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On Briot-Bouquet differential equations

Many nonlinear partial differential equations encountered in physics are autonomous, i.e. do not depend explicitly on the independent variables x (space) and t (time). In such a case, they admit a reduction, called traveling wave reduction, to an autonomous nonlinear ordinary differential equation (ODE), defined in the simplest case by $Y(x,t) = y(\xi)$, $\xi = x - ct$, with c a constant speed.
It turns out that many ODEs obtained from the traveling wave reduction is of the form

$$P(y^{(k)}, y) = 0 \ ,$$

where $P(u,v)$ is an irreducible polynomial of two variables. This is the so-called Briot-Bouquet differential equations. It was proved by Briot and Bouquet in 1856 (when $k = 1$) and Picard in 1880 (when $k = 2$) that any solution y which is meromorphic in the complex plane must belong to the class W, which consists by definition of the following functions: (i) rational functions; (ii) rational functions of $\exp(az)$, $a \in \mathbb{C}$; (iii) elliptic functions. In 1982, Eremenko proved that for every k, if the genus of the algebraic curve defined by $P(u,v) = 0$ is one, then all meromorphic solutions must be elliptic functions. He also proved that when k is odd and a solution y is meromorphic in the plane and has at least one pole, then y must belong to the class W. He then conjectured that any meromorphic solution of a Briot-Bouquet differential equation must be one of the above three types. In this talk, we shall report some recent progress on this conjecture.