positive and stable over all interval uncertainty domain in (2).*

Then, some useful results are introduced and they will be used throughout this paper.

Lemma 1 ([33]) *The system in (1) is a continuous-time positive linear system if and only if*

$$A \text{ is Metzler},\ B \geq\geq 0,\ C \geq\geq 0,\ D \geq\geq 0.$$

Proposition 1 ([34]) *The positive linear system given by (1) with input $w(t) = 0$ is stable if and only if there exists a vector $p \geq\geq 0$ (or $p >> 0$) satisfying*

$$p^T A << 0. \tag{3}$$

In the following, the definition of L_1-induced norm is presented. For a stable positive linear system given in (1), its L_1-induced norm is defined as

$$\|\Im\|_{(L_1,L_1)} \triangleq \sup_{w\neq 0,\ w\in L_1(\bar{\mathbb{R}}^n_+)} \frac{\|y\|_{L_1}}{\|w\|_{L_1}}, \tag{4}$$

where $\Im : L_1 \to L_1$ denotes the convolution operator, that is, $y(t) = (\Im * w)(t)$. We say that system (1) has L_1-induced performance at the level γ if, under zero initial conditions,

$$\|\Im\|_{(L_1,L_1)} < \gamma, \tag{5}$$

where $\gamma > 0$ is a given scalar.

The following lemma serves as a fundamental characterization on the stability of system (1) with the L_1-induced performance in (5).

Lemma 2 ([32]) *The positive linear system in (1) is stable and satisfies $\|y\|_{L_1} < \gamma\|w\|_{L_1}$ if and only if there exists a vector $p \geq\geq 0$ satisfying*

$$\mathbf{1}^T C + p^T A \quad << \quad 0, \tag{6}$$
$$p^T B + \mathbf{1}^T D - \gamma \mathbf{1}^T \quad << \quad 0. \tag{7}$$

Then, an important theorem is provided as the performance characterization for positive system (1) over all interval uncertainty domain in (2).

Theorem 1 *The positive linear system in (1) is robustly stable and satisfies $\|y\|_{L_1} < \gamma\|w\|_{L_1}$ for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $C \in [\underline{C}, \overline{C}]$ and $D \in [\underline{D}, \overline{D}]$ under zero initial conditions if and only if there exists a vector $p \geq\geq 0$ satisfying*

$$\mathbf{1}^T \overline{C} + p^T \overline{A} \quad << \quad 0, \tag{8}$$
$$p^T \overline{B} + \mathbf{1}^T \overline{D} - \gamma \mathbf{1}^T \quad << \quad 0. \tag{9}$$

Proof: (*Sufficiency*) For any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $C \in [\underline{C}, \overline{C}]$ and $D \in [\underline{D}, \overline{D}]$,

$$\mathbf{1}^T C + p^T A \leq\leq \mathbf{1}^T \overline{C} + p^T \overline{A} << 0,$$
$$p^T B + \mathbf{1}^T D - \gamma \mathbf{1}^T \leq\leq p^T \overline{B} + \mathbf{1}^T \overline{D} - \gamma \mathbf{1}^T << 0,$$

which, by Lemma 2, implies that system (1) is robust stable and satisfies $\|y\|_{L_1} < \gamma\|w\|_{L_1}$ over all interval uncertainty domain under zero initial conditions. The sufficiency is proved.

(*Necessity*) Assume that system (1) is robustly stable and satisfies $\|y\|_{L_1} < \gamma\|w\|_{L_1}$ under zero initial conditions. From Lemma 2, we have

$$\mathbf{1}^T C + p^T A \quad << \quad 0,$$
$$p^T B + \mathbf{1}^T D - \gamma \mathbf{1}^T \quad << \quad 0,$$

which implies that (8) and (9) hold. This completes the whole proof. □

3 Problem Formulation

It is noted that we cannot obtain the information of the transient state by designing conventional observers, since they only give an estimate of the state in an asymptotic way. To design a observer which can be used to estimate the state at all times, we intend to find a lower-bounding estimate $\check{x}(t)$ and an upper-bounding one $\hat{x}(t)$. With the two estimates, the signal $x(t)$ can be encapsulated at all times. In the following, a pair of observers is proposed as follows:

$$\dot{\check{x}}(t) \quad = \quad \check{F}\check{x}(t) + \check{G}y(t) + \check{K}w(t), \tag{10}$$

and

$$\dot{\hat{x}}(t) = \hat{F}\hat{x}(t) + \hat{G}y(t) + \hat{K}w(t), \quad (11)$$

where $\hat{x}(t) \in \mathbb{R}^n$, $\check{x}(t) \in \mathbb{R}^n$. $\check{F}$, $\check{G}$, $\check{K}$, $\hat{F}$, $\hat{G}$ and $\hat{K}$ are observer parameters to be determined.

