A system of coupled partial differential equations exhibiting both elevation and depression rogue wave modes

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ABSTRACT

Analytical solutions are obtained for a coupled system of partial differential equations involving hyperbolic differential operators. Oscillatory states are calculated by the Hirota bilinear transformation. Algebraically localized modes are derived by taking a Taylor expansion. Physically these equations will model the dynamics of water waves, where the dependent variable (typically the displacement of the free surface) can exhibit a sudden deviation from an otherwise tranquil background. Such modes are termed ‘rogue waves’ and are associated with ‘extreme and rare events in physics’. Furthermore, elevations, depressions and ‘four-petal’ rogue waves can all be obtained by modifying the input parameters.

KEYWORDS: Breathers, algebraic solitons, rogue waves.

1. Introduction

Rogue waves are unexpectedly large displacements from an equilibrium position or an otherwise tranquil background [1, 2]. Even though such dangerous waves have been known to the maritime community for nearly a century, large scale theoretical studies in hydrodynamics began only recently [1]. With the observation of rogue wave modes in optical fibers as waveguides, studies of such large amplitude motions have been pursued across a broad spectrum of physical disciplines, under the general category of 'extreme and rare events in physics' [2].

The widely used model for rogue waves is the nonlinear Schrödinger (NLS) equation (γ = a real parameter, * = complex conjugate),

$$i\Psi_t + \Psi_{xx} + \gamma\Psi^2\Psi^* = 0, \qquad (1)$$

where the complex valued, slowly varying wave envelope Ψ evolves under the influence of quadratic dispersion and cubic nonlinearity (t, x being slow time and group velocity coordinate in fluid mechanics respectively) [3]. Rogue wave modes (Peregrine solitons) are analytical solutions algebraically localized in x and t [4]:

$$\Psi = r\exp\left(i\gamma r^2 t\right)\left\{1-\frac{2\left(1+2i\gamma r^2 t\right)}{\gamma r^2\left(x^2+2\gamma r^2 t^2+\dfrac{1}{2\gamma r^2}\right)}\right\}, \qquad r \text{ real}, \tag{2}$$

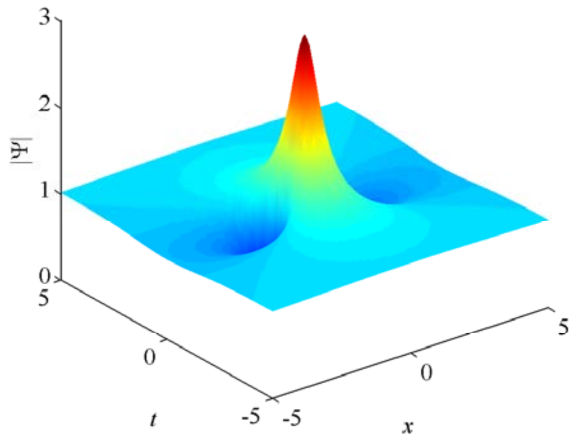

Figure 1 (Color online) Peregrine soliton [Eq. (2)] for the nonlinear Schrödinger equation [Eq. (1)] with $\gamma = r = 1$.

and are only nonsingular for $\gamma > 0$. The main displacement occurs near $x = t = 0$ as an elevation above the background plane (or continuous) wave $\Psi = r\exp(i\gamma r^2 t)$. Eq. (1) conserves the intensity $\int_{-\infty}^{\infty} |\Psi|^2 dx$ for localized boundary conditions, and hence there will be accompanying depressions nearby (Figure 1).

For special coupled NLS equations (commonly known as the Manakov system) with two components (Ψ and Φ),

$$i\Psi_t + \Psi_{xx} + \gamma(\Psi\Psi^* + \Phi\Phi^*)\Psi = 0, \quad i\Phi_t + \Phi_{xx} + \gamma(\Psi\Psi^* + \Phi\Phi^*)\Phi = 0,$$

nonsingular algebraically localized modes can also occur for $\gamma < 0$, in sharp contrast with the single component case Eq. (1) [5]. The main displacement is then a depression below the mean level in the center of the x, t plane. The character of the rogue wave mode (elevation or depression) thus appears to depend critically on the parameters of the partial differential equations.

Other than the NLS systems, many other evolution equations exhibit rogue wave modes, e.g. the Hirota equation [6], the Kadomtsev-Petviashvili equation [7], the long wave-short wave resonance model [8], and systems displaying $\mathscr{PT}$-symmetry [9]. The goal of the present work is to propose still another system of partial differential equations (PDEs) which possesses rogue wave modes. The novel characters include

● Formulations for breathers (pulsating modes) and rogue waves are given.

