DE GRUYTER DOI 10.1515/gmj-2014-0039 — Georgian Math. J. 2014; 21 (4):415–419

Research Article

Zhao Changjian and Wing Sum Cheung

On improvements of Opial-type inequalities

Abstract: In the present paper, we establish some new Opial-type integral inequalities in two variables. The results in special cases yield some of the interrelated results on Godunova–Levin's and Mitrinović–Pečarić's inequalities. These results provide new estimates on inequalities of this type.

Keywords: Opial's inequality, Hölder's inequality, Jensen's inequality

MSC 2010: 26D15

Zhao Changjian: Department of Mathematics, China Jiliang University, Hangzhou 310018, P. R. China, e-mail: chjzhao@163.com
Wing Sum Cheung: Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong,
e-mail: wscheung@hkucc.hku.hk

1 Introduction

In 1960, Opial [15] established the following inequality:

Theorem 1.1. *Suppose* $f \in C^1[0, h]$ *satisfies* $f(0) = f(h) = 0$ *and* $f(x) > 0$ *for all* $x \in (0, h)$. *Then*

$$\int_0^h |f(x)f'(x)|dx \le \frac{h}{4}\int_0^h (f'(x))^2 dx. \tag{1.1}$$

The Opial-type inequality was first established by Willett [16]:

Theorem 1.2. *Let* $x(t)$ *be absolutely continuous on* $[0, a]$, *and* $x(0) = 0$. *Then*

$$\int_0^a |x(t)x'(t)|dt \le \frac{a}{2}\int_0^a |x'(t)|^2 dt. \tag{1.2}$$

A non-trivial generalization of Theorem 1.2 was established by Hua [12]:

Theorem 1.3. *Let* $x(t)$ *be absolutely continuous in* $[0, a]$, *and* $x(0) = 0$. *Further, let* l *be a positive integer. Then*

$$\int_0^a |x(t)x'(t)|dt \le \frac{a^l}{l+1}\int_0^a |x'(t)|^{l+1} dt. \tag{1.3}$$

A sharper inequality was established by Godunova [9]:

Theorem 1.4. *Let* $f(t)$ *be a convex and increasing function on* $[0, \infty)$ *with* $f(0) = 0$. *Further, let* $x(t)$ *be absolutely continuous on* $[0, \tau]$, *and* $x(\alpha) = 0$. *Then, the following inequality holds:*

$$\int_\alpha^\tau f'(|x(t)|)|x'(t)|dt \le f\left(\int_\alpha^\tau |x'(t)|dt\right). \tag{1.4}$$

Opial's inequality and its generalizations, extensions and discretizations play an important role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations, see for example [1, 4–8, 10, 13, 17]. For Opial-type integral inequalities involving high-order partial derivatives, see [3, 18]. For an extensive survey on these inequalities, see [2]. Mitrinović and Pečarić [14] proved some new extensions of Opial-type inequalities. The aim of the present paper is to establish some Opial-type inequalities, which are some extensions of Godunova–Levin's and Mitrnović–Pečarić inequalities.

2 Statement of the results

We shall extend some of the previous results for the functions which have an integral representation. For this, we say that a function $x(s,t)$ belongs to the class $U(y,K)$ if it can be represented in the form

$$x(s,t)=\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}K(s,t,\sigma,\tau)y(\sigma,\tau)d\sigma d\tau,\quad (s,t)\in[\alpha_1,\beta_1]\times[\alpha_2,\beta_2], \tag{2.1}$$

where $y(s,t)$ is a continuous function on $[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$, and $K(s,t,\sigma,\tau)$ is an arbitrary non-negative kernel function defined on $[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\times[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$ such that $x(s,t)>0$ if $y(s,t)>0$, $(s,t)\in[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$.

