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Session 2
Extremal dependence structures and bounds of Tail Value-at-Risk

Extremal dependence structures and bounds of Tail Value-at-Risk

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Current Topics on Actuarial Models
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Tail Value-at-Risk (aka Expected Shortfall)

- The Value-at-Risk (VaR) of X at probability level α is

$$\text{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{t \in \mathbb{R} | F_X(t) \geq \alpha\}.$$

- The Tail Value-at-Risk (TVaR) at probability level α is

$$\text{TVaR}_\alpha(X) := \frac{1}{1-\alpha} \int_\alpha^1 F_X^{-1}(t)\, dt.$$

- In two recent consultative documents BCBS (2013, 2013), the Basel Committee on Banking Supervision proposed to take a move from VaR to TVaR for the measurement of market risk in banking.

Tail Value-at-Risk is Coherent

- *Translation invariant:* for any constant c,
$$\mathsf{TVaR}_\alpha(X+c) = \mathsf{TVaR}_\alpha(X) + c$$
- *Positive homogeneous:* for any positive constant λ,
$$\mathsf{TVaR}_\alpha(\lambda X) = \lambda \mathsf{TVaR}_\alpha(X)$$
- *Monotonic:* if $X \leq Y$,
$$\mathsf{TVaR}_\alpha(X) \leq \mathsf{TVaR}_\alpha(Y)$$
- *Subadditive:*
$$\mathsf{TVaR}_\alpha(X+Y) \leq \mathsf{TVaR}_\alpha(X) + \mathsf{TVaR}_\alpha(Y)$$

Given a portfolio of risks $X_1, \ldots, X_n$ with fixed marginals,

- The *dependence structure* among individual risks is difficult to obtain from a statistical point of view, while the marginal distributions of the individual risks may typically be easier to model (Embrechts et al. (2013) and Bernard et al. (2014))

  $\Rightarrow$ Dependence uncertainty in risk aggregation

Given a portfolio of risks $X_1, \ldots, X_n$ with fixed marginals,

- The *dependence structure* among individual risks is difficult to obtain from a statistical point of view, while the marginal distributions of the individual risks may typically be easier to model (Embrechts et al. (2013) and Bernard et al. (2014))

  $\Rightarrow$ Dependence uncertainty in risk aggregation

- **Objective 1** Find upper and lower bound of

$$\text{TVaR}_\alpha(X_1 + \cdots + X_n)$$

- **Objective 2** Identify corresponding dependence structures

Outline

(Part 1) Review some classical results of extremal dependence structures: comonotonicity ($n \geq 2$) and counter-monotonicity ($n = 2$)

(Part 2) Study a high-dimensional ($n > 2$) notion of counter-monotonicity: generalized mutual exclusivity (GME)

(Part 3) Lower bounds of TVaR, Haezendonck-Goovaerts risk measures and convex expectation of a sum

(Part 4) Mutual Exclusivity in the Tail (MET)

(Part 5) MET and TVaR Lower bounds

Part 1

Review some classical results of extremal dependence structures: comonotonicity ($n \geq 2$) and counter-monotonicity ($n = 2$)

Classical result I (Fréchet-Hoeffding bounds)

For any $\mathbf{X}$ inside the Fréchet space $\mathcal{R}(F_1, \dots, F_n)$:

$$W_n(x_1, \dots, x_n) \leq F_{\mathbf{X}}(x_1, \dots, x_n) \leq M_n(x_1, \dots, x_n)$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$

$W_n(x_1, \dots, x_n) := \left(\sum_{i=1}^n F_i(x_i) - n + 1\right)_+$

$M_n(x_1, \dots, x_n) := \min F_i(x_i)$

Recall: $\mathbf{X} \in \mathcal{R}(F_1, \dots, F_n)$ means that $X_i \sim F_i$ for all i

Classical result II (Fréchet-Hoeffding upper bound)

(a) $M_n(x_1, \ldots, x_n) := \min F_i(x_i)$ is **always** a proper joint cdf

(b) $F_{\mathbf{X}} = M_n \Longleftrightarrow \mathbf{X}$ is comonotonic

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Recall:

(1) $\mathbf{X}$ *is comonotonic if*

$$(X_i(\omega) - X_i(\omega'))(X_j(\omega) - X_j(\omega')) \geq 0$$

for any i, j*, and any* ω, ω' *outside a null set (the strongest positive dependence structure)*

(2) Comonotonicity $\Longleftrightarrow$ Pairewise comonotonicity

Idea of comonotonicity

Comonotonic random variables are *increasing functions* of a common random variable

Examples

- Stock price S_T and payoff of European call $(S_T - K)_+$
- Present value of the payments in the respective years for a life annuity:

$$\begin{aligned} Y_1 &= v \cdot 1_{\{T>1\}} \\ Y_2 &= v^2 \cdot 1_{\{T>2\}} \\ Y_3 &= v^3 \cdot 1_{\{T>3\}} \\ &\vdots \end{aligned}$$

Classical result III (Fréchet-Hoeffding lower bound, $n = 2$)

(a) $W_2(x_1, x_2) := (F_1(x_1) + F_2(x_2) - 1)_+$ is always a joint cdf in $\mathcal{R}(F_1, F_2)$

(b)$F_{\mathbf{X}} = W_2 \Longleftrightarrow \mathbf{X}$ is counter-monotonic

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Recall: (X_1, X_2) *is counter-monotonic if*

$$(X_1(\omega) - X_1(\omega'))(X_2(\omega) - X_2(\omega')) \leq 0$$

for any ω, ω' *outside a null set (the strongest negative dependence structure)*

Remark: Counter-monotonicity is a **two-dimensional** *concept*

Idea of counter-comonotonicity

Two random variables are counter-comonotonicity if one is an *increasing function* and the other is a *decreasing function* of a common random variable

