Optimal Insurance Risk Control With Multiple Reinsurers

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Abstract

An optimal insurance risk control problem is discussed in a general situation where several reinsurance companies enter into a reinsurance treaty with an insurance company. These reinsurance companies adopt variance premium principles with different parameters. Dividends with fixed costs and taxes are paid to shareholders of the insurance company. Under certain conditions, a combined proportional reinsurance treaty is shown to be optimal in a class of plausible reinsurance treaties. Within the class of combined proportional reinsurance strategy, analytical expressions for the value function and the optimal strategies are obtained.

Keywords: variance premium principle; reinsurance strategy; multiple reinsurers; fixed costs; taxes; HJB equation; Lagrangian function
MSC2000: Primary 93E20; Secondary 60J25; 91B30

1 Introduction

Reinsurance is one of the practical means adopted by insurance companies to transfer insurance risk. It provides a way to control risk, and so, enhance the financial stability of an insurance company. From both the theoretical and practical perspectives, it may be interesting to discuss what is an optimal level of reinsurance that an insurance company should acquire. This problem is widely known as an optimal reinsurance problem. This problem has also captured the attention of many academic researchers in actuarial science and insurance. This might be partly attributed to the intellectual challenge of the problem. A popular approach to study an optimal reinsurance problem is to use stochastic optimal control theory in continuous-time to discuss the optimization problem. Some works in this direction are, for example, Asmussen et al. [2], Bai et al. [4], Meng and Zhang [13], Hipp and Taksar [12], Meng and Siu [14], [15], [16], Cadenillas et al. [6], Meng et al. [18], Meng et al. [19], [20] and Schmidli [22], amongst others. Two major types of reinsurance strategies such as the excess of loss reinsurance strategy and the proportional

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reinsurance strategy have been the main focuses of these works. Furthermore, for the sake of mathematical convenience, the expected value premium principle is widely used for calculating premiums in the literature on optimal reinsurance. However, the use of expected value premium principle may be questioned on both theoretical and practical grounds. From the theoretical point of view, the expected value premium principle cannot incorporate the volatility of claims losses which describes fluctuations in claims losses and can be measured by the standard deviation or variance of the claims losses. From the practical perspective, it has been noted in Wang [23] that "*In insurance practice, the most widely used method is to base calculation on the first two moments.*" The variance premium principle can capture the first two moments in premium calculations. It has been used by some authors to investigate an optimal insurance risk control problem, see, for example, Chi [7], Guerra and Centeno [10], Hipp and Taksar [12], Zhou and Yuen [24], Meng [17] and Meng et al. [19], [20].

Much attention has been paid to the situation where an insurer transfer the risk exposure to only one reinsurer. Relatively little attention has been given to a general situation that multiple reinsurers participate in a reinsurance treaty. Under the criterion of minimizing value at risk (VaR) or conditional value at risk (CVaR) of an insurer's total risk exposure, Chi and Meng [8] studied an optimal reinsurance arrangement in the presence of two reinsurers, where the first reinsurer adopts the expected value premium principle while the second reinsurer uses the premium principle satisfying threes axioms: distributional invariance, risk loading and preserving stop-loss order. Asimit et al. [1] also supposed that an insurance company may be able to share the risk with two reinsurers, where the first reinsurer uses the expected value premium principle and the second reinsurer adopts a distorted premium principle. The two papers refer to a static, single-period, insurance risk model. In a continuous-time set up, Meng [17] studied an optimal risk control problem with two reinsurers who calculate premiums by the variance premium principle with different parameters. Meng et al. [19] also considered an optimal reinsurance problem with two reinsurers in a continuous-time set up, where the two reinsurers adopted an expected value premium principle and a variance premium principle, and the optimization criterion was the probability of ruin. It seems that little attention has been given to studying an optimal reinsurance problem with more than two reinsurers in a continuous-time set up. For the sake of generality, it may be of interest to consider the situation where more than two reinsurers participate in the reinsurance treaty. In a continuous-time set up, an optimal reinsurance problem with more than two reinsurers is of theoretical interest and intellectual challenge since it is a high-dimensional stochastic optimal control problem.

In this paper, an optimal insurance risk control problem is studied in a general situation where several reinsurance companies enter into a reinsurance treaty with an insurance company. These reinsurance companies adopt variance premium principles with different parameters. In addition to determining an optimal reinsurance level, another key problem for an insurance company is to determine an optimal level of dividend payments to its shareholders. This is known as an optimal dividend problem. De Finetti [9] pioneered a formal study of an optimal dividend problem, where the expected present value of all dividends before possible ruin was maximized. The seminal work of De Finetti [9] has stimulated a lot of interest among researchers in actuarial science. A combination of an optimal reinsurance problem with multiple reinsurers and an optimal dividend problem is discussed, where dividends with fixed costs and taxes are paid to shareholders of the insurance company. Mathematically, the optimal dividend problem is related to an impulse control problem and has been studied in the literature, see for example, Bai and Guo [3], Cadenillas et al. [6], Meng and Siu [14], [15], [16], Meng [17], Meng et al. [18], Sotomayor and Cadenillas et al. [21] and Meng et al. [20]. Under certain conditions, a combined proportional reinsurance treaty is shown to be optimal in the class of plausible reinsurance treaties. Within the class of combined proportional reinsurance strategy, analytical expressions for the value function and the optimal strategies are provided.

The rest of the paper is organized as follows. In the next section, the dynamic risk control problem with m reinsurers with the variance premium principles is formulated. The corresponding optimization problem is presented. In Section 3, the optimality of a combined proportional reinsurance strategy is discussed. In Section 4, analytical expressions for the value function, the optimal reinsurance and dividend strategy are derived. The final section summarizes the paper.

2 Model formulation

Uncertainty is resolved over time according to a complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions (i.e., the right continuity and $\mathbb{P}$-completeness) and $\mathbb{P}$ is a real-world probability measure. According to the Cramér-Lundberg model, the surplus of an insurance company is given by:

$$P(t) = x + pt - \sum_{i=1}^{N(t)} Z_i \ ,$$

where $x \geq 0$ is the initial surplus; $\{N_t, t \geq 0\}$ is a Poisson process with constant intensity parameter $\lambda > 0$; The claims $Z_i, i = 1, 2, \cdots$ are independent and identically distributed (*i.i.d.*) random variables, and are independent of $\{N_t, t \geq 0\}$. Assume, for each $i = 1, 2, \cdots$, Z_i has a finite mean μ and a finite second moment σ^2; p adopts the variance premium principle with the parameter $\theta_0 > 0$, i.e.,

$$pt = \mathbb{E}\left[\sum_{i=1}^{N(t)} Z_i\right] + \theta_0 \mathbb{D}\left[\sum_{i=1}^{N(t)} Z_i\right] = \lambda(\mu + \theta_0\sigma^2)t,$$

where $\mathbb{E}[\cdot]$ and $\mathbb{D}[\cdot]$ are the expectation and variance operators, respectively. To control its risk exposures, the insurance company can cede part of the loss for each claim by acquiring reinsurance. We assume that m reinsurance companies participate in a reinsurance treaty and these reinsurance companies adopt the variance premium principle with different parameters, say $\theta_j,\ j = 1, 2, \cdots, m$. Without loss of generality, we assume that $\theta_i \geq \theta_j,\ i < j$. For each claim Z_i, the j^{th} reinsurance company undertakes $g_j(Z_i)$. Then $g_0(Z_i) := Z_i - \sum_{j=1}^m g_j(Z_i)$ is the remaining part of the claim Z_i, which is retained by the insurance company. The aggregate premium that is received by these m reinsurance companies from the insurance company is given by:

$$\sum_{j=1}^{m}\left\{\mathbb{E}\left[\sum_{i=1}^{N(t)} g_j(Z_i)\right]+\theta_j\mathbb{D}\left[\sum_{i=1}^{N(t)} g_j(Z_i)\right]\right\}=\lambda t\sum_{j=1}^{m}\{\mathbb{E}[g_j(Z_i)]+\theta_j\mathbb{E}[g_j(Z_i)]^2\}.$$

