

On drift parameter estimation for reflected fractional Ornstein-Uhlenbeck processes

Chihoon Lee*

School of Business

Stevens Institute of Technology

Hoboken, NJ 07030; USA

On leave from Colorado State University

Fort Collins, CO 80523, USA

Jian Song[†]

Department of Mathematics

Department of Statistics & Actuarial Science

University of Hong Kong, Hong Kong

Abstract

We consider a reflected Ornstein-Uhlenbeck process X driven by a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Our goal is to estimate an unknown drift parameter $\alpha \in (-\infty, \infty)$ on the basis of continuous observation of the state process. We establish Girsanov theorem for the process X , derive the standard maximum likelihood estimator of the drift parameter α , and prove its strong consistency and asymptotic normality. As an improved estimator, we obtain the explicit formulas for the sequential maximum likelihood estimator and its mean squared error by assuming the process is observed until a certain information reaches a specified precision level. The estimator is shown to be unbiased, uniformly normally distributed, and efficient in the mean square error sense.

Keywords: Reflected fractional Ornstein-Uhlenbeck processes, fractional Brownian motion, fractional calculus, parameter estimation, maximum likelihood estimator, sequential maximum likelihood estimator.

AMS Subject Classifications: Primary 60G22; secondary 60H30, 62M09, 90B22

1 Introduction

We consider a drift parameter estimation problem for a one-dimensional reflected fractional Ornstein-Uhlenbeck (RFOU) process with infinitesimal drift $-\alpha x$ and infinitesimal variance σ^2 , where $\alpha \in (-\infty, \infty)$ and $\sigma > 0$. The RFOU process can serve as approximating models in diverse applications such as in physical, biological, and mathematical finance models (see, e.g., [33, 1, 5, 6] and also Section 2 below). The RFOU process behaves as a standard FOU process in the interior of

*E-mail: Chihoon.Lee@stevens.edu

[†]E-mail: txjsong@hku.hk

its domain $(0, \infty)$. However, when it reaches its boundary at zero, then the sample path returns to the interior in a manner exercising with minimal “pushing” force. Our main interest in this model stems from the fact that the RFOU process arises as the key approximating process for queueing systems with reneging or balking customers with long range dependent inter-arrival and/or service time processes (see [36, 21, 38] and the references therein). In such cases, the drift parameter α carries the physical meaning of customers’ reneging (or, balking) rate from the system. More details are provided in Section 2 with regards to how the RFOU model can arise in the applications. In this paper, we expand the previously known results on the parameter estimation problems for a reflected Ornstein-Uhlenbeck (ROU) process to the case when the noise process is given by a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Such results require several nontrivial technical efforts, since the fractional Brownian motion is not a Markov process nor a semimartingale (unless $H = 1/2$) and the classical stochastic calculus is inapplicable in its analysis.

We describe the RFOU model more precisely. Let $\Lambda := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a complete filtered probability space with the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the *usual* conditions. Define the RFOU process $\{X_t : t \geq 0\}$ reflected at zero on Λ as follows. Let $\{X_t : t \geq 0\}$ be the strong solution to the stochastic differential equation:

$$\left. \begin{aligned} dX_t &= -\alpha X_t dt + \sigma dW_t^H + dL_t, \\ X_t &\geq 0 \quad \text{for all } t \geq 0, \\ X_0 &= x, \end{aligned} \right\} \quad (1.1)$$

where $\alpha \in (-\infty, \infty)$, $\sigma \in (0, \infty)$, $x \in [0, \infty)$ and $W^H = (W_t^H)_{t \geq 0}$ is a one-dimensional standard fractional Brownian motion on Λ with a known Hurst index $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Here, the process $L = (L_t)_{t \geq 0}$ is defined to be the minimal, non-decreasing and non-negative process with initial value $L_0 = 0$, which increases only when X hits the boundary 0, so that

$$\int_{[0, \infty)} I(X_t > 0) dL_t = 0, \quad (1.2)$$

where $I(\cdot)$ is the indicator function. That is, L_t represents the (cumulative) local time of X at the boundary zero up to time $t \geq 0$. The pair (X, L) is called a solution to the *linear drift* Skorohod problem (1.1) for W^H .

Let $C(\mathbb{R}_+, \mathbb{R})$ denote the space of continuous functions on $[0, \infty)$. Consider the classical Skorohod problem for $Z \in C(\mathbb{R}_+, \mathbb{R})$, i.e., $X_t = Z_t + L_t$ where $X_t \geq 0$ for $t \geq 0$, L is nondecreasing with $L_0 = 0$, and $\int_0^\infty I(X_t > 0) dL_t = 0$. It is well-known ([35]) that the unique solution pair $(X, L) = (\Phi(Z), \Psi(Z)) \in C(\mathbb{R}_+, \mathbb{R}) \times C(\mathbb{R}_+, \mathbb{R})$ is given by

$$X_t = Z_t + L_t \quad \text{and} \quad L_t = \sup_{0 \leq s \leq t} \{-Z_s\} \vee 0. \quad (1.3)$$

For the existence, uniqueness, and Lipschitz continuity of the mapping (Φ, Ψ) , we refer the reader to Theorem 7.2 of Chen and Yao [10]. For the *generalized drift* Skorohod problem for $Z \in C(\mathbb{R}_+, \mathbb{R})$ with a Lipschitz continuous drift f , i.e., $dX_t = f(X_t)dt + dZ_t + dL_t$, the existence of a solution is given by (see, for instance, pages 16 and 17 in [31]) the following construction:

$$(X, L) = (\Phi(\mathcal{M}(Z)), \Psi(\mathcal{M}(Z))).$$

Here, $\mathcal{M} : C(\mathbb{R}_+, \mathbb{R}) \rightarrow C(\mathbb{R}_+, \mathbb{R})$ is a mapping that sets $\mathcal{M}(Z) = Y$ for Y that solves: $dY_t = f(\Phi(Y)(t))dt + dZ_t$, whose unique solution can be obtained by a standard Picard iteration since

f is a Lipschitz function and Φ is a Lipschitz mapping (Theorem 7.2 of Chen and Yao [10] or [15]). Therefore, for the Skorohod problem (1.1), there exists a unique pathwise solution $(X, L) \in C(\mathbb{R}_+, \mathbb{R}) \times C(\mathbb{R}_+, \mathbb{R})$; see also Appendix A.1 of Ward and Glynn [38] and Proposition 2 of [32]. Furthermore, by (1.3), the process L has an explicit representation as

$$L_t = \sup_{0 \leq s \leq t} \left(-x + \alpha \int_0^s X_u du - \sigma W_s^H \right) \vee 0. \quad (1.4)$$

Our main interest lies in the statistical inference for the RFOU process (1.1). More specifically, our aim is to estimate the unknown drift parameter $\alpha \in (-\infty, \infty)$ in (1.1) based on observation of the state process $\{X_t\}_{t \geq 0}$. We assume that the infinitesimal variance parameter σ^2 is known as this can be estimated via a standard approach involving the associated quadratic variation process; see Remark 4.9.

In [20], by applying the approach of fundamental martingale for fractional Brownian motion (see [27]) and computing the Laplace transform of a functional of the state process, the authors investigated an asymptotic behavior of the maximum likelihood estimator (MLE) of the drift parameter for a standard (non-reflected) FOU process when $H \in [1/2, 1)$. They obtained the strong consistency of the estimator and explicit formulas for the asymptotic bias and mean square error. In [9], the authors established long time asymptotic properties (such as consistency, asymptotic normality and convergence of the moments) of the MLE for the signal drift parameter in a partially observed fractional diffusion system, via the computations of Laplace transform. Sharp large deviation properties of the energy and the MLE for the FOU process with $H \in (1/2, 1)$ were studied in [2]. For parameters in stochastic partial differential equations (SPDEs) with additive space-time white noise, properties of MLE were investigated in [17, 18]. When the driving noise process of SPDEs is white in space and fractional in time, the properties of MLE were studied in [12]. More details on the statistical inference for SPDEs can be found in the survey paper [26] and the references therein.

The authors of [7] studied the MLE for the model (1.1) when $\alpha \in (0, \infty)$ (i.e., the ergodic case) and $H = 1/2$ (i.e., the standard Brownian motion case), and established several important properties. The MLE $\hat{\alpha}_T$ of α , based on the process $\{X_t\}$ up to a previously determined fixed time T , is given by

$$\hat{\alpha}_T := \frac{-\int_0^T X_t dX_t}{\int_0^T X_t^2 dt}. \quad (1.5)$$

The MLE $\hat{\alpha}_T$ satisfies strong consistency and asymptotic normality as $T \rightarrow \infty$. However, this estimator is biased and its mean squared error (MSE) depends on the unknown parameter to be estimated. We note that exact estimates for the bias and the MSE of the estimator $\hat{\alpha}_T$ are not available. As a remedy for this, a sequential estimation plan $(\tau(h), \hat{\alpha}_{\tau(h)})$ was proposed in [22]. It is assumed in [22] that the parameter ranges the whole real line $\alpha \in (-\infty, \infty)$ (i.e., it covers ergodic, non-ergodic, non-stationary cases) and the process $\{X_t\}$ is observed until the observed Fisher information of the process exceeds a predetermined level of precision h (see also [8]). More precisely, $\{X_t\}$ is observed over the random time interval $[0, \tau(h)]$ where the stopping time $\tau(h)$ is defined as

$$\tau(h) := \inf \left\{ t \geq 0 : \int_0^t X_s^2 ds \geq h \right\}, \quad 0 < h < \infty, \quad (1.6)$$

and the $\mathcal{F}_{\tau(h)}^X$ -measurable function $\hat{\alpha}_{\tau(h)}$ defined by

$$\hat{\alpha}_{\tau(h)} := -\frac{1}{h} \int_0^{\tau(h)} X_s dX_s \quad (1.7)$$

is a sequential estimator. Then the sequential estimation plan $(\tau(h), \hat{\alpha}_{\tau(h)})$ has shown to satisfy the following properties (cf. Chapter 17.5 of [25] or Chapter 5.2 of [4]): (a) it is unbiased; (b) the plan is closed, i.e., the time of the observation $\tau(h)$ is finite with probability 1; (c) its MSE is a constant that does not depend on the parameter to be estimated; (d) not only it provides consistent estimation plan but also $\hat{\alpha}_{\tau(h)}$ is exactly normally distributed, which makes it possible to construct an exact confidence interval for the parameter α .

Our main results are concerned with extending the aforementioned estimators (1.5), (1.7) and their respective statistical properties to the case when the system is driven by a fractional Brownian motion with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. We establish uniform exponential moment estimates of the RFOU process, which, in conjunction with certain integral representations and the fundamental martingales of fractional Brownian motions, leads to two types of fractional Girsanov formulas. Then, we obtain the standard MLE and prove its strong consistency and asymptotic normality. Furthermore, we derive the explicit expression for the sequential MLE and show that it is unbiased, uniformly normally distributed (over the entire parameter space which is the real line), and efficient in the mean square error sense.

The rest of the paper is organized as follows. In Section 2, we provide a brief discussion and derivation about how the RFOU model can naturally arise in the applications. Section 3 is devoted to preliminaries on fractional Brownian motion and fractional calculus that are necessary in our analysis. In Section 4, we establish a Girsanov theorem for the RFOU process in Theorem 4.3 and obtain the standard maximum likelihood estimator of the drift parameter, and then prove its strong consistency and asymptotic normality. We also develop an equivalent version of the Girsanov theorem in Theorem 4.7, directly from Theorem 4.3 by using the transformation result introduced in Section 3.2. In Section 5, we derive the explicit formulas for the sequential maximum likelihood estimator and its mean squared error by assuming the process is observed until a certain information reaches a specified precision level. The sequential estimator is shown to be unbiased, uniformly normally distributed, and efficient in the mean square error sense. Finally, in the Appendix we provide an independent, more direct proof for Theorem 4.7 and an auxiliary result for fractional Brownian motion.