Example

PVRV of an n-year term life $v^T \cdot 1_{\{T \leq n\}}$ (decreasing in T)

and

PVRV of an n-year pure endowment $v^n \cdot 1_{\{T > n\}}$ (increasing in T)

Classical result III (Fréchet-Hoeffding lower bound, $n \geq 3$, Dall'Aglio (1972)))

$W_n(x_1, \ldots, x_n) := \left(\sum_{i=1}^n F_i(x_i) - n + 1\right)_+$ is a joint cdf in $\mathcal{R}(F_1, \ldots, F_n)$ iff either

(i) $\sum_{i=1}^n F_i(x_i) \leq 1$ for all $\mathbf{x}$ with $0 < F_i(x_i) < 1, i = 1, 2, \ldots, n$; or

(ii) $\sum_{i=1}^n F_i(x_i) \geq n-1$ for all $\mathbf{x}$ with $0 < F_i(x_i) < 1, i = 1, 2, \ldots, n$.

Meaning of the two conditions (assuming non-degeneracy):

(i) $\sum_{i=1}^n F_i(x_i) \leq 1$ for all $\mathbf{x}$ with $0 < F_i(x_i) < 1, i = 1, 2, \ldots, n$

Meaning:

$$\begin{aligned} \mathbb{P}(X_i = \text{ess sup } X_i) &> 0 \quad \text{for all } i \\ \sum \mathbb{P}(X_i = \text{ess sup } X_i) &\geq n - 1 \end{aligned}$$

(ii) $\sum_{i=1}^n F_i(x_i) \geq n-1$ for all $\mathbf{x}$ with $0 < F_i(x_i) < 1, i = 1, 2, \ldots, n$

Meaning:

$$\begin{aligned} \mathbb{P}(X_i = \text{ess inf } X_i) &> 0 \quad \text{for all } i \\ \sum \mathbb{P}(X_i = \text{ess inf } X_i) &\geq n - 1 \end{aligned}$$

$n \geq 2$: $F_{\mathbf{X}} = M_n$(upper bound) $\Longleftrightarrow$ $\mathbf{X}$ is comonotonic

$n = 2$: $F_{\mathbf{X}} = W_2$(lower bound) $\Longleftrightarrow$ $\mathbf{X}$ is counter-monotonic

Questions: For $n \geq 3$, what is the behavior of $\mathbf{X}$ if $F_{\mathbf{X}} = W_n$? Higher-dimensional notion of counter-monotonicity?

A partial solution (Dhaene and Denuit (1999))

Non-negative random variables $X_1, \ldots, X_n$ are said to be **mutually exclusive (ME)** if

$$\mathbb{P}(X_i > 0, X_j > 0) = 0 \quad \text{for all } i \neq j.$$

Non-negative random variables $X_1, \ldots, X_n$ are ME if they are concentrated on the positive axes

$$\cup_{i=1}^{n} \{0\} \times \cdots \times \underbrace{[0, \infty)}_{i^{\text{th}}} \times \cdots \times \{0\}$$

Existence: $\mathcal{R}(F_1, \ldots, F_n)$ of non-negative distributions supports a ME random vector iff $\sum_{i=1}^{n}(1 - F_i(0)) \leq 1$

Classical result IV (mutually exclusive variables)

$\mathbf{X}^*$ is a non-negative random vector in $\mathcal{R}(F_1, \ldots, F_n)$.

(a) $\mathbf{X}^*$ is ME $\Longleftrightarrow F_{\mathbf{X}^*} = W_n$

(b) If $\mathbf{X}^*$ is ME, then

$$\sum_{i=1}^{n} X_i^* \leq_{cx} \sum_{i=1}^{n} X_i \quad \forall\, (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$$

Convex order: $W \leq_{cx} Y \Longleftrightarrow \mathbb{E}\phi(W) \leq \mathbb{E}\phi(Y)$ for any convex ϕ

Questions:

(1) Can we remove "non-negative" to obtain a general description of random vector whose joint cdf $= W_n$?

(2) Is (b) a defining property for ME random vectors? Are there other ways to characterize ME random vectors?

(3) What is the relationship between ME and counter-monotonicity?

Classical result V (characterization of (counter)-comonotonicity)

(a) (Cheung (2008, 2010)) $\mathbf{X}^*, \mathbf{X}^c$ have the same marginals, $\mathbf{X}^c$ is comonotonic

$$\sum_{i=1}^{n} X_i^* \stackrel{d}{=} \sum_{i=1}^{n} X_i^c \Longleftrightarrow \mathbf{X}^* \text{ is comonotonic}$$

(b) (Cheung et al. (2013)) $\mathbf{X}^*, \mathbf{X}^c$ have the same marginals, $\mathbf{X}^c$ is counter-monotonic

$$X_1^* + X_2^* \stackrel{d}{=} X_1^c + X_2^c \Longleftrightarrow \mathbf{X}^* \text{ is counter-monotonic}$$

Question: Generalization of (b) to higher dimensions?

Classical result VI (TVaR)

(a) $\text{TVaR}_\epsilon(X_1 + \cdots + X_n) \le \text{TVaR}_\epsilon(X_1) + \cdots + \text{TVaR}_\epsilon(X_n)$

(b) Equality holds for all $\epsilon \in (0,1) \Longleftrightarrow \mathbf{X}$ is comonotonic

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Questions:

(1) $\inf\{\text{TVaR}_\epsilon(X_1 + \cdots + X_n) \mid (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)\} = ?$

(2) Is the infimum attainable? By what kind of dependence structure?

