The nonlinear equation system approach to solving dynamic user optimal simultaneous route and departure time choice problems

by

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Paper revised and resubmitted to Transportation Research Part B

Abstract

Dynamic user optimal simultaneous route and departure time choice (DUO-SRDTC) problems are usually formulated as variational inequality (VI) problems whose solution algorithms generally require continuous and monotone route travel cost functions to guarantee convergence. However, the monotonicity of the route travel cost functions cannot be ensured even if the route travel time functions are monotone. In contrast to traditional formulations, this paper formulates a DUO-SRDTC problem (that can have fixed or elastic demand) as a system of nonlinear equations. The system of nonlinear equations is a function of generalized origin-destination (OD) travel costs rather than route flows and includes a dynamic user optimal (DUO) route choice subproblem with perfectly elastic demand and a quadratic programming (QP) subproblem under certain assumptions. This study also proposes a solution method based on the backtracking inexact Broyden-Fletcher-Goldfarb-Shanno (BFGS) method, the extragradient algorithm, and the Frank-Wolfe algorithm. The BFGS method, the extragradient algorithm, and the Frank-Wolfe algorithm are used to solve the system of nonlinear equations, the DUO route choice subproblem, and the QP subproblem, respectively. The proposed formulation and solution method can avoid the requirement of monotonicity of the route travel cost functions to obtain a convergent solution and provide a new approach with which to solve DUO-SRDTC problems. Finally, numeric examples are used to demonstrate the performance of the proposed solution method.

Keywords: Dynamic traffic assignment; Dynamic user optimal; Simultaneous route and departure time choice; Nonlinear equations; BFGS method.

1. Introduction

Dynamic traffic assignment (DTA) is a difficult problem with practical importance. Hence, it has been studied for more than four decades, but room for improvement can still be found.

According to the available route and departure time choices, DTA models can be classified into three categories (Szeto and Wong, 2011): (1) the pure departure time choice models (e.g., Vickrey, 1969; Hendrickson and Kocur, 1981; Lindsey et al., 2012), (2) the pure route choice models (e.g., Mounce and Carey, 2011; Carey and Watling, 2012; Ban et al., 2012b; Long et al., 2013, 2015a), and (3) the simultaneous route and departure time choice (SRDTC) models (e.g., Friesz et al., 1993; Huang and Lam, 2002; Szeto and Lo, 2004; Mun, 2011; Han et al., 2013b,c; Friesz and Meimand, 2014; Long et al., 2015b). To model the choice of route and/or departure time, the most commonly used principles in the DTA literature mainly include the dynamic user optimal (DUO) principle (e.g., Ran and Boyce, 1996; Lo and Szeto, 2002a,b; Nie and Zhang, 2010; Iryo, 2013; Blumberg-Nitzani and Bar-Gera, 2014), the stochastic DUO (SDUO) principle, (e.g., Han, 2003; Szeto et al., 2011; Meng and Khoo, 2012; Long et al., 2015b), and the dynamic system optimal principle (e.g., Merchant and Nemhauser, 1978a,b; Carey, 1987; Carey and Subrahmanian, 2000; Ziliaskopoulos, 2000; Chow, 2009; Ng and Waller, 2010a,b; Nie, 2011; Doan and Ukkusuri, 2012, and Han et al., 2013b). The DUO/SDUO/dynamic system optimal principle assumes that travelers select their routes and/or departure times to minimize their actual/perceived/marginal travel costs. In this paper, we mainly focus on DUO-SRDTC problems.

As with dynamic route choice problems, DUO-SRDTC problems can be modeled through both a simulation-based approach and an analytical approach. The analytical approach formulates DUO-SRDTC problems as some well-known mathematical problems, such as mathematical programming problems (e.g., Jauffred and Bernstein, 1996; Yang and Meng, 1998), nonlinear complementarity problems (NCPs) (e.g., Ban et al., 2008), and variational inequality (VI) problems (e.g., Friesz et al., 1993; Ran and Boyce, 1996; Huang and Lam, 2002; Szeto and Lo, 2004).

Because mathematical programming problems and NCPs can all be formulated into VI problems, analytical DUO-SRDTC problems can be solved by any general computational techniques developed for VI problems, provided that the convergence requirement is satisfied. The convergence of solution algorithms for VI problems is mainly determined by the mathematical properties, such as the continuity and monotonicity, of their mapping functions. For example, a projection-based method developed by Jang et al. (2005) to solve a DUO route choice problem, which is a special case of the DUO-SRDTC problem, requires the route travel cost functions to be *continuous* and *strictly monotone* with respect to the route flows to have a convergent solution. A fixed-point algorithm developed by Han et al. (2015) to solve a boundedly rational DUO-SRDTC problem requires *strong monotonicity* and *Lipschitz continuity* from the route travel cost functions. The requirement of strict monotonicity is in fact stronger than necessary, and other solution algorithms for VI problems that have a weaker convergence requirement have been used in the literature. For instance, the convergence of the alternating direction method and the descent method, which were used by Lo and Szeto (2002) and Szeto and Lo (2004) to solve the cell-based DUO route choice problem with fixed (i.e., perfectly inelastic) demand and the cell-based DUO-SRDTC problem with elastic demand, respectively, requires the mapping function of the VI problems to be *co-coercive*. According to Zhao and Hu (2007), co-coercive mappings are monotone and Lipschitz continuous but may not necessarily be strictly monotone. Algorithms with an even weaker convergence requirement, such as the day-to-day route-swapping algorithm, have also been used to solve DTA problems (Huang and Lam, 2002; Szeto and Lo, 2006; Tian et al., 2012). The convergence of the route-swapping algorithm requires the mapping functions to be *continuous* and *monotone* (Mounce and Carey, 2011). In summary, the convergence of methods for solving DUO-based DTA problems generally requires the mapping functions to be (Lipschitz) continuous and (strictly) monotone.

The requirement of monotonicity, however, is not met in general in either path-based (e.g., Szeto and Lo, 2004; Mun, 2011) or link-based analytical DUO-SRDTC problems (e.g., Chen and Hsueh, 1998; Chen et al., 2001), in which the route and link flow rates, respectively, are used as decision variables. In path-based problems, the generalized path travel cost is used to formulate the mapping functions of the corresponding VI problems. However, in this paper, we show that the generalized path travel cost can be nonmonotone with respect to the route flows, even if the travel time function is monotone, which implies that the mapping functions of the VI problems are generally nonmonotone. In link-based models, generalized origin-destination (OD) travel costs are used to formulate the mapping functions of the corresponding VI problems (see Ran and Boyce, 1996, for details). Because the generalized OD travel cost equals the minimum travel cost of all paths between the OD pair, the nonmonotonicity of generalized path travel cost functions with respect to route flows also leads to the nonmonotonicity of the mapping functions of the link-based models. As a result, the convergence requirement of some commonly used algorithms for solving the DUO-SRDTC problems cannot be met.

Some researchers have focused on the use of the notion of merely generalized monotone delay operators/mapping functions or nonmonotone delay operators to study DUE convergence and/or solve DTA problems. The pioneers may be Friesz et al. (2011), who provided a continuous time fixed-point algorithm for their DUO-SRDTC problem. This algorithm has proved to be convergent for *merely component-wise strongly pseudomonotone* delay operators that are also *component-wise weakly monotone*; such operators form a class of nonmonotone operators. Pseudomonotonicity (including strong pseudomonotonicity) is a consequence of monotonicity and thus is one type of generalized monotonicity (Han et al., 2015). A pseudomonotone operator is not necessarily monotone, but the monotonicity of a mapping implies its pseudomonotonicity (Rockafellar and Wets, 1998). Long et al. (2013) adopted the extragradient method, whose convergence requires the mapping functions to be *Lipschitz continuous* and *pseudomonotone*. To solve the boundedly rational DUO-SRDTC problem, Han et al. (2015) adopted the proximal point method (PPM), whose convergence relies on the assumption of *semistrictly quasimonotone* and *continuity* of the operator.

In contrast to traditional methods that rely on monotonicity for convergence (e.g., the route-swapping and descent methods) and methods that rely on the notion of generalized monotonicity for convergence (e.g., Friesz et al., 2011; Long et al., 2013; Han et al., 2015), we reformulate an analytical DUO-SRDTC problem as a system of nonlinear equations instead of VI problems, whose decision variables are not route flows but the equilibrium generalized travel cost of each OD pair. The system of nonlinear equations for a DUO-SRDTC problem can be formed by equating the demand functions and *the retrieved OD demand functions*, both of which are functions of generalized OD travel costs. Under certain assumptions, the retrieved OD demand function is defined by two subproblems: a DUO route choice subproblem with perfectly elastic demand and a quadratic programming (QP) subproblem. The route choice subproblem is proven to be equivalent to the DUO-SRDTC problem with perfectly elastic demand under the departure time choice equilibrium condition and a unique mapping condition between the route travel time and the generalized route cost. The route choice subproblem aims to determine the used path sets at the DUO state. The QP subproblem aims to choose the closest OD demand vector to the one defined by the demand function based on the output of the route choice subproblem.

To solve the system of nonlinear equations, we adopt one of the most popular quasi-Newton methods: the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method. The advantage of the BFGS algorithm over a Newton method is that the convergence of the former is superlinear and fast, whereas its computational complexity is significantly lower than that of the Newton method. The major difficulty in solving the system of nonlinear equations with the BFGS method is the lack of practical line search strategies. We adopt a backtracking inexact BFGS method proposed by Yuan and Lu (2008) as the main algorithm of our proposed solution method to solve the proposed system of nonlinear equations. The advantages of this method are as follows. It has a norm descent property (i.e., the search direction is descent for the norm function), and its global and superlinear convergence can be achieved under mild conditions (Yuan and Lu, 2008). In contrast to traditional line search techniques, the inexact BFGS method can avoid the computation of the Jacobian matrix during a line search and thus reduce the computation time, especially for large-scale problems. We note that the BFGS method adopted in this paper was initially developed for symmetric nonlinear equations. The Jacobian of the mapping function of the proposed system of nonlinear equations is usually asymmetric, and hence the

BFGS method is a heuristic for DUO-SRDTC problems.

At each iteration, the main algorithm provides a generalized OD travel cost vector to the algorithm for solving the DUO route choice subproblem with perfectly elastic demand. The subproblem is implicit in the mapping function of the proposed system of nonlinear equations. To solve this subproblem, we extend the extragradient method of Khobotov (1987). The main motives for the use of this method are as follows. First, it is convergent if the route travel time functions are pseudomonotone and Lipschitz continuous; it is not necessary to know the Lipschitz constant of the route travel time functions. This method has a looser convergence condition than some of the commonly used methods, such as the route-swapping method and the descent method. Second, this method can also be used to solve link-based or intersection-movement–based route choice problems (see Long et al., 2013) and does not require knowledge of all paths in the network (which can be very numerous even for medium-size networks) or a column-generation heuristic (which may not guarantee convergence) to generate required/used paths during the solution process. The use of the extragradient method allows an extension for further study in which the proposed solution method is formulated in terms of link or intersection-movement flows. Third, and most importantly, the extragradient method is only used to illustrate the advantage of the proposed approach (i.e., reformulation and hybridized solution method) in tackling the problem and can be replaced by other less-restrictive methods, e.g., the PPM, whose convergence relies on the assumption of semistrictly quasimonotone and continuity of the operator.

The outputs of the extragradient method are the equilibrium route flow vector and the information of the set of used paths; the latter is the input to the algorithm for solution of the QP subproblem. The Frank-Wolfe method is used to solve the QP subproblem. The output of the Frank-Wolfe method is the retrieved demand vector, which is used to determine the value of the mapping function in the system of equations in the iteration.

The reformulation and hybridized solution method offers a novel and efficient solution approach with which to tackle DUO-SRDTC problems for the following four reasons.

First, the proposed approach takes the generalized OD travel costs as decision variables and can be viewed as a dual algorithm for DUO-SRDTC problems. The concept of the proposed approach is very similar to that of dual algorithms for static traffic assignment problems (e.g., Fukushima, 1984; Larsson et al., 1997). The proposed algorithm may not be competitive in terms of computation speed compared to solution algorithms for the primal model, which guarantee convergence for the DUO-SRDTC problems under the monotone assumption of route travel costs. However, the existing solution algorithms cannot guarantee convergence due to the nonmonotonicity of the route travel cost functions. Compared with traditional methods of solving DUO-SRDTC problems, such as the route-swapping method and the descent method, our proposed solution approach can avoid the requirement of monotonicity of the route travel cost functions to obtain a convergent solution. It requires only that the route travel time functions satisfy looser conditions, i.e., that they are pseudomonotone and Lipschitz continuous. This is the main reason that the proposed approach performs better than some commonly used solution algorithms for the DUO-SRDTC problems in terms of convergence.

Second, our solution method relies on symmetry of Jacobian and generalized monotonicity of the route travel "time," unlike the PPM reported by Han et al. (2015), which relies on the requirement of generalized monotonicity on the path "cost" function of the SRDTC problem. Our solution method serves as a new alternative with which to solve SRDTC problems, because depending on the schedule delay function adopted, the path cost function in some SRDTC problems may not satisfy the generalized monotonicity requirement, but the travel time function (that excludes a schedule delay function) can still satisfy the generalized monotonicity requirement.

Third, as reflected in our numerical results, the solution method can be used to obtain convergent solutions for SRDTC problems that cannot be solved with some of the existing methods, such as the route-swapping method, the decent direction method, and the extragradient method.

Fourth, when the existing methods can obtain a convergent solution, the proposed approach can be more efficient than these methods in some cases, as shown in our numerical results.

The contributions of this paper include the following.

First, we provide a nonlinear equation system approach to formulate DUO-SRDTC problems and uses equilibrium-generalized OD travel costs as decision variables. This approach can be adopted to formulate other DTA or traffic assignment problems.

Second, we develop a novel and efficient solution method to solve DUO-SRDTC problems with fixed or elastic demand that serves as an alternative to the method proposed by Han et al. (2015).

Third, we illustrate the nonmonotonicity of generalized route travel cost functions in SRDTC problems. This implies that SRDTC problems can have multiple solutions and that the convergence requirement of some commonly used solution algorithms for solving SRDTC problems cannot be met.

The remainder of this paper is organized as follows: In the next section, DUO-SRDTC problems are classified into three types, and each is formulated as a VI problem. In Section 3, each DUO-SRDTC problem is further reformulated as an equivalent system of nonlinear equations. Section 4 develops a BFGS method to solve the proposed system of nonlinear equations and an extragradient method to solve the dynamic route choice subproblem with perfectly elastic demand. Numerical experiments are given in Section 5. Finally, Section 6 concludes the article.

2. Dynamic user optimal simultaneous route and departure time choice problems

We consider a network G (N, A) with multiple origins and destinations, where N and A denote the sets of nodes and links, respectively. W denotes the set of OD pairs. We discretize the modeling horizon and departure time horizon with the durations of T and T_d into finite sets of time intervals $K = \{k = 1, 2, \cdots, \underline{K}\}$ and $K_d = \{k = 1, 2, \cdots, \underline{K}_d\}$, respectively, where $\underline{K}_d < \underline{K}$. Let δ be the length of each interval such that $\delta \underline{K} = T$ and $\delta \underline{K}_d = T_d$. The demand period $[0, T_d]$ is assumed to be sufficiently long to ensure that all travelers cannot reduce their individual travel cost by changing their own departure time. The following notations are adopted throughout this paper:

P_w	set of routes connecting OD pair *w*.
P	set of routes connecting all OD pairs, $P = \bigcup_{w \in W} P_w$.
$f_p(k)$	flow rate entering route $p \in P$ during interval *k*.
$\mathbf{f}$	vector of route flows $\left(f_p(k), \forall p \in P, k \in K_d \right)$ with a dimension of $n = \underline{K}_d \mid P \mid$.
Q_w	total demand for OD pair *w* during the demand period.
$\mathbf{Q}$	vector of total demands $(Q_w, \forall w \in W)$.
$t_p(k)$	actual route travel time for travelers entering route $p \in P$ during interval *k*.
$\eta_w(k)$	minimum travel time for travelers between OD pair *w* departing during interval *k*.
$\boldsymbol{\eta}$	vector of minimum travel times between OD pairs $(\eta_w(k), \forall w \in W, k \in K_d)$.
$\psi_{wk}(t)$	generalized travel cost function with respect to travel time *t* for travelers between OD pair *w* departing during interval *k*, which is increasing with respect to *t*.
$\psi_{wk}^{-1}(\cdot)$	inverse function of $\psi_{wk}(t)$.
$c_p(k)$	generalized travel cost incurred by travelers entering route $p \in P$ during interval *k*, and $c_p(k) = \psi_{wk}(t_p(k))$ if $p \in P_w$.
π_w	generalized travel cost between OD pair *w* during the studied period.
$\boldsymbol{\pi}$	vector of generalized OD travel costs $(\pi_w, \forall w \in W)$.
$D_w(\pi_w)$	demand function for OD pair *w*, which is a function of π_w.
$\mathbf{D}(\boldsymbol{\pi})$	vector of demand functions $(D_w(\pi_w), \forall w \in W)$.
$\mathbf{D}^{-1}(\mathbf{Q})$	inverse function of $\mathbf{Q} = \mathbf{D}(\boldsymbol{\pi})$.
Ω	feasible solution set of the DUO-SRDTC problem with fixed demand.

An optimal flow pattern to a DUO-SRDTC problem must satisfy the ideal DUO-SRDTC conditions (Ran and Boyce, 1996), which can be written as follows (Huang and Lam, 2002):

$$f_p(k)[c_p(k) - \pi_w] = 0, \forall w \in W, p \in P_w, k \in K_d, \text{ and} \tag{1}$$

$$c_p(k) - \pi_w \geq 0, \forall w \in W, p \in P_w, k \in K_d. \tag{2}$$

By definition, the flow must satisfy flow conservation and non-negativity conditions:

$$\sum_{p \in P_w} \sum_{k \in K_d} f_p(k)\delta = Q_w, \forall w \in W, \text{ and} \tag{3}$$

$$f_p(k) \geq 0, \forall w \in W, p \in P_w, k \in K_d. \tag{4}$$

In the literature, the travel demands in SRDTC problems are often assumed to be a function of the generalized OD travel costs. The function is referred to as a demand function. As shown in Fig. 1, the gradient of demand function equals (i) zero when the total demand for each OD pair is *fixed* and given, (ii) negative when *elastic* demand is considered, and (iii) infinite when *perfectly elastic demand* is considered. The cases of fixed demand and perfectly elastic demand can be viewed as two special cases of elastic demand. Based on these cases, we can classify the DUO-SRDTC problems into three types, as depicted in the following three sections.

