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Session 1
On modeling dependence in claim-number processes

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Current Topics on Actuarial Models with Dependence Structure

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Outline

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1. Introduction
2. The model
3. Some actuarial issues
   - Expected discounted penalty function
   - Optimal Dividends
   - Optimal Reinsurance
4. Some practical issues

Motivation

In actuarial science, it is well known that independent assumptions in most of the models are very restrictive and somewhat unrealistic.

As the rapid growth of insurance and investment products introduces a great deal of complexity to the valuation of financial firms, careful and thorough assessment of dependent risks is crucial to the development of sophisticated tools for dynamic financial analysis.

Although the incorporation of correlation among risks makes various actuarial and financial problems difficult to deal with, the analysis of general dependence structure is still possible with the advancement of mathematical and statistical tools.

Due to the importance of the topic of study, much insightful research in this direction has been carried out in recent years.

Modeling aggregate claims distribution

In order for an insurer to ensure that accurate estimation of the underlying risks and hence appropriate corporate decisions can be made, it is very important to model the aggregate claims distribution adequately.

Under the stationary and independent increment assumptions, the surplus process of an insurance company possesses mathematically tractable properties, and hence many well-known and insightful results such as explicit solutions for some actuarial quantities of interest can be obtained in classical insurance risk theory.

However, ignoring dependencies among various attributes in insurance modeling can result in drastic differences in the assessment of various risk measures which are of particular interest to insurers and regulators.

Dependence in actuarial models

- Time dependence, Class dependence
  - Claim numbers
  - Class sizes
  - Claim arrival time and claim size
  - Claim size and premium
  - etc.

Accidents always happen

- April 19, 2015 (just outside HKU)

Events of large claims

- Natural disasters

- Sever accidents

Natural disasters

- Earthquake (Haiti, 2010)

- Flood (US, 2005)

- Tsunami (Japan, 2011)

- Tornado

Severe accidents

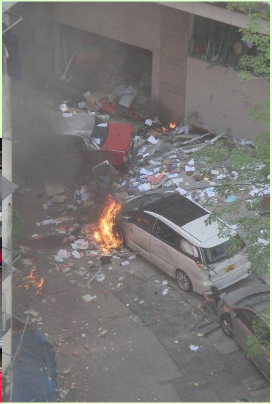
- Aircraft Crash (Taiwan, 2015)

Severe accidents

- Car Accident

Severe accidents

- Fire (Hong Kong, 2015)



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Classical risk model

Under the classical risk model, the surplus process follows

$$U(t) = u + ct - S(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \qquad U(0) = u,$$

in which $u \geq 0$ is the initial surplus, c is the premium rate, and $S(t)$ representing the aggregate claims up to time t is a compound Poisson process, the claim-number process $N(t)$ is a homogeneous Poisson process with intensity λ, and $\{X_i, i \geq 1\}$ is a sequence of positive, independent and identically distributed claim-amount random variables. It is assumed that the claim-number process $N(t)$ is independent of the claim amounts X_i.

Ruin time

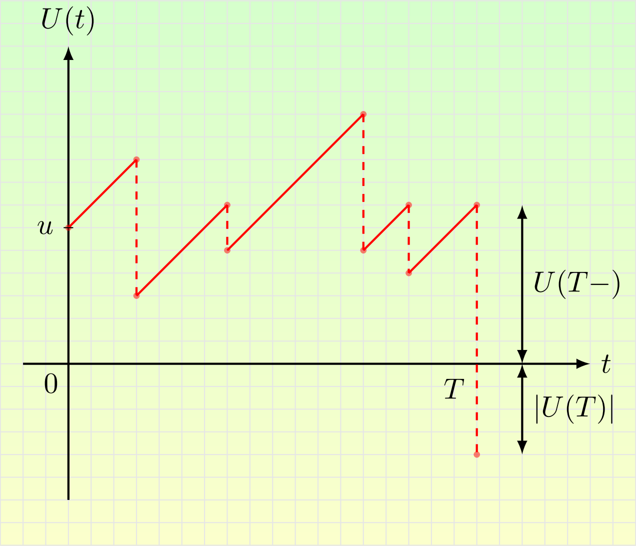
Define the time of ruin as

$$T = \inf\{t : U(t) < 0\}.$$

If $U(t) \geq 0$ for all t, then $T = \infty$. The ultimate ruin probability for the risk model given the initial surplus u is given by

$$\psi(u) = P(T < \infty | U(0) = u).$$

Sample path

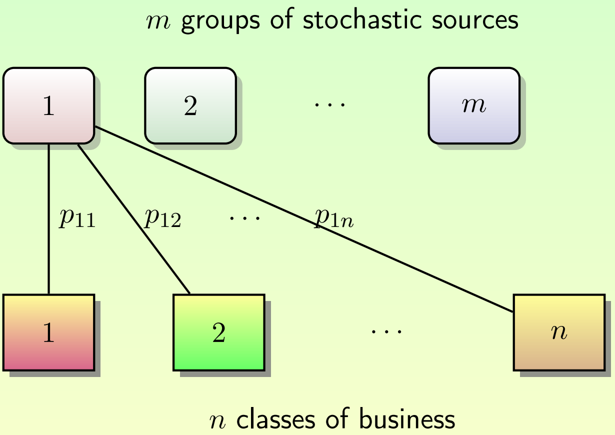


Risk model with thinning-dependence structure

Suppose that an insurance company has n $(n \geq 2)$ dependent classes of business. Stochastic sources that may cause a claim in at least one of the n classes are classified into m groups. It is assumed that each event occurred at time t in the kth group may cause a claim in the jth class with probability p_{kj} for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Denote by $N^k(t)$ the number of events from the kth group occurred up to time t. Let $N_j^k(t)$ be the number of claims of the jth class up to time t generated from the events in group k, and $X_i^{(j)}$ be the amount of the ith claim in the jth class.

m groups of stochastic sources



Aggregate claim process of individual class

Then, the total amount of claims for the jth class up to time t can be expressed as

$$S_j(t) = \sum_{i=1}^{N_j(t)} X_i^{(j)} \ ,$$

where $N_j(t) = N_j^1(t) + N_j^2(t) + \cdots + N_j^m(t)$ is the claim-number process of the jth class.

Aggregate claim process of the company

The aggregate claims process of the company is given by

$$S(t) = \sum_{j=1}^{n} S_j(t) = \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} X_i^{(j)} , \quad (1)$$

where $\{X_i^{(j)};\ i = 1, 2, \cdots\}$ is assumed to be a sequence of i.i.d. non-negative random variables with common distribution F_j for each j. As usual, we assume that the n sequences $\{X_i^{(1)};\ i = 1, 2, \cdots\}, \ldots, \{X_i^{(n)};\ i = 1, 2, \cdots\}$ are mutually independent and are independent of all claim-number processes.

Surplus process of the company

The surplus process of the insurance company is defined as

$$U(t) = u + ct - S(t) = u + ct - \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} X_i^{(j)} \, . \tag{2}$$

In order to make the analysis of $U(t)$ mathematically tractable, we need the following assumptions:

(A1) The processes $N^1(t), \ldots, N^m(t)$ are independent Poisson processes with parameters $\lambda_1, \ldots, \lambda_m$, respectively. For $k \neq k'$, the two vectors of claim-number processes, $(N^k(t), N_1^k(t), \ldots, N_n^k(t))$ and $(N^{k'}(t), N_1^{k'}(t), \ldots, N_n^{k'}(t))$, are independent.

(A2) For each k $(k = 1, \ldots, m)$, $N_1^k(t), \ldots, N_n^k(t)$ are conditionally independent given $N^k(t)$.