(3) Lower bound of $\text{TVaR}_\alpha(X_1 + \cdots + X_n)$?

Part 2

Generalized mutual exclusivity - characterization and properties

Definition (Generalized mutual exclusivity (GME))

$l_i :=$ ess infX_i, $u_i :=$ ess supX_i

$X_1, \ldots, X_n$ are said to be

(a) *mutually exclusive from below (MEB)* if

$$\mathbb{P}(X_i > l_i, X_j > l_j) = 0, \quad i \neq j$$

(b) *mutually exclusive from above (MEA)* if

$$\mathbb{P}(X_i < u_i, X_j < u_j) = 0, \quad i \neq j$$

Lemma (existence of GME variables)

$\mathcal{R}(F_1, \ldots, F_n)$ accommodates GME random variables iff either

(a) $\sum_{i=1}^{n} F_i(l_i) \geq n-1$
(in this case, MEB random variables are supported)

or

(b)$\sum_{i=1}^{n}(1 - F_i(u_i-)) \geq n-1$
(in this case, MEA random variables are supported)

Assumption (A) = condition of this lemma

Lemma (Condition (A))

$n \geq 3$: for $\mathcal{R}(F_1, \ldots, F_n)$,

$$\text{Condition (A)} \Longleftrightarrow W_n \text{ is a proper joint cdf}$$

$\therefore$ $\mathcal{R}(F_1, \ldots, F_n)$ accommodates GME random variables
$\Longleftrightarrow W_n$ is a proper joint cdf

Remark:

W_2 is always a joint cdf

Theorem (GME and Fréchet-Hoeffding lower bound)

$n \geq 3$: $\mathbf{X}$ is GME $\Longleftrightarrow F_{\mathbf{X}} = W_n$

Remarks:

(1) "$\Rightarrow$" is also true for $n = 2$

(2) "$\Leftarrow$" does not hold for $n = 2$ in general: X_1, X_2 can be counter-monotonic without being mutually exclusive

Definition (pairwise counter-monotonicity (PCM))

$\mathbf{X}$ is PCM if (X_i, X_j) is counter-monotonic whenever $i \neq j$

Theorem (GME and PCM)

$n \geq 3$: $\mathbf{X}$ is GME $\Longleftrightarrow$ $\mathbf{X}$ is PCM

Remarks:

(1) "$\Rightarrow$" is trivial, also true for $n = 2$

(2) "$\Leftarrow$" does not hold for $n = 2$

(3) "GME $\Leftarrow$ PCM + Condition (A)" is simple

(4) "Condition (A) $\Leftarrow$ GME $\Leftarrow$ PCM" is difficult

Theorem (GME and convex order)

Suppose that $\mathcal{R}(F_1, \ldots, F_n)$ satisfies Condition (A).

$\mathbf{X}^* \in \mathcal{R}(F_1, \ldots, F_n)$ is GME

$$\Longleftrightarrow \sum_{i=1}^{n} X_i^* \leq_{cx} \sum_{i=1}^{n} X_i \quad \forall (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$$

Remarks

(1) "$\Leftarrow$" is not always true if $\mathcal{R}(F_1, \ldots, F_n)$ does not satisfy Condition (A)

(2) "$\Rightarrow$" is always true, even without Condition (A)

Example

Let $F \sim \text{Bin}(m, p/n)$, where $m \geq 2$ and $p, n \in \mathbb{N}$. By Wang and Wang (2011), F is n-completely mixable: $\exists \mathbf{X}^* = (X_1^*, \ldots, X_n^*) \in \mathcal{R}(F, \ldots, F)$ such that $X_1^* + \cdots + X_n^*$ is almost surely constant. By Jensen's inequality,

$$X_1^* + \cdots + X_n^* \leq_{cx} X_1 + \cdots + X_n$$

for all $(X_1, \ldots, X_n) \in \mathcal{R}(F, \ldots, F)$. However,

$$\sum_{i=1}^{n} \mathbb{P}(X_i^* = 0) < n - 1 \quad \text{and} \quad \sum_{i=1}^{n} \mathbb{P}(X_i^* = m) < n - 1,$$

so Condition (A) is violated: $\mathbf{X}^*$ cannot be mutually exclusive.

Theorem (GME and variance order)

Suppose that $\mathcal{R}(F_1, \ldots, F_n)$ satisfies Condition (A). All marginals are square integrable.

$\mathbf{X}^* \in \mathcal{R}(F_1, \ldots, F_n)$ is GME

$$\iff \text{Var}\left(\sum_{i=1}^{n} X_i^*\right) \leq \text{Var}\left(\sum_{i=1}^{n} X_i\right) \; \forall (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$$

Theorem (GME and sum)

$\mathbf{X}^*, \mathbf{X}^M \in \mathcal{R}(F_1, \ldots, F_n)$, $\mathbf{X}^M$ is GME

$$\mathbf{X}^* \text{ is GME} \iff \sum_{i=1}^n X_i^* \stackrel{d}{=} \sum_{i=1}^n X_i^M$$
$$\iff \mathbb{E}\phi\left(\sum_{i=1}^n X_i^*\right) = \mathbb{E}\phi\left(\sum_{i=1}^n X_i^M\right)$$
$$\iff \text{Var}\left(\sum_{i=1}^n X_i^*\right) = \text{Var}\left(\sum_{i=1}^n X_i^M\right)$$

ϕ: any strictly convex function

Theorem (Distributional representation of GME variables)

Suppose $\mathcal{R}(F_1, \ldots, F_n)$ satisfies Condition (A).