2.1. The DUO-SRDTC problem with elastic demand

When elastic demand is considered, the total demand Q_w is assumed to be an invertible, continuous, decreasing function of the generalized OD travel cost π_w, i.e., $Q_w = D_w(\pi_w)$. According to Eq. (3), Q_w equals the sum of the route flows and hence is a function of the route flows. Therefore, the generalized OD travel cost can be formulated as a function of the route flows:

$$\pi_w(\mathbf{f}) = D_w^{-1}(Q_w) = D_w^{-1}\left(\sum_{k\in K_d}\sum_{p\in P_w} f_p(k)\delta\right), \forall w \in W \quad . \tag{5}$$

A DUO-SRDTC solution to the SRDTC problem with elastic demand must satisfy constraints (1) through (5). Note that $c_p(k)$ in the DUO-SRDTC conditions (1) and (2) is a function of the route travel time $t_p(k)$, which in turn is a function of the route flow vector $\mathbf{f}$. The functional form of $c_p(k)$ will be presented in Section 2.4. The route travel time $t_p(k)$ is the output of the travel time determination procedure (see, e.g., Lo and Szeto, 2002a) and the dynamic network loading (DNL) subproblem of the studied DUO-SRDTC problem, in which the DNL problem depicts the manner in which traffic propagates inside a traffic network given an input of route flows and determines both the link and route cumulative flows to compute travel times. Following Szeto and Lo (2004), the DUO-SRDTC problem with elastic demand can be formulated as a VI problem: find a vector $\mathbf{f}^* \in \Psi$ such that

$$\left\langle \mathbf{F}_1(\mathbf{f}^*), \mathbf{f} - \mathbf{f}^* \right\rangle \geq 0, \forall \mathbf{f} \in \Psi \ , \tag{6}$$

where $\mathbf{F}_1(\mathbf{f}) = \left(c_p(k) - \pi_w(\mathbf{f}), \forall w \in W, p \in P_w, k \in K_d\right)$, Ψ is a closed convex set, and

$$\Psi = \left\{\mathbf{f} \geq \mathbf{0} : f_p(k) \leq \overline{F}, \forall p \in P, k \in K_d\right\},$$

where $\overline{F}$ is a very large positive constant. Because the traffic demand during each interval δ is generated on the basis of population, the traffic demand of a zone in that interval is restricted by its population. Moreover, the number of trips made by each person in each interval must be finite in reality. Hence, the flow on any route departing during any interval should not be greater than the total population of the network multiplied by the maximum number of trips that can be made by a person in any interval divided by the length of the interval, which in turn is bounded by a very large positive constant.

2.2. The DUO-SRDTC problem with fixed demand

One special case of elastic demand is that the total demand for each OD pair during the demand period is fixed and given. Let $\bar{Q}_w$ be the total fixed demand for OD pair *w* during the demand period, and let $\bar{\mathbf{Q}}$ be the vector of total fixed demands $\left(\bar{Q}_w, \forall w \in W\right)$. We have the following lemma:

Lemma 1. The DUO-SRDTC conditions with fixed demand (i.e., perfectly inelastic demand) can be formulated as a finite-dimensional VI problem: find a vector $\mathbf{f}^* \in \Omega$ such that

$$\left\langle \mathbf{F}_2(\mathbf{f}^*), \mathbf{f} - \mathbf{f}^* \right\rangle \geq 0, \forall \mathbf{f} \in \Omega , \tag{7}$$

where $\mathbf{F}_2(\mathbf{f}) = \left(c_p(k), \forall p \in P, k \in K_d\right)$, Ω is a closed convex set, and

$$\Omega = \left\{ \mathbf{f} \geq \mathbf{0} : \sum_{p \in P_w} \sum_{k \in K_d} f_p(k)\delta = \bar{Q}_w, \forall w \in W \right\}. \tag{8}$$

The proof directly follows that in Friesz et al. (1993).

2.3. The DUO-SRDTC problem with perfectly elastic demand

Perfectly elastic demand is another extreme case of elastic demand. If demand is perfectly elastic, residual demand does not increase at all (Salop and Scheffman, 1983). The gradient of the generalized OD travel cost function (5) equals zero (or, equivalently, the demand function is vertical) when the demand is perfectly elastic. In other words, the generalized travel cost between each OD pair is fixed. VI problem (6) can be extended to formulate the DUO-SRDTC problem with perfectly elastic demand. The only difference is that the generalized travel cost between each OD pair is no longer a function of the route flows but becomes an input to the model. Therefore, the DUO-SRDTC problem with perfectly elastic demand can be formulated as a VI problem: find a vector $\mathbf{f}^* \in \Psi$ such that

$$\left\langle \mathbf{F}_3(\mathbf{f}^*), \mathbf{f} - \mathbf{f}^* \right\rangle \geq 0, \forall \mathbf{f} \in \Psi , \tag{9}$$

where $\mathbf{F}_3(\mathbf{f}) = \left(c_p(k) - \pi_w, \forall w \in W, p \in P_w, k \in K_d\right)$, and π_w is predetermined and independent with $\mathbf{f}$.

For the fixed-demand case, we always have $\sum_{p \in P_w} \sum_{k \in K_d} f_p(k)\delta = \sum_{p \in P_w} \sum_{k \in K_d} f_p^*(k)\delta = \bar{Q}_w$, and the OD-based generalized cost term π_w can be dropped into VI problem (7). However, for the case with perfectly elastic demand, we usually have $\sum_{p \in P_w} \sum_{k \in K_d} f_p(k) \neq \sum_{p \in P_w} \sum_{k \in K_d} f_p^*(k)$, i.e., $Q_w \neq Q_w^*$, because $\mathbf{f}^* \in \Psi$ and $\mathbf{f} \in \Psi$ correspond to different OD demands and generalized travel costs, and hence the term π_w cannot be dropped from VI problem (9). Specially, if π_w is selected as zero or a very small number, $c_p(k) > \pi_w$ will always be satisfied. According to the DUO conditions (1) and (2), an optimal solution to VI problem (9) will be a zero vector. Note that operator $\mathbf{F}_3(\mathbf{f})$ can be negative for some feasible route flow patterns $\mathbf{f} \in \Psi$. For example, $\mathbf{F}_3(\mathbf{0}) < \mathbf{0}$ can be satisfied if the predetermined generalized OD travel cost π_w is sufficiently large. However, when $\mathbf{f}^*$ is an optimal solution to VI problem (9), $\mathbf{F}_3(\mathbf{f}^*)$ is non-negative.

2.4. Generalized route travel cost function

The generalized travel cost for travelers between OD pair *w* consists of three components: (1) the disutility of early departure at the origin node, (2) the trip travel time, and (3) a "penalty" for reaching the destination early or late. Following most existing studies (see e.g., Huang and Lam, 2002), the disutility of early departure at the origin node is not considered in SRDTC problems. It is assumed that travelers have a desired arrival time period, which is expressed as the arrival time window $[k_w^* - \Delta_w, k_w^* + \Delta_w]$, where k_w^* is the middle point of the time window and represents the official work start time for travelers between OD pair *w,* and Δ_w is the interval of arrival time flexibility. The generalized travel cost function $\psi_{wk}(t)$ for travelers between OD pair *w* who depart during time interval *k* and require a travel time of *t* units to reach their destination is formulated as

$$\psi_{wk}(t) = \alpha t + \begin{cases} \beta[k_w^* - \Delta_w - k\delta - t], & \text{if } k\delta + t < k_w^* - \Delta_w, \\ \gamma[k\delta + t - k_w^* - \Delta_w], & \text{if } k\delta + t > k_w^* + \Delta_w, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where α is the unit cost of travel time and β and γ are the unit costs of early and late arrival, respectively. According to empirical results (Small, 1982), we have $0 < \beta < \alpha < \gamma$.

Lemma 2: $\psi_{wk}(t)$ is monotonic increasing with respect to travel time under the conditions $0 < \beta < \alpha$ and $0 < \gamma$.

The proof is presented in Appendix A.

Substituting the route travel time into Eq. (10), we can obtain the generalized route travel cost for travelers departing during time interval *k* and using path $p \in P_w$ as follows:

$$c_p(k) = \alpha t_p(k) + \begin{cases} \beta[k_w^* - \Delta_w - k\delta - t_p(k)], & \text{if } k\delta + t_p(k) < k_w^* - \Delta_w, \\ \gamma[k\delta + t_p(k) - k_w^* - \Delta_w], & \text{if } k\delta + t_p(k) > k_w^* + \Delta_w, \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

Example B.1 in Appendix B shows that a monotone function composed with another nonlinear function can lead to a nonmonotone function. This implies that route travel cost functions may not be monotone with respect to route flows, even if the route travel time functions are monotone. As a result, SRDTC problems can have multiple solutions, and the convergence requirement of some algorithms that are commonly used to solve SRDTC problems (i.e., Ran and Boyce, 1996; Huang and Lam, 2002;

Szeto and Lo, 2004) cannot be met. Although Ghali and Smith (1993) pointed out that generalized route travel cost functions are not necessarily monotone if there are *multiple active* bottlenecks per route, they did not give an example to demonstrate it. Smith and Ghali (1990) proved that route travel *time* functions are monotone with respect to route flows for the *single*-bottleneck-per-route case. Perakis and Roels (2006) derived a continuous and strictly monotone travel time function based on the theory of kinematic waves, subject to certain restrictions (including only one spillback per path). However, the above studies did not consider generalized route travel *cost* functions. Example B.2 in Appendix B illustrates that generalized route *cost* functions can be nonmonotone with respect to route flows even for the single-bottleneck-per-route case.

3. Formulating a DUO-SRDTC problem as the corresponding system of nonlinear equations

Section 3 contains three subsections. In Section 3.1, we first formulate the first subproblem of a DUO-SRDTC problem, the DUO route choice problem with perfectly elastic demand, as a VI problem based on route travel times rather than on generalized route costs. Next, we provide a method with which to retrieve the minimum route travel time from a generalized OD or route travel cost and then define the subproblem: given a generalized OD travel cost vector, the problem determines a route flow vector to satisfy the DUO route choice and flow non-negativity conditions. The optimal route flow vector is used to deduce the used path sets at the DUO state. We then show that the DUO-SRDTC problem with perfectly elastic demand is equivalent to the DUO route choice problem with perfectly elastic demand under a mapping condition between the route travel time and the generalized route cost and the departure time choice equilibrium condition. In Section 3.2, the second subproblem, the QP subproblem, is presented. It aims to retrieve an OD demand vector based on the information of used path sets obtained from the route choice subproblem and choose the closest OD demand vector to the target one defined from the demand function. In Section 3.3, the system of nonlinear equations for a DUO-SRDTC problem is presented. The system of nonlinear equations is a function of generalized OD travel costs and is formed by equating each demand function to the corresponding retrieved OD demand function defined by the two subproblems.

3.1. Dynamic route choice subproblem with perfectly elastic demand

3.1.1 Problem formulation

The optimality condition of a DUO route choice problem can be stated as follows (Ran and Boyce, 1996): *If, for each OD pair at each instant of time, the actual travel times experienced by travelers departing at the same time are equal and minimal, then the dynamic traffic flow over the network is in* *a travel-time-based DUO state*. Hence, the DUO conditions of the dynamic route choice problem with perfectly elastic demand can be written as follows:

$$t_p(k)\begin{cases} = \eta_w(k), \text{ if } f_p(k) > 0, \\ \geq \eta_w(k), \text{ if } f_p(k) = 0, \end{cases} \forall w \in W, p \in P_w, k \in K_d. \tag{12}$$

Note that *the minimum travel time* $\eta_w(k)$ *for travelers traveling between each OD pair* w *departing during any time interval k is known and fixed*. Mathematically, the DUO conditions of the dynamic route choice problem with perfectly elastic demand can be equivalently expressed as follows:

$$f_p(k)[t_p(k) - \eta_w(k)] = 0, \forall w \in W, p \in P_w, k \in K_d, \text{ and} \tag{13}$$

$$t_p(k) - \eta_w(k) \geq 0, \forall w \in W, p \in P_w, k \in K_d. \tag{14}$$

Similar to the DUO-SRDTC problem with perfectly elastic demand, the DUO route choice problem with perfectly elastic demand can be formulated as a VI problem, which may be stated as follows:

Lemma 3: The dynamic route flow vector $\mathbf{f}$ satisfies the DUO conditions (13) and (14) (or (12)) if and only if the vector satisfies the following VI problem: find a vector $\mathbf{f}^* \in \Psi$ such that

$$\left\langle \mathbf{F}_4(\mathbf{f}^*), \mathbf{f} - \mathbf{f}^* \right\rangle \geq 0, \forall \mathbf{f} \in \Psi, \tag{15}$$

where $\mathbf{F}_4(\mathbf{f}) = \left(t_p(k) - \eta_w(k), \forall w \in W, p \in P^w, k \in K_d \right)$.

The proof of Lemma 3 is similar to that of Theorem 5.2 in Ran and Boyce (1996). Similar to VI problem (9), the OD travel time term $\eta_w(k)$ cannot be dropped from VI problem (15).

In the mapping function $\mathbf{F}_4(\mathbf{f})$, the OD travel time $\eta_w(k)$ is predetermined, and hence the solution existence of VI problem (15) requires that the route travel time functions $t_p(k)$ must be continuous with respect to route flows, because the solution set is convex after inducing a ball centered at the origin with a radius of maximum OD travel demand, and solution uniqueness further requires that the route travel time functions $t_p(k)$ must be strictly monotone with respect to route flows.

3.1.2. Retrieving the minimum route travel time from a generalized OD travel cost

If the generalized travel cost associated with path $p \in P_w$ and time interval *k* is known, then the corresponding route travel time can be obtained from the inverse function of the generalized route travel cost function (11):

$$t_p(k)=\psi_{wk}^{-1}(c_p(k))=\begin{cases}[c_p(k)-\beta(k_w^*-\Delta_w-k\delta)]/(\alpha-\beta), & \text{if } c_p(k)<\alpha(k_w^*-\Delta_w-k\delta),\\ [c_p(k)-\gamma(k\delta-k_w^*-\Delta_w)]/(\alpha+\gamma), & \text{if } c_p(k)>\alpha(k_w^*+\Delta_w-k\delta),\\ c_p(k)/\alpha, & \text{otherwise.}\end{cases} \quad (16)$$

Recall that $\psi_{wk}(t)$ is the generalized travel cost function for travelers between OD pair *w*, who depart during time interval *k* and require a travel time of *t* units to their destination. If $\psi_{wk}(\eta_w(k))=\pi_w$ is satisfied for all $k\in K_d$ and the generalized OD travel cost π_w, which equals the generalized cost of used routes between OD pair *w*, is known, we can use Eq. (16) to obtain the trip travel time of each OD pair for the DUO dynamic route choice subproblem with perfectly elastic demand, given as follows:

$$\eta_w(k)=\psi_{wk}^{-1}(\pi_w)=\begin{cases}[\pi_w-\beta(k_w^*-\Delta_w-k\delta)]/(\alpha-\beta), & \text{if } \pi_w<\alpha(k_w^*-\Delta_w-k\delta),\\ [\pi_w-\gamma(k\delta-k_w^*-\Delta_w)]/(\alpha+\gamma), & \text{if } \pi_w>\alpha(k_w^*+\Delta_w-k\delta),\\ \pi_w/\alpha, & \text{otherwise.}\end{cases} \quad (17)$$

Then, the input to VI problem (15), $\eta_w(k)$, can be written as $\psi_{wk}^{-1}(\pi_w)$, and we can define the dynamic route choice subproblem as a problem of determining a route flow vector to satisfy the DUO route choice and flow non-negativity conditions given a generalized OD travel cost vector. Moreover, based on Eq. (17), the equivalency between the DUO route choice and SRDTC problems with perfectly elastic demand can be established, as presented in the next subsection.

3.1.3. The equivalency between the DUO route choice and SRDTC problems with perfectly elastic demand

Theorem 1. For a given vector of generalized OD travel costs $\boldsymbol{\pi}=[\pi_w]$, let $\eta_w(k)=\psi_{wk}^{-1}(\pi_w)$ for all $w\in W$, $k\in K_d$, if $0<\beta<\alpha$, and $0<\gamma$, then VI problem (15) is equivalent to VI problem (9).

The proof is presented in Appendix C.

According to Theorem 1, any optimal route flow solution to VI problem (15) is also optimal to VI problem (9) when $\eta_w(k)=\psi_{wk}^{-1}(\pi_w)$ for all $w\in W$, $k\in K_d$. Hence, we can obtain an optimal route flow solution to VI problem (9) by setting $\eta_w(k)=\psi_{wk}^{-1}(\pi_w)$ and solving the DUO route choice problem with perfectly elastic demand, i.e., VI problem (15).

3.2. QP subproblem

In VI problem (9), the vector of the generalized OD travel costs $\boldsymbol{\pi}$ is a model input. For any given $\boldsymbol{\pi}$, let $\Pi(\boldsymbol{\pi})$ be the set of optimal route flow solutions to VI problem (9), where the formulation of the set depends on how the DNL problem is modeled. Some examples of $\Pi(\boldsymbol{\pi})$ will be discussed later in this subsection.

For any $\mathbf{f}\in\Pi(\boldsymbol{\pi})$, we can obtain the corresponding OD demand vector $\mathbf{Q}$ according to Eq. (3).

If the demand function $\mathbf{D}(\boldsymbol{\pi})$ is adopted as the target OD demand vector, then the following optimization problem can be used to retrieve the OD demand vector from the generalized OD travel cost vector, which aims to choose the closest OD demand vector to the target vector:

$$\min_{\mathbf{f},\mathbf{Q}} (\mathbf{Q} - \mathbf{D}(\boldsymbol{\pi}))^{\mathrm{T}} (\mathbf{Q} - \mathbf{D}(\boldsymbol{\pi})) . \tag{18}$$

Subject to $\mathbf{f} \in \Pi(\boldsymbol{\pi})$, and constraint (3),

where the demand function $\mathbf{D}(\boldsymbol{\pi})$ is an input to the DUO-SRDTC problem considered and hence is given.

The following three assumptions are used:

Assumption 1: Route travel times are continuous with respect to route flows.

Assumption 2: The set $\Pi(\boldsymbol{\pi})$ is a convex polytope.

Assumption 2A: The VI problem (15) has a unique optimal link travel time vector.