Then the surplus process of the insurance company with reinsurance arrangement and dividend payments can be written as:

$$P^1(t)=x+p^1t-\sum_{i=1}^{N(t)} g_0(Z_i)-\sum_{n=1}^{\infty} I_{\{\tau_n\le t\}}\xi_n, \tag{2.1}$$

where

$$p^1 \;=\; \lambda\left(\mu+\theta_0\sigma^2-\sum_{j=1}^{m}\left\{\mathbb{E}[g_j(Z_i)]+\theta_j\mathbb{E}[g_j(Z_i)]^2\right\}\right),$$

and for each $i=1,2,\cdots$, τ_i and ξ_i are the time and amount of the i^{th} dividend payment, respectively, which will be defined precisely later.

It seems uneasy to obtain explicit results for the combined optimal reinsurance and dividend problem in the above compound Poisson process. With a view to deriving explicit results for the combined optimal reinsurance and dividend problem, we adopt here a pure diffusion approximation in Grandell [11] or Bäuerle [5] to approximate the surplus process (2.1). Using the notation defined as above, the pure diffusion approximation to the surplus process without dividend payments can be represented as:

$$dX(t)=\upsilon(g_0(Z),g_1(Z),\cdots,g_m(Z))dt+\sigma(g_0(Z))dB(t),$$

where $\{B(t),t\ge 0\}$ is a standard Brownian motion on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},\mathbb{P})$,

$$\upsilon(g_0(Z),g_1(Z),\cdots,g_m(Z)) \;=\; \lambda\left(\theta_0\sigma^2-\sum_{j=1}^{m}\theta_j\mathbb{E}[g_j(Z)]^2\right), \tag{2.2}$$

and

$$\sigma(g_0(Z)) \;=\; \sqrt{\lambda\mathbb{E}[g_0(Z)]^2}. \tag{2.3}$$

Furthermore if dividends are paid to shareholders, the diffusion surplus process of the insurance company with dividend payments is governed by:

$$R(t)=X(t)-\sum_{n=1}^{\infty}\xi_n I_{\{\tau_n\le t\}}, \tag{2.4}$$

where $\{\tau_i;i=1,2,\cdots\}$ is an increasing sequence of stopping times and $\{\xi_i;i=1,2,\cdots\}$ is a sequence of non-negative random variables associated with the amounts of dividends paid to shareholders.

Assume that the insurer dynamically adjusts it's risk position. Then (2.4) becomes:

$$\begin{aligned} dR^{\pi}(t) \;=\;& \upsilon(g_0(t,Z),g_1(t,Z),\cdots,g_m(t,Z))dt+\sigma(g_0(t,Z))dB(t)\\ &-d\left(\sum_{n=1}^{\infty} I_{\{\tau_n\le t\}}\xi_n\right), \end{aligned} \tag{2.5}$$

where the two functions υ and σ are defined by (2.2) and (2.3) respectively; the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_{t-}]$ is used, and Z is independent of $\mathcal{F}_{t-}$. In what follows, the superscript π is suppressed in $R^{\pi}(t)$ and we write $R(t)$ for $R^{\pi}(t)$.

Definition 2.1. *A*

$$\pi \equiv \{g_0, g_1, \cdots, g_m; \mathcal{S}\} \equiv \{g_0, g_1, \cdots, g_m; \tau_1, \tau_2, \cdots, \tau_n, \cdots; \xi_1, \xi_2, \cdots, \xi_n, \cdots\}$$

is called an admissible policy if it satisfies the following conditions:

(I1) for each $z \in \Re^+$, $\{g_j(t, z); t \geq 0\}$ *are* $\{\mathcal{F}_t\}_{t\geq 0}$*-predictable;*

(I2) for each $(t, \omega) \in [0, \infty) \times \Omega$, $g_j(t, z, \omega)$ *is Borel-measurable in* z*;*

(I3) $g_j(t, z) \geq 0$, $\sum_{j=0}^{m} g_j(t, z) = z$*;*

(I4) $\{\tau_n\}_{n\geq 1}$ *is an increasing sequence, and for each* $i = 1, 2, \cdots$*, each* $t \geq 0$, $\{\tau_i \leq t\} \in \mathcal{F}_t$ *and* $\xi_i \in \mathcal{F}_{\tau_i}$, $0 < \xi_i \leq R(\tau_i-)$*;*

(I5) There is at least a pair (k, l) *such that* $\sqrt{\mathbb{E}\left[\sum_{i\geq 0, i\neq k} g_i(t, Z)\right]^2} - \sum_{i\geq 0, i\neq k,l} \sqrt{\mathbb{E}[g_i(t, Z)]^2} \geq 0$*, where* $1 \leq k, l \leq m$, $k \neq l$*;*

(I6) the stochastic differential equation (2.5) has a unique strong solution.

We write $\mathcal{A}$ for the space of all admissible policies.

Remark 2.1. *The condition (I5) is purely technical, and it will be used in Lemma 3.1. Clearly when* $m = 1$ *or* $m = 2$*, the condition (I5) holds true, i.e.,*

- *for* $m = 1$, $\sqrt{\mathbb{E}(g_0(t, Z))^2} \geq 0$*;*
- *for* $m = 2$,
$$\sqrt{\mathbb{E}(g_0(t, Z) + g_1(t, Z))^2} \geq \sqrt{\mathbb{E}(g_0(t, Z))^2}$$
and
$$\sqrt{\mathbb{E}(g_0(t, Z) + g_2(t, Z))^2} \geq \sqrt{\mathbb{E}(g_0(t, Z))^2}.$$
However,
- *for* $m \geq 3$*, the condition (I5) may hold true or may not hold true. If* $\{g_i(Z), i = 0, 1, \cdots, m\}$ *adopt proportional reinsurance strategies, it can be verified that the condition (I5) must hold true.*

The following example presents some no-proportional reinsurance strategies satisfying the condition (I5).