In particular, for $\lambda>0$, we let

$$K(s,t,\sigma,\tau)=K_\lambda(t,s,\sigma,\tau)=\begin{cases}\frac{[(s-\alpha)+(t-\beta)]^{\lambda-1}}{\Gamma(\lambda)}, & s+t\geq\sigma+\tau,\\ 0, & s+t<\sigma+\tau.\end{cases}$$

Theorem 2.1. *For $i=1,2,3$, let $x_i(s,t)\in U(y_i,K)$, where $y_2(s,t)>0$ for all $(s,t)\in[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$, let $p(s,t)>0$ for all $(s,t)\in[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$, and let $f(x,y)$ be convex and increasing on $[0,\infty)\times[0,\infty)$. Then the following inequality holds:*

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}p(s,t)f\Big(\Big|\frac{x_1(s,t)}{x_2(s,t)}\Big|,\Big|\frac{x_3(s,t)}{x_2(s,t)}\Big|\Big)dsdt\leq\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\phi(s,t)f\Big(\Big|\frac{y_1(s,t)}{y_2(s,t)}\Big|,\Big|\frac{y_3(s,t)}{y_2(s,t)}\Big|\Big)dsdt, \tag{2.2}$$

where

$$\phi(t,s)=y_2(s,t)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\frac{p(\sigma,\tau)K(s,t,\sigma,\tau)}{x_2(\sigma,\tau)}d\sigma d\tau.$$

Remark 2.2. Let $x_i(s,t)$, $y_i(s,t)$ and $p(s,t)$ be reduced to $x_i(t)$, $y_i(t)$, $p(t)$, respectively, where $t\in(\alpha,\tau)$ and $i=1,2,3$, with suitable modifications in Theorem 2.1. Then (2.2) becomes the following inequality established by Mitrinović and Pečarić [14]:

$$\int_{\alpha}^{\tau}p(t)f\Big(\Big|\frac{x_1(t)}{x_2(t)}\Big|,\Big|\frac{x_3(t)}{x_2(t)}\Big|\Big)dt\leq\int_{\alpha}^{\tau}\phi(t)f\Big(\Big|\frac{y_1(t)}{y_2(t)}\Big|,\Big|\frac{y_3(t)}{y_2(t)}\Big|\Big)dt, \tag{2.3}$$

where

$$\phi(t)=y_2(t)\int_{\alpha}^{\tau}\frac{p(s)K(t,s)}{x_2(s)}ds,$$

$y(t)$ is a continuous function on $[\alpha,\tau]$ and $K(t,s)$ is an arbitrary non-negative kernel defined on $[\alpha,\tau]\times[\alpha,\tau]$ such that $x(t)>0$ if $y(t)>0$, $t\in[\alpha,\tau]$.

Remark 2.3. Taking $K(s,t,\sigma,\tau)=K_\lambda(s,t,\sigma,\tau)$ in Theorem 2.1, (2.2) reduces to

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}p(s,t)f\Big(\Big|\frac{x_1(s,t)}{x_2(s,t)}\Big|,\Big|\frac{x_3(s,t)}{x_2(s,t)}\Big|\Big)dsdt\leq\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\phi(s,t)f\Big(\Big|\frac{y_1(s,t)}{y_2(s,t)}\Big|,\Big|\frac{y_3(s,t)}{y_2(s,t)}\Big|\Big)dsdt, \tag{2.4}$$

where

$$\phi(s,t)=y_2(s,t)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\frac{[(s-\sigma)+(t-\tau)]^{\lambda-1}}{\Gamma(\lambda)}\frac{p(\sigma,\tau)}{x_2(\sigma,\tau)}d\sigma d\tau,\quad s+t\geq\sigma+\tau,$$

and $\phi(s,t)=0$ if $s+t<\sigma+\tau$.

Let us change $K_\lambda(s,t,\sigma,\tau)$ to $K_\lambda(t,s)$. Namely, for $\lambda>0$,

$$K_\lambda(t,s)=\begin{cases}\frac{(t-s)^{\lambda-1}}{\Gamma(\lambda)}, & s\leq t,\\ 0, & s>t.\end{cases}$$

Further, let $x_i(s,t)$, $y_i(s,t)$, $f(s,t)$ and $p(s,t)$ be reduced to $s_i(t)$, $y_i(t)$, $f(t)$ and $p(t)$, respectively, where $t \in (\alpha, \tau)$ and $i = 1, 2$. Taking $K(t,s) = K_\lambda(t,s)$ in (2.4), we reduce (2.4) to the result of Godunova and Levin [11].