The path travel time and cost obtained from both the linear travel time model (LTTM) (see Ban et al., 2008 and Long et al., 2013 for details) and the point queue model (PQM) (see Nie and Zhang, 2005; Ban et al., 2012a; Han et al., 2013a for details) are continuous with respect to path flows (Szeto and Lo, 2006; Han et al., 2013c). Hence, Assumption 1 is satisfied by the two models. However, Assumption 1 may not be true when queue spillback is considered in the DNL models (e.g., Daganzo, 1998; Nie, 2010). Assumption 2 or 2A is introduced to ensure that subproblem (18) has a unique optimal OD demand vector and can be solved quickly. Based on the above assumptions, we have the following propositions:

Proposition 1: Under Assumption 1, VI problems (6), (7), and (9) guarantee solution existence.

Proof. According to Huang and Lam (2002) and Szeto and Lo (2004), if route travel times are continuous with respect to route flows, then route travel cost functions are continuous with respect to route flows. Moreover, for the fixed-demand case, the route flow is bounded by the corresponding OD demand. For the cases of elastic demand and perfectly elastic demand, the route flow is bounded by $\bar{F}$. In all three cases, the solution set is convex. Hence, the solution set is a compact convex set. This, together with the continuity of the route flow, implies that VI problems (6), (7), and (9) have at least one optimal solution. This completes the proof. □

Proposition 2: Under Assumptions 1 and 2, an optimal solution exists to optimization problem (18). Moreover, the problem has a unique optimal OD demand vector $\mathbf{Q}^*$.

Proof. According to Proposition 1, VI problem (9) guarantees solution existence under Assumption 1, and hence the set $\Pi(\boldsymbol{\pi})$ is nonempty. The solution set to optimization problem (18) is hence nonempty. The set $\Pi(\boldsymbol{\pi})$ is bounded because the route flows are bounded by $\bar{F}$, and the demand vector is bounded due to the flow conservation condition. The objective function is continuous. By Weierstrass' theorem, an optimal solution exists to the problem. Moreover, under Assumption 2, optimization problem (18) is a convex QP problem; hence the solution set is convex with respect to $\mathbf{Q}$. The objective function of optimization problem (18) is quadratic and strictly convex with respect to $\mathbf{Q}$. Therefore, optimization problem (18) has a unique optimal OD demand vector $\mathbf{Q}^*$. □

In Appendix E, we showed that the set $\Pi(\boldsymbol{\pi})$ can be formulated by a group of linear constraints under Assumption 2A. Hence, we have the following two propositions:

Proposition 3: Under Assumption 2A, if the LTTM is adopted as the traffic flow model, the set $\Pi(\boldsymbol{\pi})$ is a convex polytope.

Proposition 4: Under Assumption 2A, if the PQM is adopted as the traffic flow model, the set $\Pi(\boldsymbol{\pi})$ is a convex polytope.

Propositions 3 and 4 imply that Assumption 2A leads to Assumption 2 when either the LTTM or the PQM is adopted as the DNL model. This implies that Assumption 2A is an alternative for Assumption 2. Note that Assumption 2 may be satisfied by other DNL models that *do not* satisfy Assumption 2A.

The optimal path flow patterns and OD demand patterns may not be unique for a given generalized OD travel cost vector. Hence, we develop convex QP subproblem (18) under Assumption 2 or 2A to retrieve an OD demand pattern that is the closest to the target OD demand. The convex QP subproblem does not have a unique path flow pattern. However, the optimal OD demand solution to the convex QP subproblem is unique, i.e., the retrieved OD demand vector is unique, under Assumption 1. According to Propositions 2 through 4, if Assumption 2A is satisfied, then the solution set is convex and QP subproblems (66) and (71) in Appendix E can be used to retrieve a *unique* OD demand pattern from the generalized OD travel cost vector $\boldsymbol{\pi}$ by adopting the LTTM and the PQM, respectively. However, a unique link travel time vector does not always lead to a unique link flow pattern. For example, in the PQM, under free-flow traffic conditions, the link travel time equals the free-flow travel time for a range of link flows. Hence, the free-flow travel time of a link maps to multiple link flow levels.

Neither the LTTM nor the PQM can capture important realistic traffic dynamics such as queue spillbacks that often occur in congested urban traffic networks. Queue spillbacks may introduce discontinuities into the proposed model, and Assumptions 1, 2, and 2A may not be satisfied. In addition, if queue spillbacks occur, the network may not be able to accommodate the predefined demand, which would lead to infeasibility. Such issues do not exist in the DUO-SRDTC models that use traffic flow models or DNL models without consideration of physical queues, e.g., the LTTM and the PQM. Therefore, the proposed approach is more suitable for DUO-SRDTC problems without such consideration. In Appendix E.4, we integrate the LTM (Yperman, 2007) into the proposed model. The LTM is developed on the basis of the simplified theory of traffic flow proposed by Newell (1993) and can capture realistic traffic dynamics such as queue spillbacks. The results in Section 5 show that our method can still be used to solve the problem with such consideration.

Let $\mathbf{S}(\boldsymbol{\pi})$ be an optimal OD demand vector of the optimization problem (18) or *a vector of OD demands retrieved from the vector of generalized OD travel costs*. We refer to $\mathbf{S}(\boldsymbol{\pi})$ as a ***retrieved OD demand function***. Proposition 2 implies that $\mathbf{S}(\boldsymbol{\pi})$ is an injective function of the generalized OD travel cost vector $\boldsymbol{\pi}$. According to the definition of the set $\Pi(\boldsymbol{\pi})$, any optimal solution to optimization problem (18) is an optimal solution to VI problem (9). Optimization problem (18) may not guarantee a unique optimal solution because of multiple path flow solutions, but it guarantees to have a unique optimal OD demand vector under Assumptions 1 and 2, according to Proposition 2. Let $(\mathbf{Q}^*(\boldsymbol{\pi}),\ \mathbf{f}^*(\boldsymbol{\pi}))$ be an optimal solution to optimization problem (18). Then, by the definition of retrieved OD demand function, we have $\mathbf{S}(\boldsymbol{\pi}) = \mathbf{Q}^*(\boldsymbol{\pi})$.

3.3. Reformulation of DUO-SRDTC problems

The retrieved OD demand vector and the OD demand vector defined by the demand function for the studied problem are both functions of the vector of generalized OD travel costs $\boldsymbol{\pi}$. When the retrieved OD demand function $\mathbf{S}(\boldsymbol{\pi})$ equals the demand function $\mathbf{D}(\boldsymbol{\pi})$, i.e., $\mathbf{S}(\boldsymbol{\pi}) = \mathbf{D}(\boldsymbol{\pi})$, we can solve for the generalized OD travel cost for the studied DUO-SRDTC problem. Note that the value of the retrieved OD demand function $\mathbf{S}(\boldsymbol{\pi})$ for a given vector $\boldsymbol{\pi}$ is defined by the DUO-SRDTC problem with perfectly elastic demand (9) (or equivalently, dynamic route choice subproblem (15)) and QP subproblem (18), and the demand function $\mathbf{D}(\boldsymbol{\pi})$ is an input to the DUO-SRDTC problem considered and hence is given.

Based on the preceding discussion, an SRDTC problem can be formulated as a system of nonlinear equations as follows:

$$\mathbf{Z}(\boldsymbol{\pi}) = \mathbf{S}(\boldsymbol{\pi}) - \mathbf{D}(\boldsymbol{\pi}) = \mathbf{0}\ , \tag{19}$$

where $\mathbf{Z}(\boldsymbol{\pi})$ denotes the mapping function of the system of equations and $\mathbf{S}(\boldsymbol{\pi})$ is defined by two subproblems depicted in subsections 3.1 and 3.2 (i.e., problems (15) and (18)). Note that the detailed formulation for $\Pi(\boldsymbol{\pi})$ in mathematical programming problem (18) depends on the underlying traffic flow models, and problem (18) only defines a *unique* way to obtain an OD demand pattern and compute the value of the mapping function $\mathbf{Z}(\boldsymbol{\pi})$ for all traffic flow models adopted.

Theorem 2. Let $\boldsymbol{\pi}^* = [\pi_w^*]$ be an optimal solution to the system of nonlinear equations (19) and $(\mathbf{Q}^*(\boldsymbol{\pi}^*),\ \mathbf{f}^*(\boldsymbol{\pi}^*))$ be an optimal solution to QP subproblem (18) with $\mathbf{f}^*(\boldsymbol{\pi}^*) = \mathbf{f}^* = [f_p^*(k)]$, and $\mathbf{Q}^*(\boldsymbol{\pi}^*) = \mathbf{Q}^* = [Q_w^*]$. Then, $\mathbf{f}^*$ is also an optimal solution to VI problem (6).

Proof. Because $(\mathbf{Q}^*(\boldsymbol{\pi}^*),\ \mathbf{f}^*(\boldsymbol{\pi}^*))$ must be feasible to the QP subproblem, $\mathbf{f}^*$ and $\mathbf{Q}^*$ satisfy flow conservation condition (3) . Moreover, because $\mathbf{f}^* \in \Pi(\boldsymbol{\pi})$, $\mathbf{f}^*$ is also an optimal solution to VI subproblem (9). Hence, $\mathbf{f}^*$ must satisfy non-negativity condition (4). Because an optimal solution to VI subproblem (9) must satisfy DUO-SRDTC conditions (1) and (2), we have

$$f_p^*(k)[c_p^*(k) - \pi_w^*] = 0, \forall w \in W\ , p \in P_w, k \in K_d\text{, and} \tag{20}$$

$$c^*_p(k) - \pi^*_w \geq 0, \forall w \in W, p \in P_w, k \in K_d. \tag{21}$$

By definition, we have $\mathbf{S}(\boldsymbol{\pi}^*) = \mathbf{Q}^*$. Because $\boldsymbol{\pi}^*$ is a solution to the system of nonlinear equation (19), we have

$$\mathbf{D}(\boldsymbol{\pi}^*) = \mathbf{S}(\boldsymbol{\pi}^*) = \mathbf{Q}^*. \tag{22}$$

Hence, we have $\boldsymbol{\pi}^* = \mathbf{D}^{-1}(\mathbf{Q}^*)$. This implies that Eq. (5) is satisfied. In summary, $\boldsymbol{\pi}^*$ and $\mathbf{f}^*$ satisfy conditions (1) through (5), and therefore $\mathbf{f}^*$ is a solution to VI problem (6). □

For the fixed-demand case, $\mathbf{D}(\boldsymbol{\pi}^*) = \overline{\mathbf{Q}}$. The mapping function of the system of nonlinear equations (19) then becomes $\mathbf{S}(\boldsymbol{\pi}) - \overline{\mathbf{Q}}$, and the problem becomes determination of the optimal value of $\boldsymbol{\pi}^*$ such that $\mathbf{S}(\boldsymbol{\pi}^*) = \overline{\mathbf{Q}}$. We have the following theorem:

Theorem 3. Let $\boldsymbol{\pi}^* = [\pi^*_w]$ be an optimal solution to the system of nonlinear equations (19), $(\mathbf{Q}^*(\boldsymbol{\pi}^*), \mathbf{f}^*(\boldsymbol{\pi}^*))$ be an optimal solution to QP subproblem (18) with $\mathbf{f}^*(\boldsymbol{\pi}^*) = \mathbf{f}^* = [f^*_p(k)]$, and $\mathbf{Q}^*(\boldsymbol{\pi}^*) = \mathbf{Q}^* = [Q^*_w]$. If the gradient of the demand function $\mathbf{D}(\boldsymbol{\pi})$ equals zero or equivalently $\mathbf{D}(\boldsymbol{\pi}) = \overline{\mathbf{Q}}$, then $\mathbf{f}^*$ is also an optimal solution to VI problem (7).

Proof. Similar to the proof of Theorem 2, $\mathbf{f}^*$ must satisfy (1) through (4) and (19), and we have $\mathbf{S}(\boldsymbol{\pi}^*) = \mathbf{Q}^*$. Therefore, we have $\mathbf{D}(\boldsymbol{\pi}^*) = \mathbf{S}(\boldsymbol{\pi}^*) = \mathbf{Q}^*$ and $\sum_{p \in P_w} \sum_{k \in K_d} f^*_p(k)\delta = Q^*_w$. Moreover, because the demand is fixed, $\mathbf{D}(\boldsymbol{\pi}^*) = \overline{\mathbf{Q}}$, and hence we have

$$\mathbf{Q}^* = \mathbf{S}(\boldsymbol{\pi}^*) = \mathbf{D}(\boldsymbol{\pi}^*) = \overline{\mathbf{Q}}, \text{ and} \tag{23}$$

$$\sum_{p \in P_w} \sum_{k \in K_d} f^*_p(k)\delta = Q^*_w = \overline{Q}_w, \forall w \in W. \tag{24}$$

Because of $\mathbf{f}^* \geq \mathbf{0}$ and Eq. (24), we have $\mathbf{f}^* \in \Omega$. Therefore, $\mathbf{f}^*$ is also a solution to VI problem (7). □

Theorems 2 and 3 imply that the proposed system of nonlinear equations (19) can be used to formulate the DUO-SRDTC problem with elastic demand or with fixed demand and that a solution to the system of nonlinear equations (19) can be used to deduce an optimal route flow pattern to DUO-SRDTC problems by solving VI subproblem (9) and QP subproblem (18) to evaluate the mapping function $\mathbf{S}(\boldsymbol{\pi}) - \mathbf{D}(\boldsymbol{\pi})$. For the fixed-demand case, i.e., $\mathbf{D}(\boldsymbol{\pi}) = \overline{\mathbf{Q}}$, an optimal solution to QP subproblem (18) equals the demand $\overline{\mathbf{Q}}$ only when $\boldsymbol{\pi}$ is a solution to the system of nonlinear equations (19), where the QP subproblem aims to determine the closest OD demand vector to the target OD demand $\overline{\mathbf{Q}}$. If $\boldsymbol{\pi}$ is not a solution to the system of nonlinear equations (19), the optimal solution to the QP subproblem $\mathbf{Q}^*(\boldsymbol{\pi})$ will not equal the demand $\overline{\mathbf{Q}}$.

In addition to the VI problem, a DUO-SRDTC problem can be formulated as the fixed-point (FP) problem (e.g., Szeto et al., 2011; Han et al., 2015) and the NCP (e.g., Wie et al., 2002; Ban et al., 2008), all of which can be reformulated into or viewed as the system of nonlinear equations. However, our formulation is different from a typical FP or NCP formulation because VI subproblem (9) and QP subproblem (18) are implicated in the mapping function of $\mathbf{S}(\boldsymbol{\pi})$ in our formulation. There is no analytical form for $\mathbf{S}(\boldsymbol{\pi})$. Solving these two subproblems is required to evaluate $\mathbf{S}(\boldsymbol{\pi})$ in the mapping function of the system of nonlinear equations (19).

As shown below in our numerical example, the DUO-SRDTC problem with perfectly elastic demand (9) is more difficult to solve than the equivalent DUO route choice problem with perfectly elastic demand (15). Therefore, we solve SRDTC problems by instead solving the corresponding dynamic route choice subproblem (15). Although the route choice problem is still path-based, methods have been developed to solve these types of problems efficiently. For example, INDY (e.g., Bliemer et al., 2004) and the DTA model Streamline (e.g., Raadsen et al., 2010) use path-set generation on large networks (with more than 1000 zones) with up to 1 million routes and can be handled by a regular personal computer. Some smarter ways of using route sets, such as the concept of subpaths, have also been developed to improve the efficiency of using computing resources (e.g., Chabini, 2001; Raadsen et al., 2010).

Formulation (19) requires few decision variables compared with the formulation of DUO-SRDTC problems as classical FP or NCP problems. If a DUO-SRDTC problem is directly formulated as an FP or NCP problem, the dimension is $\underline{K}_d \times |P|$, and the direct solution of the resultant problem can be very inefficient, as shown in the numerical example below. However, the dimension of the proposed system of nonlinear equations (19) equals the number of OD pairs, which is far smaller than that of direct formulations. More importantly, our solution approach divides a DUO-SRDTC problem into smaller problems, each of which can be handled efficiently. This approach allows us to solve larger DUO-SRDTC problems more quickly than with the classical methods.

4. Solution algorithms

The BFGS method is proposed to solve the system of nonlinear equations (19) to obtain the generalized OD travel cost. During each iteration of the BFGS method, the mapping function $\mathbf{Z}(\boldsymbol{\pi})$ is evaluated, in which the given vector of generalized OD travel costs $\boldsymbol{\pi}$ is converted to a vector of trip travel time $\boldsymbol{\eta} = (\eta_w(k), \forall w \in W, k \in K_d)$ by Eq. (17); the value of the retrieved OD demand function $\mathbf{S}(\boldsymbol{\pi})$ at $\boldsymbol{\pi}$ can then be obtained by two steps: 1) solve VI subproblem (15) with $\eta_w(k) = \psi_{wk}{}^{-1}(\pi_w)$ to obtain the set of used paths and 2) solve QP subproblem (18) to obtain the demand vector closest to the target demand function vector $\mathbf{D}(\boldsymbol{\pi})$ using the output of the previous step. VI subproblem (15) can be solved by any general computational techniques developed for VI problems or FP problems, such as the method of successive averages (e.g., Tong and Wong, 2000), the day-to-day swapping method (e.g., Szeto and Lo, 2006; Mounce and Carey, 2011), the projection method (e.g., Lo and Szeto, 2002; Szeto and Lo, 2004), and FP algorithms (e.g., Han et al., 2015), provided that the convergence requirement is met. We extend the extragradient method of Khobotov (1987) to solve VI subproblem (15), although other methods with even weaker conditions could be applied. Moreover, the Frank-Wolfe algorithm is adopted to solve the QP subproblem, although active set methods and gradient projection methods can also be used to solve this problem. Therefore, the BFGS method forms the main solution algorithm, and the extragradient method and the Frank-Wolfe method form two subalgorithms. The Frank-Wolfe algorithm is well known, so the details are not provided in this section. Instead, we provide the details of the BFGS and extragradient methods. First, however, some gap functions are provided to evaluate their convergence.