Example 1: Suppose the claims $\{Z_i\}$ satisfy $\mathbb{P}(Z_i = 2) = 0.9, \mathbb{P}(Z_i = 3) = 0.1$ and $g_1(Z_i) = g_2(Z_i) \equiv 0$, $g_3(Z_i) \equiv 1$, $\mathbb{P}(g_0(Z_i) = 1) = 0.9, \mathbb{P}(g_0(Z_i) = 2) = 0.1$. Thus

$$\sqrt{\mathbb{E}(g_0(Z_i) + g_1(Z_i) + g_3(Z_i))^2} = 1.14 > 1 = \sqrt{\mathbb{E}g_0^2(Z_i)} + \sqrt{\mathbb{E}g_1^2(Z_i)}.$$

However, Example 2 below gives some reinsurance strategies which don't satisfy the condition (I5).

Example 2: Suppose the claims $\{Z_i\}$ satisfy $\mathbb{P}(Z_i = 4) = 0.9, \mathbb{P}(Z_i = 1003) = 0.1$ and $g_1(Z_i) = g_2(Z_i) = g_3(Z_i) \equiv 1$, $\mathbb{P}(g_0(Z_i) = 1) = 0.9, \mathbb{P}(g_0(Z_i) = 1000) = 0.1$. Then

$$\begin{aligned}
\sqrt{\mathbb{E}(g_0(Z_i) + g_1(Z_i) + g_3(Z_i))^2} = 316.87 < 317.23 &= \sqrt{\mathbb{E}g_0^2(Z_i)} + \sqrt{\mathbb{E}g_1^2(Z_i)} \\
&= \sqrt{\mathbb{E}g_0^2(Z_i)} + \sqrt{\mathbb{E}g_3^2(Z_i)}, \\
\sqrt{\mathbb{E}(g_0(Z_i) + g_2(Z_i) + g_3(Z_i))^2} = 316.87 < 317.23 &= \sqrt{\mathbb{E}g_0^2(Z_i)} + \sqrt{\mathbb{E}g_2^2(Z_i)} \\
&= \sqrt{\mathbb{E}g_0^2(Z_i)} + \sqrt{\mathbb{E}g_3^2(Z_i)}, \\
\sqrt{\mathbb{E}(g_0(Z_i) + g_1(Z_i) + g_2(Z_i))^2} = 316.87 < 317.23 &= \sqrt{\mathbb{E}g_0^2(Z_i)} + \sqrt{\mathbb{E}g_1^2(Z_i)} \\
&= \sqrt{\mathbb{E}g_0^2(Z_i)} + \sqrt{\mathbb{E}g_2^2(Z_i)}.
\end{aligned}$$

That is, there doesn't exist a pair (k, l) satisfying (I5).

Suppose that there is a fixed amount of transaction cost attributed to the advisory and consulting fees, say $K(K > 0)$, and proportional tax, say $1 - k(0 < k < 1)$, associated with each dividend payment. Then as it is typical in the literature the optimization problem of the insurance company is to select $\pi \in \Pi(x)$ so as to maximize the following performance function:

$$J(x, \pi) := \mathbb{E}\left[\sum_{n=1}^{\infty} e^{-\delta\tau_n}(-K + k\xi_n) I_{\{\tau_n \le \tau^\pi\}}\right],$$

where $\delta > 0$ is a continuously compounded valuation interest rate and the ruin time τ^π corresponding to policy π is

$$\tau^\pi = \inf\{t : R_t^\pi < 0\}.$$

Our goal is to find an optimal strategy $\pi \in \mathcal{A}$ so as to maximize the expected present value of dividends before bankruptcy. That is, to determine the value function

$$V(x) := \sup\{J(x, \pi); \pi \in \mathcal{A}\}, \tag{2.6}$$

and an optimal strategy π^* such that $V(x) = J(x, \pi^*)$.

Remark 2.2. *The continuous-time insurance risk model with multiple reinsurers considered here may be thought of as a generalization to those models with two resinurers in, for example, Meng [17] and Meng et al. [19].*

3 Optimal reinsurance form

In this section, the combined proportional reinsurance treaty is shown to be an optimal form among the class of plausible reinsurance treaties. There are a number of papers discussing optimal forms of reinsurance in various continuous-time insurance risk models. Some examples are Meng and Zhang [13], Meng and Siu [14], [15], [16], Meng [17], and Meng et al. [19], [20], amongst others. The mathematical techniques used here to prove the optimality results are in line with those used in the literature, see, for example, Meng and Zhang [13], Meng and Siu [16], Meng [17], Meng et al. [19], Without loss of generality, we assume that $(k,l)=(1,m)$, i.e.,

$$\sqrt{\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2}-\sum_{i=0,i\neq 1}^{m-1}\sqrt{\mathbb{E}[g_i(t,Z)]^2}\geq 0,$$

Then

$$dR^{\pi}(t)=\mu(g_0(t,Z),g_2(t,Z),\cdots,g_m(t,Z))dt+\sigma(g_0(t,Z))dB(t)-d\left(\sum_{n=1}^{\infty} I_{\{\tau_n\leq t\}}\xi_n\right),$$

where

$$\mu[g_0(Z),g_2(Z),\cdots,g_m(Z)] \;=\; \lambda\left(\theta_0\sigma^2-\theta_1\mathbb{E}\left[Z-\sum_{j=0,j\neq 1}^{m} g_j(Z)\right]^2-\sum_{j=2}^{m}\theta_j\mathbb{E}[g_j(Z)]^2\right).$$

For convenience, the following functions are defined. Let

$$\begin{aligned}
g_0^{(a_0,b)}(z) &= a_0bz,\\
g_1^{(b)}(z) &= (1-b)z,\\
g_j^{(a_j,b)}(z) &= a_jbz,\\
g_m^{(a_0,a_2,\cdots,a_{m-1},b)}(z) &= \left(1-\sum_{i=0,i\neq 1}^{m-1} a_i\right)bz,
\end{aligned}$$

where $0\leq a_0,a_2,\cdots,a_{m-1},b\leq 1, a_0+a_2+\cdots+a_{m-1}\leq 1$ and $j=2,3,\cdots,m-1$.

Lemma 3.1. *For any fixed $\pi=(g_0,g_1\cdots,g_m)\in\mathcal{A}$, there are two sets of $\{\mathcal{F}_t\}_{t\geq 0}$-predictable processes $\{\tilde{a}_{0t},\tilde{a}_{2t},\cdots,\tilde{a}_{(m-1)t}\}$ and $\{\tilde{b}_t\}$ such that*

$$\begin{aligned}
\mu\left[g_0^{(\tilde{a}_{0t},\tilde{b}_t)}(Z),g_2^{(\tilde{a}_{2t},\tilde{b}_t)}(Z),\cdots,g_m^{(\tilde{a}_{0t},\tilde{a}_{2t},\cdots,\tilde{a}_{(m-1)t},\tilde{b}_t)}(Z)\right] &\geq \mu[g_0(t,Z),g_2(t,Z),\cdots,g_m(t,Z)],\\
\sigma[g_0^{(\tilde{a}_{0t},\tilde{b}_t)}(Z)] &= \sigma[g_0(t,Z)].
\end{aligned}$$

Proof: Let $\{\tilde{b}_t;t\geq 0\}$ and $\{\tilde{a}_{jt};t\geq 0\}$ be two $\{\mathcal{F}_t\}_{t\geq 0}$-predictable processes taking values in $[0,1]$ such that

$$\tilde{b}_t^2\mathbb{E}Z^2 = \mathbb{E}(\tilde{b}_tZ)^2=\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,z)\right]^2, \tag{3.1}$$

$$\tilde{a}_{jt}^2\tilde{b}_t^2\mathbb{E}Z^2 = \mathbb{E}(\tilde{a}_{jt}\tilde{b}_tZ)^2=\mathbb{E}[g_j(t,Z)]^2, \tag{3.2}$$

where $j=0,2,3,\cdots,m-1$.