In the following, we first consider the lower-bounding case.

Define the error state $\check{e}(t) = x(t) - \check{x}(t)$; then it follows from systems (1) and (10) that

$$\begin{aligned}\dot{\check{e}}(t) &= (A - \check{F} - \check{G}C)x(t) + \check{F}\check{e}(t) \\ &\quad +(B - \check{G}D - \check{K})w(t).\end{aligned} \quad (12)$$

Suppose that

$$\check{z}(t) = L\check{e}(t) \quad (13)$$

stands for the output of error states and here $L \geq\geq 0$ is known. Now, by defining

$$\begin{aligned}\check{\xi}(t) &= \begin{bmatrix} x(t) \\ \check{e}(t) \end{bmatrix}, \ \check{A}_\xi = \begin{bmatrix} A & 0 \\ A - \check{F} - \check{G}C & \check{F} \end{bmatrix}, \\ \check{B}_\xi &= \begin{bmatrix} B \\ B - \check{G}D - \check{K} \end{bmatrix}, \ \check{C}_\xi = \begin{bmatrix} 0 & L \end{bmatrix}, \end{aligned} \quad (14)$$

and with (1), (15) and (13), the augmented system is obtained as follows:

$$\begin{cases} \dot{\check{\xi}}(t) = \check{A}_\xi\check{\xi}(t) + \check{B}_\xi w(t), \\ \check{z}(t) = \check{C}_\xi\check{\xi}(t). \end{cases} \quad (15)$$

The observer in (10) is designed for the positive system in (1) to approximate $x(t)$ by $\check{x}(t)$. Consequently, the estimate $\check{x}(t)$ is required to be positive, like system state $x(t)$ itself, which implies that the observer in (10) should be a positive system. From Lemma 1, we see that $\check{F}$ is Metzler, $\check{G} \geq\geq 0$ and $\check{K} \geq\geq 0$ are needed. In the following, the positive lower-bounding observer problem is established.

Positive Lower-bounding Observer Design (PLOD): *Given a positive system (1) with $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $C \in [\underline{C}, \overline{C}]$, and $D \in [\underline{D}, \overline{D}]$, design a positive observer of the form (10) with $\check{F}$ being Metzler, $\check{G} \geq\geq 0$ and $\check{K} \geq\geq 0$ such that the augmented system (15) is positive, robustly stable and satisfies the performance $\|\check{z}\|_{L_1} < \gamma\|w\|_{L_1}$ under zero initial conditions.*

Similarly, one may define $\hat{e}(t) = \hat{x}(t) - x(t)$ and $\hat{\xi}(t) = [x^T(t),\ \hat{e}^T(t)]^T$. Suppose that

$$\hat{z}(t) = L\hat{e}(t) \quad (16)$$

stands for the output of error states. Then we have the augmented system

$$\begin{cases} \dot{\hat{\xi}}(t) = \hat{A}_\xi\hat{\xi}(t) + \hat{B}_\xi w(t), \\ \hat{z}(t) = \hat{C}_\xi\hat{\xi}(t), \end{cases} \quad (17)$$

where

$$\begin{aligned}\hat{A}_\xi &= \begin{bmatrix} A & 0 \\ \hat{F} + \hat{G}C - A & \hat{F} \end{bmatrix}, \\ \hat{B}_\xi &= \begin{bmatrix} B \\ \hat{G}D + \hat{K} - B \end{bmatrix}, \ \hat{C}_\xi = \begin{bmatrix} 0 & L \end{bmatrix}.\end{aligned}$$

In the following, the positive upper-bounding observer design (PUOD) problem is formulated.

Positive Upper-bounding Observer Design (PUOD): *Given a positive system (1) with $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $C \in [\underline{C}, \overline{C}]$, and $D \in [\underline{D}, \overline{D}]$, design a positive observer of the form (11) with $\hat{F}$ being Metzler, $\hat{G} \geq\geq 0$ and $\hat{K} \geq\geq 0$ such that the augmented system (17) is positive, robustly stable and satisfies the performance $\|\hat{z}\|_{L_1} < \gamma\|w\|_{L_1}$ under zero initial conditions.*

4 Main Results

In this section, we propose a pair of positive state-bounding observers which bound the state $x(t)$ at all times, and satisfies the L_1-induced performance. To achieve this, we first establish the performance characterization result for the lower-bounding augmented system (15). Then, necessary and sufficient conditions are presented for the design of lower-bounding observer. Finally, parallel results are obtained for upper-bounding case.