● The transition in wave profiles among ‘elevations’, ‘depressions’ and ‘four-petal configurations’ results from variation in one single parameter in the solution of the PDEs, and not the PDEs themselves. Physically this parameter is the wave number of the carrier wave packet. In other words, for a fixed system of PDEs, different families of wave profiles can be observed by changing the input wavelength.

For the widely studied NLS equation of one single complex valued component, rogue waves are possible only if dispersion and nonlinearity are of the same sign [1, 2]. The situation is more intriguing for two or more components. Similarities and differences between the proposed system and the known ones will be highlighted in the discussion on wave profiles (Section 4).

● The appropriate range of this parameter can also be predicted precisely from an analysis of modulation instability (i.e. linear stability of the plane wave).

2. A system of coupled partial differential equations

Consider the system

$$A_{xx} + \lambda A_{yy} + (u_{xx} + \lambda u_{yy})A + \mu A = 0, \tag{3a}$$

$$u_{xy} = \sigma(AA^* - C), \tag{3b}$$

where A and u are complex and real valued respectively. The parameters λ, μ, σ and C are real. Eqs. (3a, 3b) do indeed bear close resemblance to widely studied model systems in physical applications, e.g. the Davey-Stewartson equations for water waves in fluid mechanics [10, 11, 12], where A represents a slowly varying wave envelope while u typically measures the induced mean flow. The sign of λ in Eq. (3a), related to the hyperbolic or elliptic character of the equation, will be a crucial factor. In particular we shall focus later on the case of a hyperbolic differential operator.

An elegant method to find multi-soliton of special nonlinear PDEs is the Hirota bilinear method [13, 14]. We shall demonstrate here that this bilinear method is also effective in obtaining the rogue wave modes [8, 15]. A sequence of dependent variable transformations is implemented (ρ = a real amplitude parameter):

$$A = \rho \exp[i(\alpha x+\beta y)]\, g/f, \quad u = 2 \log f, \quad g \text{ complex and } f \text{ real.} \tag{4}$$

The resulting bilinear equations are

$$(D_x^2 + 2i\alpha D_x + \lambda D_y^2 + 2i\lambda\beta D_y)g.f = 0, \quad \mu=\alpha^2+\lambda\beta^2, \tag{5a}$$

$$D_x D_y f.f = \sigma\rho^2(gg^* - f^2), \quad C = \rho^2, \tag{5b}$$

where D is the Hirota bilinear operator [13, 14]:

$$D_x^m D_t^n g.f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n g(x,t)\, f(x',t')|_{x'=x,t'=t} \,. \tag{6}$$

A remark on the historical development is in order. Eqs. (5a, 5b) actually were first studied in a systematic investigation of complex bilinear equations exhibiting 2-soliton solutions [16]. The investigation then was restricted to the case of zero boundary condition in the far field. Here the scope is extended to wave pattern with nonzero far field condition. Other contributions of the present work include

● deducing the governing PDEs (Eq. 3),

● obtaining the breathers and rogue waves, and finally,

● performing an analysis on modulation instability, and thus correlating precisely with the criterion for the onset of rogue waves.

3. Breathers and rogue waves

Following earlier works in the literature [8], a family of analytical solutions termed breathers can typically be obtained through an expansion scheme (p, q complex, $\zeta^{(1)}$, $\zeta^{(2)}$ being arbitrary phase factors):

$g = 1 + a_1 \exp(px + qy + \zeta^{(1)}) + a_2 \exp(p^*x + q^*y + \zeta^{(2)})$

$+ Ma_1a_2 \exp[(p + p^*)x + (q + q^*)y + \zeta^{(1)} + \zeta^{(2)}],$ (7a)

$f = 1 + \exp(px + qy + \zeta^{(1)}) + \exp(p^*x + q^*y + \zeta^{(2)})$

$+ M \exp[(p + p^*)x+(q + q^*)y + \zeta^{(1)} + \zeta^{(2)}]$. (7b)

Case (A) $\lambda = -1$, $\alpha = -\beta$, $\sigma > 0$

The algebra simplifies considerably in the special case,

$$\lambda = -1, \alpha = -\beta, \tag{8}$$

and thus the differential operators in Eq. (3) will be hyperbolic. The parameters in Eq. (7) are

$$a_1 = -1 + \frac{4i\alpha}{p - q + 2i\alpha}, a_2 = -1 + \frac{4i\alpha}{p^* - q^* + 2i\alpha}, \; M = \frac{(p - p^*)^2 - (q - q^*)^2}{(p + p^*)^2 - (q + q^*)^2} ,$$

and the connection between p and q, i.e. a dispersion relation, is then

$$pq = -\frac{2\sigma\rho^2 (p-q)^2}{4\alpha^2 + (p-q)^2}.$$

For a nonsingular solution, q needs to be genuinely complex for purely imaginary p, implying a constraint $\alpha^2 < 2\sigma\rho^2$.