Throughout the paper, C denotes a generic constant which may vary in different places.

2 Motivation of the RFOU model

Here we provide some details about how the RFOU model could arise in the applications. Firstly, in the context of financial time series modelling, the RFOU processes can be used to describe the spot foreign exchange rate processes, the domestic interest rate processes, and even some asset price processes in a regulated financial market system (cf. [6, 5]) with long range dependence and heavy tails stylized facts, which seem to be common to a wide variety of markets, instruments and periods [13, 39]. Secondly, in engineering applications to queueing and storage systems, the RFOU model can play as the key approximating process for systems with reneging or balking customers (cf. [36, 37, 38] and the references therein), subject to their inter-arrival and/or service time processes exhibiting long range dependence characteristics in the traffic data. In such cases, the drift parameter α carries the physical meaning of customers' reneging (or, balking) rate from the system. We provide a more detailed justification below.

Consider a single-server, single-class queueing model under heavy traffic subject to their reneging behaviors. More precisely, we shall consider a sequence of single-server queueing systems indexed by

$N = 1, 2, \dots$, and assume that the sequence of interarrival times $\{T_k^{(N)} - T_{k-1}^{(N)}\}_{k=1}^\infty$ are stationary with $E(T_k^{(N)} - T_{k-1}^{(N)}) = 1/\lambda^{(N)}$ such that $\lambda^{(N)} \rightarrow \lambda \in (0, \infty)$ as $N \rightarrow \infty$. Also, we assume an invariance principle holds:

$$\frac{T_{[Nt]}^{(N)} - [Nt]/\lambda^{(N)}}{N^H} \Rightarrow \sigma W^H(t),$$

where W^H is a standard fractional Brownian motion with Hurst parameter H and $EW^H(t)^2 = t^{2H}$ and some scaling constant $\sigma > 0$. For the sake of simplicity, we will assume $\lambda = 1$. Then, under a mild assumption on $\text{Var}(T_k^{(N)} - T_{k-1}^{(N)}) \leq K < \infty$, one gets the following functional central limit theorem for the arrival process $A^{(N)}(t) = \sum_{n=1}^\infty I(T_n^{(N)} \leq t)$, $t \geq 0$:

$$\frac{A^{(N)}(Nt) - \lambda^{(N)}Nt}{N^H} \Rightarrow \sigma W^H(t), \quad (2.1)$$

where W^H is a standard fractional Brownian motion (cf. Theorem 2.1 of [21]).

With the arrival process $\{A^{(N)}(t) : t \geq 0\}$, consider the queueing system operating with a constant service rate $\mu^{(N)} > 0$, if there are customers in the buffer, and otherwise the server becomes idle. Moreover, customers faced with long waiting times will abandon the system before receiving service; let $R^{(N)}(t)$ be the total number of customers who abandoned the system during the time interval $[0, t]$. Incorporating these conditions will yield the following equation on the queue length process ($Q^{(N)}(t) : t \geq 0$):

$$\begin{aligned} Q^{(N)}(t) &= Q^{(N)}(0) + A^{(N)}(t) - R^{(N)}(t) - \int_0^t \mu^{(N)} I_{[Q^{(N)}(s) > 0]} ds \\ &= Q^{(N)}(0) + A^{(N)}(t) - \int_0^t \alpha^{(N)} Q^{(N)}(s) ds - \mu^{(N)} t + \int_0^t \mu^{(N)} I_{[Q^{(N)}(s) = 0]} ds, \end{aligned}$$

where $\alpha^{(N)} > 0$ is a proportionality factor related with the customers' abandonment rate from the system. We assume that the abandonment factor $\alpha^{(N)}$ is of $O(N^H)$, in particular, $\alpha^{(N)}/N^H \rightarrow \alpha$ as $N \rightarrow \infty$ for some constant $\alpha > 0$.

In view of the functional central limit theorem scaling used in (2.1), we have that

$$\begin{aligned} \frac{Q^{(N)}(Nt)}{N^H} &= \frac{Q^{(N)}(0)}{N^H} + \frac{A^{(N)}(Nt)}{N^H} - \frac{1}{N^H} \int_0^{Nt} \alpha^{(N)} Q^{(N)}(s) ds \\ &\quad - \frac{\mu^{(N)}Nt}{N^H} + \frac{1}{N^H} \int_0^{Nt} \mu^{(N)} I_{[Q^{(N)}(s) = 0]} ds \\ &= \frac{Q^{(N)}(0)}{N^H} + \frac{A^{(N)}(Nt) - \lambda^{(N)}Nt}{N^H} - \frac{1}{N^H} \int_0^{Nt} \alpha^{(N)} Q^{(N)}(s) ds \\ &\quad - \frac{\mu^{(N)}Nt - \lambda^{(N)}Nt}{N^H} + \frac{1}{N^H} \int_0^{Nt} \mu^{(N)} I_{[Q^{(N)}(s) = 0]} ds. \end{aligned}$$

Lastly, we impose a ‘‘heavy traffic’’ assumption implying that the system processing capacity is balanced with the system load, that is, the ‘‘drift’’ term $(\lambda^{(N)} - \mu^{(N)})$ converges to zero, at a certain rate closely related with the scaling in (2.1): $N^{1-H}(\lambda^{(N)} - \mu^{(N)}) \rightarrow 0$ as $N \rightarrow \infty$. With the initial condition $\frac{Q^{(N)}(0)}{N^H} \rightarrow x \in [0, \infty)$ and owing to the Lipschitz continuity property of the *generalized* Skorohod (reflection) map [38, 32] together with the continuous- mapping theorem, we finally obtain the weak convergence of the scaled queue length process $\{\frac{Q^{(N)}(Nt)}{N^H} : t \geq 0\}_{N=0}^\infty$ to the

RFOU process with the reflecting boundary given by zero. We note that similar derivations are possible for the offered waiting time process, in the context of state-dependent admission control setup with customers' impatient behaviors (see [23] and the references therein).

3 Preliminaries on fractional calculus and fractional Brownian motion

3.1 Fractional calculus

In this section, we recall some basic results from fractional calculus. See [34] for more details. Let $a, b \in \mathbb{R}$ with $a < b$ and let $\alpha > 0$. (The symbol α in this section should not be confused with the parameter of the RFOU process.) The left-sided and right-sided fractional Riemann-Liouville integrals of $f \in L^1([a, b])$ of order α are defined for almost all $t \in (a, b)$ by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

and

$$I_{b-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds,$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^{\infty} r^{\alpha-1} e^{-r} dr$ is the Euler gamma function.

Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p([a, b])$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}).

If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $\alpha \in (0, 1)$, then the left and right-sided fractional derivatives are defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}} ds \right) \quad (3.1)$$

and

$$D_{b-}^{\alpha} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t)-f(s)}{(s-t)^{\alpha+1}} ds \right) \quad (3.2)$$

for almost all $t \in (a, b)$.

Let $C^{\alpha}([a, b])$ denote the space of α -Hölder continuous functions of order α on the interval $[a, b]$. When $\alpha p > 1$, then we have $I_{a+}^{\alpha}(L^p) \in C^{\alpha-\frac{1}{p}}([a, b])$. On the other hand, if $\beta > \alpha$, then $C^{\beta}([a, b]) \subset I_{a+}^{\alpha}(L^p)$ for all $p > 1$.

The following inversion formulas hold:

$$\begin{aligned} I_{a+}^{\alpha}(I_{a+}^{\beta} f) &= I_{a+}^{\alpha+\beta} f, & f &\in L^1; \\ D_{a+}^{\alpha}(I_{a+}^{\alpha} f) &= f, & f &\in L^1; \\ I_{a+}^{\alpha}(D_{a+}^{\alpha} f) &= f, & f &\in I_{a+}^{\alpha}(L^1); \\ D_{a+}^{\alpha}(D_{a+}^{\beta} f) &= D_{a+}^{\alpha+\beta} f, & f &\in I_{a+}^{\alpha+\beta}(L^1), \alpha + \beta \leq 1. \end{aligned}$$

Similar inversion formulas hold for the operators I_{b-}^{α} and D_{b-}^{α} as well.

We also have the following integration by parts formula.

Proposition 3.1. *If $f \in I_{a+}^\alpha(L^p), g \in I_{b-}^\alpha(L^q)$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\int_a^b (D_{a+}^\alpha f)(s)g(s)ds = \int_a^b f(s)(D_{b-}^\alpha g)(s)ds. \quad (3.3)$$

The following proposition indicates the relationship between Young's integral and Lebesgue integral.

Proposition 3.2. *Suppose that $f \in C^\lambda(a, b)$ and $g \in C^\mu(a, b)$ with $\lambda + \mu > 1$. Let $\lambda > \alpha$ and $\mu > 1 - \alpha$. Then the Riemann-Stieltjes integral $\int_a^b f dg$ exists and it can be expressed as*

$$\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(t) D_{b-}^{1-\alpha} g_{b-}(t) dt, \quad (3.4)$$

where $g_{b-}(t) = g(t) - g(b)$.

3.2 Fractional Brownian motion

Let $H \in (0, 1)$ be a constant. A fractional Brownian motion $\{W_t^H, t \in \mathbb{R}\}$ of Hurst parameter H is a Gaussian process with zero mean and covariance function

$$\mathbb{E}(W_t^H W_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}) =: R_H(t, s).$$

Denote by \mathcal{E} the set of step functions on $[0, T]$. Let \mathcal{H} be the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product $\langle I_{[0,t]}, I_{[0,s]} \rangle = R(t, s)$. Then the mapping $I_{[0,t]} \rightarrow W_t^H$ can be extended to be an isometry $\varphi \rightarrow W^H(\varphi)$ between \mathcal{H} and the Gaussian space generated by W^H .

The covariance kernel $R_H(t, s)$ has the following integral representation

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, r) K_H(s, r) dr,$$

where

$$K_H(t, s) = b_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \quad (3.5)$$

with $b_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{\frac{1}{2}}$, where $\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$ for $x > 0$ is the Gamma function.

In particular, when $H \in (\frac{1}{2}, 1)$, $K_H(t, s)$ can be all written as

$$K_H(t, s) = C_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

with $C_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}} = b_H(H-\frac{1}{2})$, where $\beta(x, y) = \int_0^1 t^{1-x} (1-t)^{1-y} dt$ for $x > 0, y > 0$ is the Beta function.

Let K_H denote the operator on $L^2([0, T])$, which is an isomorphism from $L^2([0, T])$ onto $I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$,

$$(K_H f)(t) = \int_0^t K_H(t, s) f(s) ds.$$

We can rewrite the action of K_H as (see [14] or [29])

$$(K_H f)(t) = \begin{cases} C_H \Gamma(H - \frac{1}{2}) I_{0+}^1 s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} (s^{\frac{1}{2}-H} f)(t), & \text{if } H > \frac{1}{2}, \\ b_H \Gamma(H + \frac{1}{2}) I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} (s^{H-\frac{1}{2}} f)(t), & \text{if } H < \frac{1}{2}. \end{cases} \quad (3.6)$$

If f is absolutely continuous, for $H < \frac{1}{2}$, we can write

$$(K_H f)(t) = b_H \Gamma(H + \frac{1}{2}) I_{0+}^1 s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} (s^{\frac{1}{2}-H} f)(t).$$

Consider the operator K_H^* from \mathcal{E} to $L^2([0, T])$ defined as

$$(K_H^* \varphi)(s) = K_H(T, s) \varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r, s) dr.$$

Noting that $(K_H^* I_{[0,t]}) (s) = K_H(t, s) I_{[0,t]}(s)$, K_H^* can be extended to be an isometry between \mathcal{H} and $L^2([0, T])$,

$$\langle f, g \rangle_{\mathcal{H}} = \langle K_H^* f, K_H^* g \rangle_{L^2([0, T])}, \quad f, g \in \mathcal{H}.$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert spaces \mathcal{H} and $L^2([0, T])$. Hence the process

$$W_t = W^H((K_H^*)^{-1} I_{[0,t]}), \quad t \geq 0 \quad (3.7)$$

is a standard Brownian motion, and W^H has the following integral representation

$$W_t^H = \int_0^t K_H(t, s) dW_s. \quad (3.8)$$

We say that W is the standard Brownian motion *related with* W^H .