4.1. Gap functions

Gap functions are developed to provide a convenient measure of convergence for solution algorithms. The following two gap functions are adopted to evaluate the quality of the computed solutions to the DUO route choice problem with perfectly elastic demand and the DUO-SRDTC problem, respectively:

$$G_1(\mathbf{f}) = \max_{w \in W,\, p \in P_w,\, k \in K_d} \{\delta_p(k) \,|\, t_p(k) - \eta_w(k) \,|\}, \text{ and} \tag{25}$$

$$G_2(\mathbf{f}) = \max_{w \in W,\, p \in P_w,\, k \in K_d} \{\delta_p(k) \,|\, c_p(k) - \pi_w \,|\}, \tag{26}$$

where $\delta_p(k)$ denotes an indicator variable for route *p*; $\delta_p(k) = 1$ if $f_p(k) > 0$; otherwise, $\delta_p(k) = 0$. Eq. (25) gives the largest difference between the travel times on all used routes and the corresponding minimum route travel times for the DUO route choice problem with perfectly elastic demand. In Eq. (25), the minimum route travel time $\eta_w(k)$ is an input to the DUO route choice problem, and its value is predetermined by Eq. (17). According to DUO route choice conditions (13) and (14), gap function (25) is equal to zero if the DUO route choice conditions hold. Otherwise, the gap value is positive. Eq. (26) gives the largest difference between the route travel costs associated with all used routes and the corresponding minimum route travel costs for the DUO-SRDTC problem considered. If the fixed OD demand is considered, then we have $\pi_w = \min\{c_p(k) : p \in P_w, k \in K_d\}$. If the elastic OD demand is considered, then Eq. (5) is used to determine the generalized OD travel cost π_w. If the perfectly elastic OD demand is considered, then the generalized OD travel cost π_w is an input to the problem, and its value is predetermined. According to DUO-SRDTC conditions (1) and (2), gap function (26) is equal to zero if a route flow solution satisfies the DUO-SRDTC conditions. Otherwise, the gap function gives a positive value.

The following gap function is used to evaluate the convergence of the BFGS method for the proposed system of nonlinear equations:

$$G_3(\boldsymbol{\pi}) = \left\|\mathbf{Z}(\boldsymbol{\pi})\right\| / \left\|\mathbf{D}(\boldsymbol{\pi})\right\|. \tag{27}$$

If $\boldsymbol{\pi}$ is a solution to nonlinear equation (19), then gap function (27) is equal to zero and $\boldsymbol{\pi}$ is the equilibrium generalized OD travel cost. Otherwise, the gap value is positive.

To compare the solution quality of the existing solution methods for solution of the DUO-SRDTC problem with fixed demand, the following gap function is adopted (Huang and Lam, 2002):

$$G_4(\mathbf{f}) = \frac{\sum_{w\in W}\sum_{p\in P_w}\sum_{k\in K_d} f_p(k)\delta\left[c_p(k)-\pi_w\right]}{\sum_{w\in W}\bar{Q}_w\pi_w}. \tag{28}$$

If the DUO conditions of the SRDTC problem with fixed demand are satisfied, then the numerator on the right-hand side of Eq. (28) is equal to zero, and hence the gap function is also equal to zero. The denominator of the right-hand side of Eq. (28) is the total system travel cost, and gap function (28) is used to determine the relative gap of the current solution.

We also adopt the following gap function for the DUO-SRDTC problems with either fixed or elastic demand (Szeto and Lo, 2004):

$$G_5(\mathbf{f}) = \left\|\mathbf{f} - \mathrm{Proj}_\Phi(\mathbf{f} - \phi\mathbf{F}(\mathbf{f}))\right\|, \tag{29}$$

where ϕ is a positive constant, Φ is the feasible solution set of the studied DUO-SRDTC problem, and $\mathrm{Proj}_\Phi(\mathbf{f} - \phi\mathbf{F}(\mathbf{f}))$ is the Euclidean projection map of vector $\mathbf{f} - \phi\mathbf{F}(\mathbf{f})$ onto Φ, which can be computed by a linear projection method (see Panicucci et al., 2007 for details). If the fixed demand is considered, then $\Phi = \Omega$ and $\mathbf{F}(\mathbf{f}) = \mathbf{F}_1(\mathbf{f})$; otherwise, $\Phi = \Psi$ and $\mathbf{F}(\mathbf{f}) = \mathbf{F}_2(\mathbf{f})$. Gap function (29) measures the proximity between $\mathrm{Proj}_\Phi(\mathbf{f} - \phi\mathbf{F}(\mathbf{f}))$ and $\mathbf{f}$. Under the DUO conditions, we have $\mathbf{f} = \mathrm{Proj}_\Phi(\mathbf{f} - \phi\mathbf{F}(\mathbf{f}))$, and the residual error $\left\|\mathbf{f} - \mathrm{Proj}_\Phi(\mathbf{f} - \phi\mathbf{F}(\mathbf{f}))\right\|$ is equal to zero. Note that if the fixed demand is considered, then the route flow solution deduced from a computed solution to the system of nonlinear equations (19) may not satisfy Eq. (3), i.e., the sum of the route flows departing at any time from an origin heading to a destination is not equal to the corresponding OD demand. This is the case for two reasons. First, Eq. (3) is ignored in the DUO route choice problem with perfectly elastic demand. Second, the solution to the system of nonlinear equations obtained is not optimal but only nearly optimal. (Note that the gap is small but not equal to zero.) To ensure that the flow conservation condition is satisfied, we use $\mathrm{Proj}_\Omega(\mathbf{f})$ to replace the obtained route flow vector $\mathbf{f}$ when calculating the gap value for the case of fixed demand, because $\mathrm{Proj}_\Omega(\mathbf{f})$ satisfies Eq. (3) (i.e., $\mathrm{Proj}_\Omega(\mathbf{f}) \in \Omega$) by definition. For the elastic-demand case, flow conservation is always ensured. Hence, no extra projection for the route flows is needed.

4.2. The main solution algorithm: A BFGS method for solving DUO-SRDTC problems

The BFGS method for solving the system of nonlinear equations (19) is outlined as follows:

Step 0. *Initialization*. Select an initial solution $\boldsymbol{\pi}_0$, an initial symmetric positive definite matrix $\mathbf{H}_0$, the constants $\mu, \rho \in (0,1)$, and the convergence tolerance $\varepsilon_1 > 0$, and let $\iota = 0$.

Step 1. *Convergence criterion*. If $G_3(\boldsymbol{\pi}_\iota) < \varepsilon_1$, then terminate the algorithm; otherwise, obtain the descent direction $\mathbf{d}_\iota = -\mathbf{H}_\iota \mathbf{Z}(\boldsymbol{\pi}_\iota)$.

Step 2. *Stepsize determination*. Let i be the smallest non-negative integer such that

$$\left\|\mathbf{Z}(\boldsymbol{\pi}_\iota + \rho^i \mathbf{d}_\iota)\right\|^2 \le \left\|\mathbf{Z}(\boldsymbol{\pi}_\iota)\right\|^2 + \mu\rho^i \mathbf{Z}(\boldsymbol{\pi}_\iota)^{\mathrm{T}} \mathbf{d}_\iota . \tag{30}$$

Set the stepsize $\lambda_\iota = \rho^i$.

Step 3. *Generalized OD travel cost update*. Set $\boldsymbol{\pi}_{\iota+1} = \boldsymbol{\pi}_\iota + \lambda_\iota \mathbf{d}_\iota$.

Step 4. *Symmetric positive definite matrix update*. Set $\mathbf{s}_\iota = \boldsymbol{\pi}_{\iota+1} - \boldsymbol{\pi}_\iota = \lambda_\iota \mathbf{d}_\iota$, and $\mathbf{y}_\iota = \mathbf{Z}(\boldsymbol{\pi}_{\iota+1}) - \mathbf{Z}(\boldsymbol{\pi}_\iota)$. If $\mathbf{s}_\iota^T \mathbf{y}_\iota > 0$, then set

$$\mathbf{H}_{\iota+1} = \mathbf{H}_\iota + \frac{(\mathbf{s}_\iota^T \mathbf{y}_\iota + \mathbf{y}_\iota^T \mathbf{H}_\iota \mathbf{y}_\iota)(\mathbf{s}_\iota \mathbf{s}_\iota^T)}{(\mathbf{s}_\iota^T \mathbf{y}_\iota)^2} - \frac{\mathbf{H}_\iota \mathbf{y}_\iota \mathbf{s}_\iota^T + \mathbf{s}_\iota \mathbf{y}_\iota^T \mathbf{H}_\iota}{\mathbf{s}_\iota^T \mathbf{y}_\iota} ; \tag{31}$$

otherwise, set $\mathbf{H}_{\iota+1} = \mathbf{H}_\iota$. Set $\iota = \iota + 1$, and go to Step 1.

To solve the DUO-SRDTC problem with *fixed* or *elastic* demand, the BFGS method solves many DUO-SRDTC problems with perfectly elastic demand as subproblems. In each subproblem, the vector of generalized OD travel cost $\boldsymbol{\pi}_\iota$ is fixed as a priori. In the overall solution procedure, the vector of the generalized OD travel cost is updated by Step 3 of the BFGS method but is not obtained from an arbitrary flow vector, such as a zero vector or a vector with very small component values. The initial vector of $\boldsymbol{\pi}_\iota$ is set as $\tilde{\boldsymbol{\pi}} = (\tilde{\pi}_w, \forall w \in W)$ in Step 0, where $\tilde{\pi}_w$ is the generalized OD travel cost for travelers that depart during the first time interval with free-flow travel time. This choice can ensure that the value of π_w would not be smaller than $\tilde{\pi}_w$ throughout the solution process, ensuring that there must be a feasible flow pattern to achieve this value.

Following most of the literature, we set the initial symmetric positive definite matrix $\mathbf{H}_0$ to be an identity matrix. In Step 2, a backtracking inexact line search technique, which was originally proposed by Yuan and Lu (2008), was adopted to determine the step size. In contrast to some traditional line search techniques, technique (30) can avoid the computation of the Jacobian matrix of $\mathbf{Z}(\boldsymbol{\pi})$ and thus reduce the computation difficulty, especially for large-scale problems.

In Steps 1, 2, and 4 of each iteration, DUO route choice subproblem (15) and QP subproblem (18) are respectively solved by the extragradient method and the Frank-Wolfe algorithm to evaluate $\mathbf{S}(\boldsymbol{\pi})$ and consequently $\mathbf{Z}(\boldsymbol{\pi})$. Note that based on Theorems 2 and 3, the objective value of QP subproblem (18) is zero when $\boldsymbol{\pi}$ is an optimal solution to the system of nonlinear equation (19). Note also that in practical terms, it is very difficult to absolutely achieve a gap G_3 of zero. Usually, the solution algorithm is terminated when G_3 is sufficiently small.

The BFGS method has a norm descent property, and its global and superlinear convergence has been proved under mild conditions (Yuan and Lu, 2008):

- $\mathbf{Z}(\boldsymbol{\pi})$ is continuously differentiable on an open convex set containing the level set defined by $\Omega = \{\boldsymbol{\pi}_\iota \mid \|\mathbf{Z}(\boldsymbol{\pi}_\iota)\| \le \|\mathbf{Z}(\boldsymbol{\pi}_0)\|\}$. The Jacobian of $\mathbf{Z}(\boldsymbol{\pi})$ is symmetric, bounded, and positive definite on the convex set.
- $\mathbf{H}_\iota$ is a good approximation of $\mathbf{Z}'(\boldsymbol{\pi})$.
- $\mathbf{Z}'(\boldsymbol{\pi}_\iota)$ is Hölder continuous at an optimal solution.

However, the Jacobian matrix of $\mathbf{Z}(\boldsymbol{\pi})$ is, in general, asymmetric, and hence the global and superlinear convergence of the above BFGS method for the DUO-SRDTC problem should be further investigated. Therefore, when the above mild conditions do not hold, the BFGS algorithm is a heuristic for the DUO-SRDTC problems and may be unable to obtain a solution to the proposed system of equations or may not achieve convergence. In such a case, we may be required to use the algorithm repeatedly with different initial solutions and use a gap function to determine whether the heuristic stops and yields a solution sufficiently close to a solution to the proposed system. However, as shown in the numerical results, the proposed algorithm indeed gives better results than other existing methods if a solution of the proposed system is obtained.

Note that the proposed solution algorithm cannot be reduced to a regular extragradient method in the fixed-demand case. In every iteration of the BFGS method, two subproblems, i.e., the DUO route choice subproblem and the QP subproblem, must still be solved, because we do not know the optimal generalized OD travel costs. The extragradient method is only used to solve the route choice subproblem and obtain an optimal route flow vector and hence the used path set, and the QP subproblem is used to retrieve an OD demand pattern that is closest to the target OD demand based on the used path set of the route choice subproblem. We stop the overall algorithm when we can find generalized OD travel costs so that the demand constraint (i.e., the flow conservation condition $\sum_{p \in P_w} \sum_{k \in K_d} f_p(k)\delta = \overline{Q}_w$) is satisfied. In other words, we did not fix the demand vector $\mathbf{Q}$ at the beginning of the BFGS method when solving the fixed-demand case.

4.3. Subalgorithm: An extragradient method for solving the DUO route choice subproblem

The following extragradient method can be adopted to solve VI problem (15):

Step 0. *Initialization*. Let $\mathbf{f}_1$ be any feasible route flow vector. Set the parameters ϑ, $\xi \in (0,1)$, $\overline{\lambda} > 0$, the convergence tolerance $\varepsilon_2 > 0$, and set $\lambda_1 = \overline{\lambda}$ and the iteration index $\iota = 1$.

Step 1. *Convergence criterion*. Terminate the subalgorithm if $G_1(\mathbf{f}_\iota) < \varepsilon_2$.

Step 2. *Route flow update*.

Step 2.1. $\overline{\mathbf{f}}_\iota$ *computation*. Compute:

$$\overline{\mathbf{f}}_\iota = \mathrm{Proj}_\Psi(\mathbf{f}_\iota - \lambda_\iota \mathbf{F}_4(\mathbf{f}_\iota)) .$$

Step 2.2. *Stepsize determination*.

If $\lambda_\iota > \vartheta \dfrac{\|\mathbf{f}_\iota - \bar{\mathbf{f}}_\iota\|}{\|\mathbf{F}_4(\mathbf{f}_\iota) - \mathbf{F}_4(\bar{\mathbf{f}}_\iota)\|}$, then reduce λ_ι using:

$$\lambda_\iota = \min\left\{\xi\lambda_\iota, \vartheta \frac{\|\mathbf{f}_\iota - \bar{\mathbf{f}}_\iota\|}{\|\mathbf{F}_4(\mathbf{f}_\iota) - \mathbf{F}_4(\bar{\mathbf{f}}_\iota)\|}\right\},$$

and return to Step 2.1.

Step 2.3. *Computation of* $\mathbf{f}_{\iota+1}$ and $\lambda_{\iota+1}$. Update the route flow vector by:

$$\mathbf{f}_{\iota+1} = \mathrm{Proj}_\Psi(\mathbf{f}_\iota - \lambda_\iota \mathbf{F}_4(\bar{\mathbf{f}}_\iota)),$$

set $\lambda_{\iota+1} = \min\left\{\bar{\lambda}, \vartheta \dfrac{\|\mathbf{f}_\iota - \bar{\mathbf{f}}_\iota\|}{\|\mathbf{F}_4(\mathbf{f}_\iota) - \mathbf{F}_4(\bar{\mathbf{f}}_\iota)\|}\right\}$, $\iota = \iota + 1$, and return to Step 1.

An all-or-nothing assignment under free-flow conditions was used to determine the feasible route flow vector $\mathbf{f}_1$ in Step 0. In this algorithm, $\mathrm{Proj}_\Psi(\mathbf{f})$ is the Euclidean projection map onto Ψ. $\mathbf{f}_\iota$ and λ_ι are the route flow vector and the step size at iteration ι, respectively. A re-initialization of the step size was used in Step 2.3 to avoid the problem of the reduction of the convergence rate due to excessively small step sizes at some iterations.

This algorithm has been shown to converge to an equilibrium flow under some mild assumptions (Panicucci et al., 2007). It is convergent if the route travel time functions are pseudomonotone and Lipschitz continuous. Another merit of the algorithm is that it is not necessary to know the Lipschitz constant of the path travel time functions. The extragradient method can also be extended to solve VI problems (6), (7), and (9), provided that the convergence requirement is satisfied. If the extragradient method is applied to solve the DUO-SRDTC problems with fixed or elastic demand, we need only redefine the mapping function, replace the feasible solution set by that of the problem, and change the stopping criterion (e.g., $G_2(\mathbf{f}_\iota) < \varepsilon_2$). The extragradient method may be considered a heuristic because the required conditions (such as pseudomonontonicity) may not be satisfied.

The dimension of the system of nonlinear equations (19) equals the number of OD pairs, and hence the memory usage of the BFGS method itself is very limited. The memory usage for solution of VI subproblem (15) and QP subproblem (18) dominate. Therefore, the memory usage of the overall solution procedure for the SRDTC problem basically equals the sum of that of the extragradient method and that of the Frank-Wolfe method.

5. Numerical examples

In this section, six examples were developed to illustrate the performance of the solution algorithm. In Example 1, the LTTM was adopted and the fixed-demand case was considered. The DNL procedure of the LTTM directly follows Ban et al. (2008). In Examples 2 through 5, the PQM was adopted, and both fixed-demand and elastic-demand cases were considered. The DNL procedure of the PQM directly follows Huang and Lam (2002). In Example 6, the LTM was adopted, and the elastic-demand case was considered. The DNL procedure of the LTM directly follows Yperman (2007). All of the experiments were run on a computer with an Intel Core i5-2540 2.60-GHz CPU with 4.00 GB of RAM.

Example 1. Papageorgiou's network with fixed demand: the efficiency and convergence of the proposed solution algorithm

In this example, Papageorgiou's (1990) network was adopted (see Fig. 2), which consists of five nodes, seven links, four OD pairs, and 12 routes. The length of each time interval was set at 1 min. The OD demands were 2500 vehicles for OD pairs (1, 4) and (1, 5) and 3000 vehicles for OD pairs (2, 4) and (2, 5). The LTTM (64) was adopted to implement DNL. The free-flow travel time τ_a^0 (in minutes) and the parameter β_a are given in Fig. 2. The parameters for the generalized travel cost function were α = HK\$1.0/min, β = HK\$0.5/min, γ = HK\$1.0/min, $\Delta_w = 5$ min, and $k_w^* = 35$ min from the start of the modeling horizon for all OD pairs. The values of the parameters for the BFGS method were $\varepsilon_1 = 1.0 \times 10^{-5}$, $\mu = 0.4$, and $\rho = 0.9$. The values of the parameters for the extragradient method were $\varepsilon_2 = 1.0 \times 10^{-8}$ min, $\vartheta = 0.8$, $\xi = 0.9$, and $\overline{\lambda} = 10.0$. The value of the parameter in the gap function $G_5(\mathbf{f})$ was $\phi = 0.01$ (Szeto and Lo, 2004). The Manhattan norm (i.e., 1-norm) was adopted in gap functions $G_3(\mathbf{f})$ and $G_5(\mathbf{f})$.

The proposed approach was adopted to solve the DUO-SRDTC problem with fixed demand in the preceding setting. The values of four gap functions obtained during the implementation of the algorithm are presented in Fig. 3. We can observe that the values of the gap functions quickly decrease as the number of iterations increases.