(a) If $\mathbf{X}^M$ is MEB, then

$$\mathbf{X}^M \stackrel{d}{=} \left(F_1^{-1}(U_1^*), \ldots, F_n^{-1}(U_n^*)\right),$$

where, for $i = 1, \ldots, n$,

$$U_i^* := (1-U)1_{\left\{U \geq \sum_{j=1}^i q_j\right\}} + \left(U + 1 - \sum_{j=1}^i q_j\right) 1_{\left\{U < \sum_{j=1}^i q_j\right\}}$$

and

$$F_i^{-1}(U_i^*) = \begin{cases} F_i^{-1}\left(U + 1 - \sum_{j=1}^i q_j\right), & \sum_{j=1}^{i-1} q_j \leq U < \sum_{j=1}^i q_j, \\ l_i, & \text{otherwise} \end{cases}$$

Theorem (Distributional representation of GME variables), cont.

(b) If $\mathbf{X}^M$ is MEA, then

$$\mathbf{X}^M \stackrel{d}{=} \left(F_1^{-1}(U_1^*), \ldots, F_n^{-1}(U_n^*)\right),$$

where, for $i = 1, \ldots, n$,

$$U_i^* := (1-U)1_{\left\{U \in \left[\sum_{j=1}^{i} p_j, 1\right)\right\}} + \left(U + 1 - \sum_{j=1}^{i} p_j\right) 1_{\left\{U \in (0, \sum_{j=1}^{i} p_j)\right\}}$$

and

$$F_i^{-1}(U_i^*) = \begin{cases} F_i^{-1}\left(U + 1 - \sum_{j=1}^{i} p_j\right), & \sum_{j=1}^{i-1} p_j \le U < \sum_{j=1}^{i} p_j, \\ u_i, & \text{otherwise} \end{cases}$$

Theorem (CF of mutually exclusive sum)

Assume $l_1, \ldots, l_n = 0$ or $u_1, \ldots, u_n = 0$.

$$X_1^M, \ldots, X_n^M \text{ are ME} \Longrightarrow \varphi_{S^M}(t) = \sum_{i=1}^{n} \varphi_{X_i}(t) - (n-1)$$

Furthermore, if

$$\sum_{i=1}^{n} \varphi_{X_i}(t) - (n-1)$$

is a valid CF, then $\mathcal{R}(F_1, \ldots, F_n)$ supports GME random vectors.

Application (mutually exclusive sum of mixture distributions)

Y_i: strictly positive

$$X_i^M \stackrel{d}{=} \begin{cases} 0, & \text{with probability } p_i, \\ Y_i, & \text{with probability } 1-p_i, \end{cases}$$

$X_1^M, \dots, X_n^M$: MEB (requires $\sum_{i=1}^n p_i \geq n-1$)

$$X_1^M + \cdots + X_n^M \stackrel{d}{=} \begin{cases} 0, & \text{with probability } \sum_{i=1}^n p_i - (n-1), \\ Y_1, & \text{with probability } 1-p_1, \\ \vdots & \vdots \\ Y_n, & \text{with probability } 1-p_n. \end{cases}$$

Application (mutually exclusive sum of compound distributions)

$S_i \stackrel{d}{=} \sum_{j=1}^{N_i} X_{ij}$, $X_{ij} \sim F_X$

(a) MEB $(N_1^M, \dots, N_n^M)$ exists $\Rightarrow$ MEB $(S_1^M, \dots, S_n^M)$ exists

(b) $S_1^M + \cdots + S_n^M$ has a compound distribution

primary: $N_1^M + \cdots + N_n^M$, secondary: X

Part 3

Lower bounds of TVaR, Haezendonck-Goovaerts risk measures and convex expectation of a sum

(1) $\inf\{\text{TVaR}_\epsilon(X_1 + \cdots + X_n) \mid (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)\} =?$

(2) Is the infimum attainable? By what kind of dependence structure?

(3) Lower bound of $\text{TVaR}_\alpha(X_1 + \cdots + X_n)$?

(4) Other more general risk measures? Convex expectation?

Theorem (for bounded below risks)

$X_1, \ldots, X_n \in L^1$ with ess inf $X_i = l_i > -\infty$ for all i.

$S := X_1 + \cdots + X_n$.

(a) For any $\epsilon \in (0,1)$,

$$\text{TVaR}_{1-\epsilon}(S) \geq \sum_{i=1}^{n} l_i + \max_{(\epsilon_1,\ldots,\epsilon_n)\in S(\epsilon)} \sum_{i=1}^{n} \frac{\epsilon_i}{\epsilon}\Big(\text{TVaR}_{1-\epsilon_i}(X_i) - l_i\Big)$$

(lower bound denoted by $L_B(1-\epsilon)$)

Theorem (for bounded below risks), cont.

(b) $\text{TVaR}_{1-\epsilon}(S) = L_B(1-\epsilon)$ for all $\epsilon \in (0,1) \iff X_1, \ldots, X_n$ are MEB

(GME is characterized by the minimality of the TVaR of the sum)

Theorem (for bounded above risks)

$X_1, \dots, X_n \in L^1$ with ess sup $X_i = u_i < \infty$ for all i.

$S := X_1 + \cdots + X_n$.