Based on Theorem 1, the following result is derived to serve as a characterization on the stability of lower-bounding augmented system (15) with the performance $\|\check{z}\|_{L_1} < \gamma\|w\|_{L_1}$. The proof is omitted here.

Theorem 2 *The lower-bounding augmented system in (15) is positive, robustly stable and satisfies $\|\check{z}\|_{L_1} < \gamma\|w\|_{L_1}$ for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $C \in [\underline{C}, \overline{C}]$ and $D \in [\underline{D}, \overline{D}]$ under zero initial conditions if and only if there exist Metzler matrix $\underline{\check{A}}_\xi$, $\underline{\check{B}}_\xi \geq\geq 0$, $\check{C}_\xi \geq\geq 0$ and a vector $p \geq\geq 0$ satisfying*

$$\mathbf{1}^T\check{C}_\xi + p^T\overline{\check{A}}_\xi << 0, \quad (18)$$

$$p^T\overline{\check{B}}_\xi - \gamma\mathbf{1}^T << 0, \quad (19)$$

where

$$\begin{aligned}\underline{\check{A}}_\xi &= \begin{bmatrix} \underline{A} & 0 \\ \underline{A} - \check{F} - \check{G}\overline{C} & \check{F} \end{bmatrix}, \ \underline{\check{B}}_\xi = \begin{bmatrix} \underline{B} \\ \underline{B} - \check{G}\overline{D} - \check{K} \end{bmatrix}, \\ \overline{\check{A}}_\xi &= \begin{bmatrix} \overline{A} & 0 \\ \overline{A} - \check{F} - \check{G}\underline{C} & \check{F} \end{bmatrix}, \ \overline{\check{B}}_\xi = \begin{bmatrix} \overline{B} \\ \overline{B} - \check{G}\underline{D} - \check{K} \end{bmatrix}.\end{aligned}$$

Then, a necessary and sufficient condition is further established for the existence of the lower-bounding observer.

Theorem 3 *Given a stable continuous-time positive system (1), a lower-bounding observer (10) exists such that the augmented system (15) is positive, robustly stable and satisfies $\|\check{z}\|_{L_1} < \gamma\|w\|_{L_1}$ for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $C \in [\underline{C}, \overline{C}]$ and $D \in [\underline{D}, \overline{D}]$ under zero initial conditions if and only if there exist Metzler matrix $\check{M}_F$, $\check{M}_G \geq\geq 0$, $\check{M}_K \geq\geq 0$ and vectors $p_1 \geq\geq 0$, $p_2 \geq\geq 0$ satisfying*

$$\left[\ \check{M}_F\ \right]_{ij} \geq 0,\ i,j = 1,\ldots,n,\ i \neq j, \quad (20)$$

$$\left[\ \check{M}_G\ \right]_{il} \geq 0,\ i = 1,\ldots,n,\ l = 1,\ldots,p, \quad (21)$$

$$\left[\ \check{M}_K\ \right]_{ik} \geq 0,\ i = 1,\ldots,n,\ k = 1,\ldots,m, \quad (22)$$

$$p_{2i}^T [\ \underline{A}\ ]_{ij} - [\ \check{M}_G\ ]_{r,i} [\ \overline{C}\ ]_{c,j} - [\ \check{M}_F\ ]_{ij} \geq 0, \quad (23)$$

$$p_{2i}^T [\ \underline{B}\ ]_{ik} - [\ \check{M}_G\ ]_{r,i} [\ \overline{D}\ ]_{c,k} - [\ \check{M}_K\ ]_{ik} \geq 0, \quad (24)$$

$$p_1^T \overline{A} + p_2^T \overline{A} - \sum_{i=1}^{n} [\ \check{M}_F\ ]_{r,i} - \sum_{i=1}^{p} [\ \check{M}_G\ ]_{r,i} \underline{C} << 0, \quad (25)$$

$$\sum_{i=1}^{n} [\ \check{M}_F\ ]_{r,i} + \mathbf{1}^T L << 0, \quad (26)$$

$$p_1^T \overline{B} + p_2^T \overline{B} - \sum_{i=1}^{p} [\ \check{M}_G\ ]_{r,i} \underline{D} - \sum_{i=1}^{m} [\ \check{M}_K\ ]_{r,i} - \gamma \mathbf{1}^T << 0. \quad (27)$$