Traditionally, CPU time is widely used to evaluate the efficiency of solution algorithms. However, CPU time generally relies on factors such as the compiler, coding skills, and the performance of the computer. Because performing DNL is the most time-consuming step in the solution of DTA models, we mainly used the number of DNLs performed instead of the CPU time to evaluate the efficiency of the proposed and existing solution methods. We set the maximum number of DNLs performed at 400,000 and compared the performance of the proposed approach with three existing solution algorithms: the descent method (see Szeto and Lo, 2004 for details), the extragradient method, and the route-swapping method (See Appendix D for details). The first two methods can be used to solve all VI problems provided that the convergence requirement is satisfied, whereas the last method was developed by Huang and Lam (2002) for DUO-SRDTC problems with fixed demand.

Table 1 provides the number of DNLs required for each solution algorithm to achieve a certain value for G_4. The route-swapping method could guarantee very slow convergence and required a CPU time of 12.11 s and DNL 25,977 times to achieve a G_4 value of less than 1.0×10^{-5}, but was unable to achieve a G_4 value of less than 1.0×10^{-6} after performing DNL 400,000 times. The descent method failed to solve the DUO-SRDTC problem. In contrast to the previous two methods, the extragradient method and the proposed approach could successfully solve the DUO-SRDTC problem, and, respectively, required CPU times of 4.79 s and 31.77 s and performed DNL 7906 and 11,239 times to achieve a G_4 value of less than 1.0×10^{-7}. Only one DNL is required in both the route-swapping method and the descent method during each iteration. The average numbers of DNL performed in each iteration were 2.75 for the extragradient method and 803 for the proposed approach. The memory usage was 12.50 MB for the route-swapping method, 12.67 MB for the extragradient method, 12.41 MB for the descent method, and 21.73 MB for the proposed approach. The proposed approach required about twice as much memory as each of the others due to the Frank-Wolfe method for solving the QP subproblem. The extragradient method and the descent method had memory usage requirements that were very similar to those of the route-swapping method, and hence in the following experiments, we report only the memory usage of the route-swapping method and the proposed approach.

Example 2. A simple network with two parallel routes and fixed demand: comparing the effectiveness of the proposed algorithm to existing algorithms

As shown in Fig. 4, this example adopted a sample network with two parallel routes to demonstrate the performance of the proposed solution method. We considered a modeling period from 6 AM to 10 AM and $\delta = 0.01$ h, and thus $\underline{K} = 400$ intervals. All input data were the same as those in Arnott et al. (1990) and Huang and Lam (2002). The route index, free-flow travel time (in hours), and bottleneck exit capacity of each route are also given in Fig. 4. The total trip demand from origin A to destination B was 22,000 vehicles. The parameters for the generalized travel cost function were α = \$6.4/h, β = \$3.9/h, γ = \$15.21/h, $\Delta_B = 0.25$ h, and $k_B^* = 3$ h (i.e., the official work start time was 9 AM). The values of the parameters for the solution algorithms were the same as those in Example 1.

The proposed approach was adopted to solve the DUO-SRDTC problem with fixed demand. The gap values are presented in Fig. 5. We can observe that the values of the gap functions quickly decrease as the number of iterations increases. The values of gap functions G_2, G_3, G_4, and G_5 decreased to \$$4.0 \times 10^{-9}$, 1.0×10^{-6}, 5.0×10^{-10}, and 3.0×10^{-9} vehicles per hour, respectively, after five iterations. These values imply that when the generalized OD travel cost is very near a solution to the proposed system of nonlinear equations, the corresponding route flows are very close to an optimal solution to the DUO-SRDTC problem. We also provide the route inflow rates, queue lengths, and travel costs on the two routes in Fig. 6, which are identical to the corresponding analytical solutions derived by Arnott et al. (1990) and the numerical results of Huang and Lam (2002). We can observe from the figure that the generalized travel costs are exactly equal and minimal for all used routes and chosen departure times and that the minimum travel cost is \$6.1019.

Table 2 provides the number of DNLs required for each solution algorithm to achieve a certain value for G_4. It can be seen that both the descent method and the extragradient method failed to solve the DUO-SRDTC problem. Moreover, the route-swapping method could lead to a very slow convergence and required a CPU time of 110.25 s and performed DNL 279,263 times to achieve a G_4 value of less than 1.0×10^{-5}, but was unable to achieve a G_4 value of less than 1.0×10^{-6} after performing DNL 400,000 times. In contrast to other solution methods, the proposed approach could guarantee very quick convergence and only required a CPU time of 2.32 s, and 9996 DNLs (about 1999 times in each iteration) were required to achieve a G_4 value of less than 1.0×10^{-7}. The memory usage was 8.75 MB for the route-swapping method and 16.93 MB for the proposed approach. Again, the proposed approach required about twice as much memory as the route-swapping method.

Example 3: A simple network with two parallel routes and perfectly elastic demand: Comparing the convergence of the extragradient method in solving the DUO route choice problem and the DUO-SRDTC route and departure time choice problem

We set the generalized OD travel cost at \$6.1019 and compared the performance of the extragradient method for solving the DUO-SRDTC problem with perfectly elastic demand and the DUO route choice problem with perfectly elastic demand using the setting in Example 2. The convergence of the extragradient method for the two problems is presented in Fig. 7. It can be seen that the extragradient method for the route choice problem could converge much more quickly than that for the DUO-SRDTC problem because of the monotonicity of the mapping function of the route choice problem. We also found that the solutions of the two problems agree with the results presented in Fig. 6. This observation is consistent with the results presented in Theorem 1 in the sense that an optimal solution to the DUO-SRDTC problem with perfectly elastic demand is also an optimal solution to the DUO route choice problem with perfectly elastic demand.

Example 4. A grid network with fixed demand: Comparing the effectiveness of the proposed algorithm to existing algorithms.

In this example, a grid network was adopted (see Fig. 8) that consists of nine nodes, 12 links, two OD pairs, and eight routes (Huang and Lam, 2002). All network parameters are the same as those in Huang and Lam (2002). The link index, free-flow travel time (in hours), and bottleneck exit capacity of each link are also given in Fig. 8. The trip demands from A to C and from B to C are 20,000 and 10,000 vehicles, respectively. All other parameters are the same as those used in Example 1.

We used the proposed approach to solve the DUO-SRDTC problem with fixed demand for the grid network scenario; the values of four gap functions obtained during the implementation of the algorithm are presented in Fig. 9. We can observe that the values of the gap functions quickly decrease as the number of iterations increases. The values of gap functions G_2, G_4, and G_5 decreased to

$$1.0 \times 10^{-7}$, 1.0×10^{-8}, and 1.0×10^{-7} vehicles per hour, respectively, after 14 iterations and a CPU time of 55.75 s when the G_3 value was less than 1.0×10^{-8}. These gap values imply that when the generalized OD travel costs are very near an optimal solution to the proposed system of nonlinear equation (19), the corresponding route flows are very near an optimal solution to the DUO-SRDTC problem. Table 3 provides the numbers of DNLs performed for four solution algorithms to achieve a certain value for G_4. The results presented in Table 3 are similar to those in Table 2. The proposed solution algorithm is far more effective than other solution algorithms. However, as in other examples, the proposed approach required the implementation of many DNLs in each iteration (an average of about 2327) and required more memory (12.12 MB for the route-swapping method and 19.59 MB for the proposed approach).

The route inflow rates, queue lengths, and travel costs for each route are depicted graphically in Fig. 10. We can observe that the generalized travel costs are exactly equal and minimal for all used routes and chosen departure times and that the minimum travel costs from A to C and from B to C are $10.2010 and $7.5558, respectively. In Fig. 11, we provide the solution obtained by the route-swapping method, which performed DNL 200,000 times. It can be seen that the equilibrium OD travel costs obtained by the two algorithms are basically the same, whereas the route flow rates obtained by the two algorithms have very similar tendencies but are markedly different. The value of gap function G_4 presented in Table 3 implies that the solution to the DUO-SRDTC problem presented in Fig. 10 is much closer to an optimal solution to the problem than that in Fig. 11.

Example 5. The grid network with elastic demand: the efficiency and convergence of the proposed solution algorithm

In this example, the grid network in Example 4 was also adopted to demonstrate that the proposed solution algorithm is also effective for the DUO-SRDTC problem with elastic demand. The following linear demand function was adopted in this example:

$$\mathbf{D}(\boldsymbol{\pi}) = \mathbf{Q}_0 + \chi(\boldsymbol{\pi}_0 - \boldsymbol{\pi}), \tag{32}$$

where $\mathbf{Q}_0$ and $\boldsymbol{\pi}_0$ are the vector of reference demands and the vector of reference OD travel costs, respectively, and χ is a positive parameter that reflects the sensitivity of travel demand to travel cost. When χ approaches zero, the demand approaches $\mathbf{Q}_0$, and when χ tends to infinity, the demand tends to become perfectly elastic. In this example, we set $\mathbf{Q}_0 = [20000, 10000]^{\mathrm{T}}$ (vehicles) and $\boldsymbol{\pi}_0 = [10.2010,\ 7.5558]^{\mathrm{T}}$ ($), which equal the total OD demand and the equilibrium OD travel cost in Example 4, respectively. All other parameters were the same as those used in Example 4, except the parameter $\mu = 0.1$ in the BFGS method.

We set different values for the parameter χ and tested the convergence of the proposed solution algorithm. We used the proposed algorithm to obtain a solution to the DUO-SRDTC problem with elastic demand, and the convergence of the algorithm is demonstrated in Fig. 12. It can be observed that the values of both gap functions quickly converge to the corresponding convergence tolerances when the number of iterations increases. The results presented in Fig. 12 also show that larger the value of χ, the faster the BFGS method obtains an acceptable solution. This implies that the DUO-SRDTC problem with elastic demand can be solved more easily when the demand functions are more elastic. We also compared the performance of the proposed approach with the extragradient method for solving the DUO-SRDTC problem with elastic demand. Table 4 provides the number of DNLs required to achieve different convergence tolerances (compared with gap function $G_5(\mathbf{f})$) by the two solution algorithms. We can observe that the extragradient method does not always guarantee convergence, and the larger the value of the parameter χ, the fewer the number of DNLs required for this method to achieve a certain convergence criterion. The results presented in Table 4 also show that the proposed approach could guarantee convergence in all of our studied cases and that the convergence speed could be much faster than that of the extragradient method. We also tested the performance of the descent method (Szeto and Lo, 2004). We found that the descent method failed to solve the DUO-SRDTC problem with elastic demand in the cases considered.

Because $\boldsymbol{\pi}_0$ is the solution to the proposed system of nonlinear equations associated with fixed demand, we have $\mathbf{Q}_0 = \mathbf{S}(\boldsymbol{\pi}_0)$. Substituting $\boldsymbol{\pi} = \boldsymbol{\pi}_0$ into Eq. (32), we have $\mathbf{D}(\boldsymbol{\pi}_0) = \mathbf{Q}_0$. Therefore, $\mathbf{Z}(\boldsymbol{\pi}_0) = \mathbf{S}(\boldsymbol{\pi}_0) - \mathbf{D}(\boldsymbol{\pi}_0) = \mathbf{0}$ is satisfied, and hence $\boldsymbol{\pi}_0$ is a solution to the system of nonlinear equations (19), and the corresponding route flow vector is an optimal solution to the DUO-SRDTC problem with elastic demand. Our numerical results confirm these analytical results. We found that the obtained solution to the DUO-SRDTC problem with elastic demand in this example was the same as the solution to the DUO-SRDTC problem with fixed demand in Example 4, no matter how large the value of parameter χ is adopted.

Example 6. Papageorgiou's network with elastic demand: the efficiency and convergence of the proposed solution algorithm

In this example, Papageorgiou's (1990) network in Example 1 was also adopted to demonstrate that the proposed solution algorithm may be also effective for the DUO-SRDTC problem with elastic demand when using the LTM as the traffic flow model. The input parameters of the LTM for all of the links were the same and are given as follows:

- Jam density: 133 vehicles/km (i.e., a gap of 7.5 m between two adjacent vehicles).
- Free-flow speed: 54 km/h (i.e., 15 m/s); backward shock-wave speed: 18 km/h (i.e., 5 m/s).
- Flow capacity: 1800 vehicles/h/lane (i.e., 0.5 vehicle/s/lane).

The length of each time interval was 5 min, and the modeling horizon contained 30 intervals. The free-flow travel times were four intervals for links 1-3 and 2-3, six intervals for links 1-4 and 2-5, five intervals for links 3-4 and 3-5, and two intervals for links 4-5 and 5-4. The numbers of lanes were two for links 4-5 and 5-4, and four for the other six links. Following Szeto and Lo (2004), we assumed the inverse function of the total demand function of OD pair w in the form of $\pi_w = \bar{\pi}_w - m_w Q_w$, and $m_{14} = m_{15} = m_{24} = m_{25}$ = HK\$0.001/traveler. $\bar{\pi}_{14} = \bar{\pi}_{25}$ = HK\$50, $\bar{\pi}_{15}$ = HK\$70, and $\bar{\pi}_{24}$ = HK\$65. The parameters for the generalized travel cost function were α = HK\$1.0/min, β = HK\$0.9/min, γ = HK\$2.0/min, $\Delta_w = 15$ min, and $k_w^* = 85$ min. The values of the parameters for the solution algorithms were the same as those in Example 1.

The proposed approach was adopted to solve the DUO-SRDTC problem with elastic demand with the preceding setting. The values of the three gap functions obtained during the implementation of the algorithm are presented in Fig. 11. We can observe that the values of the gap functions quickly decrease as the number of iterations increases. It required 6.34 s of CPU time and performed DNL 33,226 times (about 2,556 times in each iteration) to achieve a G_5 value of less than 1.0×10^{-8} vehicles per hour. We also tested the performance of the descent method, the extragradient method, and the route-swapping method. We found that the three existing methods could *not* achieve a G_5 value of less than 1.0×10^{-2} vehicles per hour after performing DNL 400,000 times. Again, the memory usage of the proposed approach was about double that of the route-swapping method (8.40 MB for the route-swapping method and 15.41 MB for the proposed approach).

6. Conclusions

This paper proposes the reformulation of a DUO-SRDTC problem as a system of nonlinear equations in terms of generalized OD travel costs. The demand can be either elastic or fixed. The mapping function of the system is formed by equating the retrieved demand function and the demand function, and captures two subproblems under Assumptions 1 and 2: a DUO route choice subproblem with perfectly elastic demand and a QP subproblem. The route choice subproblem is proven to be equivalent to the DUO-SRDTC problem with perfectly elastic demand under the departure time choice equilibrium condition and a unique mapping condition between the route travel time and the generalized route cost. This subproblem aims to determine the used path set given a generalized OD travel cost vector. The QP subproblem aims to determine the closest OD demand vector to the target vector defined by the demand function using the information of the used path set obtained from the route choice subproblem. We prove that a solution to the proposed system is equivalent to an optimal solution to the corresponding DUO-SRDTC problem.

The BFGS method with a backtracking inexact line search is used to solve the proposed system of nonlinear equations. The extragradient method is developed to solve the proposed DUO route choice subproblem. The algorithm is convergent if the travel time functions are pseudomonotone and Lipschitz continuous, and it is not necessary to know the Lipschitz constant of the travel time functions in advance. However, this extragradient method can be replaced by other methods, such as the PPM, with weaker convergence conditions. The QP subproblem is solved with the Frank-Wolfe algorithm.

Assumption 2 or 2A is introduced to ensure subproblem (18) to be QP, which has a unique closest OD demand matrix to the target matrix and can be solved efficiently, for example, by the Frank-Wolfe method. If these assumptions are not satisfied, the solution set may be nonconvex. In the worst case, subproblem (18) does not have a unique OD demand matrix and the Frank-Wolfe method can only be used to obtain a local optimal OD matrix. The proposed approach may not be convergent in this case and can be used only as a heuristic. However, in other cases, subproblem (18) has a unique global optimal solution. The optimal solution can be obtained by existing global optimization methods such as α branch and bound methods (although they are always less efficient than local optimization methods such as the Frank-Wolfe method). Hence, the proposed reformulation and hybridized solution method can still be used to solve SRDTC problems as long as subproblem (18) has a unique global optimal solution and the Frank-Wolfe method is replaced by a global optimization method.

Examples are given to show the performance of the proposed approach. The results show that the generalized route travel cost function can be nonmonotone, even if the route travel time function is monotone, and the extragradient method for the DUO route choice subproblem with perfectly elastic demand can converge much more quickly than that for the DUO-SRDTC subproblem with perfectly elastic demand. The results also show that the proposed approach for solving the DUO-SRDTC problem with either fixed or elastic demand can converge more quickly than some commonly used algorithms in some cases. More importantly, the results presented in Tables 1 through 4 illustrate the convergence improvements of the proposed approach over some existing methods both for fixed-demand and elastic-demand cases. It can be observed that our proposed approach can solve all of the test cases and that the existing methods cannot. Moreover, compared with the DUO-SRDTC subproblem with perfectly elastic demand, it can be seen that solution of the corresponding DUO route choice subproblem is easier due to the monotonicity of its mapping function.

In this paper, DUO-SRDTC problems are formulated as path-based models that can be solved efficiently using a column-generation heuristic to generate paths as needed or concept of subpaths. Alternatively, link-based or intersection-movement–based models (Long et al., 2013) can be used instead to avoid solving path-based models, and they have good potential for application in large-scale networks. Future studies could use link-based or intersection-movement–based models in this framework and examine the resultant computation performance. Moreover, the BFGS method adopted was initially developed for symmetric nonlinear equations. However, the global and superlinear convergence of this method for asymmetric nonlinear equations should be further investigated. Therefore, we aim to improve both the formulation and the solution algorithm. We will also compare the performance of solving the SRDTC problem directly by the PPM and that by the proposed method with the use of the PPM to handle the route choice subproblem. Furthermore, Assumptions 2 and 2A are strong and may not be satisfied by real-world DTA models in general. Relaxing these assumptions for the proposed approach remains a subject for future study.

Acknowledgements

This work is jointly supported by the National Basic Research Program of China (2012CB725401), the National Natural Science Foundation of China (71431003, 71522001), the Fok Ying Tung Education Foundation (141081), a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (HKU 17207214), and a grant (201311159123) from the University Research Committee of the University of Hong Kong. The authors are grateful to Prof. Terry L. Friesz and the three reviewers for their constructive comments.

Appendix A

The following is the proof of Lemma 2.