Advantages of model (2)

- Risk model (2) not only possesses the thinning-dependence structure but also embraces the risk model with common shock.
- Risk model (2) is still a compound Poisson risk model:

$$\begin{aligned} U(t) &= u + ct - S(t) = u + ct - \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} X_i^{(j)} \\ &\sim \tilde{U}(t) = u + ct - \tilde{S}(t) = u + ct - \sum_{i=1}^{\tilde{N}(t)} Z_i. \end{aligned}$$

where $\tilde{S}(t)$ is a compound Poisson process with transformed Poisson claim-number process $\tilde{N}(t)$ and transformed claim-sizes Z_i.

Special cases

- If $m = n$, $p_{kk} = 1$ for $k = 1, \ldots, n$, and $p_{ij} = 0$ for $i \neq j$ and $i, j = 1, \ldots, n$, then $U(t)$ of (2) is the sum of n independent compound Poisson processes.
- If $m = n$ and $p_{kk} = 1$ for $k = 1, \ldots, n$, then $U(t)$ of (2) is the thinning risk model of Yuen and Wang (2002).
- If $n = 2$, $m = 3$, $p_{12} = p_{21} = 0$, $p_{31} = p_{32} = 1$, $p_{11} = p_{22} = 1$, then $U(t)$ of (2) is the risk model with common shock for two dependent classes of business discussed in Cossette and Marceau (2000).

MGF of $S(t)$

Denote the moment generating function of $X^{(j)}$ and $S(t)$ by $M_j(r)$ and $M_S(r)$, respectively. Then,

$$\begin{aligned} M_S(r) &= E[\exp\{rS(t)\}] \\ &= \exp\left\{t\sum_{k=1}^{m}\lambda_k\left(\prod_{j=1}^{n}\Big(p_{kj}M_j(r)+1-p_{kj}\Big)-1\right)\right\} \end{aligned}$$

which implies that $S(t)$ is a compound Poisson process with intensity

$$\lambda=\lambda_1\left(1-\prod_{j=1}^{n}(1-p_{1j})\right)+\cdots+\lambda_m\left(1-\prod_{j=1}^{n}(1-p_{mj})\right).$$

Example

For $n = 2$ and $m = 3$,

$$S(t) = \sum_{i=1}^{N_1^1(t)+N_1^2(t)+N_1^3(t)} X_i^{(1)} + \sum_{i=1}^{N_2^1(t)+N_2^2(t)+N_2^3(t)} X_i^{(2)}$$

is a compound Poisson process with intensity λ and distribution function G given by

$$\begin{aligned} \lambda &= \lambda_1\big(1-(1-p_{11})(1-p_{12})\big) + \lambda_2\big(1-(1-p_{21})(1-p_{22})\big) + \\ &\quad \lambda_3\big(1-(1-p_{31})(1-p_{32})\big) \\ &= \lambda_1(p_{11}+p_{12}-p_{11}p_{12}) + \lambda_2(p_{21}+p_{22}-p_{21}p_{22}) + \\ &\quad \lambda_3(p_{31}+p_{32}-p_{31}p_{32}) \ , \end{aligned}$$

and

$$\begin{aligned} G(x) &= \frac{\lambda_1 p_{11}(1-p_{12}) + \lambda_2 p_{21}(1-p_{22}) + \lambda_3 p_{31}(1-p_{32})}{\lambda} F_1(x) \\ &\quad + \frac{\lambda_1 p_{12}(1-p_{11}) + \lambda_2 p_{22}(1-p_{21}) + \lambda_3 p_{32}(1-p_{31})}{\lambda} F_2(x) \\ &\quad + \frac{\lambda_1 p_{11} p_{12} + \lambda_2 p_{21} p_{22} + \lambda_3 p_{31} p_{32}}{\lambda} F_1 \star F_2(x) \ , \end{aligned}$$

where $F_1 \star F_2$ represents the convolution of F_1 and F_2. □

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Some actuarial issues

- Expected discounted penalty function
- Optimal Dividends
- Optimal Reinsurance

Expected discounted penalty function

The Gerber-Shiu (1998) expected discounted penalty function under the surplus process is given by

$$m(u) = \mathsf{E}\left[e^{-\alpha T} w(U(T-), |U(T)|) I(T < \infty) | U(0) = u\right],$$

where $I(A)$ is the indicator function of event A, and the parameter $\alpha > 0$ can be interpreted as a force of interest.

Special Cases

- If $\delta = 0$ and $w(y_1, y_2) = 1$, we have
  $m(u) = \xi(u) = P(T < \infty | U_0 = u)$ - ruin probability.
- If $\delta = 0$ and $w(y_1, y_2) = I(y_1 < u_1, y_2 < u_2)$, we have
  $m(u) = P(U(T-) < u_1, U(T) < u_2 | U(0) = u)$
  - joint distribution of surplus immediately before ruin and deficit at ruin.
- If $\delta = 0$ and $w(y_1, y_2) = I(y_1 < u, y_2 < \infty)$, we have
  $m(u) = P(U(T-) < u | U(0) = u)$
  - distribution of surplus immediately before ruin.
- If $\delta = 0$ and $w(y_1, y_2) = I(y_1 < \infty, y_2 < u)$, we have
  $m(u) = P(U(T) < u | U(0) = u)$
  - distribution of deficit at ruin.

Ruin Probability

Here we shall carry out some numerical analysis for the special case of ruin probability.

To ensure that the ultimate ruin probability is less than 1, we need to set $k = (1+\theta)\lambda \mathrm{E}(Z)$ with the relative security loading $\theta > 0$.

Denote the moment generating function of $X^{(j)}$ and $S(t)$ by $M_j(r)$ and $M_S(r)$, respectively. Assume that $M_j(r)$ exists for $j = 1, 2, \cdots, n$. Hence, $M_Z(r)$ and $M_S(r)$ also exist.

Adjustment coefficient

The adjustment coefficient R is defined as the smallest positive solution of the equation

$$M_S(r) = \exp(rk).$$

From the classical risk theory, we have

$$\psi(u) = \frac{\exp(-Ru)}{\mathrm{E}\left(\exp(-RU_T)|T < \infty\right)},$$

for $u \geq 0$. It is easily seen that

$$\psi(u) \leq \exp(-Ru).$$

Models with two classes

For simplicity, we only study the case with two dependent classes of business, that is, $n = 2$.

To assess the impact of dependence on ruin probability, we consider the following risk models for comparison.

Model I: The claim-number processes for the two classes, $N_t^{\mathrm{I}(1)}$ and $N_t^{\mathrm{I}(2)}$, are mutually independent. The intensity of $N_t^{\mathrm{I}(j)}$ is given by λ_j^{I} for $j = 1, 2$. The surplus process is $U_t^{\mathrm{I}} = u + kt - S_t^{\mathrm{I}}$ with the aggregate claims process S_t^{I} given by

$$S_t^{\mathrm{I}} = S_t^{\mathrm{I}(1)} + S_t^{\mathrm{I}(2)} = \sum_{i=1}^{N_t^{\mathrm{I}(1)}} X_i^{(1)} + \sum_{i=1}^{N_t^{\mathrm{I}(2)}} X_i^{(2)}.$$