Moreover, we have the relationship between the Wiener integrals with respect to fractional Brownian motion and its related Brownian motion as

$$W^H(\varphi) := \int_0^T \varphi(s) dW_s^H = \int_0^T (K_H^* \varphi)(t) dW_t, \quad \varphi \in \mathcal{H}. \quad (3.9)$$

We refer to [3, Chapter 2] and [28, Section 5.1] for more details.

The expressions in (3.7) and (3.8) provide the transformations between fractional Brownian motion and Brownian motion. In this section, we shall establish another useful transformation using the fundamental martingale M^H defined in (3.10) below.

For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, define

$$k_H(t, s) = \kappa_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}, \quad 0 < s < t,$$

where $\kappa_H = 2H\Gamma(\frac{3}{2} - H)\Gamma(H + \frac{1}{2})$. Define

$$M_t^H = \int_0^t k_H(t, s) dW_s^H. \quad (3.10)$$

Then from (3.9), one gets that M^H is a Gaussian martingale with quadratic variation $\langle M^H \rangle_t = \lambda_H^{-1} t^{2-2H}$, where $\lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}$. The martingale M^H was introduced as the *fundamental martingale* for the fractional Brownian motion W^H in order to get a Girsanov type theorem for W^H in [27].

By Lévy Characterization Theorem, it is easy to verify that the process

$$B_t = \frac{H(2H-1)}{C_H} \int_0^t s^{H-\frac{1}{2}} dM_s^H, \quad t \geq 0 \quad (3.11)$$

is a standard Brownian motion for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

The formula (3.8) provides an integral representation for the fractional Brownian motion W^H in terms of the standard Brownian motion W given in (3.7). On the other hand, Theorem 3.1 below adopted from [27, Theorem 5.2] claims that the Brownian motion W coincides with B given in (3.11) pathwisely.

Theorem 3.1. *Let the process $B = \{B_t\}_{t \geq 0}$ be defined as in (3.11). Then B is the standard Brownian motion related with W^H for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, i.e.,*

$$W_t^H = \int_0^t K_H(t, s) dB_s,$$

and hence $B_t = W_t$ for $t \geq 0$ a.s. where W is given by (3.7).

4 Parameter estimation for the RFOU

In this section, we shall obtain exponential moment estimates for the RFOU process X (Section 4.1), then derive Girsanov theorem for X and use it to derive the MLE of the drift parameter (Section 4.2), and finally establish the strong consistency and the asymptotic normality of the MLE (Section 4.3).

4.1 Estimates on the RFOU process

For any $\beta \in (0, 1)$, we denote by $C^\beta(0, T)$ the space of β -Hölder continuous functions on the interval $[0, T]$. For $x \in C^\beta(0, T)$, we will make use of the notations

$$\|x\|_{\beta, T} = \sup_{0 \leq \theta < r \leq T} \frac{|x_r - x_\theta|}{|r - \theta|^\beta}, \quad (4.1)$$

and

$$\|x\|_{\infty, T} = \sup_{0 \leq r \leq T} |x_r|. \quad (4.2)$$

Lemma 4.1. *Let X_t be the strong solution to the following Skorohod equation*

$$dX_t = -\alpha X_t dt + \sigma dW_t^H + dL_t \quad (4.3)$$

subject to $X(0) = x \geq 0$, where L is the minimal non-decreasing process. Then X is $(H-)$ Hölder continuous. There exists $\lambda_0 > 0$ depending on T, H and ϵ such that

$$\mathbb{E} \exp(\lambda_0 \|X\|_{\infty, T}^2) < \infty \quad \text{and} \quad \mathbb{E} \exp(\lambda_0 \|X\|_{H-\epsilon, T}^2) < \infty. \quad (4.4)$$

Proof. From the explicit representation (1.4) of the local time L of the one-dimensional Skorohod reflection problem, we have

$$X_t = x - \alpha \int_0^t X_s ds + \sigma W_t^H + \sup_{0 \leq s \leq t} \{-x + \alpha \int_0^s X_u du - \sigma W_s^H\} \vee 0$$

for each pathwise solution X to (4.3).

Assume $\alpha > 0$. We have that

$$\begin{aligned} X_t &\leq x - \alpha \int_0^t X_s ds + \sigma W_t^H + \alpha \int_0^t X_s ds + \sup_{0 \leq s \leq t} \{-x - \sigma W_s^H\} \vee 0 \\ &\leq 2x + 2\sigma \|W^H\|_{\infty, T}. \end{aligned}$$

For general $\alpha \in \mathbb{R}$, we have

$$|X_t| \leq 2x + 2|\alpha| \int_0^t |X_s| ds + 2\sigma \|W^H\|_{\infty, T}.$$

By Gronwall's inequality, we have

$$\|X\|_{\infty, T} \leq 2(x + \sigma \|W^H\|_{\infty, T}) e^{2|\alpha|T}$$

Then, the first inequality in (4.4) follows from the Fernique's Theorem, which claims the exponential integrability of the square of a seminorm for a Gaussian process (see [16]).

For the second inequality, noting that the function $f(x) = x \vee 0 := \max\{x, 0\}$ is Lipschitz, and that for a general continuous function g ,

$$\begin{aligned} \sup_{0 \leq s \leq t+r} \{g_s\} \vee 0 - \sup_{0 \leq s \leq t} \{g_s\} \vee 0 &\leq \left(\sup_{t \leq s \leq t+r} \{g_s\} - \sup_{0 \leq s \leq t} \{g_s\} \right) \vee 0 \\ &\leq \sup_{t \leq s \leq t+r} \{g_s\} - g_t \\ &\leq \sup_{t \leq s, u \leq t+r} \{g_s - g_u\}, \end{aligned}$$

we have

$$|X_r - X_s| \leq 2|\alpha| \|X\|_{\infty, T} (r - s) + 2\sigma \|W\|_{H-\epsilon, T} (r - s)^{H-\epsilon}.$$

So we get that

$$\|X\|_{H-\epsilon, T} \leq 2|\alpha| \|X\|_{\infty, T} T^{1+\epsilon-H} + 2\sigma \|W\|_{H-\epsilon, T}.$$

Applying Fernique's Theorem once more, we have $\mathbb{E} e^{\lambda \|W\|_{H-\epsilon, T}^2} < \infty$ for some $\lambda > 0$. Combining with the first inequality for $\|X\|_{\infty, T}$, we have the second inequality. \square

4.2 MLE based on the Girsanov transform

Let $0 < s < t \leq T$ and $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Recall (from Section 3.2) the following notations

$$k_H(t, s) = \kappa_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}, \quad \kappa_H = 2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2});$$

$$M_t^H = \int_0^t k_H(t, s) dW_s^H.$$

The quadratic variation of the martingale M^H is $\langle M^H \rangle_t = \lambda_H^{-1} t^{2-2H}$, where $\lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}$.

The following Girsanov theorem for fractional Brownian motion was given in [27] based on the fundamental martingale M^H . We refer to [20] and [29] for Girsanov theorem for fractional Brownian motion with more general drift.

Theorem 4.2 (Girsanov theorem for shifted fractional Brownian motion). *For $a \in \mathbb{R}$ and $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $X_t = W_t^H + at, t \geq 0$ is a fractional Brownian motion with Hurst parameter H under P_a where P_a is given by*

$$\frac{dP_a}{dP} = \exp\left(-aM_t^H - \frac{1}{2}a^2\langle M^H \rangle_t\right).$$

Now we develop Girsanov theorem for our RFOU process X by using the fundamental martingale M^H .

Theorem 4.3 (Girsanov theorem for RFOU). *For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, let*

$$\eta_T = \exp\left(\frac{\alpha}{\sigma} \int_0^T \chi_s dM_s^H - \frac{\alpha^2}{2\sigma^2} \int_0^T \chi_s^2 d\langle M^H \rangle_s\right).$$

Then $\{\sigma^{-1}X_s : 0 \leq s \leq T\}$ is a reflected fractional Brownian motion under the new probability \tilde{P} defined by $d\tilde{P}/dP = \eta_T$.

Proof. It suffices to prove that $\mathbb{E}(\eta_T) = 1$. Indeed, if $\mathbb{E}(\eta_T) = 1$, then by (3.11) and classical Girsanov theorem for standard Brownian motion (see, e.g., [24]), we have that $\{\sigma M_t^H, 0 \leq t \leq T\}$ under the original probability P has the same distribution as $\{-\alpha \int_0^t \chi_s d\langle M^H \rangle_s + \sigma M_t^H, 0 \leq t \leq T\}$ under the new probability \tilde{P} . Therefore $\{\sigma^{-1}X_t : 0 \leq t \leq T\}$ is a reflected fractional Brownian motion under \tilde{P} .

Now we show that $\mathbb{E}(\eta_T) = 1$. By Lemma 4.4 below and Lemma 4.1, there exists a positive constant δ such that for

$$\mathbb{E} \exp\left(\frac{\alpha^2}{2\sigma^2} \int_t^{t+\delta} \chi_s^2 d\langle M^H \rangle_s\right) < \infty \quad (4.5)$$

for $0 \leq t < t + \delta \leq T$.

Choose n big enough such that $\frac{T}{n} \leq \delta$, and let $t_i = \frac{iT}{n}, i = 1, \dots, n$. Denoting $\frac{\alpha}{\sigma}\chi_s$ by $f(s)$, we

have

$$\begin{aligned}
\mathbb{E}[\eta_T] &= \mathbb{E} \exp \left(\int_0^T f(s) dM_s^H - \frac{1}{2} \int_0^T f^2(s) d\langle M^H \rangle_s \right) \\
&= \mathbb{E} \left(\mathbb{E} \left[\exp \left(\int_0^T f(s) dM_s^H - \frac{1}{2} \int_0^T f^2(s) d\langle M^H \rangle_s \right) \middle| \mathcal{F}_{t_{n-1}} \right] \right) \\
&= \mathbb{E} \left(\exp \left(\int_0^{t_{n-1}} f(s) dM_s^H - \frac{1}{2} \int_0^{t_{n-1}} f^2(s) d\langle M^H \rangle_s \right) \right. \\
&\quad \left. \times \mathbb{E} \left[\exp \left(\int_{t_{n-1}}^T f(s) dM_s^H - \frac{1}{2} \int_{t_{n-1}}^T f^2(s) d\langle M^H \rangle_s \right) \middle| \mathcal{F}_{t_{n-1}} \right] \right).
\end{aligned}$$