Proof: We need to prove that for any values of travel times t_x and t_y with $t_x < t_y$, we have $\psi_{wk}(t_x) < \psi_{wk}(t_y)$. After choosing any values of t_x and t_y, they satisfy the conditions depicted by one of the following six exhaustive cases:

Case 1. *t_x satisfies $k\delta + t_x < k_w^* - \Delta_w$ and t_y satisfies $k\delta + t_y < k_w^* - \Delta_w$*

Denote t_1 as the travel time that satisfies $k\delta + t_1 < k_w^* - \Delta_w$. We then obtain $\psi_{wk}(t_1) = \alpha t_1 + \beta[k_w^* - \Delta_w - k\delta - t_1] = (\alpha - \beta)t_1 + \beta(k_w^* - \Delta_w - k\delta)$. The coefficient associated with t_1 is positive (i.e., $\alpha - \beta > 0$). Hence, for any values of two travel times t_x and t_y with $t_x < t_y$, $k\delta + t_x < k_w^* - \Delta_w$, and $k\delta + t_y < k_w^* - \Delta_w$, we have $\psi_{wk}(t_x) < \psi_{wk}(t_y)$.

Case 2. *t_x satisfies $k\delta + t_x > k_w^* + \Delta_w$, and t_y satisfies $k\delta + t_y > k_w^* + \Delta_w$*

Denote t_2 as the travel time that satisfies $k\delta + t_2 > k_w^* + \Delta_w$. Then, $\psi_{wk}(t_2) = \alpha t_2 + \gamma[k\delta + t_2 - k_w^* - \Delta_w] = (\alpha + \gamma)t_2 + \gamma[k\delta - k_w^* - \Delta_w]$. The coefficient associated with t_2, $(\alpha + \gamma)$, is positive. Hence, for any values of two travel times t_x and t_y with $t_x < t_y$, $k\delta + t_x > k_w^* + \Delta_w$, and $k\delta + t_y > k_w^* + \Delta_w$, we have $\psi_{wk}(t_x) < \psi_{wk}(t_y)$.

Case 3. *t_x satisfies $k_w^* - \Delta_w \le k\delta + t_x \le k_w^* + \Delta_w$ and t_y satisfies $k_w^* - \Delta_w \le k\delta + t_y \le k_w^* + \Delta_w$*

Denote t_3 as the travel time that satisfies $k_w^* - \Delta_w \le k\delta + t_3 \le k_w^* + \Delta_w$. Then, $\psi_{wk}(t_3) = \alpha t_3$. The coefficient associated with t_3, α, is positive. Hence, for any values of two travel times t_x and t_y with $t_x < t_y$, $k_w^* - \Delta_w \le k\delta + t_x \le k_w^* + \Delta_w$, and $k_w^* - \Delta_w \le k\delta + t_y \le k_w^* + \Delta_w$, we have $\psi_k^{rs}(t_x) < \psi_k^{rs}(t_y)$.

Case 4. *t_x satisfies $k\delta + t_x < k_w^* - \Delta_w$ and t_y satisfies $k_w^* - \Delta_w \le k\delta + t_y \le k_w^* + \Delta_w$*

According to the conditions of this case, we have $\psi_{wk}(t_x) = \alpha t_x + \beta[k_w^* - \Delta_w - k\delta - t_y]$ and

$\psi_{wk}(t_y) = \alpha t_y$. Denote t_4 as the travel time that satisfies $k\delta + t_4 = k_w^* - \Delta_w$. Then, when t_x approaches to t_4, $\psi_{wk}(t_x)$ approaches $\alpha t_4 + \beta[k_w^* - \Delta_w - k\delta - t_4] = \alpha t_4 = \psi_{wk}(t_4)$. Moreover, when t_y reduces to t_4, $\psi_{wk}(t_y)$ reduces to $\psi_{wk}(t_4)$. Therefore, for any values of two travel times t_x and t_y with $t_x < t_y$, $k\delta + t_x < k_w^* - \Delta_w$, and $k_w^* - \Delta_w \le k\delta + t_y \le k_w^* + \Delta_w$, we have $\psi_{wk}(t_x) < \psi_{wk}(t_4) \le \psi_{wk}(t_y)$ and hence $\psi_{wk}(t_x) < \psi_{wk}(t_y)$.

Case 5. *t_x satisfies $k_w^* - \Delta_w \le k\delta + t_x \le k_w^* + \Delta_w$ and t_y satisfies $k\delta + t_y > k_w^* + \Delta_w$*

According to the conditions of this case, we obtain $\psi_{wk}(t_x) = \alpha t_x$ and $\psi_{wk}(t_y) = \alpha t_y + \gamma[k\delta + t_y - k_w^* - \Delta_w]$. Denote t_5 as the travel time that satisfies $k\delta + t_5 = k_w^* + \Delta_w$. Then, when t_x increases to t_5, $\psi_{wk}(t_x)$ increases to $\psi_{wk}(t_5)$. Moreover, when t_y approaches t_5, $\psi_{wk}(t_y)$ approaches $\alpha t_5 + \gamma[k\delta + t_5 - k_w^* - \Delta_w] = \alpha t_5 = \psi_{wk}(t_5)$. Hence, for any values of two travel times t_x and t_y with $t_x < t_y$, $k_w^* - \Delta_w \le k\delta + t_x \le k_w^* + \Delta_w$, and $k\delta + t_y > k_w^* + \Delta_w$, we have $\psi_{wk}(t_x) < \psi_{wk}(t_5) \le \psi_{wk}(t_y)$ and hence $\psi_{wk}(t_x) < \psi_{wk}(t_y)$.

Case 6. *t_x satisfies $k\delta + t_x < k_w^* - \Delta_w$ and t_y satisfies $k\delta + t_y > k_w^* + \Delta_w$*

Based on the previous definitions, we have $t_4 \le t_3 \le t_5$ and hence $\psi_{wk}(t_4) \le \psi_{wk}(t_5)$. Moreover, because $\psi_{wk}(t_x) < \psi_{wk}(t_4)$ (from Case 4) and $\psi_{wk}(t_5) \le \psi_{wk}(t_y)$ (from Case 5), we obtain $\psi_{wk}(t_x) < \psi_{wk}(t_4) \le \psi_{wk}(t_5) < \psi_{wk}(t_y)$, leading to $\psi_{wk}(t_x) < \psi_{wk}(t_y)$.

As we can see, the monotonicity of $\psi_{wk}(t)$ can be proven for each case. Hence, we can conclude that for any values of travel times t_x and t_y with $t_x < t_y$, we have $\psi_{wk}(t_x) < \psi_{wk}(t_y)$. This completes the proof. □

Appendix B

Example B.1. A monotone function composed with a nonlinear function can lead to a nonmonotone function

We consider the following monotone function (with positive definite Jacobian matrix) and nonlinear functions:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x}, \text{ and} \tag{33}$$

$$g_i(y) = \begin{cases} y + 0.5(6 - i - y), & \text{if } y \le 6 + i \\ y + 2(y + i - 6), & \text{otherwise} \end{cases}, \forall i = 1, 2, 3. \tag{34}$$

where $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$. Based on above functions, we can define the following function

$$\mathbf{G}(\mathbf{x}) = \begin{bmatrix} g_1(f_1(\mathbf{x})) \\ g_2(f_2(\mathbf{x})) \\ g_3(f_3(\mathbf{x})) \end{bmatrix}. \tag{35}$$

Let $\mathbf{x}_1 = (1,2,2)^{\mathrm{T}}$ and $\mathbf{x}_2 = (3,1,1)^{\mathrm{T}}$. Substituting them into Eqs. (33) and (35), we have $\mathbf{f}(\mathbf{x}_1) = (2,4,6)^{\mathrm{T}}$, $\mathbf{f}(\mathbf{x}_2) = (4,5,6)^{\mathrm{T}}$, $\mathbf{G}(\mathbf{x}_1) = (3.5,4,12)^{\mathrm{T}}$, and $\mathbf{G}(\mathbf{x}_2) = (4.5,7,12)^{\mathrm{T}}$. Therefore, we have

$$\left\langle \mathbf{f}(\mathbf{x}_2) - \mathbf{f}(\mathbf{x}_1), \mathbf{x}_2 - \mathbf{x}_1 \right\rangle = 3 \text{ and } \left\langle \mathbf{G}(\mathbf{x}_2) - \mathbf{G}(\mathbf{x}_1), \mathbf{x}_2 - \mathbf{x}_1 \right\rangle = -1. \tag{36}$$

Eq. (36) is consistent with that $\mathbf{f}(\mathbf{x})$ is a monotone function and implies that $\mathbf{G}(\mathbf{x})$ a nonmonotone function. Therefore, a monotone function composed with a nonlinear function can lead to a nonmonotone function.

Example B.2. Nonmonotonicity of generalized route travel cost functions due to schedule delay

In this example, we consider a network with a single link and a single route and adopt the PQM to implement the DNL (Nie and Zhang, 2005). The free-flow travel time and capacity of the link are 1 interval and 10 vehicles per interval, respectively. The parameters for the generalized travel cost function were set to $\alpha = 1.0$, $\beta = 0.5$, $\gamma = 1.5$, $\Delta_w = 0$, and $k_w^* = 7$. We first set the vector of route flows $\mathbf{f} = (20,20,20,4,4,4,4,4)^T$. Using the PQM and the route cost function (11), we can obtain the vectors of route travel times and costs: $\mathbf{t} = (2,3,4,3.4,2.8,2.2,1.6,1)^T$ and $\mathbf{c} = (4,4,4,4,4,4,4,4)^T$. Similarly, when we set the vector of route flows $\tilde{\mathbf{f}} = (15,15,15,7,7,7,7,7)^T$, we can obtain the corresponding vectors of route travel times and costs: $\tilde{\mathbf{t}} = (1.5,2,2.5,2.2,1.9,1.6,1.3,1.0)^T$ and $\tilde{\mathbf{c}} = (3.75,3.5,3.25,2.6,1.95,2.5,3.25,4)^T$. Therefore, we have

$$\left\langle \mathbf{t} - \tilde{\mathbf{t}}, \mathbf{f} - \tilde{\mathbf{f}} \right\rangle = 6 \text{, and} \tag{37}$$

$$\left\langle \mathbf{c} - \tilde{\mathbf{c}}, \mathbf{f} - \tilde{\mathbf{f}} \right\rangle = -9.6. \tag{38}$$

Eq. (37) is consistent with that the route travel time function is monotone with respect to departure flows (Smith and Ghali, 1990), and Eq. (38) implies that the generalized route travel *cost* functions can be nonmonotonic with respect to departure flows, even if the route travel *time* function is monotone.

Appendix C

The following is the proof of Theorem 1.

Proof. First, we prove that any optimal solution to VI problem (15) is also optimal to the VI problem (9). Let $\mathbf{f}^* \in \Psi$ be an optimal solution to VI problem (15). Then, $\mathbf{f}^*$ must satisfy non-negativity condition (4) and the following DUO conditions:

$$f_p^*(k)[t_p^*(k) - \eta_w(k)] = 0, \forall w \in W, p \in P_w, k \in K_d, \text{ and} \tag{39}$$

$$t_p^*(k) - \eta_w(k) \geq 0, \forall w \in W, p \in P_w, k \in K_d. \tag{40}$$

Because function $\psi_{wk}(t)$ is monotonically increasing under the conditions $0 < \beta < \alpha$ and $0 < \gamma$ (see Lemma 2) and inequality (40) implies $t_p^*(k) \geq \eta_w(k)$, we have

$$c_p^*(k) = \psi_{wk}(t_p^*(k)) \geq \psi_{wk}(\eta_w(k)) = \pi_w, \forall w \in W, p \in P_w, k \in K_d. \tag{41}$$

Condition (41) implies that the optimal solution $\mathbf{f}^*$ satisfies condition (2). That is,

$$c_p^*(k) - \pi_w \geq 0, \forall w \in W, p \in P_w, k \in K_d. \tag{42}$$

According to condition (12), we have

$$t_p^*(k) \begin{cases} = \eta_w(k), & \text{if } f_p^*(k) > 0, \\ \geq \eta_w(k), & \text{if } f_p^*(k) = 0, \end{cases} \forall w \in W, p \in P_w, k \in K_d. \tag{43}$$

Because function $\psi_{wk}(t)$ is monotonically increasing with respect to *t*, we have

$$c_p^*(k) = \psi_{wk}(t_p^*(k)) \begin{cases} = \psi_{wk}(\eta_w(k)) = \pi_w, & \text{if } f_p^*(k) > 0, \\ \geq \psi_{wk}(\eta_w(k)) = \pi_w, & \text{if } f_p^*(k) = 0, \end{cases} \forall w \in W, p \in P_w, k \in K_d. \tag{44}$$

Equivalently, condition (44) can be rewritten as follows:

$$c_p^*(k) \begin{cases} = \pi_w, & \text{if } f_p^*(k) > 0, \\ \geq \pi_w, & \text{if } f_p^*(k) = 0, \end{cases} \forall w \in W, p \in P_w, k \in K_d. \tag{45}$$

According to condition (45), we have

$$f_p^*(k)[c_p^*(k) - \pi_w] = 0, \forall w \in W, p \in P_w, k \in K_d. \tag{46}$$

Conditions (46) and (42) imply that $\mathbf{f}^*$ satisfies the DUO conditions of the SRDTC problem with perfectly elastic demand, and hence $\mathbf{f}^*$ is also an optimal solution to VI problem (9).

We then prove that any optimal solution to VI problem (9) is also optimal to VI problem (15). Let $\mathbf{f}^* \in \Psi$ be an optimal solution to VI problem (9). Then, $\mathbf{f}^*$ must satisfy non-negativity condition (4) of the DUO problem with perfectly elastic demand and the DUO conditions of the SRDTC problem with perfectly elastic demand (i.e., conditions (46) and (42)). Because the function $\psi_{wk}(t)$ is monotonically increasing with respect to *t*, its inverse function $\psi_{wk}^{-1}(\pi)$ is also monotonically increasing. According to condition (42), we have

$$c_p^*(k) - \pi_w \geq 0 \Rightarrow c_p^*(k) \geq \pi_w, \forall w \in W, p \in P_w, k \in K_d, \text{ and hence} \tag{47}$$

$$t_p^*(k) = \psi_{wk}^{-1}(c_p^*(k)) \geq \psi_{wk}^{-1}(\pi_w) = \eta_w(k), \forall w \in W, p \in P_w, k \in K_d. \tag{48}$$

Similarly, condition (45) implies:

$$t_p^*(k) = \psi_{wk}^{-1}(c_p^*(k)) \begin{cases} = \psi_{wk}^{-1}(\pi_w) = \eta_w(k), \text{ if } f_p^*(k) > 0, \\ \geq \psi_{wk}^{-1}(\pi_w) = \eta_w(k), \text{ if } f_p^*(k) = 0, \end{cases} \forall w \in W, p \in P_w, k \in K_d. \tag{49}$$

According to condition (49), $\mathbf{f}^* \in \Psi$ satisfies condition (39). Also, condition (48) implies that condition (40) is satisfied. Therefore, $\mathbf{f}^* \in \Psi$ satisfies the DUO condition of the route choice problem with perfectly elastic demand and is also an optimal solution to VI problem (15). □

Appendix D

The following is the route-swapping method proposed by Huang and Lam (2002) for the DUO-SRDTC problem with fixed demand:

Step 1. *Initialization*. Choose an initial vector of route flows $\mathbf{f}_1$, set the iteration index $\iota = 1$, and select the convergence tolerance $\varepsilon > 0$.

Step 2. *Generalized route travel cost update*. Implement the DNL model and obtain the route travel time $t_p(k)_\iota$ and the generalized route travel cost $c_p(k)_\iota$. Obtain the minimum travel cost for each OD pair and the corresponding path set of each OD pair, respectively, by

$$\pi_{w\iota} = \min\{c_p(k)_\iota : p \in P_w, k \in K_d\}, \text{ and} \tag{50}$$

$$\hat{P}_{w\iota} = \min\{(p,k) : c_p(k)_\iota = \pi_{w\iota}, p \in P_w, k \in K_d\}. \tag{51}$$

Step 3. *Route flow update*. Calculate the route flow vector $\mathbf{f}_{\iota+1}$ by:

$$f_p(k)_{\iota+1} = f_p(k)_\iota - \rho_\iota f_p(k)_\iota [c_p(k)_\iota - \pi_{w\iota}], \forall p \in P_w, k \in K_d, \tag{52}$$

$$f_p(k)_{\iota+1} = f_p(k)_\iota + \frac{\psi_\iota}{|\hat{P}_{w\iota}|}, \forall (p,k) \in \hat{P}_{w\iota}, \tag{53}$$

where

$$\psi_\iota = \sum_{p \in P_w, k \in K_d} \rho_\iota f_p(k)_\iota [c_p(k)_\iota - \pi_w].$$

Step 4. *Convergence criterion*. If $G_4(\mathbf{f}_\iota) < \varepsilon$, then stop the algorithm; otherwise, set $\iota = \iota + 1$ and go to Step 2.

In Eq. (52), ρ_ι is the step size at the ι th iteration. Huang and Lam (2002) and Tian et al. (2012) adopted the following step-size methods, respectively:

$$\rho_\iota = 0.0001\{1^{(1 \to 1000)}, 1/2^{(1001 \to 2000)}, 1/3^{(2001 \to 3000)}, \cdots\}, \text{ and} \tag{54}$$

$$\rho_\iota = 0.012\{1^{(1 \to 1000)}, 1/2^{(1001 \to 2000)}, 1/3^{(2001 \to 3000)}, \cdots\}. \tag{55}$$

Because the performance of the step-size sequence depicted by Eq. (55) was better than that depicted by Eq. (54) according to our preliminary studies, we adopted Eq. (55) to calculate the step size in Step 3.

Appendix E

E.1. General constraints for retrieving OD demands

Dynamic network constraints are generally used to formulate the feasible domain of the DTA problems. Those constraints can be classified into five categories (Ran and Boyce, 1996; Ban et al., 2008): mass balance constraints, first-in-first-out (FIFO) constraints, flow conservation constraints, flow propagation constraints, and definitional constraints. The mass balance constraints define the relationship among the link flow (i.e., the number of vehicles on the link), the inflow rate, and the outflow rate; the change in the link flow at each time instant is equal to the difference between the inflow rate and the outflow rate at that time. The FIFO constraints require that the vehicles that enter the link earlier leave it sooner.