Using the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}\left[Z-\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2 \\
=\ & \mathbb{E}Z^2-2\mathbb{E}\left[Z\left(\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right)\right]+\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2 \\
\geq\ & \mathbb{E}Z^2-2\sqrt{\mathbb{E}Z^2}\sqrt{\mathbb{E}\left(\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right)^2}+\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2 \\
=\ & \mathbb{E}Z^2-2\tilde{b}_t\mathbb{E}Z^2+\mathbb{E}(\tilde{b}_tZ)^2 \\
=\ & \mathbb{E}[Z-\tilde{b}_tZ]^2.
\end{aligned}$$

If $\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2\neq 0$, from (3.1) and (3.2), we have

$$\tilde{a}_{jt}=\sqrt{\frac{\mathbb{E}[g_j(t,Z)]^2}{\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2}},\ j=0,2,\cdots,m-1.$$

$$\begin{aligned}
&\left(1-\sum_{j=0,j\neq 1}^{m-1}\tilde{a}_{jt}\right)^2\tilde{b}_t^2\mathbb{E}Z^2 \\
=\ & \left(1-\sum_{j=0,j\neq 1}^{m-1}\sqrt{\frac{\mathbb{E}[g_j(t,Z)]^2}{\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2}}\right)^2\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2 \\
=\ & \left(\sqrt{\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2}-\sum_{j=0,j\neq 1}^{m-1}\sqrt{\mathbb{E}[g_j(t,Z)]^2}\right)^2 \\
\leq\ & \left(\sqrt{\mathbb{E}[g_m(t,Z)]^2}\right)^2 \\
=\ & \mathbb{E}[g_m(t,Z)]^2.
\end{aligned}$$

If $\mathbb{E}\left[\sum_{i=0,i\neq 1}^{m} g_i(t,Z)\right]^2=0$, then $\tilde{b}_t=0$. In this case we select $\tilde{a}_{jt}\in[0,1]$ to be any constant for $j=0,2,\cdots,m-1$. This completes the proof. $\square$

Remark 3.1. *From (3.1) and (3.2), we can assure* $\tilde{a}_{0t}^2+\tilde{a}_{2t}^2+\cdots+\tilde{a}_{(m-1)t}^2\in[0,1]$*; however the case* $\tilde{a}_{0t}+\tilde{a}_{2t}+\cdots+\tilde{a}_{(m-1)t}>1$ *may occur. Nevertheless under the condition (I5),* $\tilde{a}_{0t}+\tilde{a}_{2t}+\cdots+\tilde{a}_{(m-1)t}\in[0,1]$ *holds true.*

Theorem 3.1. *For any fixed*

$$\pi=(g_0,g_1,\cdots,g_m,\tau_1,\tau_2,\cdots,\tau_n,\cdots;\xi_1,\xi_2,\cdots,\xi_n,\cdots)\in\mathcal{A},$$

there exists

$$\begin{aligned}
\pi_1\ =\ & \Big(g_0^{(a_{0t}^1(t),b_t^1)}(Z),g_1^{(b_t^1)}(Z),g_2^{(a_{2t}^1,b_t^1)}(Z),\cdots,g_m^{(a_{0t}^1,a_{2t}^1,\cdots,a_{(m-1)t}^1,b_t^1)}(Z),\tau_1^1,\tau_2^1, \\
& \cdots,\tau_n^1,\cdots;\xi_1^1,\xi_1^1,\cdots,\xi_n^1,\cdots\Big)\in\mathcal{A}
\end{aligned}$$

such that $V^{\pi_1}(x) \geq V^{\pi}(x)$.

Proof. Let $\pi = (g_0, g_1, \cdots, g_m, \tau_1, \tau_2, \cdots, \tau_n, \cdots; \xi_1, \xi_2, \cdots, \xi_n, \cdots) \in \mathcal{A}$. Take

$$(a^1_{0t}, a^1_{2t}, \cdots, a^1_{(m-1)t}, b^1_t) = (\tilde{a}_{0t}, \tilde{a}_{2t}, \cdots, \tilde{a}_{(m-1)t}, \tilde{b}_t),$$

$\tau^1_n = \tau_n$ and

$$\begin{aligned} \xi^1_n = \xi_n + & \int_{\tau_{n-1}}^{\tau_n -} \left\{ \mu \left[g_0^{(\tilde{a}_{0t}, \tilde{b}_t)}(Z), g_2^{(\tilde{a}_{2t}, \tilde{b}_t)}(Z), \cdots, g_m^{(\tilde{a}_{0t}, \tilde{a}_{2t}, \cdots, \tilde{a}_{(m-1)t}, \tilde{b}_t)}(Z) \right] \right. \\ & \left. - \mu[g_0(t, Z), g_2(t, Z), \cdots, g_m(t, Z)] \right\} dt, \ \ n = 1, 2, \cdots. \end{aligned}$$

Let

$$\begin{aligned} \pi_1 \ = \ & \left(g_0^{(a^1_{0t}, b^1_t)}(Z), g_1^{(b^1_t)}(Z), g_2^{(a^1_{2t}, b^1_t)}(Z), \cdots, g_m^{(a^1_{0t}, a^1_{2t}, \cdots, a^1_{(m-1)t}, b^1_t)}(Z)\ , \tau^1_1, \tau^1_2, \right. \\ & \left. \cdots, \tau^1_n, \cdots; \xi^1_1, \xi^1_1, \cdots, \xi^1_n, \cdots \right) \in \mathcal{A} \end{aligned}$$

Then

$$\begin{aligned} dR^{\pi_1}(t) &= \mu \left[g_0^{(a^1_{0t}, b^1_t)}(Z), g_2^{(a^1_{2t}, b^1_t)}(Z), \cdots, g_m^{(a^1_{0t}, a^1_{2t}, \cdots, a^1_{(m-1)t}, b^1_t)}(Z) \right] dt \\ & \quad + \sigma(g_0^{(a^1_{0t}, b^1_t)}(Z)) dB(t) - d\left(\sum_{n=1}^{\infty} I_{\{\tau^1_n \leq t\}} \xi^1_n \right) \\ &= \mu[g_0(t, Z), g_2(t, Z), \cdots, g_m(t, Z)] dt + \sigma(g_0(t, Z)) dB(t) - d\left(\sum_{n=1}^{\infty} I_{\{\tau_n \leq t\}} \xi_n \right) \\ &= dR^{\pi}(t). \end{aligned}$$

Since $\xi^1_n \geq \xi_n$, $\sum_{n=1}^{\infty} I_{\{\tau^1_n \leq t\}} \xi^1_n \geq \sum_{n=1}^{\infty} I_{\{\tau_n \leq t\}} \xi_n$. Therefore, $V^{\pi_1}(x) \geq V^{\pi}(x)$. $\square$

4 Explicit solution

In this section, an explicit solution to the optimal dividend problem (2.6) will be derived within the class of combined proportional reinsurance treaties given in Section 3. The Lagrange multipler approach is adopted to solve the Hamiltonian-Jacobi-Bellman (HJB) equation arising from the control problem. The mathematical techniques used in the derivations here are similar to those used in Cadenillas et al. [6], Meng and Siu [14], [16], Meng [17], and Meng et al. [20], for example.