(a) For any $\epsilon \in (0, 1)$,

$$\text{TVaR}_\epsilon(S) \geq \sum_{i=1}^{n} u_i + \max_{(\epsilon_1,\dots,\epsilon_n)\in S(\epsilon)} \sum_{i=1}^{n} \frac{1-\epsilon_i}{1-\epsilon} \left(\text{TVaR}_{\epsilon_i}(X_i) - u_i\right)$$

(lower bound denoted by $L_A(\epsilon)$)

(b) $\text{TVaR}_\epsilon(S) = L_A(\epsilon)$ for all $\epsilon \in (0, 1) \iff X_1, \dots, X_n$ are MEA.

Proof:

Use the previous theorem, and the following identity:

$$\mathbb{E}(X) = (1-\alpha)\text{TVaR}_\alpha(X) - \alpha\text{TVaR}_{1-\alpha}(-X)$$

Theorem (for general risks)

$X_1, \ldots, X_n \in L^1$, $S := X_1 + \cdots + X_n$.

For any $\epsilon \in (0,1)$,

$$\mathsf{TVaR}_\epsilon(S) \geq \max_{(\epsilon_1,\ldots,\epsilon_n)\in S(\epsilon)} \sum_{i=1}^n \frac{1}{1-\epsilon} \int_{\epsilon_i}^{1-\epsilon+\epsilon_i} F_{X_i}^{-1}(t)\,\mathsf{d}t$$

(lower bound denoted by $L_G(\epsilon)$)

Sketch of proof:

Step 1:

$$\mathsf{TVaR}_\epsilon(S) \geq \mathsf{TVaR}_\epsilon(X_1 \wedge c_1 + \cdots + X_n \wedge c_n)$$

for any $c_1, \ldots, c_n$

Step 2:

$$\begin{aligned} &\mathsf{TVaR}_\epsilon(X_1 \wedge c_1 + \cdots + X_n \wedge c_n) \\ &\quad \geq \sum_{i=1}^n c_i + \max_{(\epsilon_1,\ldots,\epsilon_n)\in S(\epsilon)} \sum_{i=1}^n \frac{1-\epsilon_i}{1-\epsilon} \left(\mathsf{TVaR}_{\epsilon_i}(X_i \wedge c_i) - c_i\right) \\ &\quad \geq \sum_{i=1}^n c_i + \sum_{i=1}^n \frac{1-\epsilon_i}{1-\epsilon} \left(\mathsf{TVaR}_{\epsilon_i}(X_i \wedge c_i) - c_i\right) \end{aligned}$$

for any $(\epsilon_1, \ldots, \epsilon_n) \in S(\epsilon)$

Step 3:

$$f_i(c_i, \epsilon_i) := c_i + \frac{1-\epsilon_i}{1-\epsilon} \left(\mathsf{TVaR}_{\epsilon_i}(X_i \wedge c_i) - c_i \right)$$

$$\Rightarrow \mathsf{TVaR}_\epsilon(S) \geq \sup_{(\epsilon_1,\ldots,\epsilon_n) \in S(\epsilon)} \sup_{c_1,\ldots,c_n} \sum_{i=1}^{n} f_i(c_i, \epsilon_i)$$

Step 4:

For any fixed $\epsilon_i \in [0, \epsilon]$,

$$\sup_{c_i} f_i(c_i, \epsilon_i) = \frac{1}{1-\epsilon} \int_{\epsilon_i}^{1-\epsilon+\epsilon_i} F_{X_i}^{-1}(t) \, \mathrm{d}t$$

$$\therefore \mathsf{TVaR}_\epsilon(S) \geq \max_{(\epsilon_1,\ldots,\epsilon_n) \in S(\epsilon)} \sum_{i=1}^{n} \frac{1}{1-\epsilon} \int_{\epsilon_i}^{1-\epsilon+\epsilon_i} F_{X_i}^{-1}(t) \, \mathrm{d}t$$

Summary I

For risks that are bounded from below:

$$\mathsf{TVaR}_\epsilon(S) \geq \sum_{i=1}^{n} l_i + \max_{(\epsilon_1,\ldots,\epsilon_n) \in S(1-\epsilon)} \sum_{i=1}^{n} \frac{\epsilon_i}{1-\epsilon} \Big(\mathsf{TVaR}_{1-\epsilon_i}(X_i) - l_i \Big)$$

Notation: $L_B(\epsilon)$

Summary II

For risks that are bounded from above:

$$\mathsf{TVaR}_\epsilon(S) \geq \sum_{i=1}^n u_i + \max_{(\epsilon_1,\ldots,\epsilon_n)\in S(\epsilon)} \sum_{i=1}^n \frac{1-\epsilon_i}{1-\epsilon} \left(\mathsf{TVaR}_{\epsilon_i}(X_i) - u_i\right)$$

Notation: $L_A(\epsilon)$

Summary III

For general risks:

$$\mathsf{TVaR}_\epsilon(S) \geq \max_{(\epsilon_1,\ldots,\epsilon_n)\in S(\epsilon)} \sum_{i=1}^n \frac{1}{1-\epsilon} \int_{\epsilon_i}^{1-\epsilon+\epsilon_i} F_{X_i}^{-1}(t)\,\mathrm{d}t$$