The flow conservation constraints describe that all flows that enter any node (except the destination node) and the demand generated at this node must leave this node. Thus, the corresponding flow conservation constraints can be formulated as follows:

$$U_{ap}(k) = \zeta_{ap} U_p(k) + \sum_{b \in A} \zeta_{bap} V_{bp}(k), \forall a \in A, p \in P, k \in K \ , \tag{56}$$

where $\zeta_{ap} = 1$ if link *a* is the first link on route *p*; otherwise, $\zeta_{ap} = 0$. $\zeta_{bap} = 1$ if link *a* is the next link after leaving link *b* along route *p*; otherwise, $\zeta_{bap} = 0$. $U_p(k)$ is the cumulative flow on route $p \in P$ at the end of interval *k*, $U_{ap}(k)$ and $V_{ap}(k)$ are cumulative flows on route $p \in P$ entering and leaving link *a* at the end of interval *k*, respectively.

The flow propagation constraints represent the consistent evolution of traffic flows in both temporal and spatial domains. They describe the relationship between link inflows and outflows, which are connected by the time-dependent link travel times. Following Ban et al. (2008), we use an "inverse link travel time function" $p_a(k) = \tau_a(k - p_a(k))$, which denotes the travel time on link *a* for vehicles that exit the link at the end of time interval *k*. Under Assumption 2A, all optimal solutions to VI problem (15) have a unique link travel time $\tau_a^*(k)$, and the optimal inverse link travel time $p_a^*(k)$ is also unique, where the variables in this appendix with asterisks indicate their values at the DUO state. Therefore, we can reformulate the flow propagation constraints in terms of cumulative flows as follows:

$$V_{ap}(k) = (1 - \lambda^*_{a,k}) U_{ap}(\ell^*_{a,k}) + \lambda^*_{a,k} U_{ap}(\ell^*_{a,k} + 1), \forall a \in A, p \in P, k \in K \ , \tag{57}$$

where $\ell^*_{a,k}$ is the entry interval of vehicles that leave link *a* at the end of interval *k*, and

$\ell^*_{a,k} = \lfloor k - p^*_a(k) \rfloor$; $\lambda^*_{a,k}$ is the cumulative outflow parameter, and $\lambda^*_{a,k} = k - p^*_a(k) - \ell^*_{a,k}$.

To ensure that the DUO route choice condition is satisfied, i.e., that no travelers use the route with a travel time greater than the minimum OD travel time, we have

$$U_p(k) - U_p(k-1) = 0, \forall (w, p, k) \in \{w \in W, p \in P_w, k \in K_d \mid t^*_p(k) > \eta_w(k)\}. \tag{58}$$

The definitional constraints are given as follows:

$$f_p(k)\delta = U_p(k) - U_p(k-1), \forall p \in P, k \in K_d, \tag{59}$$

$$U_a(k) = \sum_{p \in P} U_{ap}(k), \forall a \in A, k \in K, \tag{60}$$

$$V_a(k) = \sum_{p \in P} V_{ap}(k), \forall a \in A, k \in K, \tag{61}$$

$$U_{ap}(k) - U_{ap}(k-1) \geq 0, \forall a \in A, p \in P, k \in K, \text{ and} \tag{62}$$

$$U_{ap}(0) = 0, \forall a \in A, p \in P, \tag{63}$$

where constraint (59) describes the relationship between path flows and cumulative path flows. Constraints (60) and (61) provide the relationship between cumulative path flows and cumulative link flows. Constraint (62) ensures that the cumulative flows are nondecreasing, and the constraint implies flow non-negative condition (4). Constraint (63) requires the cumulative flows to be zero at the beginning of the studied period.

E.2. Using the linear travel time model to retrieve OD demands

The LTTM can be expressed as follows (Ban et al., 2008):

$$\tau_a(k) = \tau^0_a[1 + \beta^x_a x_a(k)], \forall a \in A, k \in K, \tag{64}$$

where τ^0_a is the free-flow travel time of link *a*, $\beta^x_a > 0$ is a constant, and $x_a(k)$ is the number of vehicles on link *a* at the end of interval *k*. This model implicitly captures FIFO.

Substituting the mass balance constraint $x_a(k) = U_a(k) - V_a(k)$ and the equilibrium link travel time $\tau^*_a(k)$ into Eq. (64) and rearranging the resultant expression, we have

$$U_a(k) - V_a(k) = \left[\tau^*_a(k) - \tau^0_a\right] / (\tau^0_a \beta^x_a), \forall a \in A, k \in K. \tag{65}$$

Under Assumption 2A, if the LTTM is adopted as the traffic flow model, then the following QP can be used to retrieve a *unique* OD demand pattern from the generalized OD travel cost vector $\boldsymbol{\pi}$:

$$\min \sum_{w \in W} \left(Q_w - D_w(\pi_w)\right)^2, \tag{66}$$

subject to constraints (3), (56) through (63), and (65).

In this problem, $\Pi(\boldsymbol{\pi})$ is defined by constraints (56) through (63) and (65).

E.3. Using the point queue model to retrieve OD demands

Following Huang and Lam (2002), we use the following link travel time function for the PQM:

$$\tau_a(k) = \tau^0_a + q_a(k) / C_a, \forall a \in A, k \in K, \tag{67}$$

where C_a is the outflow capacity of link *a*, $q_a(k)$ is the queue length at the end of interval *k*, and

$$q_a(k) = \max\{q_a(k-1) + U_a(k) - U_a(k-1) - C_a, 0\}, \forall a \in A, k \in K \ . \tag{68}$$

Huang and Lam (2002) proved that the calculation of travel times by Eq. (67) and (68) guarantees FIFO and that Eq. (68) captures the mass balance constraint.

Under Assumption 2A, Eq. (67) implies that the queue lengths under DUO condition (12) are unique. According to Eq. (68), if $q_a(k) = 0$, the first term on the right-hand side of Eq. (68) is nonpositive; otherwise, $q_a(k)$ is equal to the first term on the right-hand side of Eq. (68). Therefore, under DUO condition (12), we have

$$U_a(k) - U_a(k-1) \begin{cases} = C_a + q_a^*(k) - q_a^*(k-1), \forall (a,k) \in \{a,k \mid q_a^*(k) > 0\}, \\ \le C_a - q_a^*(k-1), \forall (a,k) \in \{a,k \mid q_a^*(k) = 0\}. \end{cases} \tag{69}$$

In the PQM, the link outflows are no more than the link outflow capacity, and hence the following constraint is imposed:

$$V_a(k) - V_a(k-1) \le C_a, \forall a \in A, k \in K \ . \tag{70}$$

Under Assumption 2A, if the PQP is adopted as the traffic flow model, then the following QP can be used to retrieve a *unique* OD demand pattern from the generalized OD travel cost vector $\boldsymbol{\pi}$:

$$\min \sum_{w \in W} \left(Q_w - D_w(\pi_w) \right)^2 , \tag{71}$$

subject to constraints (3), (56) through (63), (69), and (70),

where $\Pi(\boldsymbol{\pi})$ is defined by constraints (56) through (63), (69), and (70).

E.4. Using the link transmission model to retrieve OD demands

Following Long et al. (2011), we use the following link travel time function for the LTM:

$$\tau_a(k) = (n_k - k)\delta - \sum_{l=n_{k-1}}^{n_k - 1} \delta[V_a(l) - U_a(k-1)] / [U_a(k) - U_a(k-1)], \forall a \in A, k \in K \ , \tag{72}$$

where n_k is the critical outflow interval, and defined as follows:

$$n_k = \min\{l \mid U_a(k) \le V_a(l), l > k + \tau_a^0 / \delta\}, \forall k \in K \ .$$

Long et al. (2011) proved that the calculation of travel times by Eq. (72) guarantees FIFO. For the LTM, we do not treat link flows (i.e., the number of vehicles on the link), and hence the mass balance constraints are omitted. Similar to Assumption 2A, we also assume that there is a unique optimal vector of critical outflow intervals to VI problem (15). Substituting the equilibrium link travel time $\tau_a^*(k)$ and n_k^* into (72) and rearranging the resultant expression, we have

$$\left[(n_k^* - k)\delta - \tau_a^*(k)\right][U_a(k) - U_a(k-1)] = \sum_{l=n_{k-1}^*}^{n_k^* - 1} \delta[V_a(l) - U_a(k-1)], \forall a \in A, k \in K \ . \tag{73}$$

In the LTM, link outflows are constrained both by the boundary conditions at the upstream end of the link and the outflow capacity of the link (Yperman, 2007). According to the simplified theory of Newell (1993), if a free-flow traffic state occurs at the downstream link boundary at the end of interval *k*, then this state must have been emitted from the upstream boundary L_a / v_a time units earlier (i.e., a free-flow travel time τ_a^0), where L_a and v_a are, respectively, the length and the free-flow speed of link *a*. Hence, the following constraints should be satisfied:

$$V_a(k) - V_a(k-1) \leq C_a, \forall a \in A, k \in K \text{ , and} \tag{74}$$

$$V_a(k) \leq U_a\left(k - \frac{L_a}{v_a \delta}\right), \forall a \in A, k \in K \text{ .} \tag{75}$$

Similarly, the link inflows are also constrained both by the boundary conditions at the downstream end of the link and the inflow capacity of the link (Yperman, 2007). According to the simplified theory of Newell (1993), if a congested traffic state occurs at the upstream boundary at the end of interval *k*, then this state must have been emitted from the downstream boundary $(-L_a / w_a)$ time units earlier, because a congested traffic state travels with a (negative) speed w_a. Hence, the following constraints should be satisfied:

$$U_a(k) - U_a(k-1) \leq Q_a(k), \forall a \in A, k \in K \text{ , and} \tag{76}$$

$$U_a(k) \leq V_a\left(k + \frac{L_a}{\delta w_a}\right) + L_a \rho_{jam}, \forall a \in A, k \in K \text{ ,} \tag{77}$$

where ρ_{jam} is the jam density.

Under Assumption 2A, if the LTM is adopted as the traffic flow model, then the following QP can be used to retrieve a *unique* OD demand pattern from the generalized OD travel cost vector $\boldsymbol{\pi}$:

$$\min \sum_{w \in W} \left(Q_w - D_w(\pi_w)\right)^2 \text{ ,} \tag{78}$$

subject to constraints (3), (56) through (63), and (73) through (77).

In this problem, $\Pi(\boldsymbol{\pi})$ is defined by constraints (56) through (63) and (73) through (77).

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Table 1: The number of DNLs required for each solution algorithm to solve Example 1.

Convergence tolerance (G_4)	Route-swapping method	Descent method	Extragradient method	The proposed method
1.00E-01	34	-	13	6340
1.00E-02	114	-	232	6742
1.00E-03	393	-	853	7383
1.00E-04	1460	-	1326	9433
1.00E-05	25977	-	2319	10822
1.00E-06	-	-	4899	10822
1.00E-07	-	-	7906	11239

Table 2: The number of DNLs required for each solution algorithm to solve Example 2.

Convergence tolerance (G_4)	Route-swapping method	Descent method	Extragradient method	The proposed method
1.00E-01	21188	-	-	6132
1.00E-02	33621	-	-	6132
1.00E-03	45037	-	-	8775
1.00E-04	83163	-	-	9996
1.00E-05	279263	-	-	9996
1.00E-06	-	-	-	9996
1.00E-07	-	-	-	9996

Table 3: The number of DNLs required for each solution algorithm to solve Example 4.

Convergence tolerance (G_4)	Route-swapping algorithm	Descent method	Extragradient method	The proposed method
1.00E-01	15124	-	-	15664
1.00E-02	43275	-	-	20464
1.00E-03	99766	-	-	23138
1.00E-04	-	-	-	23138
1.00E-05	-	-	-	28357
1.00E-06	-	-	-	30203
1.00E-07	-	-	-	32579

Table 4: The number of DNLs required for each solution algorithm to solve Example 5.

Convergence tolerance (G_5)	Extragradient method			The proposed method		
	$\chi = 10000$	$\chi = 50000$	$\chi = 100000$	$\chi = 10000$	$\chi = 50000$	$\chi = 100000$
1.00E-01	4,097	1171	1103	14782	11898	11305
1.00E-02	-	5305	3131	15454	12642	12261
1.00E-03	-	10764	7435	19321	12642	12838
1.00E-04	-	16245	11667	22238	16767	16583
1.00E-05	-	26741	18104	24457	18779	18622
1.00E-06	-	44446	31623	24457	19875	18622
1.00E-07	-	64539	46699	25219	19875	19518

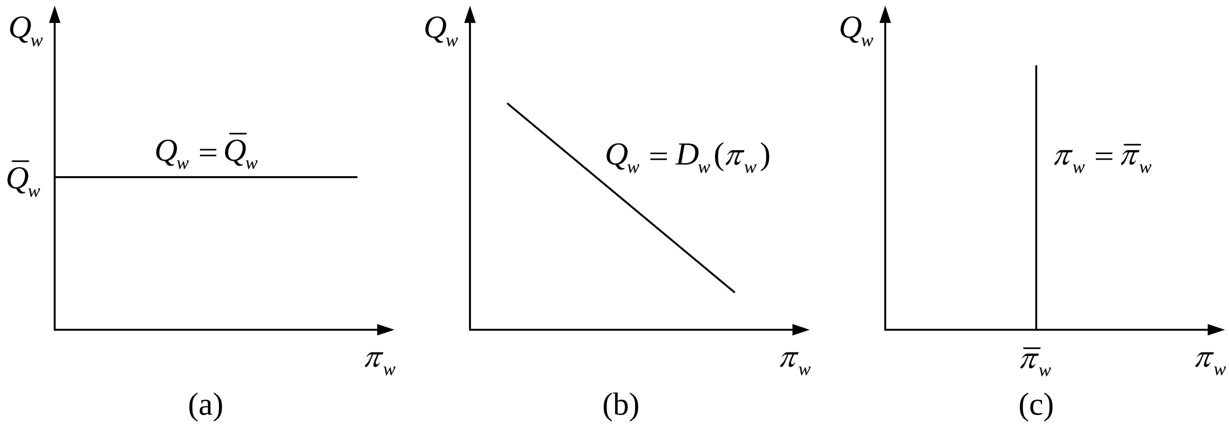

Fig. 1. Demand functions with respect to (a) fixed demand, (b) elastic demand, and (c) perfectly elastic demand.

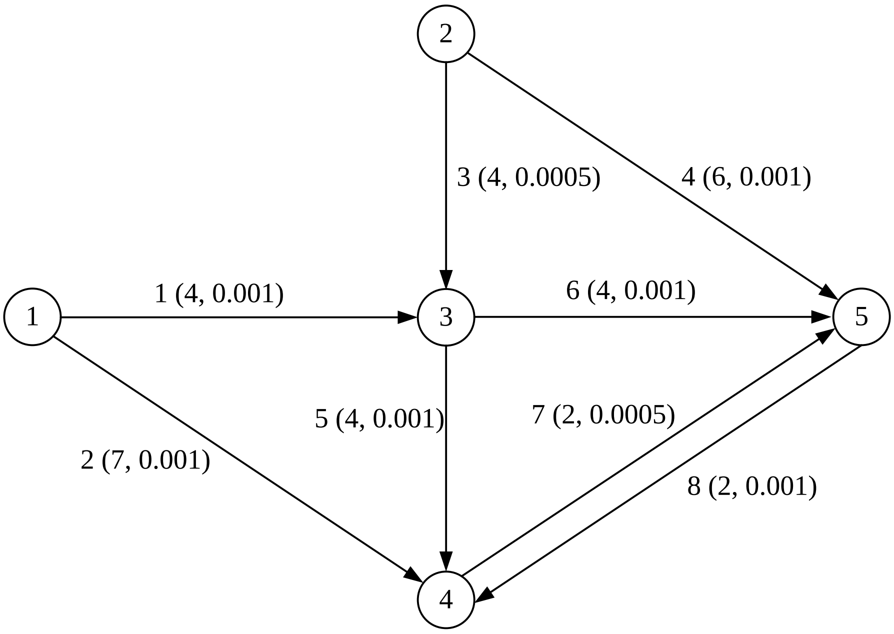

Fig. 2. Papageorgiou's network for Examples 1 and 6.

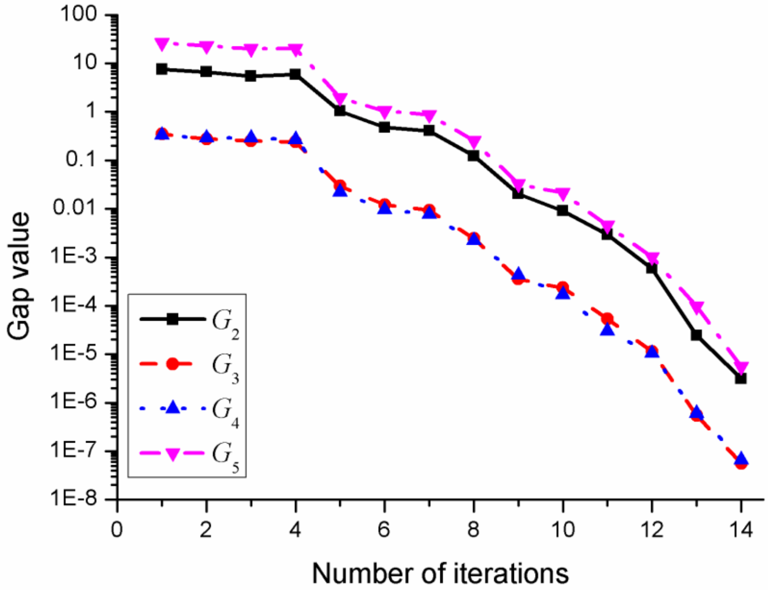

Fig.3. Convergence of the BFGS method for solving the DUO-SRDTC problem with fixed demand in Example 1.

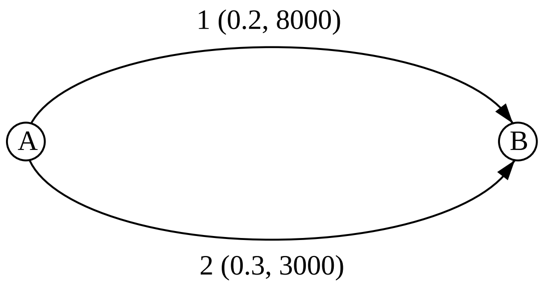

Fig. 4. An example network with two parallel routes for Example 2.