Moreover, a suitable set of $\check{F}$, $\check{G}$ and $\check{K}$ is given by

$$\begin{aligned} [\ \check{F}\ ]_{ij} &= p_{2i}^{-1} [\ \check{M}_F\ ]_{ij}, \\ [\ \check{G}\ ]_{il} &= p_{2i}^{-1} [\ \check{M}_G\ ]_{il}, \\ [\ \check{K}\ ]_{ik} &= p_{2i}^{-1} [\ \check{M}_K\ ]_{ik}. \end{aligned} \quad (28)$$

Proof: (*Sufficiency*) Note that $p_2 \geq\geq 0$, it follows from (20)–(22) and (28) that $\check{F}$ is Metzler, $\check{G} \geq\geq 0$ and $\check{K} \geq\geq 0$, which implies that the lower-bounding observer (10) is positive.

From (28) and $p_2 \geq\geq 0$, (23)–(24) become

$$[\ \underline{A}\ ]_{ij} - [\ \check{G}\ ]_{r,i} [\ \overline{C}\ ]_{c,j} - [\ \check{F}\ ]_{ij} \geq 0, \quad (29)$$

$$[\ \underline{B}\ ]_{ik} - [\ \check{G}\ ]_{r,i} [\ \overline{D}\ ]_{c,k} - [\ \check{K}\ ]_{ik} \geq 0, \quad (30)$$

and we have

$$\underline{A} - \check{G}\overline{C} - \check{F} \geq\geq 0,\ \underline{B} - \check{G}\overline{D} - \check{K} \geq\geq 0. \quad (31)$$

Combining (31) with $\check{G} \geq\geq 0$ yields the following: for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $C \in [\underline{C}, \overline{C}]$ and $D \in [\underline{D}, \overline{D}]$,

$$\begin{aligned} A - \check{G}C - \check{F} \geq\geq \underline{A} - \check{G}\overline{C} - \check{F} \geq\geq 0, \\ B - \check{G}D - \check{K} \geq\geq \underline{B} - \check{G}\overline{D} - \check{K} \geq\geq 0. \end{aligned}$$

Together with $\check{F}$ being Metzler and $L \geq\geq 0$, from (14), it shows that the augmented system (15) is positive.

From (28), we have

$$\begin{aligned} \sum_{i=1}^{n} [\ \check{M}_F\ ]_{r,i} &= p_2^T \check{F},\ \sum_{i=1}^{p} [\ \check{M}_G\ ]_{r,i} = p_2^T \check{G}, \\ \sum_{i=1}^{m} [\ \check{M}_K\ ]_{r,i} &= p_2^T \check{K}. \end{aligned} \quad (32)$$

With (32), inequalities (25)–(27) equal to

$$\begin{aligned} p_1^T \overline{A} + p_2^T \overline{A} - p_2^T \check{F} - p_2^T \check{G}\underline{C} << 0, \\ p_2^T \check{F} + \mathbf{1}^T L << 0, \\ p_1^T \overline{B} + p_2^T \overline{B} - p_2^T \check{G}\underline{D} - p_2^T \check{K} - \gamma \mathbf{1}^T << 0, \end{aligned}$$

which further imply that

$$\mathbf{1}^T \left[\ 0 \quad L\ \right] + p^T \begin{bmatrix} \overline{A} & 0 \\ \overline{A} - \check{F} - \check{G}\underline{C} & \check{F} \end{bmatrix} << 0,$$

$$p^T \begin{bmatrix} \overline{B} \\ \overline{B} - \check{G}\underline{D} - \check{K} \end{bmatrix} - \gamma \mathbf{1}^T << 0, \quad (33)$$

where $p^T = \left[\ p_1^T \quad p_2^T\ \right]$.

Therefore, by Theorem 2, we have that the lower-bounding augmented system (15) is robustly stable and satisfies $\|\check{z}\|_{L_1} < \gamma \|w\|_{L_1}$. The sufficiency is proved.