We have that

$$\mathbb{E} \left[\exp \left(\int_{t_{n-1}}^T f(s) dM_s^H - \frac{1}{2} \int_{t_{n-1}}^T f^2(s) d\langle M^H \rangle_s \right) \middle| \mathcal{F}_{t_{n-1}} \right] \leq 1 \text{ a.s.}, \quad (4.6)$$

since $\exp \left(\int_0^{\cdot} f(s) dM_s^H - \frac{1}{2} \int_0^{\cdot} f^2(s) d\langle M^H \rangle_s \right)$ is a positive local martingale and hence a supermartingale. On the other hand, the Novikov's condition $\mathbb{E} \left[\exp \left(\frac{1}{2} \int_{t_{n-1}}^T f^2(s) d\langle M^H \rangle_s \right) \right] < \infty$ is fulfilled by (4.5), and hence $\mathbb{E} \left[\exp \left(\int_{t_{n-1}}^T f(s) dM_s^H - \frac{1}{2} \int_{t_{n-1}}^T f^2(s) d\langle M^H \rangle_s \right) \right] = 1$. Combining this fact with (4.6), we have

$$\mathbb{E} \left[\exp \left(\int_{t_{n-1}}^T f(s) dM_s^H - \frac{1}{2} \int_{t_{n-1}}^T f^2(s) d\langle M^H \rangle_s \right) \middle| \mathcal{F}_{t_{n-1}} \right] = 1 \text{ a.s.}$$

We may repeat the above procedure for $\mathbb{E} \exp \left(\int_0^{t_{i-1}} f(s) dM_s^H - \frac{1}{2} \int_0^{t_{i-1}} f^2(s) d\langle M^H \rangle_s \right)$, $i = n-1, n-2, \dots, 2, 1$, and finally we obtain

$$\mathbb{E}(\eta_T) = \mathbb{E} \exp \left(\int_0^{t_1} f(s) dM_s^H - \frac{1}{2} \int_0^{t_1} f^2(s) d\langle M^H \rangle_s \right) = 1.$$

□

Lemma 4.4. *There exists a constant $C > 0$ depending on H, T and $\varepsilon \in (0, H \wedge \frac{1}{2})$ such that for $0 \leq a < b \leq T$*

$$\int_a^b \chi_s^2 d\langle M^H \rangle_s \leq \begin{cases} C \left[\|X\|_{\infty, T}^2 (b-a)^{2-2H} + \|X\|_{H-\varepsilon, T}^2 (b-a) \right], & H \in (\frac{1}{2}, 1); \\ C \|X\|_{\infty, T}^2 (b-a), & H \in (0, \frac{1}{2}). \end{cases} \quad (4.7)$$

Proof. We begin by noting that

$$\chi_t = \kappa_H^{-1} \frac{d}{dt} \left(\int_0^t s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} X_s ds \right) \frac{dt}{d\langle M^H \rangle_t}.$$

Let $Y_t = \int_0^t s^{\frac{1}{2}-H} X_s ds$. Note that for fixed t , $s^{\frac{1}{2}-H} X_s ds = dY_s = d(Y_s - Y_t)$.

Case 1: $H > \frac{1}{2}$. We have that

$$\begin{aligned}
& \int_0^t (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds \\
&= \int_0^t (t-s)^{\frac{1}{2}-H} d(Y_s - Y_t) \\
&= \lim_{s \uparrow t} (t-s)^{\frac{1}{2}-H} (Y_s - Y_t) + t^{\frac{1}{2}-H} Y_t + (H - \frac{1}{2}) \int_0^t (Y_t - Y_s) (t-s)^{-\frac{1}{2}-H} ds \\
&= t^{\frac{1}{2}-H} Y_t + (H - \frac{1}{2}) \int_0^t (Y_t - Y_s) (t-s)^{-\frac{1}{2}-H} ds \\
&= t^{\frac{1}{2}-H} Y_t + (H - \frac{1}{2}) \int_0^t (Y_t - Y_{t-s}) s^{-\frac{1}{2}-H} ds, \tag{4.8}
\end{aligned}$$

where in the third equality we used the fact

$$\begin{aligned}
0 &\leq \lim_{s \uparrow t} (t-s)^{\frac{1}{2}-H} (Y_t - Y_s) = \lim_{s \uparrow t} (t-s)^{\frac{1}{2}-H} \int_s^t r^{\frac{1}{2}-H} X_r dr \\
&\leq (\frac{3}{2} - H)^{-1} \lim_{s \uparrow t} (t-s)^{\frac{1}{2}-H} (t^{\frac{3}{2}-H} - s^{\frac{3}{2}-H}) \|X\|_{\infty, T} = 0.
\end{aligned}$$

Also, observe that

$$\begin{aligned}
& \frac{d}{dt} \int_0^t (Y_t - Y_{t-s}) s^{-\frac{1}{2}-H} ds \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(\int_0^{t+\varepsilon} (Y_{t+\varepsilon} - Y_{t+\varepsilon-s}) s^{-\frac{1}{2}-H} ds - \int_0^t (Y_t - Y_{t-s}) s^{-\frac{1}{2}-H} ds \right) \\
&=: A(t) + B(t), \tag{4.9}
\end{aligned}$$

where

$$A(t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(\int_0^{t+\varepsilon} (Y_{t+\varepsilon} - Y_{t+\varepsilon-s}) s^{-\frac{1}{2}-H} ds - \int_0^t (Y_{t+\varepsilon} - Y_{t+\varepsilon-s}) s^{-\frac{1}{2}-H} ds \right),$$

and

$$B(t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(\int_0^t (Y_{t+\varepsilon} - Y_{t+\varepsilon-s}) s^{-\frac{1}{2}-H} ds - \int_0^t (Y_t - Y_{t-s}) s^{-\frac{1}{2}-H} ds \right).$$

It is clear that

$$A(t) = 0, \tag{4.10}$$

when $t > 0$ since Y_t is continuous in t .

For $B(t)$, we have

$$\begin{aligned}
B(t) &= \int_0^t (Y'_t - Y'_{t-s}) s^{-\frac{1}{2}-H} ds \\
&= \int_0^t \left(t^{\frac{1}{2}-H} X_t - (t-s)^{\frac{1}{2}-H} X_{t-s} \right) s^{-\frac{1}{2}-H} ds \\
&= t^{\frac{1}{2}-H} \int_0^t (X_t - X_{t-s}) s^{-\frac{1}{2}-H} ds + \int_0^t X_{t-s} \left(t^{\frac{1}{2}-H} - (t-s)^{\frac{1}{2}-H} \right) s^{-\frac{1}{2}-H} ds.
\end{aligned}$$

Observe that

$$0 \leq \int_0^t |X_t - X_{t-s}| s^{-\frac{1}{2}-H} ds \leq \int_0^t \|X\|_{H-\varepsilon, T} s^{H-\varepsilon} s^{-\frac{1}{2}-H} ds = \frac{2\|X\|_{H-\varepsilon, T} t^{\frac{1}{2}-\varepsilon}}{1-2\varepsilon},$$

and also that

$$\int_0^t \left| t^{\frac{1}{2}-H} - (t-s)^{\frac{1}{2}-H} \right| s^{-\frac{1}{2}-H} ds = t^{1-2H} \int_0^1 \left| 1 - (1-s)^{\frac{1}{2}-H} \right| s^{-\frac{1}{2}-H} ds.$$

Note that $\int_0^1 \left| 1 - (1-s)^{\frac{1}{2}-H} \right| s^{-\frac{1}{2}-H} ds < \infty$, since the integrand function is continuous on $(0, 1)$, while bounded by $2(1-s)^{\frac{1}{2}-H}$ when s is close to 1, and by $(1-s)^{-\frac{1}{2}-H} s^{\frac{1}{2}-H}$ when s is close to 0. Therefore

$$|B(t)| \leq C (\|X\|_{H-\varepsilon, T} t^{1-H-\varepsilon} + \|X\|_{\infty, T} t^{1-2H}). \quad (4.11)$$

Combining equations (4.8), (4.9), (4.10) and (4.11), we have

$$\begin{aligned} \left| \frac{d}{dt} \int_0^t (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds \right| &\leq C \left| t^{\frac{1}{2}-H} t^{\frac{1}{2}-H} X_t + t^{-\frac{1}{2}-H} Y_t + B(t) \right| \\ &\leq C (\|X\|_{\infty, T} t^{1-2H} + \|X\|_{H-\varepsilon, T} t^{1-H-\varepsilon}). \end{aligned}$$

Noting that $\frac{dt}{d\langle M^H \rangle_t} = \frac{\lambda_H}{2-2H} t^{2H-1}$, for χ_t we have

$$|\chi_t| \leq C (\|X\|_{\infty, T} + \|X\|_{H-\varepsilon, T} t^{H-\varepsilon}).$$

Hence, we conclude that

$$\int_a^b \chi_s^2 d\langle M^H \rangle_s = \lambda_H^{-1} (2-2H) \int_a^b \chi_s^2 s^{1-2H} ds \leq C (\|X\|_{\infty, T}^2 (b-a)^{2-2H} + \|X\|_{H-\varepsilon, T}^2 T^{1-2\varepsilon} (b-a)).$$

Case 2: $H < \frac{1}{2}$. Notice that

$$\begin{aligned} \frac{d}{dt} \int_0^t (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds &= \Gamma\left(\frac{3}{2}-H\right) \frac{d}{dt} \left(I_{0+}^{\frac{3}{2}-H} (s^{\frac{1}{2}-H} X_s)(t) \right) \\ &= \Gamma\left(\frac{3}{2}-H\right) I_{0+}^{\frac{1}{2}-H} (s^{\frac{1}{2}-H} X_s)(t), \end{aligned}$$

and also

$$\left| I_{0+}^{\frac{1}{2}-H} (s^{\frac{1}{2}-H} X_s)(t) \right| = \frac{1}{\Gamma(\frac{1}{2}-H)} \left| \int_0^t (t-s)^{-\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds \right| \leq C \|X\|_{\infty, T} t^{1-2H}.$$

Therefore, $|\chi_t| \leq C \|X\|_{\infty, T}$. Hence, we get

$$\int_a^b \chi_s^2 d\langle M^H \rangle_s = \lambda_H^{-1} (2-2H) \int_a^b \chi_s^2 s^{1-2H} ds \leq C \|X\|_{\infty, T}^2 T^{1-2H} (b-a).$$

□

Remark 4.5. From the proof of the above lemma, we have that

- if $H > \frac{1}{2}$,

$$\begin{aligned} \frac{d}{dt} \int_0^t (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds &= t^{\frac{1}{2}-H} Y_t + (H - \frac{1}{2})(A(t) + B(t)) \\ &= t^{1-2H} X_t + (H - \frac{1}{2})B(t) \\ &= t^{1-2H} X_t + (H - \frac{1}{2}) \int_0^t \left(t^{\frac{1}{2}-H} X_t - (t-s)^{\frac{1}{2}-H} X_{t-s} \right) s^{-\frac{1}{2}-H} ds \\ &= \Gamma\left(\frac{3}{2} - H\right) D_{0+}^{H-\frac{1}{2}} (u^{\frac{1}{2}-H} X_u)(t), \end{aligned}$$

and

- if $H < \frac{1}{2}$,

$$\frac{d}{dt} \int_0^t (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds = \Gamma\left(\frac{3}{2} - H\right) I_{0+}^{\frac{1}{2}-H} (u^{\frac{1}{2}-H} X_u)(t).$$

If we define D_{0+}^α as $I_{0+}^{-\alpha}$ for $\alpha < 0$, then for all $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, χ_t can be represented as

$$\chi_t = \frac{\Gamma\left(\frac{3}{2} - H\right) \lambda_H}{2 - 2H} \kappa_H^{-1} t^{2H-1} D_{0+}^{H-\frac{1}{2}} (u^{\frac{1}{2}-H} X_u)(t). \quad (4.12)$$

Remark 4.6. By equation (3.6), we have

$$(K_H^{-1} \varphi)(t) = \begin{cases} \frac{1}{C_H \Gamma(H - \frac{1}{2})} t^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} (u^{\frac{1}{2}-H} \varphi')(t), & \text{if } H > \frac{1}{2} \\ \frac{1}{b_H \Gamma(H + \frac{1}{2})} t^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} (u^{\frac{1}{2}-H} \varphi')(t), & \text{if } H < \frac{1}{2}. \end{cases}$$

Noting that $C_H = b_H(H - \frac{1}{2})$ when $H > \frac{1}{2}$, and using the convention $D_{0+}^\alpha := I_{0+}^{-\alpha}$ for $\alpha < 0$, we have a uniform representation for K_H^{-1} :

$$(K_H^{-1} \varphi)(t) = \frac{1}{b_H \Gamma(H + \frac{1}{2})} t^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} (u^{\frac{1}{2}-H} \varphi')(t), \quad H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1). \quad (4.13)$$

Using (4.12), (4.13) and (3.11), by direct computations we can get

$$\int_0^T \chi_t dM_t^H = \int_0^T (K_H^{-1} \int_0^\cdot X_u du)(s) dB_s,$$

where B is the Wiener process defined in (3.11). By Theorem 3.1, we know that B coincides with W , the Wiener process related with the fractional Brownian motion W^H . Therefore, we conclude that Theorem 4.3 has the following equivalent version.