Notation: $L_G(\epsilon)$

Observation (for risks that are bounded from below)

$$\begin{aligned}
\mathsf{TVaR}_\epsilon(S) &= \frac{1}{1-\epsilon}\{\mathbb{E}(S) + \epsilon \mathsf{TVaR}_{1-\epsilon}(-S)\} \\
&\geq \frac{1}{1-\epsilon}\left\{\mathbb{E}(S) + \max_{(\epsilon_1,\ldots,\epsilon_n)\in S(1-\epsilon)} \sum_{i=1}^{n} \int_{\epsilon_i}^{\epsilon+\epsilon_i} F_{-X_i}^{-1}(t)\,\mathrm{d}t\right\} \\
&= \frac{1}{1-\epsilon} \max_{(\epsilon_1,\ldots,\epsilon_n)\in S(1-\epsilon)} \sum_{i=1}^{n} \left\{\int_0^{1-\epsilon_i-\epsilon} + \int_{1-\epsilon_i}^{1} F_{X_i}^{-1}(t)\,\mathrm{d}t\right\}
\end{aligned}$$

"$\geq$" by $L_G(1-\epsilon)$

$\longrightarrow$ a new lower bound $L_{B'}(\epsilon)$

Refinement 1 (for risks that are bounded from below)

$L_{B'}(\epsilon) \geq L_B(\epsilon)$, i.e., $L_{B'}(\epsilon)$ is a better lower bound

Refinement 2 (for risks that are bounded from above)

Recall: $\mathsf{TVaR}_\epsilon(S) \geq L_A(\epsilon)$ and $\mathsf{TVaR}_\epsilon(S) \geq L_G(\epsilon)$

$L_G(\epsilon) \geq L_A(\epsilon)$, i.e., $L_G(\epsilon)$ is a better lower bound

Haezendonck-Goovaerts risk measures

Given X non-negative, $\Phi : [0, \infty) \to [0, \infty)$ convex and strictly increasing with $\Phi(0) = 0$, $\Phi(1) = 1$ and $\lim_{x\to\infty} \Phi(x) = \infty$. Define $H_{1-\epsilon}(X)$ implicitly by

$$\mathbb{E}\left[\Phi\left(\frac{X}{H_{1-\epsilon}(X)}\right)\right] = \epsilon.$$

The HG risk measure of X is defined as

$$\pi_{1-\epsilon}(X) := \inf_{x\in\mathbb{R}} \left\{x + H_{1-\epsilon}[(X - x)_+]\right\}.$$

(Haezendonck and Goovaerts (1982), further investigated in Bellini and Gianin (2008, 2012), Goovaerts et al. (2012), Tang and Yang (2012))

Haezendonck-Goovaerts risk measures

Remarks:

(1) No closed-form for $\pi_{1-\epsilon}(X)$ in general

(2) If $\Phi(x) = x$, then $\pi_{1-\epsilon}(X) = \text{TVaR}_{1-\epsilon}(X)$

(3) The infimum, denoted by $x^*_{1-\epsilon,X}$, is always attained for all $\epsilon \in (0, 1)$; it can be negative even if X is non-negative

Theorem (lower bound for HG risk measures)

$X_1, \ldots, X_n$: non-negative

Suppose that the minimizer $x^*_{1-\epsilon,S}$ is non-negative. Then

$$\pi_{1-\epsilon}(S) \geq \max_{(\epsilon_1,\ldots,\epsilon_n)\in S(\epsilon)} \min_{1\leq i\leq n} \pi_{1-\epsilon_i}(X_i)$$

Furthermore, if Φ is strictly convex and equality prevails, then $X_1, \ldots, X_n$ are mutually exclusive.

Remark: A mutually exclusive sum may not attain the lower bound!

Lower bound of convex expectation

Problem of interest:

$$\inf_{X_i\sim F_i} \mathbb{E}[f(X_1 + \cdots + X_n)]$$

where f is convex

Theorem (convex lower bound)

$X_1, \ldots, X_n$ are non-negative, f is convex

(i)

$$\mathbb{E}[f(S)] \geq \sum_{i=1}^{n} \mathbb{E}[f(X_i)] - (n-1)f(0)$$

(ii) If f is strictly convex, then equality holds iff $X_1, \ldots, X_n$ are mutually exclusive random variables.

Remarks:

(1) Other convex lower bounds have been proposed (e.g. Wang and Wang (2011), Bernard et al. (2014)), but they all require strong assumptions on the marginals.

(2) Our lower bound is considerably simpler, more general and its sharpness can be characterized easily even in the heterogeneous case.

(3) As noted in Bernard et al. (2014), a universal solution, which applies to any marginals and any convex f, is not available. This explains why the convex lower bound problem has been intractable for a very long time.

Part 4

Mutual exclusivity in the tail

Brief summary, for $n \geq 3$:

- If each X_i has an essential infimum of 0, then

$$\text{TVaR}_{1-\varepsilon}(X_1 + \cdots + X_n) \geq \max_{\sum \varepsilon_i = \varepsilon} \sum_{i=1}^{n} \frac{\varepsilon_i}{\varepsilon} \text{TVaR}_{1-\varepsilon_i}(X_i) =: LB$$

- This lower bound is attained iff $X_1, \ldots, X_n$ are mutually exclusive (ME)

$$\Pr[X_i > 0, X_j > 0] = 0 \text{ for all } i \neq j$$

- ME random variables exist when and only when

$$F_1(0) + \cdots + F_n(0) \geq n - 1 \qquad (*)$$

Difficulty

The existence of ME random variables in $\mathcal{R}(F_1, \ldots, F_n)$ requires

$$\sum_{i=1}^{n} F_i(0) \geq n - 1 \qquad (*)$$

- rather stringent

Objectives

(A) Is LB still tight without the validity of (*)?

(B) How to relax (*) to maintain the tightness of LB at least for *probability level close to one*? What is the corresponding dependence structure?