(*Necessity*) Assume that the augmented system (15) is robustly stable and satisfies $\|\check{z}\|_{L_1} < \gamma \|w\|_{L_1}$. Then, according to Theorem 2, the inequalities (33) hold. Denote $p^T \triangleq \left[\ p_1^T \quad p_2^T\ \right]$ and we have that the following inequalities hold

$$\begin{aligned} p_1^T \overline{A} + p_2^T \overline{A} - p_2^T \check{F} - p_2^T \check{G}\underline{C} << 0, \\ p_2^T \check{F} + \mathbf{1}^T L << 0, \\ p_1^T \overline{B} + p_2^T \overline{B} - p_2^T \check{G}\underline{D} - p_2^T \check{K} - \gamma \mathbf{1}^T << 0, \end{aligned}$$

Noting that

$$\begin{aligned} p_2^T \check{F} &= \sum_{i=1}^{n} p_{2i} [\ \check{F}\ ]_{r,i},\ p_2^T \check{G} = \sum_{i=1}^{p} p_{2i} [\ \check{G}\ ]_{r,i}, \\ p_2^T \check{K} &= \sum_{i=1}^{m} p_{2i} [\ \check{K}\ ]_{r,i}, \end{aligned}$$

it turns out that the change of variables

$$\begin{aligned} [\ \check{M}_F\ ]_{ij} &= p_{2i} [\ \check{F}\ ]_{ij},\ [\ \check{M}_G\ ]_{il} = p_{2i} [\ \check{G}\ ]_{il}, \\ [\ \check{M}_K\ ]_{ik} &= p_{2i} [\ \check{K}\ ]_{ik} \end{aligned}$$

linearizes the problem and yields (25)–(28).

Since the lower-bounding observer (10) is positive, we have $\check{F}$ is Metzler, $\check{G} \geq\geq 0$ and $\check{K} \geq\geq 0$. With $p_2 \geq\geq 0$, the conditions $\check{F}$ is Metzler, $\check{G} \geq\geq 0$ and $\check{K} \geq\geq 0$ equal to (20)–(22).

In addition, if the lower-bounding augmented system (15) is positive, we have

$$A - \check{G}C - \check{F} \geq\geq 0,\ B - \check{G}D - \check{K} \geq\geq 0,$$

which, together with $p_2 \geq\geq 0$, implies (23)–(24). This completes the whole proof. □

For the performance analysis in the upper-bounding case, the parallel result is presented as follows.

Theorem 4 *The upper-bounding augmented system in (17) is positive, robustly stable and satisfies $\|\hat{z}\|_{L_1} < \gamma \|w\|_{L_1}$ for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $C \in [\underline{C}, \overline{C}]$ and $D \in [\underline{D}, \overline{D}]$ under zero initial conditions if and only if there exist Metzler matrix $\underline{\hat{A}}_\xi$, $\underline{\hat{B}}_\xi \geq\geq 0$, $\hat{C}_\xi \geq\geq 0$ and a vector $p \geq\geq 0$ satisfying*

$$\mathbf{1}^T \hat{C}_\xi + p^T \overline{\hat{A}}_\xi << 0, \quad (34)$$

$$p^T \overline{\hat{B}}_\xi - \gamma \mathbf{1}^T << 0, \quad (35)$$

where

$$\underline{\hat{A}}_\xi = \begin{bmatrix} \underline{A} & 0 \\ \hat{F} + \hat{G}\underline{C} - \overline{A} & \hat{F} \end{bmatrix},\ \underline{\hat{B}}_\xi = \begin{bmatrix} \underline{B} \\ \hat{G}\underline{D} + \hat{K} - \overline{B} \end{bmatrix},$$

$$\overline{\hat{A}}_\xi = \begin{bmatrix} \overline{A} & 0 \\ \hat{F} + \hat{G}\overline{C} - \underline{A} & \hat{F} \end{bmatrix},\ \overline{\hat{B}}_\xi = \begin{bmatrix} \overline{B} \\ \hat{G}\overline{D} + \hat{K} - \underline{B} \end{bmatrix}.$$

In the following, a necessary and sufficient condition is further established for the existence of the upper-bounding observer. The proof is omitted here.

Theorem 5 *Given a stable continuous-time positive system (1), an upper-bounding observer (11) exists such that the augmented system (17) is positive, robustly stable and satisfies* $\|\hat{z}\|_{L_1} < \gamma\|w\|_{L_1}$ *for any* $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $C \in [\underline{C}, \overline{C}]$ *and* $D \in [\underline{D}, \overline{D}]$ *under zero initial conditions if and only if there exist vectors* $p_1 \geq\geq 0$, $p_2 \geq\geq 0$ *and matrices* $\hat{M}_F$, $\hat{M}_G$, $\hat{M}_K$ *satisfying*

$$\begin{aligned}
&\left[\ \hat{M}_F\ \right]_{ij} \geq 0,\ i,j = 1,\dots,n,\ i \neq j,\\
&\left[\ \hat{M}_G\ \right]_{il} \geq 0,\ i = 1,\dots,n,\ l = 1,\dots,p,\\
&\left[\ \hat{M}_K\ \right]_{ik} \geq 0,\ i = 1,\dots,n,\ k = 1,\dots,m,
\end{aligned}$$