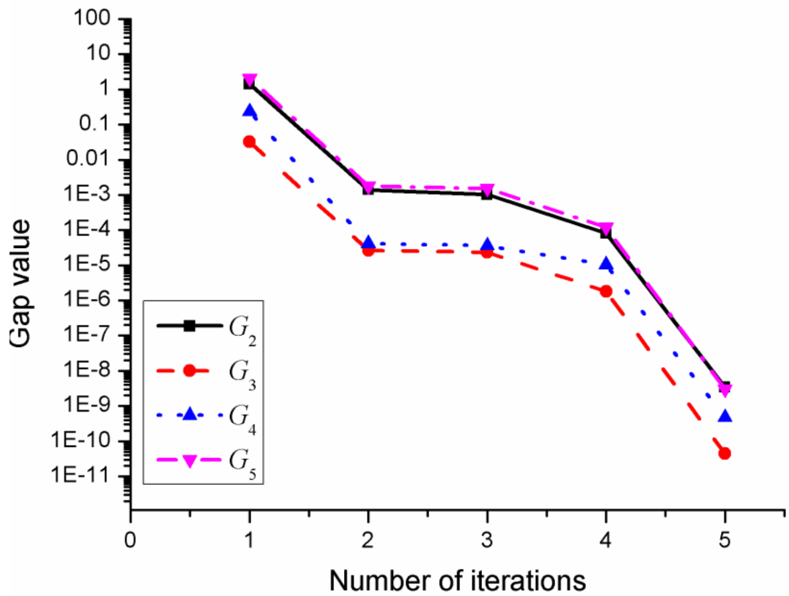

Fig.5. Convergence of the BFGS method for the DUO-SRDTC problem with fixed demand in Example 2.

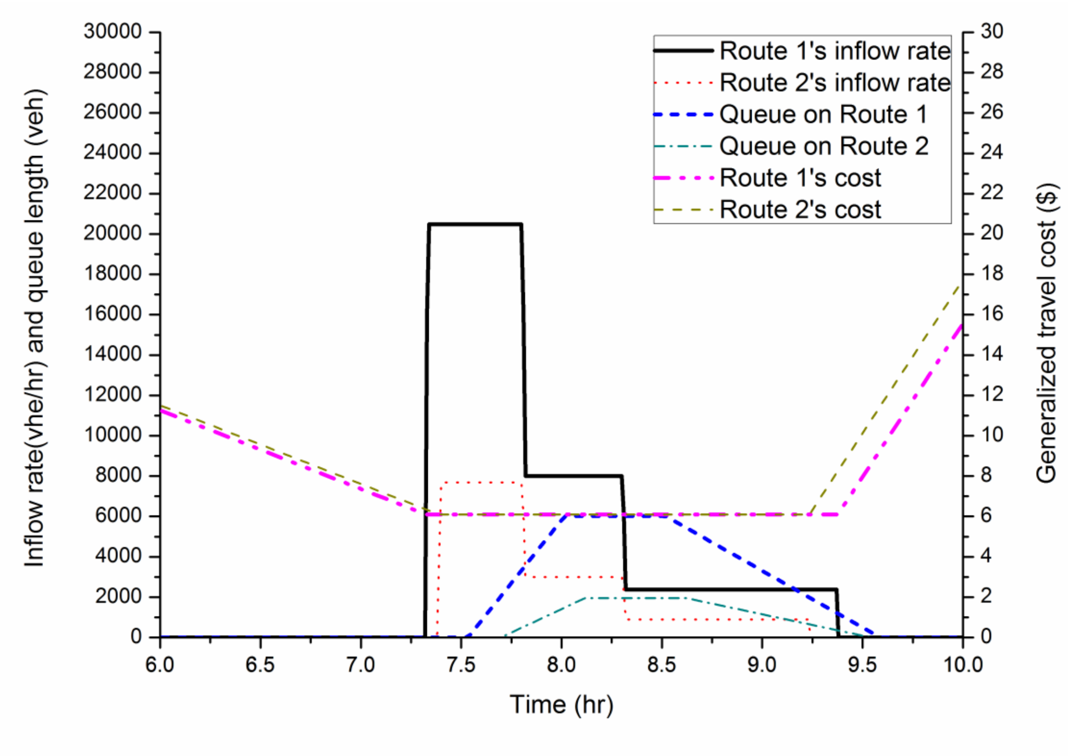

Fig. 6. Optimal solution of Example 2.

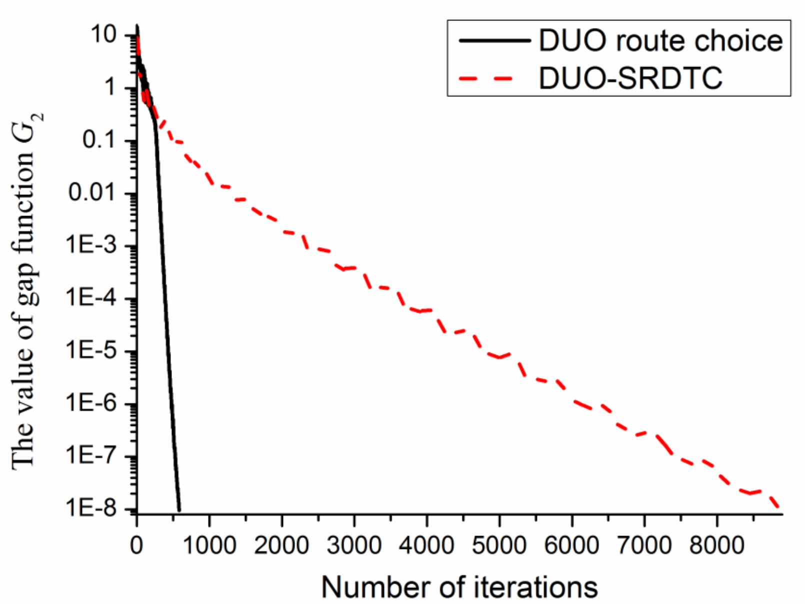

Fig. 7. Convergence of the extragradient method for solving the DUO-SRDTC problem and the DUO route choice problem with perfectly elastic demand.

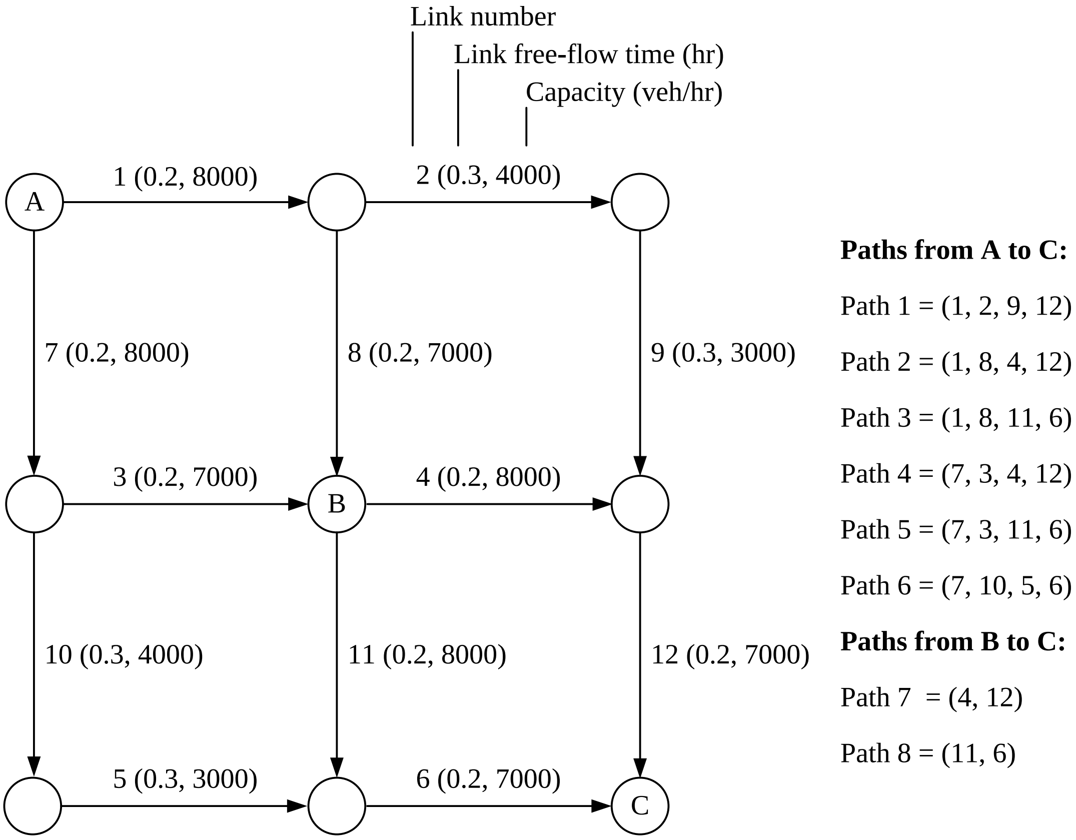

Fig. 8. A grid network for Examples 4 and 5.

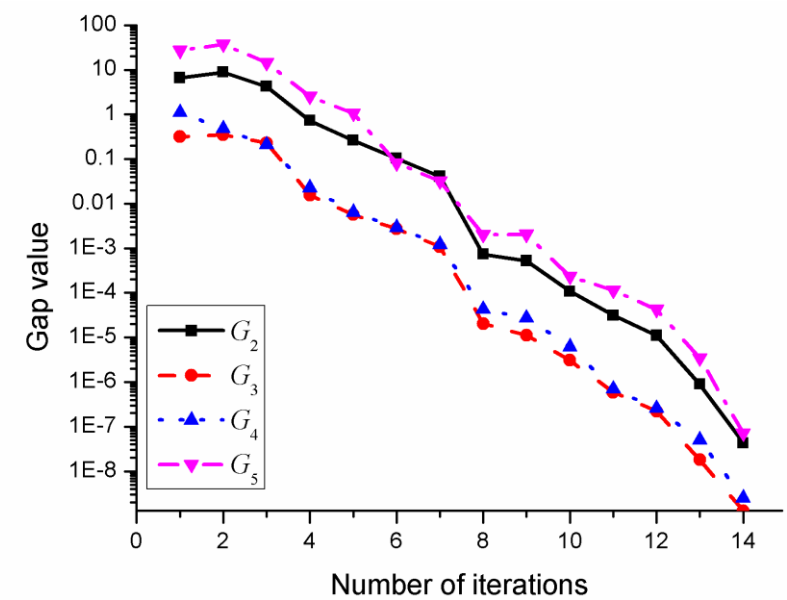

Fig.9. Convergence of the BFGS method for solving the DUO-SRDTC problem with fixed demand in Example 4.

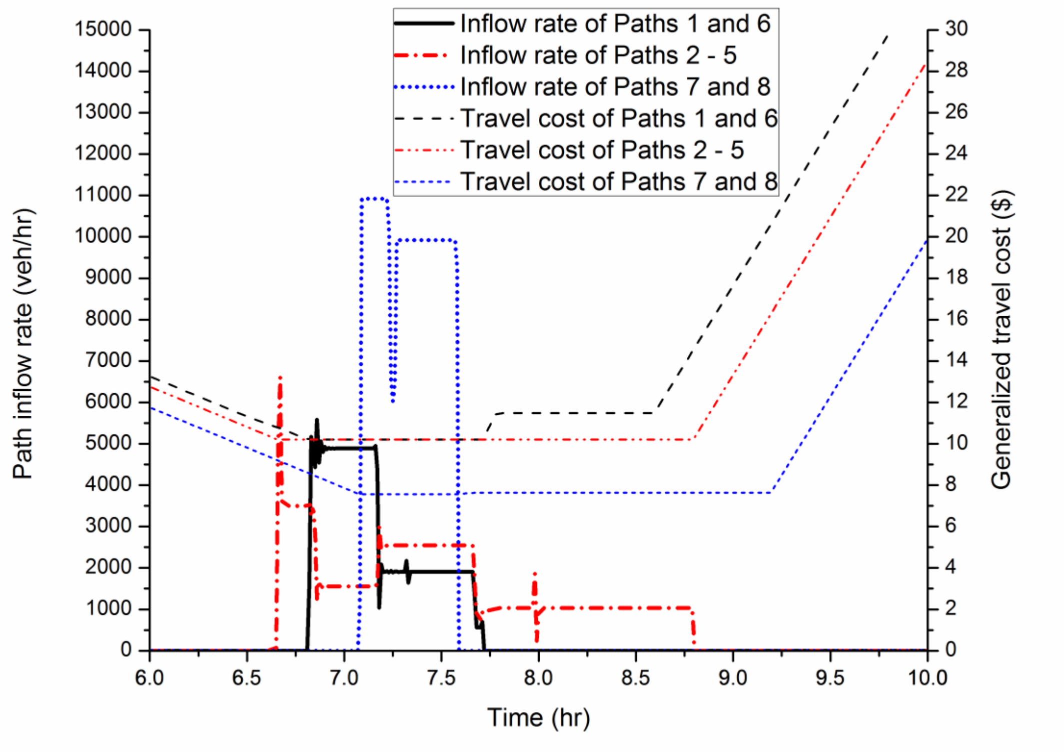

Fig. 10. Path flow rates and travel costs obtained by the proposed method in Example 4.

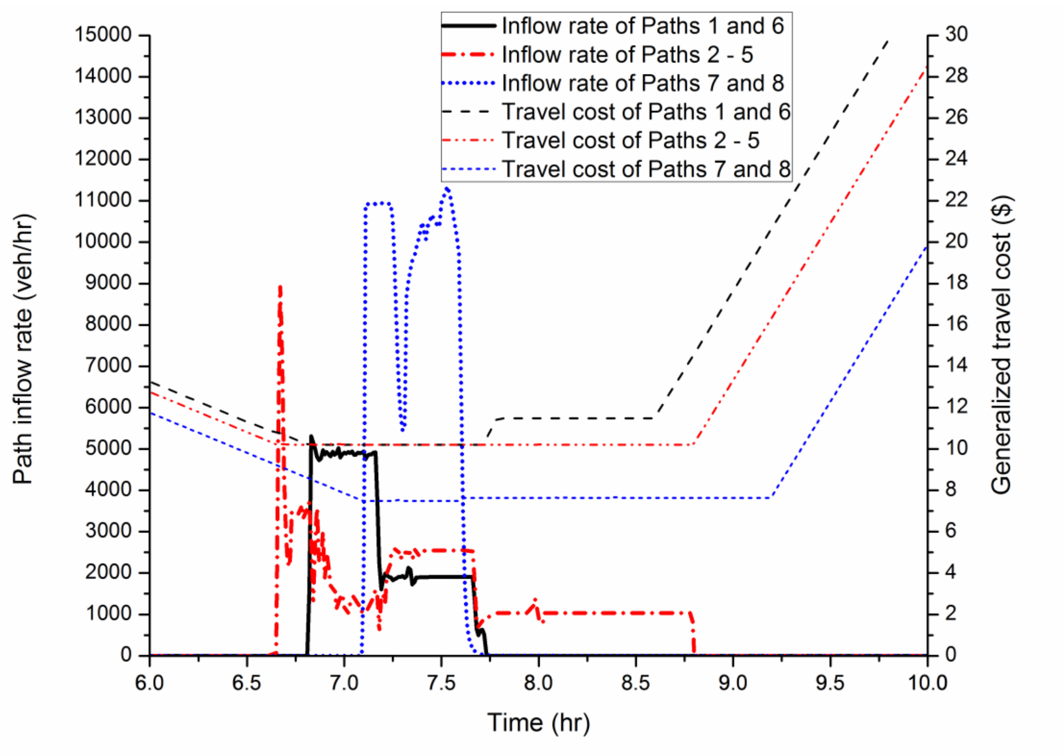

Fig. 11. Path flow rates and travel costs obtained by the route-swapping method in Example 4.

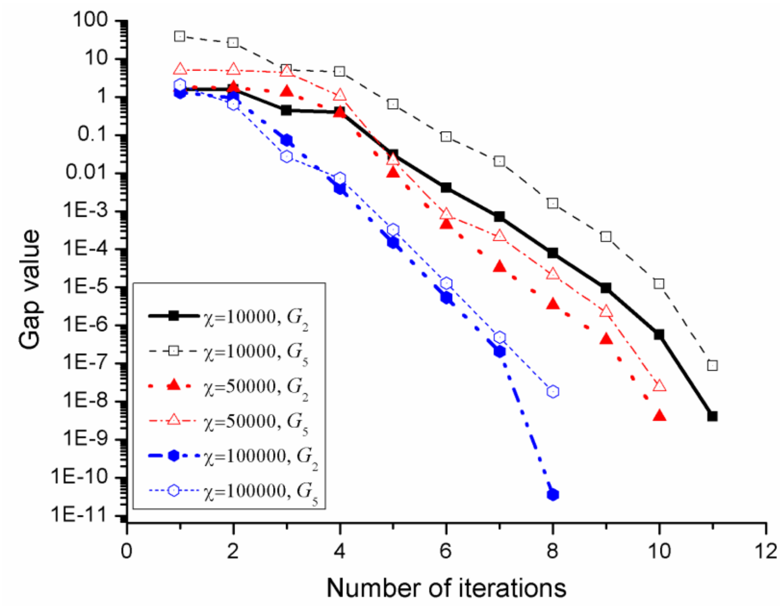

Fig. 12. Convergence of the solution algorithm for solving the DUO-SRDTC problem with elastic demand in Example 5.

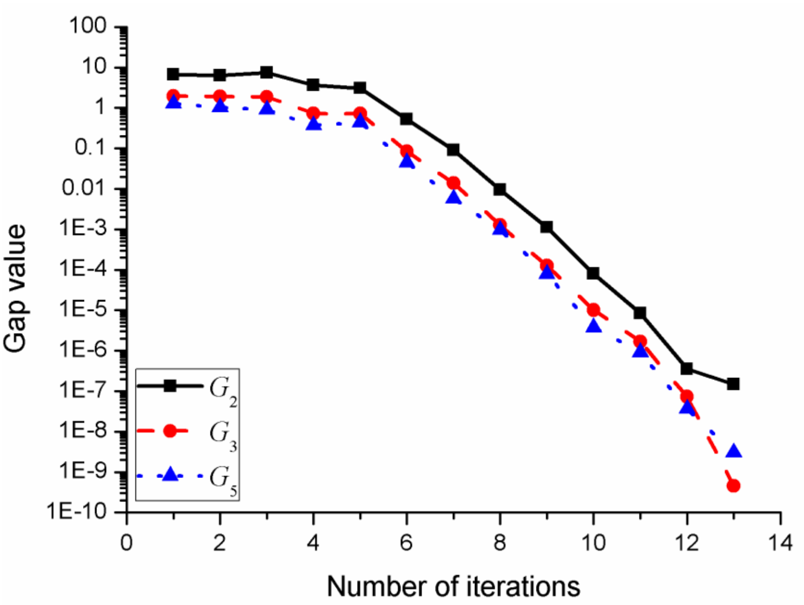

Fig. 13. Convergence of the solution algorithm for solving the DUO-SRDTC problem with elastic demand in Example 6.