Definition

A random vector $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$ is said to be *mutually exclusive in the tail, abbreviated as MET*, if there exists a probability vector $\boldsymbol{p} = (p_1, \ldots, p_n) \in (0, 1)^n$ such that

$$\Pr[X_i > F_i^{-1}(p_i), X_j > 0] = 0 \quad \text{for any } i \neq j$$

$\boldsymbol{X}$ is also called *$\boldsymbol{p}$-mutually exclusive, abbreviated as $\boldsymbol{p}$-ME*

"0" can be replaced by ess inf X_i

Support of a $\boldsymbol{p}$-ME random vector

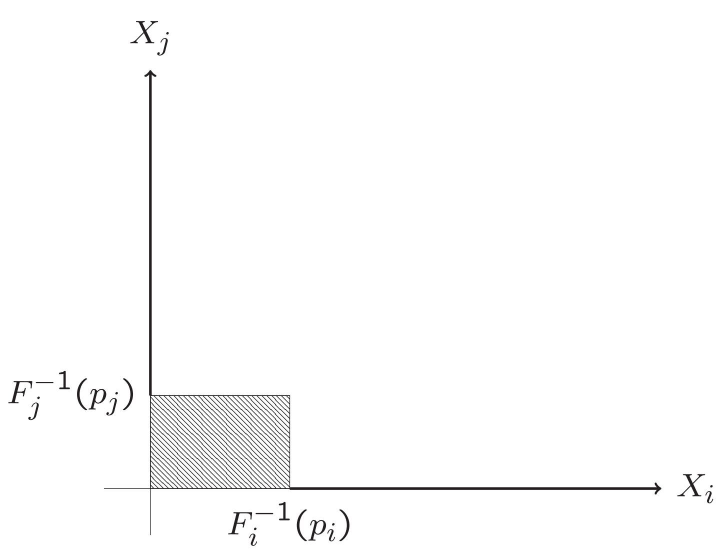
$\boldsymbol{X} \in \mathcal{R}(F_1, \ldots, F_n)$ is $\boldsymbol{p}$-ME if and only if it is supported by the box

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq F_i^{-1}(p_i) \text{ for all } i\}$$

and the half lines

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid x_i > F_i^{-1}(p_i), x_j = 0 \text{ for } j \neq i\},\ i = 1, \ldots, n$$

Support of a $\boldsymbol{p}$-ME random vector



Some observations

- if $\boldsymbol{X}$ is $\boldsymbol{p}$-ME and if $\boldsymbol{p}' \geq \boldsymbol{p}$, then $\boldsymbol{X}$ is $\boldsymbol{p}'$-ME too

  $\Rightarrow$ one can always increase the probability level p_i without destroying the MET structure.

- $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$ is $\boldsymbol{p}$-ME if and only if it is $\boldsymbol{p}'$-ME, where $p_i{}' = F_i(F_i^{-1}(p_i))$

  $\Rightarrow$ under this convention, $p_i \geq F_i(0)$ for all i

- if $p_i = F_i(0)$ for all i, $\boldsymbol{X}$ is $\boldsymbol{p}$-ME if and only if it is ME in the classical sense

Proposition (properties of MET)

Suppose that $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$ is $\boldsymbol{p}$-ME.

(i) The random vector

$$((X_1 - F_1^{-1}(p_1))_+, \ldots, (X_n - F_n^{-1}(p_n))_+)$$

is mutually exclusive.

(ii) For $i = 1, \ldots, n$, let $f_i : [0, \infty) \to [0, \infty)$ be increasing, left-continuous, and satisfying $f_i(0) = 0$. Then $(f_1(X_1), \ldots, f_n(X_n))$ is also $\boldsymbol{p}$-ME.

Proposition (existence)

(i) Suppose that $p_i \geq m_i := F_i(0)$ for all i. There exists a $\boldsymbol{p}$-ME random vector in $\mathcal{R}(F_1, \dots, F_n)$ if and only if

$$\sum_{i=1}^{n} (1 - p_i) + \max_{1 \leq i \leq n} (p_i - m_i) \leq 1.$$

In this case, $m_i > 0$ for all i.

(ii) There exists a MET random vector in $\mathcal{R}(F_1, \dots, F_n)$ if and only if $m_i > 0$ for all i.

Proposition (construction)

Step 1 Define $\Delta = \max_i(p_i - m_i)$ and $R = 1 - \sum_{i=1}^{n}(1 - p_i) - \Delta$

Step 2 Let $L, K, K_1, \dots, K_n$ be disjoint open intervals in $(0, 1)$ with $|L| = \Delta$, $|K| = R$, and $|K_i| = 1 - p_i$

Step 3 Fix any $U \sim U(0, 1)$, and let $U_i \sim U(0, 1)$ be obtained from a shuffling of U according to the following rules:

$$\begin{cases} U_i \in (p_i, 1) & \Leftrightarrow U \in K_i \\ U_i \in (p_i - \Delta, p_i) & \Leftrightarrow U \in L \\ U_i \in (0, p_i - \Delta) & \Leftrightarrow U \in \left(\left(\cup_{j \neq i} K_j \right) \cup K \right) \end{cases}$$

Step 4 Define $X_i := F_i^{-1}(U_i)$ for all $i \Rightarrow (X_1, \dots, X_n)$ is $\boldsymbol{p}$-ME

An explicit way to construct $U_1, \ldots, U_n$ in Step 3:

Let

$$s_0 := R \quad \text{and} \quad s_i = R + \sum_{j=1}^{i} (1 - p_j), \quad i = 1, \ldots, n.$$