$$\begin{aligned}
\left[\ \hat{M}_F\ \right]_{ij} + \left[\ \hat{M}_G\ \right]_{r,i}\left[\ \underline{C}\ \right]_{c,j}&\\
-p_{2i}^T\left[\ \overline{A}\ \right]_{ij} &\geq 0,\\
\left[\ \hat{M}_K\ \right]_{ik} + \left[\ \hat{M}_G\ \right]_{r,i}\left[\ \overline{D}\ \right]_{c,k}&\\
-p_{2i}^T\left[\ \underline{B}\ \right]_{ik} &\geq 0,\\
p_1^T\overline{A} + \sum_{i=1}^{n}\left[\ \hat{M}_F\ \right]_{r,i}&\\
+\sum_{i=1}^{p}\left[\ \hat{M}_G\ \right]_{r,i}\overline{C} - p_2^T\underline{A} &<< 0,\\
\sum_{i=1}^{n}\left[\ \hat{M}_F\ \right]_{r,i} + \mathbf{1}^T L &<< 0,\\
p_1^T\overline{B} + \sum_{i=1}^{p}\left[\ \hat{M}_G\ \right]_{r,i}\overline{D}&\\
+\sum_{i=1}^{m}\left[\ \hat{M}_K\ \right]_{r,i} - p_2^T\underline{B} - \gamma\mathbf{1}^T &<< 0.
\end{aligned}$$

Moreover, a suitable set of $\hat{F}$, $\hat{G}$ *and* $\hat{K}$ *is given by*

$$\begin{aligned}
\left[\ \hat{F}\ \right]_{ij} &= p_{2i}^{-1}\left[\ \hat{M}_F\ \right]_{ij},\\
\left[\ \hat{G}\ \right]_{il} &= p_{2i}^{-1}\left[\ \hat{M}_G\ \right]_{il},\\
\left[\ \hat{K}\ \right]_{ik} &= p_{2i}^{-1}\left[\ \hat{M}_K\ \right]_{ik}.
\end{aligned}$$

5 Illustrative Example

An illustrative example is presented in this section to illustrate the effectiveness of the theoretical results.

Consider system

$$\left\{\begin{array}{rcl}\dot{x}(t) &=& Ax(t) + Bw(t),\\ y(t) &=& Cx(t) + Dw(t),\end{array}\right. \tag{36}$$

with

$$\begin{aligned}
A &= \begin{bmatrix} -1.8 \pm 0.03 & 0.1 & 0.1 \pm 0.03\\ 0.2 & -1.6 \pm 0.01 & 0.6\\ 0.5 & 0.2 & -1.4 \pm 0.03 \end{bmatrix},\\
B &= \begin{bmatrix} 0.2 \pm 0.02\\ 0.5\\ 0.1 \pm 0.03 \end{bmatrix},\\
C &= \begin{bmatrix} 0.1 & 0.5 \pm 0.01 & 0.2 \end{bmatrix},\ D = 0.1 \pm 0.03.
\end{aligned}$$

Here, we choose $L = \begin{bmatrix} 0.2 & 0.5 & 0.2 \end{bmatrix}$ and assume that $\gamma = 0.15$. By solving the conditions in Theorem 3 via Yalmip, we obtain a feasible solution as follows:

$$\begin{aligned}
p_1 &= \begin{bmatrix} 0.0937 & 0.0470 & 0.0223 \end{bmatrix}^T,\\
p_2 &= \begin{bmatrix} 0.4424 & 0.4350 & 0.9906 \end{bmatrix}^T,
\end{aligned}$$

which further yields the matrices of the lower-bounding observer as

$$\begin{aligned}
\check{F} &= \begin{bmatrix} -1.9063 & 0.0561 & 0.0300\\ 0.1361 & -1.7054 & 0.2606\\ 0.4592 & 0.1570 & -1.1489 \end{bmatrix},\\
\check{G} &= \begin{bmatrix} 0.0454\\ 0.1272\\ 0.0567 \end{bmatrix},\ \check{K} = \begin{bmatrix} 0.1641\\ 0.4730\\ 0.0582 \end{bmatrix}.
\end{aligned}$$