Theorem 4.7 (Girsanov theorem for RFOU). For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, let

$$\xi_T = \exp \left(\int_0^T \left(K_H^{-1} \int_0^\cdot \frac{\alpha}{\sigma} X_r dr \right) (s) dW_s - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot \frac{\alpha}{\sigma} X_r dr \right)^2 (s) ds \right).$$

Then $\{\sigma^{-1} X_s : 0 \leq s \leq T\}$ is a reflected fractional Brownian motion under the new probability \hat{P} defined by $d\hat{P}/dP = \xi_T$.

In fact, $\xi_T = \eta_T$ a.s., and hence $\tilde{P} = \hat{P}$. In the Appendix, we shall provide a direct proof of Theorem 4.7 by using fractional calculus.

Let P^α and P^R denote the probability measures on $(C([0, T]), \mathcal{B}_T)$ induced by X and X^R respectively, where X^R is the reflected fractional Brownian motions (i.e. $\alpha = 0$). Similar to the proof of [24, Theorem 7.1], using Theorem 4.3, we can show that $P^\alpha \sim P^R$, and

$$\begin{aligned} \frac{dP^\alpha}{dP^R} = \eta_T^{-1} &= \exp\left(-\frac{\alpha}{\sigma} \int_0^T \chi_s dM_s^H + \frac{\alpha^2}{2\sigma^2} \int_0^T \chi_s^2 d\langle M^H \rangle_s\right) \\ &= \exp\left(-\frac{\alpha}{\sigma^2} \int_0^T \chi_s d\tilde{X}_s - \frac{\alpha^2}{2\sigma^2} \int_0^T \chi_s^2 d\langle M^H \rangle_s + \frac{\alpha}{\sigma^2} \int_0^T \chi_s d\tilde{L}_s\right). \end{aligned}$$

Hence the MLE for α is given by

$$\tilde{\alpha}_T := \frac{-\int_0^T \chi_s d\tilde{X}_s + \int_0^T \chi_s d\tilde{L}_s}{\int_0^T \chi_s^2 d\langle M^H \rangle_s}. \quad (4.14)$$

Remark 4.8. *Although Theorem 4.7 is equivalent to Theorem 4.3, it is not practical to get MLE for α by applying Theorem 4.7 directly as above. This is because, to get an estimator just involving the information of X and L , one needs to transform $\int_0^T (K_H^{-1} \int_0^\cdot \frac{\alpha}{\sigma} X_r dr)(s) dW_s$ to $\int_0^T ((K_H^*)^{-1} K_H^{-1} \int_0^\cdot \frac{\alpha}{\sigma} X_r dr)(s) dW_s^H$ (and then use X and L to represent W^H). The integral $\int_0^T ((K_H^*)^{-1} K_H^{-1} \int_0^\cdot \frac{\alpha}{\sigma} X_r dr)(s) dW_s^H$ is in Skorohod sense but not in Stratonovich sense, and hence if we replace dW_s^H by $\frac{1}{\sigma} dX_s + \frac{\alpha}{\sigma} X_s ds - \frac{1}{\sigma} dL_s$ in the integral, we have to deal with the computations of Skorohod integrals against X and L , which is rather complex and impractical for an estimator.*

Remark 4.9. *Let*

$$\tilde{X}_t = \int_0^t k_H(t, s) dX_s, \quad \tilde{L}_t = \int_0^t k_H(t, s) dL_s, \quad \text{and} \quad \chi_t = \frac{d}{d\langle M^H \rangle_t} \int_0^t k_H(t, s) X_s ds, \quad (4.15)$$

where $dX_t = -\alpha X_t dt + \sigma dW_t^H + dL_t$ and dL_t can be understood as a measure on $[0, \infty)$ induced by the non-decreasing function L_t . Then we have

$$d\tilde{X}_t = -\alpha \chi_t d\langle M^H \rangle_t + \sigma dM_t^H + d\tilde{L}_t. \quad (4.16)$$

We note that the infinitesimal variance parameter σ^2 of RFOU process (1.1) can be estimated by using the quadratic variation of \tilde{X} since $\langle \tilde{X} \rangle_t = \sigma^2 \langle M^H \rangle_t$ for $t > 0$ by (4.16), where \tilde{X} given in (4.15) is a functional of X and M^H defined in (3.10) is the fundamental martingale.

4.3 Properties of the MLE

The following lemma will play an essential role in establishing the strong consistency of the MLE $\tilde{\alpha}_T$ in Theorem 4.11.

Lemma 4.10. *For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, let $\{Y_t, t \geq 0\}$ be the fractional Ornstein-Uhlenbeck (FOU) process satisfying*

$$dY_t = -\alpha Y_t dt + \sigma dW_t^H$$

with initial value $Y_0 = x > 0$. Then we have $P\{X_t \geq |Y_t|, t \geq 0\} = 1$. Furthermore, we also have

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |Y_t| dt &= \mathbb{E} \left[\left| \sigma \int_{-\infty}^0 e^{\alpha s} dW_s^H + x \right| \right], \quad \text{a.s.}, & \text{when } \alpha > 0, \\ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |Y_t| dt &= \infty, \quad \text{a.s.}, & \text{when } \alpha \leq 0. \end{aligned}$$

Proof. Since $d(X_t - Y_t) = -\alpha(X_t - Y_t)dt + dL_t$ with $X_0 - Y_0 = 0$, we have $X_t \geq Y_t$ a.s. for all t . Let Z_t be the FOU process satisfying $dZ_t = -\alpha Z_t dt - \sigma dW_t^H$ with initial value $Z_0 = -x$. Then we have $Z_t = -Y_t$. On the other hand, $d(X_t + Z_t) = -\alpha(X_t + Z_t)dt + dL_t$ with $X_0 + Z_0 = 0$, and hence $-X_t \leq Z_t = -Y_t$. So we have $|Y_t| \leq X_t$ a.s. for all t . By the countability of the set \mathbb{Q}^+ of positive rational numbers, we have that $P\{X_t \geq |Y_t|, t \in \mathbb{Q}^+\} = 1$, and $P\{X_t \geq |Y_t|, t \geq 0\} = 1$ just follows from the fact that both X and Y have continuous trajectories.

Note that Y_t has the following expression,

$$Y_t = x + \sigma \int_0^t e^{-\alpha(t-s)} dW_s^H = x + \sigma \int_{-\infty}^t e^{-\alpha(t-s)} dW_s^H - e^{-\alpha t} \sigma \int_{-\infty}^0 e^{\alpha s} dW_s^H.$$

When $\alpha > 0$, the process $\left\{ \tilde{Y}_t = \sigma \int_{-\infty}^t e^{-\alpha(t-s)} dW_s^H, t \in \mathbb{R} \right\}$ is Gaussian, stationary and ergodic (see [11]), and by ergodic theorem,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\tilde{Y}_t + x| dt = \mathbb{E}(|\tilde{Y}_0 + x|),$$

which implies that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |Y_t| dt = \mathbb{E}(|\tilde{Y}_0 + x|) > 0.$$

When $\alpha = 0$, $Y_t = x + \sigma W_t^H$. It suffices to show

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |W_t^H| dt = \infty, \quad \text{a.s.}, \quad (4.17)$$

which follows from Lemma 6.1 in the Appendix.

When $\alpha < 0$, let $\beta = -\alpha > 0$, and then

$$Y_t = x + \sigma \int_0^t e^{\beta(t-s)} dW_s^H = x + \sigma \int_0^\infty e^{\beta(t-s)} dW_s^H - \sigma \int_t^\infty e^{\beta(t-s)} dW_s^H.$$

By change of variable, we have $\int_t^\infty e^{\beta(t-s)} dW_s^H = -\int_{-\infty}^{-t} e^{\beta(t+u)} dW_{-u}^H$. On the other hand, $\left\{ -\int_{-\infty}^{-t} e^{\beta(t+u)} dW_{-u}^H, t \in \mathbb{R} \right\}$ is equal to in law $\left\{ \int_{-\infty}^{-t} e^{\beta(t+u)} dW_u^H, t \in \mathbb{R} \right\}$ which is stationary and ergodic by the result in [11]. Hence $\left\{ \hat{Y}_t = \sigma \int_t^\infty e^{\beta(t-s)} dW_s^H, t \in \mathbb{R} \right\}$ is stationary and ergodic. Note that now $Y_t = x + e^{\beta t} \hat{Y}_0 - \hat{Y}_t$ and hence

$$\begin{aligned} \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T |Y_t| dt &\geq |\hat{Y}_0| \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{\beta t} dt - \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |x - \hat{Y}_t| dt \\ &\geq \infty \quad \text{a.s.} \end{aligned}$$

□

Theorem 4.11. For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, the MLE $\tilde{\alpha}_T$ in (4.14) is strongly consistent.

Proof. To prove $\lim_{T \rightarrow \infty} \tilde{\alpha}_T = \alpha$ a.s., it suffices to show that

$$\lim_{T \rightarrow \infty} \frac{\int_0^T \chi_s dM_s^H}{\int_0^T \chi_s^2 d\langle M^H \rangle_s} = 0 \quad a.s.$$

Since $\int_0^T \chi_s dM_s^H$ is a martingale, by Lepingle's law of large numbers (see [24, Lemma 17.4]), it suffices to show that

$$\lim_{T \rightarrow \infty} \int_0^T \chi_s^2 d\langle M^H \rangle_s = \infty \quad a.s. \quad (4.18)$$

From Remark 4.5, we know that for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$,

$$\chi_t = Ct^{2H-1} D_{0+}^{H-\frac{1}{2}} (u^{\frac{1}{2}-H} X_u)(t),$$

for some constant C depending on H only. Note that $D_{0+}^{H-\frac{1}{2}}$ means $I_{0+}^{\frac{1}{2}-H}$ when $H < \frac{1}{2}$.

Let $c = 1 - 2H$ and $d = 4H - 2$. Then $c + d = 2H - 1$ and $c + \frac{d}{2} = 0$. Notice that

$$\begin{aligned} \int_0^T \chi_s^2 d\langle M^H \rangle_s &= C \int_0^T s^{2H-1} \left(D_{0+}^{H-\frac{1}{2}} u^{\frac{1}{2}-H} X_u \right)^2 (s) ds \\ &= C \int_0^T s^c \left(s^{\frac{d}{2}} \left(D_{0+}^{H-\frac{1}{2}} u^{\frac{1}{2}-H} X_u \right) (s) \right)^2 ds \\ &\geq C \left[\frac{1}{T^{\frac{1+c}{2}}} \int_0^T s^{c+\frac{d}{2}} \left(D_{0+}^{H-\frac{1}{2}} u^{\frac{1}{2}-H} X_u \right) (s) ds \right]^2 \\ &= C \left[\frac{1}{T^{\frac{1+c}{2}}} \int_0^T \left(D_{T-}^{H-\frac{1}{2}} 1 \right) (s) s^{\frac{1}{2}-H} X_s ds \right]^2 \\ &= C \left[\frac{1}{T^{1-H}} \int_0^T (T-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds \right]^2, \end{aligned}$$

where the only inequality above follows by Jensen's inequality or Hölder's inequality, and the third equality follows by equation (3.3).