Now, let $x(s,t) \in U(y,K)$, where $K(s,t,\sigma,\tau) = 0$ for $\sigma + \tau > s + t$. We shall say that such functions belong to the class $U_1(y,K)$. It is clear that in this case, (2.1) reduces to

$$x(s,t) = \int_{\alpha_1}^{s}\int_{\alpha_2}^{t} K(s,t,\sigma,\tau)y(\sigma,\tau)d\sigma d\tau. \tag{2.5}$$

Theorem 2.4. *Let the function $f(x)$ be differentiable on $[0,\infty)$ such that, for $v > 1$, the function $f(x^{1/v})$ is convex and $f(0) = 0$. Let $\frac{1}{\mu} + \frac{1}{v} = 1$, and let $x(s,t) \in U_1(y,K)$, where*

$$\left(\int_{\alpha_1}^{s}\int_{\alpha_2}^{t} (K(s,t,\sigma,\tau))^{\mu} d\sigma d\tau\right)^{1/\mu} \le M.$$

Then

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} |x(s,t)|^{2(1-v)} \frac{\partial^2 f}{\partial s \partial t}(|x(s,t)|)|y(s,t)|^v dsdt \le \frac{v^2}{M^{2v}} f\left(M\left(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} |y(s,t)|^v dsdt\right)^{1/v}\right). \tag{2.6}$$

Remark 2.5. Let $x(s,t)$ and $y(s,t)$ be reduced to $x(t)$, $y(t)$, respectively, where $t \in (\alpha, \tau)$, with suitable modifications in Theorem 2.4. Then (2.6) becomes the following inequality:

$$\int_{\alpha}^{\tau} |x(t)|^{1-v} f'(|x(t)|)|y(t)|^v dsdt \le \frac{v}{M^v} f\left(M\left(\int_{\alpha}^{\tau} |y(t)|^v dt\right)^{1/v}\right).$$

This is just a new result established by Mitrinović and Pečarić [14].

3 Proofs of the results

Proof of Theorem 2.1. From the hypotheses of Theorem 2.1, it turns out that

$$\begin{aligned}
&\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} p(s,t) f\left(\left|\frac{x_1(s,t)}{x_2(s,t)}\right|, \left|\frac{x_3(s,t)}{x_2(s,t)}\right|\right) dsdt \\
&= \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} p(s,t) f\left(\left|\frac{\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} K(s,t,\sigma,\tau)y_2(\sigma,\tau)\frac{y_1(\sigma,\tau)}{y_2(\sigma,\tau)}d\sigma d\tau}{x_2(s,t)}\right|, \left|\frac{\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} K(s,t,\sigma,\tau)y_2(\sigma,\tau)\frac{y_3(\sigma,\tau)}{y_2(\sigma,\tau)}d\sigma d\tau}{x_2(s,t)}\right|\right) dsdt \\
&\le \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} p(s,t) f\left(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \frac{K(s,t,\sigma,\tau)y_2(\sigma,\tau)}{x_2(s,t)}\left|\frac{y_1(\sigma,\tau)}{y_2(\sigma,\tau)}\right| d\sigma d\tau, \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \frac{K(s,t,\sigma,\tau)y_2(\sigma,\tau)}{x_2(s,t)}\left|\frac{y_3(\sigma,\tau)}{y_2(\sigma,\tau)}\right| d\sigma d\tau\right) dsdt.
\end{aligned}$$

By using Jensen's integral inequality, we have

$$\begin{aligned}
\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} p(s,t) f\left(\left|\frac{x_1(s,t)}{x_2(s,t)}\right|, \left|\frac{x_3(s,t)}{x_2(s,t)}\right|\right) dsdt &\le \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} p(s,t)\left(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \frac{K(s,t,\sigma,\tau)y_2(\sigma,\tau)}{x_2(s,t)} f\left(\left|\frac{y_1(\sigma,\tau)}{y_2(\sigma,\tau)}\right|, \left|\frac{y_3(\sigma,\tau)}{y_2(\sigma,\tau)}\right|\right) d\sigma d\tau\right) dsdt \\
&= \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} f\left(\left|\frac{y_1(\sigma,\tau)}{y_2(\sigma,\tau)}\right|, \left|\frac{y_3(\sigma,\tau)}{y_2(\sigma,\tau)}\right|\right) y_2(\sigma,\tau)\left(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \frac{p(s,t)K(s,t,\sigma,\tau)}{x_2(s,t)} dsdt\right) d\sigma d\tau \\
&= \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \phi(\sigma,\tau) f\left(\left|\frac{y_1(\sigma,\tau)}{y_2(\sigma,\tau)}\right|, \left|\frac{y_3(\sigma,\tau)}{y_2(\sigma,\tau)}\right|\right) d\sigma d\tau,
\end{aligned}$$