Similarly, by solving the conditions in Theorem 5, with $\gamma = 0.15$, a feasible solution is achieved with

$$\begin{aligned}
p_1 &= \begin{bmatrix} 0.0486 & 0.0870 & 0.0930 \end{bmatrix}^T,\\
p_2 &= \begin{bmatrix} 0.2825 & 0.1911 & 0.7894 \end{bmatrix}^T,
\end{aligned}$$

which further yields the matrices of the upper-bounding observer as

$$\begin{aligned}
\hat{F} &= \begin{bmatrix} -1.7845 & 0.0074 & 0.0941\\ 0.2023 & -1.0645 & 0.6040\\ 0.4608 & 0.0006 & -1.4504 \end{bmatrix},\\
\hat{G} &= \begin{bmatrix} 0.1940\\ 0.0074\\ 0.4091 \end{bmatrix},\ \hat{K} = \begin{bmatrix} 0.2084\\ 0.5025\\ 0.1021 \end{bmatrix}.
\end{aligned}$$

With input $w(t) = 4.5e^{-t}|\cos(2t)|$ and zero initial conditions, Figure 1 depicts the state $x(t)$, the lower estimate $\check{x}(t)$ and the upper estimate $\hat{x}(t)$.

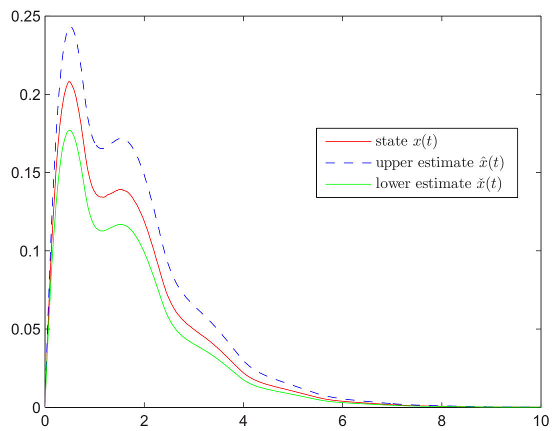

Fig. 1: State x(t) and its state-bounding estimate.

6 Conclusion

In this paper, the problem of positive observers for interval positive systems with L_1-induced performance has been studied. A new characterization on the L_1-induced performance of the augmented system has been established. Based on the novel performance characterization, conditions have been derived for the existence of state-bounding positivity preserving observers. Moreover, the observers designed in this paper can provide an estimate of the state in an asymptotic way. In addition, all the conditions are given under the LP framework and thus can be easily verified. Finally, we have proposed an example to demonstrate the effectiveness of the proposed approach.

References

[1] W. Haddad and V. Chellaboina, "Stability and dissipativity theory for nonnegative dynamical systems: a unified analysis framework for biological and physiological systems," *Nonlinear Analysis: Real World Applications*, vol. 6(1), pp. 35–65, 2005.

[2] S. Muratori and S. Rinaldi, "Equilibria, stability, and reachability of leslie systems with nonnegative inputs," *IEEE Trans. Automat. Control*, vol. 35(9), pp. 1065–1068, 1990.

[3] D. G. Luenberger, *Introduction to Dynamic Systems: Theory, Models, and Applications*. New York: Wiley, 1979.

[4] C. Wang and T. Huang, "Static output feedback control for positive linear continuous-time systems," *Int. J. Robust & Nonlinear Control*, vol. 23, pp. 1537–1544, 2013.

[5] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability of switched positive linear systems with average dwell time switching," *Automatica*, vol. 48, pp. 1132–1137, 2012.

[6] C. Wang, "Bounded real lemma for positive discrete systems," *IET Control Theory & Applications*, vol. 7, pp. 502–507, 2013.

[7] Z. Shu, J. Lam, H. Gao, B. Du, and L. Wu, "Positive observers and dynamic output-feedback controllers for interval positive linear systems," *IEEE Trans. Circuits and Systems (I)*, vol. 55(10), pp. 3209–3222, 2008.

[8] L. Benvenuti and L. Farina, "A tutorial on the positive realization problem," *IEEE Trans. Automat. Control*, vol. 49(5), pp. 651–664, 2004.

[9] L. Farina, "On the existence of a positive realization," *Systems & Control Letters*, vol. 28, pp. 219–226, 1996.

[10] M. P. Fanti, B. Maione, and B. Turchiano, "Controllability of multi-input positive discrete-time systems," *Int. J. Control*, vol. 51(6), pp. 1295–1308, 1990.

[11] M. E. Valcher, "Controllability and reachability criteria for discrete time positive systems," *Int. J. Control*, vol. 65(3), pp. 511–536, 1996.