Model A: The claim-number process in class j has the form $N_t^{\mathrm{A}(j)} = N_t^{(1j)} + N_t^{(2j)}$ (thinning dependence only) with intensity $\lambda_j^{\mathrm{A}} = \lambda_1 p_{1j} + \lambda_2 p_{2j}$ for $j = 1, 2$. The surplus process is $U_t^{\mathrm{A}} = u + kt - S_t^{\mathrm{A}}$ with the aggregate claims process S_t^{A} given by

$$S_t^{\mathrm{A}} = S_t^{\mathrm{A}(1)} + S_t^{\mathrm{A}(2)} = \sum_{i=1}^{N_t^{\mathrm{A}(1)}} X_i^{(1)} + \sum_{i=1}^{N_t^{\mathrm{A}(2)}} X_i^{(2)}.$$

Model C: The claim-number process in class j has the form $N_t^{C(j)} = N_t^{(jj)} + N_t^{(c)}$ (common shock only) with intensity $\lambda_j^C = \lambda_j + \lambda_c$ for $j = 1, 2$. The surplus process is $U_t^C = u + kt - S_t^C$ with the aggregate claims process S_t^C given by

$$S_t^C = S_t^{C(1)} + S_t^{C(2)} = \sum_{i=1}^{N_t^{C(1)}} X_i^{(1)} + \sum_{i=1}^{N_t^{C(2)}} X_i^{(2)}.$$

Model B: The claim-number process is rewritten as

$$N_t^{\mathrm{B}(j)} = N_t^{\mathrm{B}(1j)} + N_t^{\mathrm{B}(2j)} + N_t^{\mathrm{B}(c)},$$

with intensity $\lambda_j^{\mathrm{B}} = \tilde{\lambda}_1 \tilde{p}_{1j} + \tilde{\lambda}_2 \tilde{p}_{2j} + \tilde{\lambda}_c$ for $j = 1, 2$. To compare Model B to Models I, A and C, we select the parameters, $\tilde{p}_{12}$, $\tilde{p}_{21}$, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and $\tilde{\lambda}_c$ such that

$$\lambda_j^{\mathrm{B}} = \lambda_j^{\mathrm{I}} = \lambda_j^{\mathrm{A}} = \lambda_j^{\mathrm{C}}.$$

Hence, we can compare the four models on a fair basis as they have the same expected aggregate claims. There are more than one set of parameters $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{p}_{12}, \tilde{p}_{21})$.

Here, we consider the following two cases of Model B.

Model B1: Let $\tilde{p}_{12} = 0.5p_{12}$, $\tilde{p}_{21} = 0.5p_{21}$, $\tilde{\lambda}_1 = \lambda_1$, $\tilde{\lambda}_2 = \lambda_2$ and $\tilde{\lambda}_c = 0.5\lambda_c$. Therefore, we have

$$\lambda_1^{\mathrm{B1}} = \lambda_1 + 0.5\lambda_2 p_{21} + 0.5\lambda_c,$$

$$\lambda_2^{\mathrm{B1}} = \lambda_2 + 0.5\lambda_1 p_{12} + 0.5\lambda_c,$$

Model B2: Let $\tilde{p}_{12} = p_{12}$, $\tilde{p}_{21} = p_{21}$ and $\tilde{\lambda}_c = \lambda_c$. The intensities $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are determined by

$$\lambda_1^{\mathrm{B2}} = \tilde{\lambda}_1 + \tilde{\lambda}_2 p_{21} + \lambda_c, \tag{3}$$

$$\lambda_2^{\mathrm{B2}} = \tilde{\lambda}_1 p_{12} + \tilde{\lambda}_2 + \lambda_c, \tag{4}$$

For fair comparison, we need to make sure that the expected aggregate claims of the five models are the same. This can be done by setting

$$\lambda_j^{\mathrm{B2}} = \lambda_j^{\mathrm{B1}} = \lambda_j^{\mathrm{I}} = \lambda_j^{\mathrm{A}} = \lambda_j^{\mathrm{C}}.$$

Let the adjustment coefficients of Models I, A, C, B1 and B2 be R_{I}, R_{A}, R_{C}, R_{B1} and R_{B2}, respectively. One can show that

$$R_{\mathrm{B2}} < R_{\mathrm{A}} < R_{\mathrm{B1}} < R_{\mathrm{C}} < R_{\mathrm{I}}, \tag{5}$$

Exponential claims

Assume that the claim amounts $X_i^{(j)}$ follow an exponential distribution with $F_j(z) = 1 - e^{-\theta_j z}$ for $j = 1, 2$. It is easy to check that

$$F_1 * F_2(z) = \frac{\theta_2}{\theta_2 - \theta_1} F_1(z) + \frac{\theta_1}{\theta_1 - \theta_2} F_2(z).$$

Hence,

$$\begin{aligned} F_Z(z) &= \frac{1}{\lambda}\Bigg(\left(\lambda_1(1-p_{12}) + \frac{\theta_2(\lambda_1 p_{12} + \lambda_2 p_{21} + \lambda_c)}{\theta_2 - \theta_1}\right) F_1(z) \\ &\qquad + \left(\lambda_2(1-p_{21}) + \frac{\theta_1(\lambda_1 p_{12} + \lambda_2 p_{21} + \lambda_c)}{\theta_1 - \theta_2}\right) F_2(z)\Bigg), \end{aligned}$$

which is a mixed exponential distribution.

Numerical Example 1

In the exponential case, the method introduced by Gerber (1979) allows us to calculate the exact value of the ultimate ruin probability $\psi(u)$.

In our numerical study, we compare the ultimate ruin probabilities, $\psi^{\mathrm{I}}(u)$, $\psi^{\mathrm{C}}(u)$, $\psi^{\mathrm{B1}}(u)$, $\psi^{\mathrm{A}}(u)$ and $\psi^{\mathrm{B2}}(u)$, of Models I, C, B1, A and B2.

Let the means of $X_i^{(1)}$ and $X_i^{(2)}$ be $\mu_1 = 1$ and $\mu_2 = 3$, respectively.

We set $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_c = 2$, $p_{12} = 2/5$ and $p_{21} = 2/3$ so that $\tilde{\lambda}_1 = 45/11$ and $\tilde{\lambda}_2 = 15/11$ in Model B2.

The expected aggregate claims per unit time equals 22 in all the models. The constant rate of premium per unit time is arbitrarily chosen as 24.2 with $\theta = 0.1$.

Ultimate ruin probabilities

u	$\psi^{\mathrm{I}}(u)$	$\psi^{\mathrm{A}}(u)$	$\psi^{\mathrm{B1}}(u)$	$\psi^{\mathrm{B2}}(u)$	$\psi^{\mathrm{C}}(u)$
0	0.9091	0.9091	0.9091	0.9091	0.9091
10	0.6128	0.6642	0.6527	0.6701	0.6403
30	0.2871	0.3559	0.3399	0.3644	0.3231
50	0.1346	0.1907	0.1770	0.1982	0.1630
70	0.0631	0.1022	0.0922	0.1078	0.0822
90	0.0295	0.0548	0.0480	0.0586	0.0415
110	0.0138	0.0294	0.0250	0.0319	0.0209
130	0.0065	0.0157	0.0130	0.0173	0.0106
150	0.0030	0.0084	0.0068	0.0094	0.0053
200	0.0005	0.0018	0.0013	0.0021	0.0010

It is well known that the ultimate ruin probability with $u = 0$ for a compound Poisson model has the form

$$\psi(0) = \frac{1}{1+\theta},$$

which depends only on the relative security loading θ. For example, see Bowers et al. (1997). This explains the observation that the ultimate ruin probabilities in the first row for $u = 0$ are level at 0.9091 even though the transformed claim-amount distributions for the five models are different.

For other values of u, the values of the ultimate ruin probability for Model I are the lowest among all as Model I does not introduce any correlation between the two classes of business.

The ultimate ruin probabilities for Model C are smaller than other models with thinning dependence. This reflects that the impact of thinning dependence is larger than that of common shock.