If $H > \frac{1}{2}$, we have

$$\left[\frac{1}{T^{1-H}} \int_0^T (T-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds \right]^2 = \left[\frac{1}{T^H} \int_0^T \left(1 - \frac{s}{T}\right)^{\frac{1}{2}-H} \left(\frac{s}{T}\right)^{\frac{1}{2}-H} X_s ds \right]^2 \geq C \left[\frac{1}{T^H} \int_0^T X_s ds \right]^2.$$

Therefore, to prove (4.18), it suffices to show that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T X_s ds > 0 \quad a.s., \quad (4.19)$$

and it is an immediate consequence of Lemma 4.10.

If $H < \frac{1}{2}$, we can show that

$$\liminf_{T \rightarrow \infty} \frac{1}{T^{\frac{3}{2}-H}} \int_0^T (T-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds > 0 \quad a.s., \quad (4.20)$$

which is a sufficient condition for inequality (4.18).

In fact, we have

$$\begin{aligned} & \int_0^T (T-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s ds \\ &= \frac{2}{1-2H} \int_0^T \left(\int_s^T (T-r)^{-\frac{1}{2}-H} dr \right) s^{\frac{1}{2}-H} X_s ds \\ &= \frac{2}{1-2H} \int_0^T \left(\int_0^r s^{\frac{1}{2}-H} X_s ds \right) (T-r)^{-\frac{1}{2}-H} dr. \end{aligned}$$

For the term $\int_0^r s^{\frac{1}{2}-H} X_s ds$, we have

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \int_0^r s^{\frac{1}{2}-H} X_s ds \geq \liminf_{r \rightarrow \infty} \frac{1}{r} \int_1^r s^{\frac{1}{2}-H} X_s ds \geq \liminf_{r \rightarrow \infty} \frac{1}{r} \int_1^r X_s ds = \liminf_{r \rightarrow \infty} \frac{1}{r} \int_0^r X_s ds \quad a.s.$$

By Lemma 4.10, there exists a positive number a such that $\liminf_{r \rightarrow \infty} \frac{1}{r} \int_0^r X_s ds \geq a$ a.s. Choose $\varepsilon \in (0, a)$, then for almost all $\omega \in \Omega$, there exists $N(\omega) \in (0, \infty)$, such that $\frac{1}{r} \int_0^r s^{\frac{1}{2}-H} X_s(\omega) ds \geq a - \varepsilon$ for all $r > N(\omega)$.

Therefore, for almost all $\omega \in \Omega$,

$$\begin{aligned} & \liminf_{T \rightarrow \infty} \frac{1}{T^{\frac{3}{2}-H}} \int_0^T (T-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s(\omega) ds \\ &= \frac{2}{1-2H} \liminf_{T \rightarrow \infty} \frac{1}{T^{\frac{3}{2}-H}} \int_{N(\omega)}^T \left(\int_0^r s^{\frac{1}{2}-H} X_s(\omega) ds \right) (T-r)^{-\frac{1}{2}-H} dr \\ &\geq \frac{2}{1-2H} \liminf_{T \rightarrow \infty} \frac{1}{T^{\frac{3}{2}-H}} \int_{N(\omega)}^T (a-\varepsilon) r (T-r)^{-\frac{1}{2}-H} dr \\ &= \frac{2}{1-2H} \beta\left(2, \frac{1}{2} - H\right) (a - \varepsilon), \end{aligned}$$

which implies (4.20). □

Theorem 4.12. For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, the MLE $\tilde{\alpha}_T$ of α admits the asymptotic normality, i.e.,

$$\frac{\tilde{\alpha}_T - \alpha}{\sigma} \sqrt{\int_0^T \chi_s^2 d\langle M^H \rangle_s} \xrightarrow{\mathcal{L}} N(0, 1) \quad \text{as } T \rightarrow \infty.$$

Proof. This is an immediate consequence of the Central Limit Theorem for martingales (see, for instance, [30, Theorem B.10].) □

5 Sequential MLE

Recall the strong consistency and asymptotic normality of the MLE $\tilde{\alpha}_T$, established in Theorem 4.11 and Theorem 4.12. Such results are valuable from statistical analysis viewpoint of applications, however, there are some drawbacks. Generally speaking, the MLE is a biased estimator and its mean squared error (MSE) depends on the parameter to be estimated (Theorem 4.12). Therefore, useful estimates for the bias and the MSE are not available (or very difficult to derive), and as a consequence, there is no guarantee that the bias (or, variance) decays fast enough (as $T \rightarrow \infty$) to achieve the Cramer-Rao bound. Hence, we are unable to verify whether the classical Cramer-Rao lower bound can be attained or not for the MLE $\tilde{\alpha}_T$. To overcome such limitations, we consider the sequential estimation plan and verify that the proposed plan is significantly helpful both in asymptotic and non-asymptotic short time observation.

In contrast to the MLE, the proposed sequential maximum likelihood estimator (SMLE) is unbiased, exactly normally distributed (on the finite time observation), and its MSE has an explicit, simple expression that does not depend on the parameter to be estimated (see Theorem 5.1 below). The SMLE is uniformly normally distributed over the entire parameter space which is the real line. Such results would be of ample use in applications to several areas such as engineering, financial and biological modeling where unknown parameter estimation is based on relatively shorter time observation. Furthermore, an analog of the Cramer-Rao lower bound is proved and the SMLE is shown to be efficient among all unbiased estimation plans in the mean squared error sense (see Theorem 5.2 below).

Define the stopping time $\tau^H(h)$ as

$$\tau^H(h) := \inf \left\{ t \geq 0 : \int_0^t \chi_s^2 d\langle M^H \rangle_s \geq h \right\}, \quad 0 < h < \infty. \quad (5.1)$$

Then, $\mathcal{F}_{\tau^H(h)}^X$ -measurable function $\hat{\alpha}_{\tau^H(h)}$

$$\hat{\alpha}_{\tau^H(h)} := \frac{1}{h} \left[\int_0^{\tau^H(h)} \chi_s d\tilde{L}_s - \int_0^{\tau^H(h)} \chi_s d\tilde{X}_s \right] \quad (5.2)$$

is a sequential estimator. From (4.15)–(4.16), it can be seen that

$$\hat{\alpha}_{\tau^H(h)} = \alpha - \frac{\sigma}{h} \int_0^{\tau^H(h)} \chi_s dM_s^H. \quad (5.3)$$

A proof of the next theorem follows along the similar lines of [25] by accommodating our basic model assumptions involving the reflection (i.e., state space) constraint and fractional Brownian noise with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. In what follows, the index α in \mathbb{P} and \mathbb{E} emphasizes the fact that the distribution of the state process is being considered for the prescribed value α .

Theorem 5.1. *For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, the sequential estimation plan $(\tau^H(h), \hat{\alpha}_{\tau^H(h)})$, $0 < h < \infty$, has the following properties:*

- (i) $\mathbb{P}_\alpha(\tau^H(h) < \infty) = 1$ for all $\alpha \in (-\infty, \infty)$,
- (ii) $\mathbb{E}_\alpha(\hat{\alpha}_{\tau^H(h)}) = \alpha$ for all $\alpha \in (-\infty, \infty)$,

$$(iii) \mathbb{E}_\alpha(\widehat{\alpha}_{\tau^H(h)} - \alpha)^2 = \frac{\sigma^2}{h},$$

$$(iv) \widehat{\alpha}_{\tau^H(h)} - \alpha \stackrel{\mathcal{L}}{=} N(0, \frac{\sigma^2}{h}),$$

$$(v) \widehat{\alpha}_{\tau^H(h)} \rightarrow \alpha \text{ a.s. as } h \rightarrow \infty.$$

Proof. We first show that the $\mathcal{F}_{\tau^H(h)}^X$ -measurable random variable $\widehat{\alpha}_{\tau^H(h)}$ is indeed the SMLE as follows. Consider an arbitrary stopping time τ with respect to the filtration $\{\mathcal{F}_t^X\}_{t \geq 0}$ generated by the process X . Similar to [24, Theorem 7.1], the probability measures $\mathbb{P}_{\tau, X}^\theta$ and $\mathbb{P}_{\tau, X}^\alpha$ induced by the processes stopped at time τ

$$\left. \begin{aligned} dX_t^\theta &= -\theta X_t^\theta dt + \sigma dW_t^H + dL_t^\theta, & X_t^\theta &\geq 0 \text{ for all } t \geq 0, \\ dX_t^\alpha &= -\alpha X_t^\alpha dt + \sigma dW_t^H + dL_t^\alpha, & X_t^\alpha &\geq 0 \text{ for all } t \geq 0, \end{aligned} \right\}$$

respectively, are equivalent and their Radon-Nikodym derivative is given by

$$\frac{d\mathbb{P}_{\tau, X}^\alpha}{d\mathbb{P}_{\tau, X}^\theta} \Big|_{\mathcal{F}_{\tau, \theta}^X} = \exp \left\{ -\frac{1}{\sigma} \int_0^\tau (\alpha - \theta) \chi_t^\alpha dM_t^H + \frac{1}{2\sigma^2} \int_0^\tau (\alpha - \theta)^2 (\chi_t^\alpha)^2 d\langle M^H \rangle_t \right\}, \quad (5.4)$$

where $\mathcal{F}_{\tau, \theta}^X$ is the natural filtration generated by $\{X_t^\theta : 0 \leq t \leq \tau\}$, and

$$\chi^\alpha := \frac{d}{d\langle M^H \rangle_t} \int_0^t k_H(t, s) X_s^\alpha ds.$$

Let X_t^0 be the reflected fractional Brownian motion (RFBM) satisfying $dX_t^0 = \sigma dW_t^H + dL_t$, $t \geq 0$ and \mathbb{P}_τ^0 be the measure induced by the RFBM X^0 . Then the log likelihood function $\ell_\tau(\alpha)$ is given by

$$\ell_\tau(\alpha) := \sigma^2 \log \frac{d\mathbb{P}_{\tau, X}^\alpha}{d\mathbb{P}_{\tau, X}^0} = -\alpha \int_0^\tau \chi_t^\alpha d\tilde{X}_t - \frac{\alpha^2}{2} \int_0^\tau (\chi_t^\alpha)^2 d\langle M^H \rangle_t + \alpha \int_0^\tau \chi_t d\tilde{L}_t.$$

Then, by solving the equation

$$\dot{\ell}_\tau(\alpha) := \frac{d}{d\alpha} \left(\sigma^2 \log \frac{d\mathbb{P}_{\tau, X}^\alpha}{d\mathbb{P}_{\tau, X}^0} \right) = 0,$$

we obtain the SMLE given by

$$\widehat{\alpha}_\tau = \frac{-\int_0^\tau \chi_s^\alpha d\tilde{X}_s + \int_0^\tau \chi_s^\alpha d\tilde{L}_s}{\int_0^\tau (\chi_s^\alpha)^2 d\langle M^H \rangle_s} = \alpha - \frac{\sigma \int_0^\tau \chi_s^\alpha dM_s^H}{\int_0^\tau (\chi_s^\alpha)^2 d\langle M^H \rangle_s}. \quad (5.5)$$

Now, setting $\tau = \tau^H(h)$ in (5.5), we obtain for $\widehat{\alpha}_\tau = \widehat{\alpha}_{\tau^H(h)}$ the representation given by (5.3).