Let U be a uniform$(0, 1)$ random variable, and define

$$U_i = U + (1 - s_i) 1_{\{s_{i-1} < U < s_i\}} - (1 - p_i) 1_{\{U > s_i\}}, \quad i = 1, \ldots, n.$$

Proposition (characterization in terms of cdf)

Let $\boldsymbol{X} \in \mathcal{R}(F_1, \ldots, F_n)$, with $p_i \geq F_i(0)$ for all i. Then $\boldsymbol{X}$ is $\boldsymbol{p}$-ME if and only if

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \sum_{i=1}^{n} (F_i(x_i) - p_i)_+ + F_{\boldsymbol{X}}(\boldsymbol{x} \wedge \boldsymbol{a}), \quad \text{for any } \boldsymbol{x} \geq 0,$$

where $\boldsymbol{a} := (F_1^{-1}(p_1), \ldots, F_n^{-1}(p_n))$.

Proposition (decomposition of survival function)

Let $\boldsymbol{X}^*$ be MET, and $S^* := X_1 + \cdots + X_n$. Then there exists some ϕ s.t.

$$\Pr[S^* > t] = \sum_{i=1}^{n} \Pr[X_i^* > t] \quad \text{for any } t \geq \phi.$$

If $\boldsymbol{X}^*$ is ME, then

$$\Pr[S^* > t] = \sum_{i=1}^{n} \Pr[X_i^* > t] \quad \text{for any } t \geq 0.$$

Corollary (decomposition of stop-loss premium)

Let $\boldsymbol{X}^*$ be MET, and $S^* := X_1 + \cdots + X_n$. Then there exists some ϕ such that

$$\mathbb{E}\left[(S^* - d)_+\right] = \sum_{i=1}^{n} \mathbb{E}\left[(X_i^* - d)_+\right] \quad \text{for any } d \geq \phi.$$

Tail convex order (Cheung and Vanduffel (2013))

X is said to precede Y in the *tail convex order*, denoted as $X \leq_{tcx} Y$, if there exists a real number k such that $\Pr[Y > k] > 0$ and $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \geq k$.

Corollary (minimal tail convex order)

Let $\boldsymbol{X}^*$ be MET. For any random vector $\boldsymbol{X}$ with the same marginals,

$$\sum_{i=1}^{n} X_i^* \leq_{tcx} \sum_{i=1}^{n} X_i.$$

Part 5

MET and TVaR Lower bounds

Theorem (tightness of LB)

Suppose that

(i) $F_i(0) > 0$ for all i, and

(ii) $F_i(t) \neq 1$ for all t, for at least one i.

Then, there exist a MET random vector $(X_1^*, \ldots, X_n^*) \in \mathcal{R}(F_1, \ldots, F_n)$ and some probability level $\varepsilon^* \in (0, 1)$ such that

$$\begin{aligned}\mathrm{TVaR}_\varepsilon\left(\sum_{i=1}^n X_i^*\right) &= \min_{\boldsymbol{X}\in\mathcal{R}(F_1,\ldots,F_n)} \mathrm{TVaR}_\varepsilon\left(\sum_{i=1}^n X_i\right) \\ &= \max_{\sum \varepsilon_i = 1-\varepsilon} \sum_{i=1}^n \frac{\varepsilon_i}{1-\varepsilon} \mathrm{TVaR}_{1-\varepsilon_i}(X_i^*)\end{aligned}$$

for any $\varepsilon > \varepsilon^*$.

The Theorem above means that under the two stated hypotheses on the marginal distributions, the TVaR lower bound LB is reachable in $\mathcal{R}(F_1, \ldots, F_n)$ when the probability level is higher than some threshold ε^*, and mutual exclusivity in the tail is the corresponding dependence structure.

With slightly more effort, one can dispense with the assumption that $F_i(0) > 0$ for all i and show that the lower bound LB is *asymptotically tight*. The only required condition is that at least one of the risks is unbounded above.

Theorem (asymptotic tightness of LB)

If $F_i(t) \neq 1$ for all t for at least one i, then

$$\lim_{\varepsilon \to 0} \left\{ \inf_{Y_i \sim F_i} \mathrm{TVaR}_{1-\varepsilon}\left(\sum_{i=1}^{n} Y_i\right) - \max_{\sum \varepsilon_i = \varepsilon} \sum_{i=1}^{n} \frac{\varepsilon_i}{\varepsilon} \mathrm{TVaR}_{1-\varepsilon_i}(F_i) \right\} = 0.$$

In other words,

$$\inf_{Y_i \sim F_i} \mathrm{TVaR}_{1-\varepsilon}\left(\sum_{i=1}^{n} Y_i\right) \quad \text{and} \quad \max_{\sum \varepsilon_i = \varepsilon} \sum_{i=1}^{n} \frac{\varepsilon_i}{\varepsilon} \mathrm{TVaR}_{1-\varepsilon_i}(F_i)$$

can be made arbitrarily close when ε is sufficiently close to 1.

Recall:

The expression

$$\max_{\sum \varepsilon_i = \varepsilon} \sum_{i=1}^{n} \frac{\varepsilon_i}{\varepsilon} \mathrm{TVaR}_{1-\varepsilon_i}(F_i)$$

is the TVaR of an MET random vector in $\mathcal{R}(F_1, \ldots, F_n)$

Conclusion:

Under the very mild condition that $F_i(t) \neq 1$ for all t for at least one i, the dependence structure mutual exclusivity in the tail gives rise to an asymptotic TVaR lower bound

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THE END