Furthermore, the values for Model A are greater than those for Model B1 but less than those for Model B2. Thus, the ultimate ruin probabilities of the five models can be ordered in the following way:

$$\psi^{\mathrm{I}}(u) < \psi^{\mathrm{C}}(u) < \psi^{\mathrm{B1}}(u) < \psi^{\mathrm{A}}(u) < \psi^{\mathrm{B2}}(u). \tag{6}$$

Non-exponential claims

For non-exponential claims, numerical results can only be obtained via simulations.

In this case, two simulation studies are performed.

One is for gamma and Weibull claim-amount distributions while the other is for lognormal and Weibull claim-amount distributions.

In both cases, the parameters in the claim-number processes are chosen to be $\lambda_1^{\mathrm{I}} = 7$, $\lambda_2^{\mathrm{I}} = 6$, $\lambda_1 = 5$, $\lambda_2 = 4$, $\lambda_c = 2$, $p_{12} = 0.4$ and $p_{21} = 0.5$.

With these parameter values, we get $\tilde{\lambda}_1 = 3.75$ and $\tilde{\lambda}_2 = 2.5$ in Model B2.

Define the N-year finite-time ruin probability defined as

$$\psi_N(u) = \Pr(T < N | U_0 = u). \tag{7}$$

For illustration purpose, we only use $\psi_N(u)$ with a large N to approximate the ultimate ruin probabilities in our simulation studies. We shall see in the two simulation studies that $N = 1,000$ is large enough to get reasonable estimates of the ultimate ruin probabilities.

Numerical example 2

We first consider the case that $X_i^{(1)}$ and $X_i^{(2)}$ follow gamma and Weibull distributions, respectively. Their respective density functions are given by

$$f_1(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta},$$

$$f_2(x) = \frac{\tau (x/\omega)^\tau \exp(-(x/\omega)^\tau)}{x}, \tag{8}$$

with $\alpha = 0.5$, $\beta = 6$, $\omega = 1.5$ and $\tau = 0.5$.

Thus, the means and variances of the claim amounts are $\mathsf{E}(X_i^{(1)}) = \mathsf{E}(X_i^{(2)}) = 3$, $\mathsf{Var}(X_i^{(1)}) = 18$, and $\mathsf{Var}(X_i^{(2)}) = 45$. The expected aggregate claims per unit time for each of the five models is 39 and the premium rate is set to be $k = 46.8$ with $\theta = 0.2$.

Finite-time ruin probabilities with $u = 20$

N	$\psi_N^{\mathrm{I}}(u)$	$\psi_N^{\mathrm{A}}(u)$	$\psi_N^{\mathrm{B1}}(u)$	$\psi_N^{\mathrm{B2}}(u)$	$\psi_N^{\mathrm{C}}(u)$
200	0.4372	0.5243	0.5047	0.5456	0.4939
	(0.0209)	(0.0233)	(0.0216)	(0.0325)	(0.0179)
400	0.4376	0.5245	0.5048	0.5460	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0325)	(0.0180)
600	0.4376	0.5246	0.5048	0.5463	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0325)	(0.0180)
800	0.4376	0.5246	0.5048	0.5465	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0325)	(0.0180)
1,000	0.4376	0.5246	0.5048	0.5466	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0324)	(0.0180)
1,200	0.4376	0.5246	0.5048	0.5466	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0324)	(0.0180)

Estimated ultimate ruin probabilities

u	$\hat{\psi}^{\mathrm{I}}(u)$	$\hat{\psi}^{\mathrm{A}}(u)$	$\hat{\psi}^{\mathrm{B1}}(u)$	$\hat{\psi}^{\mathrm{B2}}(u)$	$\hat{\psi}^{\mathrm{C}}(u)$
20	0.4376	0.5246	0.5048	0.5466	0.4941
	(0.0210)	(0.0234)	(0.0217)	(0.0324)	(0.0180)
30	0.3323	0.4267	0.4078	0.4524	0.3938
	(0.0196)	(0.0226)	(0.0203)	(0.0313)	(0.0179)
40	0.2591	0.3490	0.3326	0.3788	0.3166
	(0.0190)	(0.0206)	(0.0181)	(0.0293)	(0.0170)
50	0.2058	0.2881	0.2722	0.3182	0.2575
	(0.0163)	(0.0183)	(0.0160)	(0.0272)	(0.0152)
60	0.1646	0.2377	0.2249	0.2684	0.2101
	(0.0144)	(0.0171)	(0.0146)	(0.0264)	(0.0149)
70	0.1338	0.1982	0.1876	0.2275	0.1727
	(0.0133)	(0.0161)	(0.0125)	(0.0237)	(0.0141)
80	0.1106	0.1655	0.1561	0.1933	0.1436
	(0.0122)	(0.0141)	(0.0119)	(0.0232)	(0.0133)

Numerical example 3

$X_i^{(1)}$ is lognormal with density function

$$g_1(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right), \tag{9}$$

with $\mu = 0.434044$ and $\sigma = 1.1528816$ while $X_i^{(2)}$ is Weibull with $\omega = 0.902703$ and $\tau = 0.4$. Then, $\mathsf{E}(X_i^{(1)}) = \mathsf{E}(X_i^{(2)}) = 3$, $\mathsf{Var}(X_i^{(1)}) = 25$ and $\mathsf{Var}(X_i^{(1)}) = 88.78$.

Similar to the previous example, the expected aggregate claims per unit time is 39 for each model and k is equal to 46.8.

Finite-time ruin probabilities with $u = 20$

N	$\psi_N^{\mathrm{I}}(u)$	$\psi_N^{\mathrm{A}}(u)$	$\psi_N^{\mathrm{B1}}(u)$	$\psi_N^{\mathrm{B2}}(u)$	$\psi_N^{\mathrm{C}}(u)$
200	0.4785	0.5353	0.5269	0.5548	0.5003
	(0.0269)	(0.0189)	(0.0193)	(0.0448)	(0.0262)
400	0.4787	0.5354	0.5270	0.5553	0.5010
	(0.0270)	(0.0188)	(0.0192)	(0.0444)	(0.0262)
600	0.4787	0.5354	0.5270	0.5554	0.5012
	(0.0270)	(0.0188)	(0.0192)	(0.0445)	(0.0262)
800	0.4787	0.5354	0.5270	0.5555	0.5013
	(0.0270)	(0.0188)	(0.0192)	(0.0445)	(0.0262)
1,000	0.4787	0.5354	0.5270	0.5555	0.5013
	(0.0270)	(0.0188)	(0.0192)	(0.0445)	(0.0262)

Estimated ultimate ruin probabilities

u	$\hat{\psi}^{\mathrm{I}}(u)$	$\hat{\psi}^{\mathrm{A}}(u)$	$\hat{\psi}^{\mathrm{B1}}(u)$	$\hat{\psi}^{\mathrm{B2}}(u)$	$\hat{\psi}^{\mathrm{C}}(u)$
20	0.4787	0.5354	0.5270	0.5555	0.5013
	(0.0270)	(0.0188)	(0.0192)	(0.0445)	(0.0262)
30	0.4008	0.4533	0.4454	0.4750	0.4256
	(0.0248)	(0.0192)	(0.0205)	(0.0437)	(0.0263)
40	0.3370	0.3892	0.3806	0.4104	0.3635
	(0.0247)	(0.0194)	(0.0205)	(0.0414)	(0.0262)
50	0.2878	0.3361	0.3286	0.3564	0.3142
	(0.0257)	(0.0193)	(0.0190)	(0.0400)	(0.0258)
60	0.2470	0.2914	0.2837	0.3108	0.2742
	(0.0238)	(0.0183)	(0.0176)	(0.0371)	(0.0266)
70	0.2142	0.2542	0.2467	0.2720	0.2401
	(0.0227)	(0.0177)	(0.0168)	(0.0350)	(0.0266)
80	0.1857	0.2221	0.2153	0.2386	0.2120
	(0.0218)	(0.0161)	(0.0163)	(0.0322)	(0.0258)

Optimal dividends

- Due to its practical importance, the issue of dividend strategies has received remarkable attention in the actuarial literature.
- For a Bernoulli model, the so-called barrier strategy was first proposed by De Finetti (1957). If the ultimate goal is to maximize the expectation of the discounted dividends paid to the shareholders of the company, he found that the optimal strategy must be a barrier strategy and showed how the optimal level of the barrier can be determined.
- Since then, research on dividend strategies has been carried out extensively.