We shall use χ instead of χ^α . To verify part (i), it suffices to show that $\lim_{T \rightarrow \infty} \int_0^T \chi_s^2 d\langle M^H \rangle_s = \infty$ a.s., which has already been proved in the proof of Theorem 4.11. The claims in (ii), (iii) and (iv) follow from the fact that the process $(\int_0^{\tau^H(h)} \chi_s dM_s^H, h \geq 0)$ is a standard Brownian motion indexed by h (see, for instance, [19, Theorem 4.6]). Also, the result (v) is an immediate consequence of (iv). \square

Next, we show that the proposed sequential estimation plan $(\tau^H(h), \hat{\alpha}_{\tau^H(h)})$ is efficient among all unbiased estimation plans in the mean squared error sense. More precisely, we prove an analog of the Cramer-Rao lower bound for arbitrary unbiased estimation plans.

Theorem 5.2. *Let the sequential plan $(\tau, \hat{\alpha}_\tau(X))$ be an arbitrary unbiased estimation plan for the RFOU process $\{X_t\}$ with the parameter $\alpha \in (-\infty, \infty)$, namely,*

$$\mathbb{E}_\alpha(\hat{\alpha}_\tau(X)) = \alpha \quad \text{for all } \alpha \in (-\infty, \infty). \quad (5.6)$$

Suppose also that $0 < \mathbb{E}_\alpha[\int_0^\tau \chi_s^2 d\langle M^H \rangle_s] < \infty$. Then,

$$\text{Var}_\alpha(\hat{\alpha}_\tau) = \mathbb{E}_\alpha[\hat{\alpha}_\tau - \alpha]^2 \geq \frac{\sigma^2}{\mathbb{E}_\alpha[\int_0^\tau \chi_s^2 d\langle M^H \rangle_s]}. \quad (5.7)$$

Remark 5.3. *A sequential estimation plan $(\tau, \hat{\alpha}_\tau)$ is said to be efficient in the MSE sense if for which (5.7) becomes an equality for all $\alpha \in (-\infty, \infty)$. Notice that since $\mathbb{E}_\alpha[\hat{\alpha}_{\tau^H(h)} - \alpha]^2 = \frac{\sigma^2}{h}$ as established in Theorem 5.1 (iii) and $\mathbb{E}_\alpha[\int_0^{\tau^H(h)} \chi_s^2 d\langle M^H \rangle_s] = h$ by the definition of $\tau^H(h)$, the sequential estimation plan $(\tau^H(h), \hat{\alpha}_{\tau^H(h)})$ is efficient in the MSE sense.*

Proof. Without loss of generality, we assume that $\sigma = 1$. In view of the Radon-Nikodym derivative expression in (5.4) with $\theta = 0$, differentiating both sides of (5.6) with respect to α yields that

$$\mathbb{E}_\alpha \left[\hat{\alpha}_\tau \left\{ - \int_0^\tau \chi_s d\tilde{X}_s - \alpha \int_0^\tau \chi_s^2 d\langle M^H \rangle_s + \int_0^\tau \chi_s d\tilde{L}_s \right\} \right] = 1 \quad (5.8)$$

(cf. the proof of Theorem 7.22 in [25]). Then, since

$$\begin{aligned} \mathbb{E}_\alpha \left[\int_0^\tau \chi_s d\tilde{X}_s + \alpha \int_0^\tau \chi_s^2 d\langle M^H \rangle_s - \int_0^\tau \chi_s d\tilde{L}_s \right] &= \mathbb{E}_\alpha \left[\int_0^\tau \chi_s dM_s^H \right] \\ &= 0, \end{aligned}$$

it follows that

$$\mathbb{E}_\alpha \left[(\hat{\alpha}_\tau - \alpha) \left(- \int_0^\tau \chi_s d\tilde{X}_s - \alpha \int_0^\tau \chi_s^2 d\langle M^H \rangle_s + \int_0^\tau \chi_s d\tilde{L}_s \right) \right] = 1. \quad (5.9)$$

Applying Cauchy-Schwarz inequality in (5.9), we obtain

$$\begin{aligned} 1 &\leq \mathbb{E}_\alpha[\hat{\alpha}_\tau - \alpha]^2 \mathbb{E}_\alpha \left[\left(\int_0^\tau \chi_s d\tilde{X}_s + \alpha \int_0^\tau \chi_s^2 d\langle M^H \rangle_s - \int_0^\tau \chi_s d\tilde{L}_s \right)^2 \right] \\ &= \mathbb{E}_\alpha[\hat{\alpha}_\tau - \alpha]^2 \mathbb{E}_\alpha \left[\left(\int_0^\tau \chi_s dM_s^H \right)^2 \right] \\ &= \mathbb{E}_\alpha[\hat{\alpha}_\tau - \alpha]^2 \mathbb{E}_\alpha \left[\int_0^\tau \chi_s^2 d\langle M^H \rangle_s \right], \end{aligned} \quad (5.10)$$

where the first equality follows from the state equation (4.16). Since $0 < \mathbb{E}_\alpha[\int_0^\tau \chi_s^2 d\langle M^H \rangle_s] < \infty$, we can divide both sides of (5.10) by this factor, and then the desired result follows. \square

6 Appendix

The following Girsanov theorem for the fractional Brownian motion was given in [29].

Theorem 6.1. *Let $\tilde{W}_t^H = W_t^H + \int_0^t u_s ds$, where $\{u_t, t \in [0, T]\}$ is an adapted process with respect to the filtration \mathcal{F}^{W^H} generated by W^H with integrable trajectories. Denote*

$$\xi_T = \exp \left(- \int_0^T \left(K_H^{-1} \int_0^\cdot u_r dr \right) (s) dW_s - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^\cdot u_r dr \right)^2 (s) ds \right),$$

where W is the Wiener Process defined by (3.7). Assume that

(i) $\int_0^\cdot u_s ds \in I_{0^+}^{H+\frac{1}{2}}(L^2([0, T]))$, almost surely.

(ii) $\mathbb{E}(\xi_T) = 1$.

Then the shifted process \tilde{W}^H is an $\mathcal{F}_t^{W^H}$ -fractional Brownian motion with Hurst parameter H under the new probability \tilde{P} defined by $d\tilde{P}/dP = \xi_T$.

We shall use the above Girsanov theorem and fractional calculus to provide an independent proof for Theorem 4.7.

Proof of Theorem 4.7. If we can show that $\mathbb{E}(\xi_T) = 1$, then by Theorem 6.1, $\widehat{W}_t^H = W_t^H - \frac{\alpha}{\sigma} \int_0^t X_s ds$ is a fractional Brownian motion under \widehat{P} . Hence $\sigma^{-1}X_t = \sigma^{-1}X_0 + \widehat{W}_t^H + \sigma^{-1}L_t$ is a reflected fractional Brownian motion under \widehat{P} . We now show that for Δt small enough, we have

$$\mathbb{E} \exp \left(\int_t^{t+\Delta t} \left(K_H^{-1} \int_0^\cdot \lambda_0 X_r dr \right)^2 (s) ds \right) < \infty. \quad (6.1)$$

We shall prove this in two cases $H < \frac{1}{2}$ and $H > \frac{1}{2}$, respectively.

Case $H < \frac{1}{2}$: Observe that

$$\begin{aligned} \left| \left(K_H^{-1} \int_0^\cdot X_r dr \right) (s) \right| &= \left| s^{H-\frac{1}{2}} I_{0^+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} X_s \right| \\ &= C s^{H-\frac{1}{2}} \left| \int_0^s (s-r)^{-\frac{1}{2}-H} r^{\frac{1}{2}-H} X_r dr \right| \\ &\leq C(1 + \|X\|_{\infty, T}). \end{aligned}$$

Then when Δt small enough, (6.1) follows by Lemma 4.1.

Case $H > \frac{1}{2}$: We have that

$$\begin{aligned} &\left(K_H^{-1} \int_0^\cdot X_r dr \right) (s) \\ &= s^{\frac{1}{2}-H} X_s + \left(H - \frac{1}{2} \right) s^{H-\frac{1}{2}} \left(X_s \int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} dr + \int_0^s \frac{X_s - X_r}{(s-r)^{\frac{1}{2}+H}} r^{\frac{1}{2}-H} dr \right). \end{aligned}$$

Noting that

$$\int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{\frac{1}{2}+H}} dr = Cs^{1-2H},$$

and

$$|X_s - X_r| \leq \|X\|_{H-\epsilon, T} (s-r)^{H-\epsilon}, \quad 0 \leq r < s \leq T,$$

we have, if we choose $\epsilon < \frac{1}{2}$,

$$\begin{aligned} \left| \left(K_H^{-1} \int_0^\cdot X_r dr \right) (s) \right| &\leq C \left(s^{\frac{1}{2}-H} s^{H-\epsilon} \|X\|_{H-\epsilon, T} + s^{H-\frac{1}{2}} \|X\|_{H-\epsilon, T} \int_0^s (s-r)^{H-\epsilon} (s-r)^{-\frac{1}{2}-H} r^{\frac{1}{2}-H} dr \right) \\ &\leq C \|X\|_{H-\epsilon, T} (s^{\frac{1}{2}-\epsilon} + s^{H-\frac{1}{2}}) \\ &\leq C \|X\|_{H-\epsilon, T}. \end{aligned}$$

Then when Δt small enough, (6.1) follows from Lemma 4.1.

Now choose n big enough such that $\frac{T}{n} \leq \Delta t$, and let $t_i = \frac{iT}{n}, i = 1, \dots, n$. Denoting $(K_H^{-1} \int_0^\cdot X_r dr) (s)$ by $f(s)$, we obtain

$$\begin{aligned} \mathbb{E}[\xi_T] &= \mathbb{E} \exp \left(\int_0^T f(s) dW_s - \frac{1}{2} \int_0^T f^2(s) ds \right) \\ &= \mathbb{E} \left(\mathbb{E} \left[\exp \left(\int_0^T f(s) dW_s - \frac{1}{2} \int_0^T f^2(s) ds \right) \middle| \mathcal{F}_{t_{n-1}} \right] \right) \\ &= \mathbb{E} \left(\exp \left(\int_0^{t_{n-1}} f(s) dW_s - \frac{1}{2} \int_0^{t_{n-1}} f^2(s) ds \right) \right. \\ &\quad \left. \times \mathbb{E} \left[\exp \left(\int_{t_{n-1}}^T f(s) dW_s - \frac{1}{2} \int_{t_{n-1}}^T f^2(s) ds \right) \middle| \mathcal{F}_{t_{n-1}} \right] \right). \end{aligned}$$

We have that $\mathbb{E} \left[\exp \left(\int_{t_{n-1}}^T f(s) dW_s - \frac{1}{2} \int_{t_{n-1}}^T f^2(s) ds \right) \middle| \mathcal{F}_{t_{n-1}} \right] \leq 1$ a.s., since

$\exp \left(\int_{t_{n-1}}^\cdot f(s) dW_s - \frac{1}{2} \int_{t_{n-1}}^\cdot f^2(s) ds \right)$ is a positive local martingale and hence a supermartingale.