[12] H. Gao, J. Lam, C. Wang, and S. Xu;, "Control for stability and positivity: equivalent conditions and computation," *IEEE Trans. Circuits and Systems (II)*, vol. 52, pp. 540–544, 2005.

[13] M. Ait Rami and F. Tadeo, "Controller synthesis for positive linear systems with bounded controls," *IEEE Trans. Circuits and Systems (II)*, vol. 54(2), pp. 151–155, 2007.

[14] W. Haddad and V. Chellaboina, "Stability theory for nonnegative and compartmental dynamical systems with time delay," *Systems & Control Letters*, vol. 51(5), pp. 355–361, 2004.

[15] P. Li, J. Lam, and Z. Shu, "H_∞ positive filtering for positive linear discrete-time systems: An augmentation approach," *IEEE Trans. Automat. Control*, vol. 55, pp. 2337– 2342, 2010.

[16] F. Mazenc and O. Bernard, "Interval observers for linear time-invariant systems with disturbances," *Automatica*, vol. 47, no. 1, pp. 140–147, 2011.

[17] T. Kaczorek, *Positive 1D and 2D Systems*. Berlin, Germany: Springer-Verlag, 2002.

[18] X. Liu, W. Yu, and L. Wang, "Stability analysis of positive systems with bounded time-varying delays," *IEEE Trans. Circuits and Systems (II)*, vol. 56, pp. 600–604, 2009.

[19] X. Liu, W. Yu, and L. Wang, "Stability analysis for continuous-time positive systems with time-varying delays," *IEEE Trans. Automat. Control*, vol. 55, pp. 1024–1028, 2010.

[20] X. Zhao, L. Zhang, and P. Shi, "Stability of a class of switched positive linear time-delay systems," *Int. J. Robust & Nonlinear Control*, vol. 23, pp. 578–589, 2013.

[21] L. Wu, J. Lam, Z. Shu, and B. Du, "On stability and stabilizability of positive delay systems," *Asian Journal of Control*, vol. 11, pp. 226–234, 2009.

[22] S. Li, Z. Xiang, and H. R. Karimi, "Stability and L_1-gain controller design for positive switched systems with mixed time-varying delays," *Applied Mathematics and Computation*, vol. 222, pp. 507–518, 2013.

[23] J. Feng, J. Lam, Z. Shu, and Q. Wang, "Internal positivity preserved model reduction," *Int. J. Control*, vol. 83(3), pp. 575–584, 2010.

[24] P. Li, J. Lam, Z. Wang, and P. Date, "Positivity-preserving H_∞ model reduction for positive systems," *Automatica*, vol. 47, no. 7, pp. 1504–1511, 2011.

[25] M. Ait Rami, C. H. Cheng, and C. de Prada, "Tight robust interval observers: an LP approach," in *Proceedings of the 47th IEEE Conference on Decision and Control*, 2008.

[26] M. Ait Rami, U. Helmke, and F. Tadeo, "Positive observation problem for linear time-lag positive systems," in *Proceedings of the Third IFAC Symposium on System Structure and Control*, 2007.

[27] P. Li and J. Lam, "Positive state-bounding observer for positive interval continuous-time systems with time delay," *Int. J. Robust & Nonlinear Control*, vol. 22, pp. 1244–1257, 2012.

[28] X. Chen, J. Lam, P. Li, and Z. Shu, "Controller synthesis for positive systems under ℓ_1-induced performance," in *Proceedings of the 24th Chinese Control and Decision Conference*, 2012.

[29] X. Chen, J. Lam, P. Li, and Z. Shu, "ℓ_1-induced norm and controller synthesis of positive systems-induced norm and controller synthesis of positive systems," *Automatica*, vol. 49, pp. 1377–1385, 2013.

[30] C. Briat, "Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: L_1-gain and L_∞-gain characterization," *Int. J. Robust & Nonlinear Control*, vol. 23, no. 17, pp. 1932–1954, 2012.

[31] X. Chen, J. Lam, P. Li, and Z. Shu, "L_1-induced performance analysis and sparse controller synthesis for interval positive systems," in *Proceedings of the International Conference of Applied and Engineering Mathematics*, 2013.

[32] X. Chen, J. Lam, P. Li, and Z. Shu, "Output-feedback control for continuous-time interval positive systems under L_1 performance," *Asian Journal of Control*, vol. doi: 10.1002/asjc.808, 2013.

[33] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. Wiley-Interscience, 2000.

[34] W. Haddad, V. Chellaboina, and Q. Hui, *Nonnegative and Compartmental Dynamical Systems*. Princeton, New Jersey: Princeton University Press, 2010.