Surplus process under threshold strategy

The surplus process under the threshold strategy is given by

$$dU(t) = \begin{cases} cdt - dS(t), & U(t) \le b, \\ (c-\alpha)dt - dS(t), & U(t) > b, \end{cases} \tag{10}$$

where c is the premium rate, the aggregate claims process for the whole book $S(t)$ is a compound Poisson process, b is the threshold value, α is the dividend rate for $U(t) \ge b$ with $c \ge \alpha$. In this process, we see that dividends are paid at rate α when the surplus is above the barrier b. Hence, the net premium rate after dividend payments is $c - \alpha \ge 0$.

Optimal threshold

Let D be the present value of all dividends until ruin. Then,

$$D = \int_0^T e^{-\delta t} dD(t),$$

where δ is the force of interest, $D(t)$ is the total dividends paid up to time t, and $T = \inf\{t : S(t) < 0\}$ is the time of ruin. Let $V(u; b) = E(D)$ be the expected discounted value of all dividends until ruin, which is a function of the initial surplus u and the threshold b.

The objective of the insurer is to select the optimal threshold b^* that maximizes the expected total discounted dividend payments until ruin.

Models for comparison

- We carry out a numerical study to empirically assess the impact of the previously introduced dependence structures on the optimal threshold for the compound Poisson model under the threshold strategy.
- In our numerical study, we consider the five previously defined risk models with two classes of business $(k = 2)$.
- These risk models include one with two independent classes while the other three processes are either the common shock dependence, or the thinning dependence, or both.

In order to compare the optimal dividends for the four risk models on an equal footing, we need to set the values of the parameters in the way that all the four models have the same expected aggregate claims.

A natural way to do it is to use the same claim-size distributions for the four models, and to set

$$\lambda_j^I = \lambda_j^C = \lambda_j^A = \lambda_j^{B1} = \lambda_j^{B2},$$

for $j = 1, 2$.

Numerical study

- We assume that the claim sizes in each class follow an exponential distribution.
- For fair comparison, we again set the parameters in the way that the five risk models have the same expected aggregate claims.
- In our numerical study, we investigate how the five dependence structures between claim classes affect the expected discounted values of all dividends until ruin.
- We also present some figures to show that the optimal threshold is independent of the initial surplus.
- Finally, we examine the effect of dependence on both the optimal thresholds and the maximized expected discounted values of all dividends until ruin.

Parameter set-up

- Set the dividend rate $\alpha = 10$ and the force of interest $\delta = 0.009$.
- $\beta_1 = 0.5$, $\beta_2 = 2$.
- $\lambda_1 = 3$, $\lambda_2 = 4$, and $\lambda_c = 2$, $p_{12} = 0.4$, $p_{21} = 0.7$.
- Set the premium rate to be $c = 10.75(1 + \theta)$, where θ is called the relative security loading in the case that $c - \alpha > 0$.

Expected discounted dividends with $u = 70$ and $b = 60$

	$V(70; 60)$				
θ	Model I	Model C	Model B1	Model A	Model B2
1.01	492.9621	492.2007	491.8244	491.4528	491.2145
1.02	493.7494	493.0367	492.6835	492.3339	492.1094
1.03	494.4337	493.768	493.4371	493.1089	492.8979
1.04	495.0293	494.4084	494.0989	493.7912	493.5932
1.05	495.5488	494.9701	494.6809	494.3928	494.2071
1.06	496.0029	495.4639	495.1937	494.9242	494.7502
1.07	496.4008	495.8988	495.6466	495.3944	495.2315
1.08	496.7505	496.2829	496.0473	495.8115	495.6589
1.09	497.0586	496.6229	496.4028	496.1822	496.0392
1.1	497.3308	496.9246	496.719	496.5124	496.3785
1.11	497.572	497.1931	497.0008	496.8074	496.6819
1.12	497.7862	497.4325	497.2526	497.0714	496.9537
1.13	497.977	497.6465	497.4781	497.3083	497.1978
1.14	498.1474	497.8384	497.6806	497.5213	497.4175
1.15	498.2999	498.0108	497.8628	497.7132	497.6157

Expected discounted dividends with $u = 50$ and $b = 60$

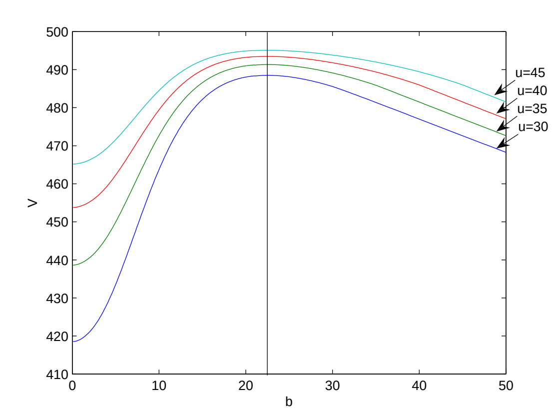
	$V(50;60)$				
θ	Model I	Model C	Model B1	Model A	Model B2
1.01	478.249	477.3731	476.9458	476.527	476.26
1.02	479.3029	478.4667	478.0582	477.6574	477.4017
1.03	480.2492	479.452	479.062	478.679	478.4345
1.04	481.1015	480.342	479.9699	479.6042	479.3706
1.05	481.8717	481.1481	480.7933	480.4442	480.2211
1.06	482.57	481.8807	481.5423	481.2092	480.9961
1.07	483.2054	482.5485	482.2257	481.9077	481.7042
1.08	483.7857	483.1592	482.8512	482.5475	482.3531
1.09	484.3174	483.7196	483.4255	483.1353	482.9495
1.1	484.8063	484.2354	483.9543	483.6769	483.4992
1.11	485.2574	484.7117	484.4429	484.1774	484.0074
1.12	485.6748	485.1528	484.8955	484.6413	484.4784
1.13	486.0624	485.5624	485.316	485.0724	484.9163
1.14	486.4231	485.9439	485.7076	485.474	485.3243
1.15	486.7599	486.3002	486.0734	485.8492	485.7054

- For the threshold $b = 60$, we calculate the expected discounted values of all dividends until ruin with two initial surpluses, $u = 50 < b$ and $u = 70 > b$. The following tables summarize the results for $u = 70$ and $u = 50$, respectively.

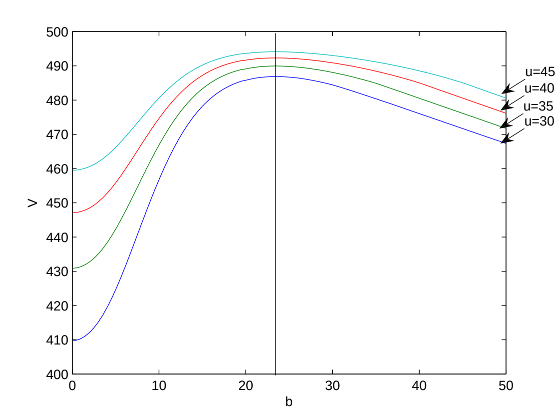
- From the two tables, we see that the expected discounted value of all dividends until ruin increases as θ increases. This is simply because the increase in premium allows the company to pay more dividends.