On the other hand, since $\mathbb{E} \left[\exp \left(\frac{1}{2} \int_{t_{n-1}}^T f^2(s) ds \right) \right] < \infty$, then by Novikov Criterion, we know that $\mathbb{E} \left[\exp \left(\int_{t_{n-1}}^T f(s) dW_s - \frac{1}{2} \int_{t_{n-1}}^T f^2(s) ds \right) \right] = 1$. This implies that

$$\mathbb{E} \left[\exp \left(\int_{t_{n-1}}^T f(s) dW_s - \frac{1}{2} \int_{t_{n-1}}^T f^2(s) ds \right) \middle| \mathcal{F}_{t_{n-1}} \right] = 1 \quad \text{a.s.}$$

We use the above procedure n times, and obtain that $\mathbb{E}(\xi_T) = 1$. □

Lemma 6.1. For $H \in (0, 1)$, we have

$$\limsup_{T \rightarrow \infty} \frac{1}{T^{H+1} \ln T} \int_0^T |W_t^H| dt \leq \mathbb{E}[|W_1^H|], \quad \text{a.s.}, \quad (6.2)$$

and

$$\liminf_{T \rightarrow \infty} \frac{1}{T \ln T} \int_0^T |W_t^H| dt \geq \frac{H}{H+1} \mathbb{E}[|W_1^H|], \quad a.s. \quad (6.3)$$

Proof. The Gaussian process $\{e^{-Ht}W_{e^t}^H, t \geq 0\}$ is stationary and ergodic. Indeed, the covariance function is given by

$$\rho(t, s) := \mathbb{E}[e^{-Ht}W_{e^t}^H e^{-Hs}W_{e^s}^H] = \frac{1}{2} \left(e^{H(t-s)} + e^{H(s-t)} - [e^{t-s} + e^{s-t} - 2]^H \right).$$

The covariance function ρ is a function of $|t-s|$, and hence the Gaussian process $\{e^{-Ht}W_{e^t}^H, t \geq 0\}$ is stationary. On the other hand, the covariance function ρ vanishes to 0 as $|t-s|$ goes to ∞ , because

$$\lim_{x \rightarrow \infty} (x^H + x^{-H} - [x + x^{-1} - 2]^H) = 0.$$

This implies that the Gaussian process is ergodic. By the ergodic theorem, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-Ht} |W_{e^t}^H| dt = \mathbb{E}[|W_1^H|], \quad a.s.,$$

which is equivalent to

$$\lim_{T \rightarrow \infty} \frac{1}{\ln T} \int_1^T s^{-H-1} |W_s^H| ds = \mathbb{E}[|W_1^H|], \quad a.s. \quad (6.4)$$

Therefore,

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{1}{T^{H+1} \ln T} \int_0^T |W_s^H| ds &= \limsup_{T \rightarrow \infty} \frac{1}{\ln T} T^{-(H+1)} \int_1^T |W_s^H| ds \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{\ln T} \int_1^T s^{-H-1} |W_s^H| ds \\ &= \mathbb{E}[|W_1^H|] \quad a.s., \end{aligned}$$

and (6.2) is proved.

Now we show (6.3). For $\alpha \in (0, 1)$, the equality (6.4) implies

$$\lim_{T \rightarrow \infty} \frac{1}{\ln T^\alpha} \int_1^{T^\alpha} s^{-H-1} |W_s^H| ds = \mathbb{E}[|W_1^H|], \quad a.s.,$$

and hence

$$\lim_{T \rightarrow \infty} \frac{1}{\ln T} \int_1^{T^\alpha} s^{-H-1} |W_s^H| ds = \alpha \mathbb{E}[|W_1^H|], \quad a.s. \quad (6.5)$$

By the relations (6.4) and (6.5),

$$\lim_{T \rightarrow \infty} \frac{1}{\ln T} \int_{T^\alpha}^T s^{-H-1} |W_s^H| ds = (1 - \alpha) \mathbb{E}[|W_1^H|], \quad a.s. \quad (6.6)$$

If we choose $\alpha = \frac{1}{H+1}$, then

$$\begin{aligned} \liminf_{T \rightarrow \infty} \frac{1}{T \ln T} \int_0^T |W_s^H| ds &\geq \liminf_{T \rightarrow \infty} \frac{1}{\ln T} T^{-\alpha(H+1)} \int_{T^\alpha}^T |W_s^H| ds \\ &\geq \liminf_{T \rightarrow \infty} \frac{1}{\ln T} \int_{T^\alpha}^T s^{-H-1} |W_s^H| ds \\ &= \frac{H}{H+1} \mathbb{E}[|W_1^H|] \quad a.s., \end{aligned}$$

where the last equality follows from (6.6). □

Acknowledgment

The authors would like to thank Professors David Nualart and Yaozhong Hu for their helpful comments and suggestions on an earlier draft of this paper. The authors also would like to thank two anonymous referees for their valuable comments which helped them to improve the paper. The research work of C. Lee is supported in part by Army Research Office (W911NF-14-1-0216).

References

- [1] F. A. Attia. On a reflected Ornstein-Uhlenbeck process with an application. *Bull. Austral. Math. Soc.*, 43(3):519–528, 1991.
- [2] B. Bercu, L. Coutin, and N. Savy. Sharp large deviations for the fractional Ornstein-Uhlenbeck process. *Teor. Veroyatn. Primen.*, 55(4):732–771, 2010.
- [3] Francesca Biagini, Yaozhong Hu, Bernt Øksendal, and Tusheng Zhang. *Stochastic calculus for fractional Brownian motion and applications*. Probability and its Applications. Springer-Verlag London Ltd., London, 2008.
- [4] Jaya P. N. Bishwal. *Parameter estimation in stochastic differential equations*, volume 1923 of *Lecture Notes in Mathematics*. Springer, Berlin, 2008.
- [5] L. Bo, D. Tang, Y. Wang, and X. Yang. On the conditional default probability in a regulated market: a structural approach. *Quantitative Finance*, 11(12):1695–1702, 2011.
- [6] L. Bo, Y. Wang, and X. Yang. Some integral functionals of reflected SDEs and their applications in finance. *Quantitative Finance*, 11(3):343–348, 2011.
- [7] Lijun Bo, Yongjin Wang, Xuewei Yang, and Guannan Zhang. Maximum likelihood estimation for reflected Ornstein-Uhlenbeck processes. *J. Statist. Plann. Inference*, 141(1):588–596, 2011.
- [8] Lijun Bo and Xuewei Yang. Sequential maximum likelihood estimation for reflected generalized Ornstein-Uhlenbeck processes. *Statist. Probab. Lett.*, 82(7):1374–1382, 2012.
- [9] Alexandre Brouste and Marina Kleptsyna. Asymptotic properties of MLE for partially observed fractional diffusion system. *Stat. Inference Stoch. Process.*, 13(1):1–13, 2010.

- [10] Hong Chen and David D. Yao. *Fundamentals of queueing networks*, volume 46 of *Applications of Mathematics*. Springer-Verlag, New York, 2001.
- [11] Patrick Cheridito, Hideyuki Kawaguchi, and Makoto Maejima. Fractional Ornstein-Uhlenbeck processes. *Electron. J. Probab.*, 8:no. 3, 14 pp. (electronic), 2003.
- [12] Igor Cialenco, Sergey V. Lototsky, and Jan Pospíšil. Asymptotic properties of the maximum likelihood estimator for stochastic parabolic equations with additive fractional Brownian motion. *Stoch. Dyn.*, 9(2):169–185, 2009.
- [13] Rama Cont. Long range dependence in financial markets. In *Fractals in Engineering*, pages 159–180. Springer, 2005.
- [14] L. Decreusefond and A. S. Üstünel. Stochastic analysis of the fractional Brownian motion. *Potential Anal.*, 10(2):177–214, 1999.
- [15] Paul Dupuis and Hitoshi Ishii. On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications. *Stochastics Stochastics Rep.*, 35(1):31–62, 1991.
- [16] X. Fernique. Régularité des trajectoires des fonctions aléatoires gaussiennes. In *École d'Été de Probabilités de Saint-Flour, IV-1974*, pages 1–96. Lecture Notes in Math., Vol. 480. Springer, Berlin, 1975.
- [17] M. Hübner, R. Khasminskii, and B. L. Rozovskii. Two examples of parameter estimation for stochastic partial differential equations. In *Stochastic processes*, pages 149–160. Springer, New York, 1993.
- [18] M. Huebner and B. L. Rozovskii. On asymptotic properties of maximum likelihood estimators for parabolic stochastic PDE's. *Probab. Theory Related Fields*, 103(2):143–163, 1995.
- [19] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [20] M. L. Kleptsyna and A. Le Breton. Statistical analysis of the fractional Ornstein-Uhlenbeck type process. *Stat. Inference Stoch. Process.*, 5(3):229–248, 2002.
- [21] Takis Konstantopoulos and Si-Jian Lin. Fractional Brownian approximations of queueing networks. In *Stochastic networks*, volume 117 of *Lecture Notes in Statist.*, pages 257–273. Springer, New York, 1996.
- [22] Chihoon Lee, Jaya P. N. Bishwal, and Myung Hee Lee. Sequential maximum likelihood estimation for reflected Ornstein-Uhlenbeck processes. *J. Statist. Plann. Inference*, 142(5):1234–1242, 2012.
- [23] Chihoon Lee and Ananda Weerasinghe. Convergence of a queueing system in heavy traffic with general patience-time distributions. *Stochastic Process. Appl.*, 121(11):2507–2552, 2011.
- [24] Robert S. Liptser and Albert N. Shiryaev. *Statistics of random processes. I*, volume 5 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, expanded edition, 2001.
- [25] Robert S. Liptser and Albert N. Shiryaev. *Statistics of random processes. II*, volume 6 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, expanded edition, 2001.

- [26] S. V. Lototsky. Statistical inference for stochastic parabolic equations: a spectral approach. *Publ. Mat.*, 53(1):3–45, 2009.
- [27] Ilkka Norros, Esko Valkeila, and Jorma Virtamo. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli*, 5(4):571–587, 1999.
- [28] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.
- [29] David Nualart and Youssef Ouknine. Regularization of differential equations by fractional noise. *Stochastic Process. Appl.*, 102(1):103–116, 2002.
- [30] B. L. S. Prakasa Rao. *Statistical inference for diffusion type processes*, volume 8 of *Kendall's Library of Statistics*. Edward Arnold, London, 1999.
- [31] Josh Reed, Amy Ward, and Dongyuan Zhan. On the generalized drift Skorokhod problem in one dimension. *J. Appl. Probab.*, 50(1):16–28, 2013.
- [32] Josh Reed and Amy R. Ward. A diffusion approximation for a generalized Jackson network with reneging. *Proceedings of the 42nd Annual Allerton Conference on Communication, Control, and Computing*, 2004.
- [33] L. M. Ricciardi. *Stochastic population models II. Diffusion models*. Lecture Notes at the International School on Mathematical Ecology. 1985.
- [34] Stefan G. Samko, Anatoly A. Kilbas, and Oleg I. Marichev. *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, Yverdon, 1993. Theory and applications, Edited and with a foreword by S. M. Nikol'skiĭ, Translated from the 1987 Russian original, Revised by the authors.
- [35] A. V. Skorohod. Stochastic equations for diffusions in a bounded region. *Theoret. Prob. Appl.*, 6:264–274, 1961.
- [36] Amy R. Ward and Peter W. Glynn. A diffusion approximation for a Markovian queue with reneging. *Queueing Syst.*, 43(1-2):103–128, 2003.
- [37] Amy R. Ward and Peter W. Glynn. Properties of the reflected Ornstein-Uhlenbeck process. *Queueing Syst.*, 44(2):109–123, 2003.
- [38] Amy R. Ward and Peter W. Glynn. A diffusion approximation for a $GI/GI/1$ queue with balking or reneging. *Queueing Syst.*, 50(4):371–400, 2005.
- [39] W. Willinger, M. S. Taqqu, and V. Teverovsky. Stock market prices and long-range dependence. *Finance and Stochastics*, 3(1):1–13, 1999.