For simplification, let

$$c_0 = a_0 b, \ c_1 = 1 - b, \ c_j = a_j b, \ c_m = \left(1 - \sum_{i=0, i \neq 1}^{m-1} a_i \right) b,$$

where $c_j \geq 0$ and $\sum_{i=0}^{m} c_i = 1$.

Under the combined proportional reinsurance strategy, the controlled surplus process $\{R^{\pi}(t)\}$ satisfies the following stochastic differential equation:

$$dR^{\pi}(t) = \mu_1(c_{1t}, c_{2t}, \cdots, c_{mt}) dt + \sigma_1(c_{0t}) dB(t) - d\left(\sum_{n=1}^{\infty} I_{\{\tau_n \leq t\}} \xi_n \right), \tag{4.1}$$

where

$$\begin{aligned}\mu_1(c_1, c_2, \cdots, c_m) &= \lambda\sigma^2\left[\theta_0 - \sum_{i=1}^{m}\theta_i c_i^2\right], \\ \sigma_1(c_0) &= \sqrt{\lambda}\sigma c_0.\end{aligned}$$

To be precise, the set of admissible strategies, denoted by $\mathcal{O}$, is re-defined. For each $\pi = (c_0, c_1, \cdots, c_m, \mathcal{S}) \in \mathcal{O}$,

(II1) $c_i = \{c_{it}, t \geq 0\}$ are $\{\mathcal{F}_t\}_{t\geq 0}$-predictable processes;

(II2) $c_{it} \in [0, 1]$ and $\sum_{i=0}^{m} c_{it} = 1$.

(II3) for each $i = 1, 2, \cdots$ and each $t \geq 0$, $\{\tau_i \leq t\} \in \mathcal{F}_t$ and $\xi_i \in \mathcal{F}_{\tau_i}$; $0 < \xi_i \leq R(\tau_i-)$;

(II4) the stochastic differential equation (4.1) has a unique strong solution.

When the volatility of the controlled surplus process is zero, i.e., $c_{0t} \equiv 0$,

$$\begin{aligned}\max_{c_i\in[0,1], c_1+\cdots+c_m=1}\mu_1(c_1, \cdots, c_m) &= \mu_1\left(\frac{1}{\theta_1\Delta_m}, \frac{1}{\theta_2\Delta_m}, \cdots, \frac{1}{\theta_m\Delta_m}\right) \\ &= \frac{\lambda\sigma^2(\theta_0\Delta_m - 1)}{\Delta_m},\end{aligned}$$

where

$$\Delta_m = \sum_{i=1}^{m}\frac{1}{\theta_i}. \tag{4.2}$$

If $\theta_0\Delta_m - 1 \geq 0$, we can choose $c_{0t} \equiv 0, c_{it} \equiv \frac{1}{\theta_i\Delta_m}$, $i = 1, \cdots, m$. This implies that the market may have arbitrage opportunities. To avoid this situation, we assume that

$$1 - \theta_0\Delta_m > 0. \tag{4.3}$$

Let $\mathcal{H}$ be the space of real-valued, twice continuously differentiable functions on $\Re^+$. For each $c_j \in [0, 1]$, the following differential operator $\mathcal{L}^{c_0,c_1,\cdots,c_m}$ acting on $\phi \in \mathcal{H}$ corresponding to the controlled surplus process is defined:

$$\mathcal{L}^{c_0,c_1,\cdots,c_m}\phi(x) = \lambda\sigma^2\left[\theta_0 - \sum_{i=1}^{m}\theta_i c_i^2\right]\phi'(x) + \frac{1}{2}\lambda\sigma^2 c_0^2\phi''(x).$$

In addition, the following operator is defined:

$$\mathcal{M}\phi(x) = \sup_{0<\xi\leq x}\{\phi(x - \xi) + k\xi - K\}.$$

Using arguments similar to Cadenillas et al. [6], the value function $V(x)$ satisfies the following HJB equation:

$$\max\left\{\max_{c_i\in[0,1], c_0+\cdots+c_m=1}[\mathcal{L}^{c_0,c_1,\cdots,c_m}\phi(x) - \delta\phi(x)], \mathcal{M}\phi(x) - \phi(x)\right\} = 0, \quad x > 0, \tag{4.4}$$

In what follows, the use of the Lagrange multiplier approach to solve the HJB equation will be discussed.

Let

$$x_1 = \inf\{x \geq 0 : \phi(x) = \mathcal{M}\phi(x)\}.$$

Then

$$\max_{c_i \in [0,1], c_0 + \cdots + c_m = 1} [\mathcal{L}^{c_0, c_1, \cdots, c_m}\phi(x) - \delta\phi(x)] = 0, \; 0 < x \leq x_1. \tag{4.5}$$

For each scalar $\beta \geq 0$, (i.e., β is the Lagrange mutlipler), the following Lagrangian function is defined.

$$\mathcal{A}^{c_0, c_1, \cdots, c_m, \beta}\phi(x) := \mathcal{L}^{c_0, c_1, \cdots, c_m}\phi(x) - \delta\phi(x) + \beta(c_0 + c_1 + \cdots + c_m - 1). \tag{4.6}$$

From

$$\frac{\partial}{\partial c_i}\left(\mathcal{A}^{c_0, c_1, \cdots, c_m, \beta}\phi(x)\right) = 0, \, i = 1, \cdots, m,$$

we have

$$\phi'(x) = \frac{\beta}{2\lambda\sigma^2\theta_i c_i(x)}, \tag{4.7}$$

which results in

$$c_i(x) = \frac{\theta_1 c_1(x)}{\theta_i}, i = 1, \cdots, m. \tag{4.8}$$

From

$$\frac{\partial}{\partial \beta}\left(\mathcal{A}^{c_0, c_1, \cdots, c_m, \beta}\phi(x)\right) = 0,$$

with (4.8), we have

$$c_1(x) = \frac{1 - c_0(x)}{\theta_1 \Delta_m}. \tag{4.9}$$

Combining (4.7) and (4.9) yields

$$\beta = \beta_\phi(x) := \frac{2\lambda\sigma^2(1 - c_0(x))\phi'(x)}{\Delta_m}. \tag{4.10}$$

From

$$\frac{\partial}{\partial c_0}\left(\mathcal{A}^{c_0, c_1, \cdots, c_m, \beta}\phi(x)\right) = 0,$$

with (4.10), we have

$$c_0(x)\phi''(x) + \frac{2(1 - c_0(x))\phi'(x)}{\Delta_m} = 0. \tag{4.11}$$

Putting (4.8), (4.9) and (4.11) into (4.5) yields

$$\lambda\sigma^2\left[\theta_0\Delta_m - 1 + c_0(x)\right]\phi'(x) - \delta\Delta_m\phi(x) = 0. \tag{4.12}$$

Suppose that $\phi'(x) > 0$. From (4.12) with $\phi(0) = 0$ we have $c_0(0) = 1 - \Delta_m\theta_0$.