- For fixed θ, we also see in both tables that the expected discounted value of all dividends until ruin decreases as the degree of dependence increases.

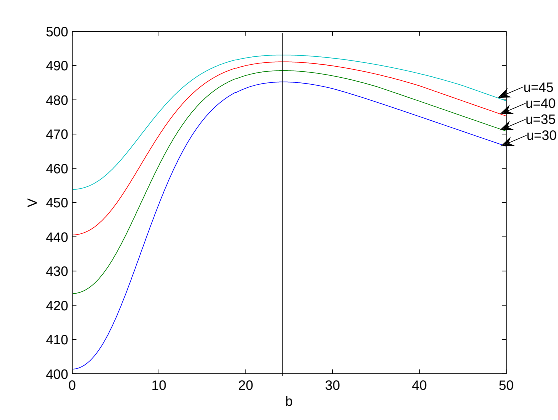
$V(u;b)$ with $\theta = 1$ and $u = 30, 35, 40, 45$ for Model I



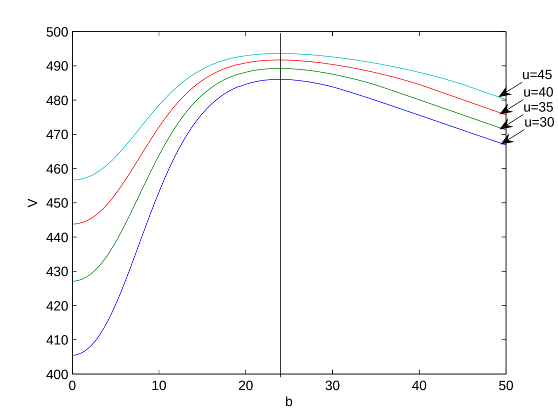
$V(u;b)$ with $\theta = 1$ and $u = 30, 35, 40, 45$ for Model C



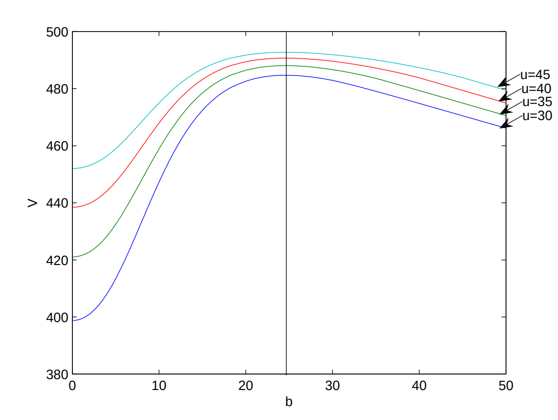
$V(u;b)$ with $\theta = 1$ and $u = 30, 35, 40, 45$ for Model A



$V(u;b)$ with $\theta = 1$ and $u = 30, 35, 40, 45$ for Model B1



$V(u;b)$ with $\theta = 1$ and $u = 30, 35, 40, 45$ for Model B2



- Figures 1-5, it is assumed that $\theta = 1$.
- We see from these figures that no matter how the initial surplus u changes, it does not affect the level of optimal threshold in each risk model.
- The following table displays the optimal values with $u = 40$. From the table, we see that $b_I^* < b_C^* < b_{B1}^* < b_A^* < b_{B2}^*$, where b_I^*, b_C^*, b_{B1}^*, b_A^*, and b_{B2}^* are the optimal thresholds for the five models.
- The table also shows that $V_I^* > V_C^* > V_{B1}^* > V_A^* > V_{B2}^*$, where V_I^*, V_C^*, V_{B1}^*, V_A^*, and V_{B2}^* are the maximized expected discounted value of all dividends until ruin for the five models.
- These suggest the fact that the effect of a higher optimal threshold is in general not enough to offset the effect of the corresponding higher dependence on the maximized value.

Optimal thresholds and expected discounted dividends with $u = 40$

	Model I		Model C		Model B1		Model A		Model B2	
θ	b_I^*	V_I^*	b_C^*	V_C^*	b_{B1}^*	V_{B1}^*	b_A^*	V_A^*	b_{B2}^*	V_{B2}^*
1.01	22.3	494.3965	23.1	493.3169	23.6	492.7599	24	492.1941	24.3	491.8239
1.02	22	495.1848	22.8	494.1954	23.3	493.6815	23.7	493.1577	24	492.8139
1.03	21.7	495.8571	22.5	494.9531	23	494.4806	23.4	493.997	23.7	493.6787
1.04	20.9	496.4334	22.3	495.6064	22.7	495.1731	23.1	494.7277	23.4	494.4337
1.05	20.8	496.9236	22	496.1697	22.4	495.773	22.8	495.3637	23.1	495.0927
1.06	20.7	497.3407	21.7	496.6556	22.1	496.293	22.6	495.9174	22.8	495.6681
1.07	20.5	497.6964	21.5	497.0751	21.9	496.7441	22.3	496.4	22.5	496.1708
1.08	20.2	498.0006	21.2	497.4375	21.6	497.1358	22	496.8207	22.3	496.6103
1.09	20	498.2613	21	497.7513	21.4	497.4763	21.8	497.188	22.1	496.995
1.1	19.8	498.4849	20.8	498.0232	21.2	497.7726	21.6	497.5091	21.8	497.3321
1.11	19.6	498.6771	20.6	498.2591	21	498.0309	21.4	497.79	21.6	497.6278
1.12	19.4	498.8426	20.4	498.4642	20.8	498.2564	21.1	498.0362	21.4	497.8876
1.13	19.2	498.9854	20.2	498.6427	20.6	498.4535	20.9	498.2523	21.2	498.1161
1.14	19	499.1088	19.5	498.7987	20.4	498.6261	20.7	498.4422	21	498.3174
1.15	18.9	499.2157	19.5	498.9357	20.2	498.7775	20.6	498.6094	20.8	498.495

Optimal reinsurance

Under the classical risk model with proportional reinsurance, the surplus process follows

$$dU(t) = (c - \delta(q_t))\, dt - q_t dS(t), \quad U(0) = x, \tag{11}$$

$q_t \in [0, 1]$ represents the retention level. That is, for a claim X_i, the insurer pays $q_t X_i$, and the reinsurer pays $(1 - q_t)X_i$. $\delta(q_t)$ is the reinsurance premium rate at time t, then the premium rate remaining for insurer at time t is $c - \delta(q_t)$.

Classical risk model with common shock

Let $S_l(t)$ be the aggregate claims amounts for the lth class risk, with

$$S_l(t) = \sum_{i=1}^{\tilde{N}_l(t)} X_i^{(l)},$$

where $\tilde{N}_l(t)$ is the claim number process for class l $(l = 1, 2, ..., n)$. It is assumed that $\{X_i^{(l)}, i \geq 1, l = 1, 2, ..., n\}$ are independent claim size random variables, and that they are independent of $\{\tilde{N}_l(t), l = 1, 2, ..., n\}$.