A popular method to derive the rogue wave is the Darboux transformation [17, 18], but Eqs. (4 – 7) have clearly demonstrated that the Hirota bilinear method is also a feasible scheme in computing breathers (and subsequently rogue waves) [8, 19]. This alternative is especially valuable as most integrable systems possess Hirota forms. To generate the rogue waves, a long wave limit (p tending to zero) is now taken with the provision $\exp(\zeta^{(1)}) = \exp(\zeta^{(2)}) = -1$ [8, 15, 19].

The rogue wave mode is given by

$$A = \rho \exp\left\{ i\alpha\left(x-y\right)\right\} \times \left\{ 1 + \frac{2\left[-1+i\alpha\left(x-y\right)\right]}{\sigma\rho^2\left[\left(x+\frac{y(\sigma\rho^2-\alpha^2)}{\sigma\rho^2}\right)^2 + \frac{\alpha^2\left(2\sigma\rho^2-\alpha^2\right)}{\left(\sigma\rho^2\right)^2}y^2 + \frac{1}{2\sigma\rho^2-\alpha^2}\right]}\right\}, \tag{9}$$

and the constraint in the dispersion relation for complex q, i.e. $0 < \frac{\alpha^2}{\sigma\rho^2} < 2$, will ensure that Eq. (9) is nonsingular.

Case (B) $\lambda = -1$, $\alpha = \beta$, $\sigma = -\sigma_0 < 0$ (or $\sigma_0 > 0$)

Historically from earlier studies in fluid mechanics [3], it is instructive to consider a different range for the parameters (even though mathematically the results can be deduced by shifting the independent variable). In this case the system becomes

$$A_{xx} - A_{yy} + (u_{xx} - u_{yy})A + \mu A = 0,$$

$$u_{xy} = -\sigma_0(AA^* - C).$$

A similar mechanism can be developed to arrive at the rogue wave solution:

$$A = \rho \exp\{i\alpha(x+y)\} \times$$

$$\left\{ 1 - \frac{2[1 - i\alpha(x+y)]}{\sigma_0\rho^2 \left[\left(x + \frac{y(\alpha^2 - \sigma_0\rho^2)}{\sigma_0\rho^2} \right)^2 + \frac{\alpha^2 y^2 (2\sigma_0\rho^2 - \alpha^2)}{\sigma_0{}^2\rho^4} + \frac{1}{2\sigma_0\rho^2 - \alpha^2} \right]} \right\}, \qquad (10)$$

$$u = 2\log\left[\left(x + \frac{y(\alpha^2 - \sigma_0\rho^2)}{\sigma_0\rho^2} \right)^2 + \frac{\alpha^2 y^2 (2\sigma_0\rho^2 - \alpha^2)}{\sigma_0{}^2\rho^4} + \frac{1}{2\sigma_0\rho^2 - \alpha^2} \right],$$

provided that $0 < \alpha^2 < 2\sigma_0\rho^2$ and $C = \rho^2$.

4. Wave Profiles

The constraint concerning 'elevations' and 'depressions' must be clarified first. From Eq. (3b), if u_y is localized in x and y, then the integral

$$\int_{-\infty}^{+\infty} \left(AA^* - \rho^2 \right) dx$$

vanishes, implying that any 'elevations' above the mean level ρ must be accompanied by nearby 'depressions'. Nevertheless, a rogue wave is termed an 'elevation' here if the main displacement in the center is above the mean level, despite the fact that several depressions below the mean level must be present nearby to conserve the integral.

For Eq. (3a, 3b) the rogue wave profiles exhibit an intriguing sequence of transformations upon changes in the parameters. It will first be instructive to review the known properties of the widely studied NLS equation. For the single component case, rogue waves can only occur when dispersion and nonlinearity are of the same sign. Only modes with an elevation in the center are possible, and the maximum displacement is always three times the background. For the two-component case (two complex valued envelopes), a change in the relative frequency (or wave number) between the waveguides can lead to transformations among elevations, depressions and ‘four-petal’ configurations, where two elevations co-exist with two depressions [20]. Similar transitions can also be observed in the long wave-short wave resonance system (LWSW) [21], with one complex valued envelope and one real valued function. However, the derivative in one independent variable in such LWSW system is only of the first order. The major differences between the present efforts and earlier works are that the present system (Eqs. (3a, 3b)) involves (a) just *one* complex valued envelope (A), and (b) a *second* order derivative for both independent variables.