Taking derivative with respect to x on both sides of (4.12)

$$(\lambda\sigma^2 c_0'(x) - \delta\Delta_m)\phi'(x) + \lambda\sigma^2 [\theta_0\Delta_m - 1 + c_0(x)] \phi''(x) = 0. \tag{4.13}$$

Substituting (4.11) into (4.13) gives:

$$\{(\lambda\sigma^2 c_0'(x) - \delta\Delta_m)\Delta_m c_0(x) - 2\lambda\sigma^2(1 - c_0(x)) [\theta_0\Delta_m - 1 + c_0(x)]\}\phi'(x) = 0.$$

Then

$$c_0'(x) = \Gamma(c_0(x)), \tag{4.14}$$

where

$$\Gamma(s) = \frac{2\lambda\sigma^2(1 - s) [\theta_0\Delta_m - 1 + s] + \delta\Delta_m^2 s}{\lambda\sigma^2\Delta_m s}.$$

It can be easily shown that for $s \in [1 - \Delta_m\theta_0, 1]$, $\Gamma(s) > 0$. Let

$$G(y) = \int_{1-\Delta_m\theta_0}^{y} \frac{1}{\Gamma(s)} ds,$$

Obviously, $G(y)$ is a strictly increasing function. Thus, the inverse function of $G(y)$ exists, say $G^{-1}(y)$. Then, from the equation (4.14) with $c_0(0) = 1 - \Delta_m\theta_0$

$$c_0(x) = \bar{c}_0(x) := G^{-1}(x). \tag{4.15}$$

Thus, from (4.7) and (4.9),

$$c_i(x) = \bar{c}_i(x) := \frac{1 - G^{-1}(x)}{\theta_i\Delta_m}, i = 1, 2, \cdots, m. \tag{4.16}$$

Putting (4.15) into (4.11) and solving (4.11) with $\phi(0) = 0$ give:

$$\phi(x) = q_1 \int_0^x H(z)dz,\ 0 \le x \le G(1), \tag{4.17}$$

where

$$H(z) = \exp\left\{\int_z^{G(1)} \frac{2(1 - G^{-1}(y))}{\Delta_m G^{-1}(y)} dy\right\}. \tag{4.18}$$

For $G(1) \le x \le x_1$,we guess $c_0(x) = 1$ and $c_j(x) = 0$ for $i = 1, \cdots, m$. Thus (4.5) becomes

$$\frac{1}{2}\lambda\sigma^2\phi''(x) + \lambda\sigma^2\theta_0\phi'(x) - \delta\phi(x) = 0,$$

and the solution is

$$\phi(x) = q_2 e^{r_+(x-G(1))} + q_3 e^{r_-(x-G(1))},\ G(1) \le x \le x_1, \tag{4.19}$$

where

$$r_+ = \frac{-\lambda\sigma\theta_0 + \sqrt{\lambda^2\sigma^2\theta_0^2 + 2\lambda\delta}}{\lambda\sigma}$$

and

$$r_- = \frac{-\lambda\sigma\theta_0 - \sqrt{\lambda^2\sigma^2\theta_0^2 + 2\lambda\delta}}{\lambda\sigma}.$$

For $x \geq G(1)$, we guess that

$$\phi(x) = \phi(x_1) + k(x - x_1). \tag{4.20}$$

First, by the continuity of $\phi'(x)$ and $\phi''(x)$ at $x = G(1)$,

$$\begin{aligned} q_2 r_+ + q_3 r_- &= q_1, \\ q_2 r_+^2 + q_3 r_-^2 &= 0. \end{aligned}$$

Solving then gives:

$$q_2 = q_1 b_1, \quad q_3 = q_1 b_2,$$

where

$$b_1 = \frac{r_-}{r_+(r_- - r_+)}, \quad b_2 = \frac{r_+}{r_-(r_+ - r_-)}.$$

Then the suggested solution is given by:

$$\phi(x) = \begin{cases} q_1 \int_0^x H(z)dz, & 0 \leq x \leq G(1), \\ q_1(b_1 e^{r_+(x-G(1))} + b_2 e^{r_-(x-G(1))}), & G(1) \leq x \leq x_1 \ , \\ \phi(x_1) + k(x - x_1), & x \geq x_1. \end{cases} \tag{4.21}$$

The unknown constants q_1 and x_1 are determined in the sequel.

Define

$$U(x) = \begin{cases} H(x), & 0 \leq x \leq G(1), \\ b_1 r_+ e^{r_+(x-G(1))} + b_2 r_- e^{r_-(x-G(1))}, & x \geq G(1). \end{cases} \tag{4.22}$$

Obviously,

$$\phi(x) = q_1 \int_0^x U(s)ds, \ 0 \leq x \leq x_1.$$

It is not difficult to show that $U(x)$ is a convex function.

Let

$$\begin{aligned} I_1(q_1) : &= \int_{\tilde{x}_{q_1}}^{x_{q_1}^*} (k - q_1 U(y))dy, \quad q_1 \in [\bar{q}, k], \\ I_2(q_1) : &= \int_0^{x_{q_1}^*} (k - q_1 U(y))dy, \quad q_1 \in (0, \bar{q}], \end{aligned}$$

where

$$\bar{q} = \frac{k}{H(0)},$$

and $x_{q_1}^*$ and $\tilde{x}_{q_1}$ satisfy:

$$q_1 U(x_{q_1}^*) = q_1 U(\tilde{x}_{q_1}) = k.$$

Similarly to, for example, Meng [17], it can be shown that there exist a $\hat{q}_1$ such that $I_1(\hat{q}_1) = K$ or $I_2(\hat{q}_1) = K$. Thus in (4.21) letting $x_1 = x^*_{\hat{q}_1}$ and $q_1 = \hat{q}_1$ give:

$$\phi'(\tilde{x}_{\hat{q}_1}) = \phi'(x^*_{\hat{q}_1}) = k, \quad \phi(x^*_{\hat{q}_1}) = \phi(\tilde{x}_{\hat{q}_1}) + k(x^*_{\hat{q}_1} - \tilde{x}_{\hat{q}_1}) - K,$$

or

$$\phi'(x^*_{\hat{q}_1}) = k, \quad \phi(x^*_{\hat{q}_1}) = kx^*_{\hat{q}_1} - K.$$

Let

$$\hat{T}(x) = \begin{cases} \hat{q}_1 U(x), & 0 \le x \le x^*_{\hat{q}_1}, \\ k, & x \ge x^*_{\hat{q}_1}, \end{cases}$$

and

$$\varphi(x) = \int_0^x \hat{T}(y)dy. \tag{4.23}$$

Theorem 4.1. *Suppose the function $\varphi(x)$ is defined by (4.23). Then*

1. *$\varphi(x)$ is continuously differentiable for $x \in \Re^+$ and twice continuously differentiable for $x \in \Re^+ \backslash x^*_{\hat{q}_1}$;*
2. *$\varphi(x)$ is a solution of the HJB equation (4.4).*

Proof. The differentiability of $\varphi(x)$ follows directly from its construction in (4.23). It remains to show that $\varphi(x)$ is a solution of the HJB equation (4.4).