The claim number processes are correlated in the way that

$$\tilde{N}_l(t) = N_l(t) + N(t)$$

with $N_l(t)$, $l = 1, 2, ..., n$ and $N(t)$ being $n + 1$ independent Poisson processes with intensities $\lambda_1, ..., \lambda_n$ and λ, respectively. Therefore, the aggregate claim size process generated from m classes of business is given by

$$S(t) = \sum_{l=1}^{n} S_l(t) = \sum_{l=1}^{n} \left(\sum_{i=1}^{N_l(t)+N(t)} X_i^{(l)} \right). \qquad (12)$$

It follows from Yuen et al. (2002) or Wang and Yuen (2005) that $S(t)$ has the same distribution of a compound Poisson process with parameter $\tilde{\lambda} = \sum_{l=1}^{m} \lambda_l + \lambda$, and that the common distribution of the claim size random variable X' is given by

$$F_{X'}(x) = \sum_{l=1}^{n} \frac{\lambda_l}{\tilde{\lambda}} F_{X^{(l)}}(x) + \frac{\lambda}{\tilde{\lambda}} F_{\sum_{l=1}^{n} X^{(l)}}(x).$$

Dynamic reinsurance with dependent risks

Let $\{U^q(t), t \geq 0\}$ denote the associated surplus process, i.e., $U^q(t)$ is the wealth of the insurer at time t, if he (or she) follows strategy q_t.

Furthermore, the company is allowed to invest all its surplus in a risk-free asset (bond or bank account) with interest rate $r \geq 0$.

This process then evolves as

$$dU^q(t) = [rU^q(t) + (c - \delta(q_t))]dt - \sum_{l=1}^{n} q_{lt} dS_l(t). \tag{13}$$

Where $q_{lt} \in [0, 1]$ is the retention level for X^l $(l = 1, 2, ..., n)$.

Objective function

Assume now that the insurer is interested in maximizing the expected utility from terminal wealth, say at time T. The utility function is $u(x)$, which satisfies $u' > 0$ and $u'' < 0$. Then the objective function is

$$J^q(t,x) = E[u(R_T^q)|R_t^q = x], \tag{14}$$

The corresponding value function is

$$V(t,x) = \sup_q J^q(t,x). \tag{15}$$

We assume that the insurer has an exponential utility function

$$u(x) = -\frac{m}{\nu} e^{-\nu x},$$

for $m > 0$ and $\nu > 0$. This utility has constant absolute risk aversion (CARA) parameter ν.

Such an utility function plays an important role in insurance mathematics and actuarial practice as this is the only function under which the principle of "zero utility" gives a fair premium that is independent of the level of reserves of an insurance company (see Gerber (1979)).

Optimal Reinsurance

We restrict our attention to the model with two dependent classes of insurance business. The reinsurance premium is calculated according to the expected value principle. That is,

$$\delta(q) = \sum_{l=1}^{2}(1+\eta_l)(1-q_l)(\lambda_l+\lambda)E(X^{(l)}),$$

where $\eta_l (l = 1, 2)$ are the reinsurer's safety loading of the m classes of insurance business.

In Examples 1-4, we assume that the claim sizes $\{X_i^{(1)}\}$ and $\{X_i^{(2)}\}$ are exponentially distributed with parameters α_1 and α_2, respectively. That is, $X^{(1)} \sim \exp(\alpha_1)$ and $X^{(2)} \sim \exp(\alpha_2)$. For computational convenience, we set $\eta_1 = \eta_2 = \eta$.

Example 1. Let $\lambda_1 = 3$, $r = 0.3$, $T = 10$, $\lambda_2 = 4$, $\lambda = 2$, $\nu = 0.5$, $\alpha_1 = 2$, $\alpha_2 = 3$, $t = 1$. The results are shown in Table 1.

Table 1 The effect of η on the optimal reinsurance strategies

η	t_1	t_2	q_1^*	q_2^*	t_3	t_4	$\bar{q}_1^*$	$\bar{q}_2^*$
1	10	8.5766	0.0661	0.1030	8.3203	6.8327	0.1112	0.1738
2	8.8024	7.3303	0.0963	0.1497	6.0098	4.5222	0.2225	0.3476
3	8.2205	6.7529	0.1146	0.1780	4.6582	3.1707	0.3337	0.5214
4	7.8689	6.4048	0.1274	0.1976	3.6993	2.2117	0.4450	0.6952
5	7.6278	6.1667	0.1369	0.2122	2.9555	1.4679	0.5562	0.8690
6	7.4496	5.9910	0.1444	0.2237	2.3477	0.8602	0.6731	1
7	7.3110	5.8545	0.1506	0.2331	1.8339	0.3463	0.8075	1

From Table 1, we see that the optimal reinsurance strategies (q_1^*, q_2^*) increase with η.

This phenomenon reflects that the insurer would rather retain a greater share of each claim as the reinsurance premium increases.

Example 2. Let $\eta = 2$, $r = 0.3$, $T = 10$, $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda = 2$, $\nu = 0.5$, $t = 2.5$. The results are shown in Tables 2 and 3.

Table 2 The effect of α_1 on the optimal reinsurance strategies

α_1	t_1	t_2	q_1^*	q_2^*	t_3	t_4	$\bar{q}_1^*$	$\bar{q}_2^*$
1	10	7.3303	0.0755	0.2348	8.3203	4.5222	0.1745	0.5452
2	8.8024	7.3303	0.1510	0.2348	6.0098	4.5222	0.3489	0.5452
3	7.4508	7.3303	0.2264	0.2348	4.6582	4.5222	0.5234	0.5452
4	6.4919	7.3303	0.3019	0.2348	3.6993	4.5222	0.6978	0.5452
5	5.7481	7.3303	0.3774	0.2348	2.9555	4.5222	0.8723	0.5452
6	5.1403	7.3303	0.4529	0.2348	2.3477	4.5222	1	0.5491
7	4.6265	7.3303	0.5284	0.2348	1.8339	4.5222	1	0.5610

Table 3 The effect of α_2 on the optimal reinsurance strategies

α_2	t_1	t_2	q_1^*	q_2^*	t_3	t_4	$\bar{q}_1^*$	$\bar{q}_2^*$
1	8.8024	10	0.1510	0.0783	6.0098	8.1768	0.3489	0.1821
2	8.8024	8.6819	0.1510	0.1565	6.0098	5.8736	0.3489	0.3635
3	8.8024	7.3303	0.1510	0.2348	6.0098	4.5222	0.3489	0.5452
4	8.8024	6.3714	0.1510	0.3130	6.0098	3.5633	0.3489	0.7269
5	8.8024	5.6276	0.1510	0.3913	6.0098	2.8194	0.3489	0.9086
6	8.8024	5.0198	0.1510	0.4696	6.0098	2.2117	0.3549	1
7	8.8024	4.5060	0.1510	0.5478	6.0098	1.6979	0.3645	1

From Tables 2 ($\alpha_2 = 3$) and 3 ($\alpha_1 = 2$), we can see that a greater value of α_i yields greater values of the optimal reinsurance strategies q_i^*.

As expected, since a smaller value of α_i implies "more risky" claim size with a larger expected value, the insurer would rather retain a less share of each claim.

Also, it is easy to observe from the tables that q_i^*, $i = 1, 2$, is independent of α_j, $j = 2, 1$, that is, when the claim sizes are exponentially distributed, $X^{(1)}$ ($X^{(2)}$) has no effect on the optimal reinsurance strategies q_2^* (q_1^*).

This interesting phenomenon may only appear in the case with exponential claim sizes and the reinsurance premium calculated under the expected value principle.