The dynamics of the transition can now be described more precisely. For the present system, elevation, depression and 'four-petal' types rogue wave modes can be found by varying the wave number only, without changing the sign of σ or other parameters of the PDEs Eqs. (3a, 3b). This contrast with the case of one component NLS system can be illustrated through a typical example say $\sigma\rho^2 = 1$, where the existence condition then requires the restriction

$$0 < \alpha < \sqrt{2}\,. \tag{11}$$

As α decreases slightly from the upper bound, a small amplitude rogue wave will emerge from the plane wave background. The main displacement is below the plane wave background (Figure 2a) and is thus a depression type rogue wave. As α decreases further, the depression deepens and splits into two smaller 'valleys', forming a four-petal pattern (Figure 2b). Just before the splitting occurs, the intensity hits a minimum value of zero. In the optical context, this state may be termed a 'black' rogue wave due to the zero intensity of light at the center.

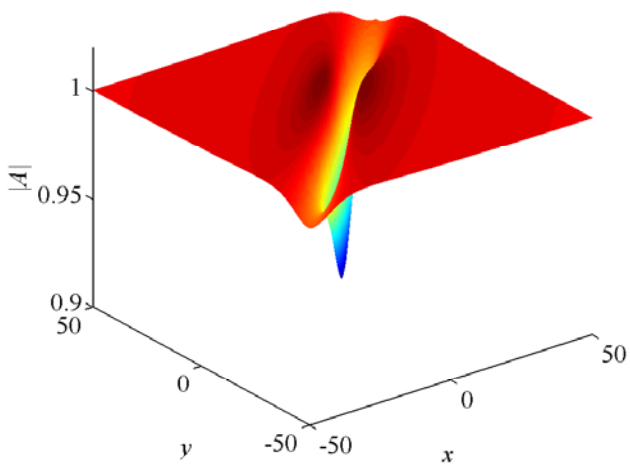

Figure (2a) (Color online) Rogue waves as given in Eq. (9) with $\sigma = \rho = 1$:

Depression type rogue wave when $\alpha = 1.4$

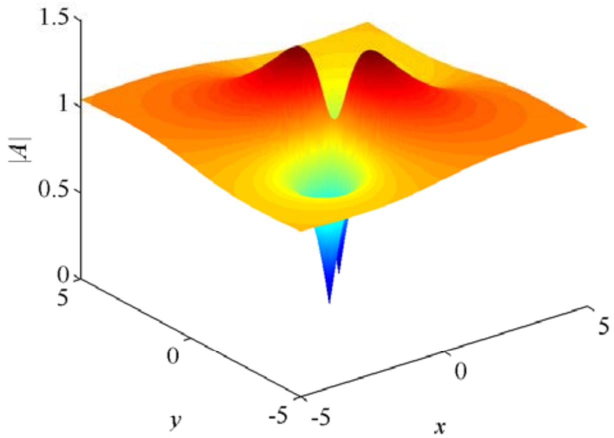

Figure (2b) (Color online) Rogue waves as given in Eq. (9) with $\sigma = \rho = 1$:

Four-petal shaped rogue wave when $\alpha = 1$.

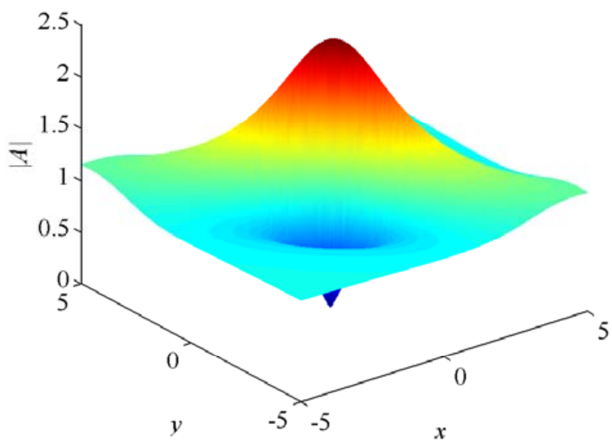

Figure (2c) (Color online) Rogue waves as given in Eq. (9) with $\sigma = \rho = 1$: Elevation type rogue wave when $\alpha = 0.5$.