Similar to Theorem 5.1 of Cadenillas et al. [6], we can easily show

$$\begin{cases} \mathcal{M}\varphi(x) - \varphi(x) < 0, & 0 \le x \le x^*_{\hat{q}_1}, \\ \mathcal{M}\varphi(x) - \varphi(x) = 0, & x \ge x^*_{\hat{q}_1}. \end{cases}$$

In what follows, we will verify

$$\begin{cases} \max_{c_i \in [0,1], c_0 + \cdots + c_m = 1} [\mathcal{L}^{c_0, c_1, \cdots, c_m} \varphi(x) - \delta\varphi(x)] = 0, & 0 \le x \le x^*_{\hat{q}_1}, \\ \max_{c_i \in [0,1], c_0 + \cdots + c_m = 1} [\mathcal{L}^{c_0, c_1, \cdots, c_m} \varphi(x) - \delta\varphi(x)] < 0, & x \ge x^*_{\hat{q}_1}. \end{cases}$$

(I) <u>For $0 \le x \le G(1)$</u>

Taking the Lagrangian multiplier to be

$$\beta(x) = \frac{2\lambda\sigma^2(1 - G^{-1}(x))H(x)}{\Delta_m}$$

and substituting (4.23) into $\mathcal{L}^{c_0, c_1, \cdots, c_m}\varphi(x) - \delta\varphi(x) + \beta(x)(c_0 + c_1 + \cdots + c_m - 1)$ give:

$$\begin{aligned} &\mathcal{L}^{c_0, c_1, \cdots, c_m}\varphi(x) - \delta\varphi(x) + \beta(x)(c_0 + c_1 + \cdots + c_m - 1) \\ = \ & -\lambda\sigma^2 c_0^2 \hat{q}_1 \frac{(1 - G^{-1}(x))H(x)}{\Delta_m G^{-1}(x)} + \lambda\sigma^2\hat{q}_1 \Bigg[\frac{2(1 - G^{-1}(x))(c_0 + c_1 + \cdots + c_m - 1)}{\Delta_m} \\ & + \theta_0 - \sum_{i=1}^m \theta_i c_i^2 \Bigg] H(x) - \delta\hat{q}_1 \int_0^x H(s)ds \\ = \ & -\lambda\sigma^2\hat{q}_1 \frac{(1 - G^{-1}(x))H(x)}{\Delta_m G^{-1}(x)} (c_0 - \bar{c}_0(x))^2 - \lambda\sigma^2\hat{q}_1\theta_i \sum_{i=1}^m (c_i - \bar{c}_i(x))^2 \\ & + \lambda\sigma^2\hat{q}_1 \frac{(1 - G^{-1}(x))\left(G^{-1}(x) - 2\right) H(x) + \theta_0\Delta_m}{\Delta_m} - \delta\hat{q}_1 \int_0^x H(s)ds, \end{aligned}$$

where $\bar{c}_0(x)$ and $\bar{c}_i(x)$ are given by (4.15) and (4.16), respectively.

Thus for any $c_i \in (-\infty, +\infty)$, $i = 0, 1, \cdots, m$,

$$\begin{aligned}
& \mathcal{L}^{c_0, c_1, \cdots, c_m} \varphi(x) - \delta\varphi(x) + \beta(x)(c_0 + c_1 + \cdots + c_m - 1) \\
\leq\ & \mathcal{L}^{\bar{c}_0(x), \bar{c}_1(x), \cdots, \bar{c}_m(x)} \varphi(x) - \delta\varphi(x) + \beta(x)(\bar{c}_0(x) + \bar{c}_1(x) + \cdots + \bar{c}_m(x) - 1) \\
=\ & \mathcal{L}^{\bar{c}_0(x), \bar{c}_1(x), \cdots, \bar{c}_m(x)} \varphi(x) - \delta\varphi(x).
\end{aligned}$$

This shows

$$\arg \max_{c_i \in [0,1], c_0 + c_1 + \cdots + c_m = 1} \{\mathcal{L}^{c_0, c_1, \cdots, c_m} \varphi(x) - \delta\varphi(x)\} = (\bar{c}_0(x), \bar{c}_1(x), \cdots, \bar{c}_m(x)).$$

In what follows, we will show

$$\mathcal{L}^{\bar{c}_0(x), \bar{c}_1(x), \cdots, \bar{c}_m(x)} \varphi(x) - \delta\varphi(x) = 0.$$

By its construction, $\varphi(x)$ satisfies (4.13), that is,

$$\frac{d[\mathcal{L}^{\bar{c}_0(x), \bar{c}_1(x), \cdots, \bar{c}_m(x)} \varphi(x) - \delta\varphi(x)]}{dx} = 0,$$

This shows $\mathcal{L}^{\bar{c}_0(x), \bar{c}_1(x), \cdots, \bar{c}_m(x)} \varphi(x) - \delta\varphi(x) \equiv C$(constant). In what follows, we will show $C = 0$.

Obviously

$$\mathcal{L}^{\bar{c}_0(x), \bar{c}_1(x), \cdots, \bar{c}_m(x)} \varphi(x) - \delta\varphi(x) = \lambda\sigma^2 \hat{q}_1 \left[\theta_0 - \frac{1 - G^{-1}(x)}{\Delta_m}\right] - \delta\varphi(x) \to 0 \,\text{as}\, x \to 0,$$

which shows that $C = 0$.