This guess is partly due to the following two facts:

(i) it was shown in Tables 4 and 5 in Liang and Yuen (2014) that the optimal results under the variance principle are indeed affected by both claim size distributions in the compound Poisson risk model;

(ii) we see later in Example 5 and Table 6 that the claim sizes $X^{(1)}$ ($X^{(2)}$) does affect the values of the optimal reinsurance strategies q_2^* and q_1^*, when $X^{(1)}$ ($X^{(2)}$) follows a gamma distribution with parameters $\beta \neq 1$ and α.

However, when one of the optimal strategies hits the boundary one, we see from Tables 2 and 3 that the other optimal strategy is forced to change regardless to the choice of the claim size distributions.

Example 3. Let $r = 0.3$, $T = 10$, $\lambda_2 = 4$, $\eta = 2$, $\nu = 0.5$, $\alpha_1 = 2$, $\alpha_2 = 3$, $t = 4.6$. The results are shown in Table 4.

Table 4 The effect of λ on the optimal reinsurance strategies

λ	t_1	t_2	q_1^*	q_2^*	t_3	t_4	$\bar{q}_1^*$	$\bar{q}_2^*$
1	8.6176	7.1790	0.2996	0.4613	5.7821	4.3367	0.7083	1
2	8.8024	7.3303	0.2835	0.4408	6.0098	4.5222	0.6583	1
3	8.9134	7.3678	0.2742	0.4359	6.1542	4.6478	0.6273	0.9858
4	8.9871	7.4993	0.2682	0.4190	6.2537	4.7395	0.6089	0.9590
5	9.0391	7.5526	0.2640	0.4124	6.3251	4.8095	0.5960	0.9391
6	9.0778	7.5946	0.2610	0.4072	6.3796	4.8651	0.5863	0.9236
7	9.1075	7.6286	0.2587	0.4031	6.4212	4.9105	0.5791	0.9110

From Table 4, we see that the optimal reinsurance strategies decrease while λ increases.

Since a greater value of λ implies a greater value of expected claim number, the insurer would rather retain a less share of each claim.

Example 4. Let $r = 0.3$, $T = 10$, $\lambda_2 = 4$, $\eta = 2$, $\lambda = 2$, $\nu = 0.5$, $\alpha_1 = 2$, $\alpha_2 = 3$, $t = 4.5$. The results are shown in Table 5.

Table 5 The effect of λ_1 on the optimal reinsurance strategies

λ_1	t_1	t_2	q_1^*	q_2^*	t_3	t_4	$\bar{q}_1^*$	$\bar{q}_2^*$
1	9.2321	7.2525	0.2418	0.4379	6.5400	4.4448	0.5460	1
2	8.9572	7.2998	0.2626	0.4317	6.1961	4.4935	0.6015	1
3	8.8024	7.3303	0.2751	0.4278	6.0098	4.5222	0.6358	0.9934
5	8.6340	7.3672	0.2893	0.4231	5.8127	4.5549	0.6745	0.9837
6	8.5832	7.3792	0.2938	0.4216	5.7540	4.5650	0.6865	0.9807
7	8.5442	7.3886	0.2972	0.4204	5.7092	4.5729	0.6958	0.9784
8	8.5134	7.3963	0.3000	0.4194	5.6740	4.5791	0.7031	0.9765

From Table 5, we see that a greater value of λ_1 yields greater values of the optimal reinsurance strategies q_1^* but smaller values of the optimal reinsurance strategy q_2^*.

Along the same lines, one can numerically show that a greater value of λ_2 yields greater values of q_2^* but smaller values of q_1^*.

In the following example, we show that q_1^* (q_2^*) does depend on the claim sizes $X^{(2)}$ ($X^{(1)}$) for some distributions. We assume that the claim sizes $X^{(2)}$ has a gamma distribution with parameters $\beta \neq 1$ and α, that is, $X^{(2)} \sim \Gamma(\beta, \alpha)$.

Then, we have

$$\begin{cases} \mu_2 & = E(X^{(2)}) = \frac{\beta}{\alpha}, \\ \sigma_2^2 & = (\lambda_2 + \lambda)(\beta^2 + \beta)/\alpha^2, \\ M_2(r) & = (\frac{\alpha}{\alpha - r})^\beta, \quad for \quad r < \alpha. \end{cases}$$

Example 5. Assume that $X^{(1)} \sim \exp(\alpha_1)$ and $X^{(2)} \sim \Gamma(\beta, \alpha)$. Let $\lambda = 2$, $\lambda_1 = 3$, $r = 0.3$, $T = 10$, $\lambda_2 = 4$, $\eta = 2$, $\nu = 0.5$, $\alpha_1 = 2$, $\alpha = 3$, $t = 1.8$. The results are shown in Table 6.

Table 6 The effect of β on the optimal reinsurance strategies

β	t_1	t_2	q_1^*	q_2^*	t_3	t_4	$\bar{q}_1^*$	$\bar{q}_2^*$
0.6	8.6290	6.7988	0.1289	0.2232	5.8364	3.8070	0.2979	0.5477
0.7	8.6769	6.9395	0.1271	0.2140	5,8861	4.0007	0.2935	0.5167
0.8	8.7216	7.0747	0.1254	0.2055	5.9312	4.1837	0.2896	0.4891
0.9	8.7633	7.2049	0.1238	0.1976	5.9723	4.3573	0.2860	0.4643
1	8.8024	7.3303	0.1224	0.1903	6.0098	4.5222	0.2828	0.4419
2	9.0886	8.3819	0.1123	0.1388	6.2613	5.8352	0.2623	0.2980
3	9.2627	9.1830	0.1066	0.1092	6.3970	6.7747	0.2518	0.2248

From Table 6, we see that a greater value of β yields smaller values of the optimal reinsurance strategies q_i^* $(i = 1, 2)$.

Since a greater value of β implies "more risky" claim sizes with a larger expected value, the insurer would rather retain a less share of each claim in general.

It is also reasonable to expect that the optimal reinsurance strategy q_2^* decreases faster than q_1^* as the claim size distribution $X^{(2)}$ has a more direct impact on q_2^*.

Outline

Solvency II

Regulatory requirements for the insurance industry are referred to as the Solvency II which adopts a three-pillar approach with the aim to align risk measurement and risk management.

One of the three pillars requires insurers to hold sufficient regulatory capital such that they are protected against adverse events with a high probability over a one-year period. Subject to regulatory approval, this task can be done by using the following technical tools:

- Stochastic modeling
- Internal models

Operational Risk

- Basel II regulatory requirements
  - Credit risk, Market risk, Operational risk
  - Basel Committee on Banking Supervision defines operational risk as

    "the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events".

- Banks are required to hold adequate capital against operational risk

Operational Risk

- Three methods to estimate required capital
  - Basic Indicator Approach
  - Standard Approach, and
  - Advanced Measurement Approach (AMA)
- Banks can calculate the capital charge using internally developed model subject to regulatory approval
- Risk measure used for capital charge should be comparable to a 99.9% confidence level for a one-year holding period
- Under the AMA, a popular method is the Loss Distribution Approach (LDA)
  - banks quantify distributions for frequency and severity of operational risk losses for each business line over a one-year time horizon
  - use their own dependence structure among business lines.

Operational Risk

- Suppose that there are m business lines in a company.
- Under the LDA, the commonly used model for calculating the total annual loss L can be formulated as $L = \sum_{i=1}^{m} L_i$ where $L_i = \sum_{j=1}^{N_i} = X_{ij}$ is the annual loss in the ith business line, N_i is the number of operational risk losses (frequency) in the ith business line, and X_{ij}'s are the sizes of losses (severity) in the ith business line.
- Hence, the aggregate annual loss distribution function is then given by $F(x) = P(L \leq x)$.
- In this set-up, copulas and Monte Carlo simulation are often used to estimate F so that capital requirements can be computed.

The End