When α is reduced further, the two maxima (instead of the two valleys) will eventually merge, giving rise to a typical elevation type rogue wave (Figure 2c). Unlike the NLS Peregrine solitons, the maximum displacement is not always three times the background amplitude, but is instead just bounded above by this constraint. Finally as α tends to zero in Eq. (11), the solution Eq. (9) loses the character of a rogue wave and becomes a traveling wave (Figure 3).

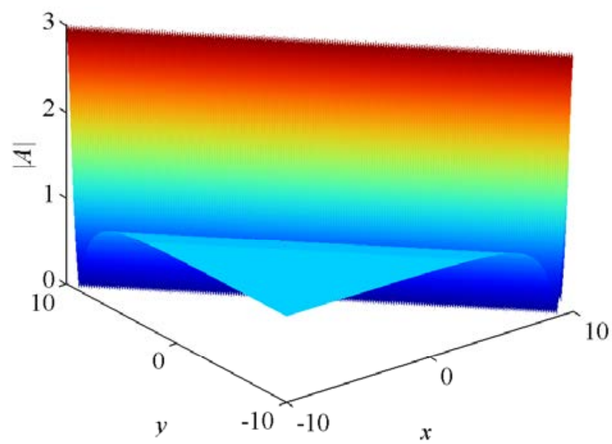

Figure 3 (Color online) Traveling wave solution as given in Eq. (9) with $\sigma = \rho = 1$ and $\alpha = 0$.

5. Modulation instability

Modulation instability refers to the exponential growth of background noise in the propagation of a plane wave in a system where dispersion and nonlinearity interact. Detailed discussions have been given in monographs in the literature of hydrodynamics [22] and optics [23], and hence the presentation here will be brief. Starting with the plane wave

$A = \rho\exp[i\alpha(x-y)], \quad u = 0$

of Eq. (3), one introduces perturbations

$A = \rho\exp[i\alpha(x-y)](1 + A'), \quad u = u'.$

Linearization and seeking modes of the form $\exp[i(kx - sy)]$ will yield

$(s^2 - k^2)^2(ks + 2\sigma\rho^2) = 4ks\alpha^2(s - k)^2\,.$

Following the reasoning that the rogue waves are the long wave disturbances, we look for $s = \xi k + O(k^2)$ as k tends to 0. Complex values of ξ (or instability) will arise if Eq. (11) holds, implying that the onset of modulation instability correlates precisely with the existence criterion of rogue wave. This result confirms again similar trends found in the literature for NLS equation and other systems [5, 8].

6. Discussions and conclusions

A system of partial differential equations with localized analytical solutions is proposed. The recently studied mechanisms for rogue waves, e.g. the Hirota bilinear method and the connection with modulation instability, are demonstrated to be applicable. A very valuable feature for this system is that transitions among various classes of wave profiles, e.g. 'elevations', 'depressions' and 'four-petal' configurations, can be observed by modifying the input parameters (or physically the incident wave). In contrast with earlier results in the literature, such transformations in the present system are attained with just one complex valued envelope [20] and in a system with a second order derivative for both independent variables [21]. The Hirota method, demonstrated to be effective for discrete systems [24], is thus useful for rogue wave too, an entity of importance in fluids, optics and plasma [25, 26].

Equations exhibiting similar analytic structures have been used extensively in modeling water waves in fluid mechanics. Several examples will be discussed:

- A slowly varying envelope subject to long wavelength modulations in two mutually perpendicular, horizontal dimensions will lead to the Davey-Stewartson equations, where the complex valued envelope is coupled to the real valued induced mean flow [10]. The governing differential operator for the mean flow (corresponding to the real valued u of Eqs. (3a, 3b)) may be elliptic or hyperbolic, depending on the properties of the fluid like depth or surface tension. The character of the differential equations may have significant implications in terms of singularity formation (or building up of large amplitude motion physically) [11]. Similar Davey-Stewartson type equations also arise in a two-layer fluid [12].

● In hydrodynamic or optical waveguide media with inhomogeneous material properties, variable-coefficient type nonlinear Schrödinger equations will occur [27, 28]. Whether such variable-coefficient equations can generate rogue waves will need further investigations.

● Instead of considering waves on fluids of finite depth, dynamical systems which elucidate motion on shallow water (long waves), e.g. Kadomtsev Petviashvili and Boiti-Leon-Manna-Pempinelli equations also exhibit intriguing features regarding solitons [29, 30, 31].

Besides hydrodynamic waves [22], the subject of modulation instability has also been studied extensively in optical physics [23], since such instability will lead to growth and saturation of plane waves in temporal and spatial optical waveguides. All these indicators will make Eqs. (3a, 3b) a valuable system in future modeling studies in applied mathematics [32].

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