(II) <u>For $G(1) \leq x \leq x^*_{\hat{q}_1}$</u>

Substituting (4.23) into $\mathcal{L}^{c_0, c_1, \cdots, c_m} \varphi(x) - \delta\varphi(x)$ gives

$$\begin{aligned}
& \mathcal{L}^{c_0, c_1, \cdots, c_m} \varphi(x) - \delta\varphi(x) \\
=\ & \frac{1}{2} \lambda\sigma^2 c_0^2 \hat{q}_1 (b_1 r_+^2 e^{r_+(x - G(1))} + b_2 r_-^2 e^{r_-(x - G(1))}) \\
& + \lambda\sigma^2 \hat{q}_1 \left[\theta_0 - \sum_{i=1}^{m} \theta_i c_i^2\right] \left(b_1 r_+ e^{r_+(x - G(1))} + b_2 r_- e^{r_-(x - G(1))}\right) \\
& - \delta \hat{q}_1 (b_1 e^{r_+(x - G(1))} + b_2 e^{r_-(x - G(1))}).
\end{aligned}$$

Noting that

$$b_1 r_+^2 e^{r_+(x - G(1))} + b_2 r_-^2 e^{r_-(x - G(1))} > 0,$$

and

$$b_1 r_+ e^{r_+(x - G(1))} + b_2 r_- e^{r_-(x - G(1))} > 0,$$

we have

$$\arg \max_{c_i \in [0,1], c_0 + c_1 + \cdots + c_m = 1} \{\mathcal{L}^{c_0, c_1, \cdots, c_m} \varphi(x) - \delta\varphi(x)\} = (1, 0, \cdots, 0, 0),$$

that is

$$\max_{c_i\in[0,1],c_0+c_1+\cdots+c_m=1}\{\mathcal{L}^{c_0,c_1,\cdots,c_m}\varphi(x)-\delta\varphi(x)\}=\mathcal{L}^{1,0,\cdots,0,0}\varphi(x)-\delta\varphi(x)=0, G(1)\le x\le x^*_{\hat{q}_1}.$$

(III) <u>For $x \ge x^*_{\hat{q}_1}$</u>

Since $\varphi''(x^*_{\hat{q}_1}-)\ge 0$, then

$$\begin{aligned}
& \lambda\sigma^2\left[\theta_0-\sum_{i=1}^m\theta_i c_i^2\right]k-\delta\varphi(x^*_{\hat{q}_1})\\
\le\; & \lambda\sigma^2\left[\theta_0-\sum_{i=1}^m\theta_i c_i^2\right]k-\delta\varphi(x^*_{\hat{q}_1})+\frac{1}{2}\lambda\sigma^2c_0^2\varphi''(x^*_{\hat{q}_1}-)\\
\le\; & 0.
\end{aligned}$$

Consequently, for $x > x^*_{\hat{q}_1}$,

$$\begin{aligned}
& \frac{1}{2}\lambda\sigma^2c_0^2\varphi''(x)+\lambda\sigma^2\left[\theta_0-\sum_{i=1}^m\theta_i c_i^2\right]\varphi'(x)-\delta\varphi(x)\\
=\; & \lambda\sigma^2\left[\theta_0-\sum_{i=1}^m\theta_i c_i^2\right]k-\delta\varphi(x)\\
<\; & \lambda\sigma^2\left[\theta_0-\sum_{i=1}^m\theta_i c_i^2\right]k-\delta\varphi(x^*_{\hat{q}_1})\le 0,
\end{aligned}$$

that is,

$$\max_{c_i\in[0,1],c_0+c_1+\cdots+c_m=1}\{\mathcal{L}^{c_0,c_1,\cdots,c_m}\varphi(x)-\delta\varphi(x)\}<0, x\ge x^*_{\hat{q}_1}. \tag{4.24}$$

This completes the proof. □

Theorem 4.2. *Let*

$$\begin{aligned}
c_0^*(x) &= \begin{cases} G^{-1}(x), & 0\le x\le G(1),\\ 1, & x\ge G(1),\end{cases}\\
c_j^*(x) &= \begin{cases} \frac{1-G^{-1}(x)}{\theta_j\Delta_m}, & 0\le x\le G(1),\\ 0, & x\ge G(1),\end{cases}
\end{aligned}$$

where $j=1,\cdots,m$.

Then the value function $V(x)$ is $V(x)=\varphi(x)$, (i.e., $\varphi(x)$ is defined by (4.23)), and the optimal strategy

$$\pi^*=(c^*_{0t},c^*_{1t},\cdots,c^*_{mt},\tau^*_1,\tau^*_2,\cdots,\tau^*_n,\cdots;\xi^*_1,\xi^*_2,\cdots,\xi^*_n,\cdots)\in\mathcal{O}$$

is given as follows:

(1) If $I_1(\bar{q})>K$, then

(1-1) for $0<x<x^*_{\hat{q}_1}$, $c^*_{jt}=c^*_j(R^{\pi^*}(t-)),\tau^*_0:=0,\tau^*_i:=\inf\{t>\tau^*_{i-1}:R^{\pi^*}(t-)=x^*_{\hat{q}_1}\}$, $\xi^{\pi^*}_i:=x^*_{\hat{q}_1}-\tilde{x}_{\hat{q}_1}$, $j=0,2,\cdots,m-1$ and $i=1,2,\cdots$.

(1-2) *for* $x \geq x^*_{\hat{q}_1}$, $c^*_{jt} = c^*_j(R^{\pi^*}(t-)), \tau^*_1 := 0, \tau^*_i := \inf\{t > \tau^*_{i-1} : R^{\pi^*}(t-) = x^*_{\hat{q}_1}\}$, $\xi^{\pi^*}_1 := x - \tilde{x}_{\hat{q}_1}, \xi^{\pi^*}_i := x^*_{\hat{q}_1} - \tilde{x}_{\hat{q}_1}, j = 0, 2, \cdots, m-1$ *and* $i = 2, \cdots$.

(2) *If* $I_1(\bar{q}) \leq K$, *then*

(2-1) *for* $0 < x < x^*_{\hat{q}_1}$, $c^*_{jt} = c^*_j(R^{\pi^*}(t-)), \tau^*_1 := \inf\{t > 0 : R^{\pi^*}(t-) = x^*_{\hat{q}_1}\}$, $\xi^{\pi^*}_1 := x^*_{\hat{q}_1}$, $\tau^*_i = +\infty, i \geq 2$, $j = 0, 2, \cdots, m-1$.

(2-2) *for* $x \geq x^*_{\hat{q}_1}$, $c^*_{jt} = c^*_j(R^{\pi^*}(t-)), \tau^*_1 := 0$, $\xi^{\pi^*}_1 := x$, $\tau^*_i = +\infty, i \geq 2$, $j = 0, 2, \cdots, m-1$.

Proof. The proof is standard, so we omit it. □

5 Conclusion

A combined optimal reinsurance and dividend problem of an insurer was considered, where multiple reinsurers participate in a reinsurance treaty. These reinsurers adopted the variance premium principle with different parameters. The insurer paid dividends to its shareholders, and each dividend incurred a fixed amount of transaction costs and taxes. Using a pure diffusion approximation to the surplus process, under certain assumptions, the combined proportional reinsurance treaty was shown to be optimal among the class of plausible reinsurance treaties. Furthermore using the HJB dynamic programming approach, explicit characterizations for the value function and optimal reinsurance and impulse dividend strategies were obtained. However, if the condition (I5) is not satisfied, what is an optimal form of optimal reinsurance strategy? This may represent an interesting topic for further research.

Acknowledgements We would like to thank Jingmin He for help discussion on this paper. H. Meng is supported by the Nature Science Foundation of China (11271385), the MOE Project of Key Research Institute of Humanities and Social Science in universities (14JJD790001) and the Program for Innovation Research in Central University of Finance and Economics. H. Yang's research was supported by Research Grants Council of the Hong Kong Special Administrative Region (project No. HKU 705313P).

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