where

$$\phi(\sigma,\tau) = y_2(\sigma,\tau)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\frac{p(s,t)K(s,t,\sigma,\tau)}{x_2(s,t)}dsdt.$$

Hence

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} p(s,t)f\left(\left|\frac{x_1(s,t)}{x_2(s,t)}\right|,\left|\frac{x_3(s,t)}{x_2(s,t)}\right|\right)dsdt \le \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\phi(s,t)f\left(\left|\frac{y_1(s,t)}{y_2(s,t)}\right|,\left|\frac{y_3(s,t)}{y_2(s,t)}\right|\right)dsdt,$$

where

$$\phi(s,t) = y_2(s,t)\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\frac{p(\sigma,\tau)K(s,t,\sigma,\tau)}{x_2(\sigma,\tau)}d\sigma d\tau.$$

This completes the proof. ☐

Proof of Theorem 2.4. From the hypotheses of Theorem 2.4 and in view of Hölder's inequality, we obtain

$$\begin{aligned}|x(s,t)| &\le \int_{\alpha_1}^{s}\int_{\alpha_2}^{t} K(s,t,\sigma,\tau)|y(\sigma,\tau)|d\sigma d\tau\\ &\le \left(\int_{\alpha_1}^{s}\int_{\alpha_2}^{t}(K(s,t,\sigma,\tau))^{\mu}d\sigma d\tau\right)^{1/\mu}\left(\int_{\alpha_1}^{s}\int_{\alpha_2}^{t}|y(\sigma,\tau)|^{\upsilon}d\sigma d\tau\right)^{1/\upsilon}\\ &\le M\left(\int_{\alpha_1}^{s}\int_{\alpha_2}^{t}|y(\sigma,\tau)|^{\upsilon}d\sigma d\tau\right)^{1/\upsilon}.\end{aligned}$$

Now, let

$$z(s,t) = \int_{\alpha_1}^{s}\int_{\alpha_2}^{t}|y(\sigma,\tau)|^{\upsilon}d\sigma d\tau.$$

Hence

$$\frac{\partial^2 z(s,t)}{\partial s\partial t} = |y(s,t)|^{\upsilon}.$$

Moreover, it is easy to see that

$$|x(s,t)| \le M(z(s,t))^{1/\upsilon}.$$

Therefore

$$\begin{aligned}&\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}|x(s,t)|^{2(1-\upsilon)}\frac{\partial^2 f}{\partial s\partial t}(|x(s,t)|)|y(s,t)|^{\upsilon}dsdt\\ &\quad\le \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}M^{2(1-\upsilon)}(z(s,t))^{2(1/\upsilon-1)}\frac{\partial^2 f}{\partial s\partial t}(Mz(s,t)^{1/\upsilon})\frac{\partial z(s,t)}{\partial s}\frac{\partial z(s,t)}{\partial t}dsdt\\ &\quad= \frac{\upsilon^2}{M^{2\upsilon}}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\frac{\partial^2 f}{\partial s\partial t}(Mz(s,t)^{1/\upsilon})\cdot\frac{M}{\upsilon}(z(s,t))^{1/\upsilon-1}\frac{\partial z(s,t)}{\partial s}\cdot\frac{M}{\upsilon}(z(s,t))^{1/\upsilon-1}\frac{\partial z(s,t)}{\partial t}dsdt\\ &\quad= \frac{\upsilon^2}{M^{2\upsilon}}\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\frac{\partial^2 f}{\partial s\partial t}(Mz(s,t)^{1/\upsilon})d(M(z(s,t))^{1/\upsilon})_s d(M(z(s,t))^{1/\upsilon})_t\\ &\quad= \frac{\upsilon^2}{M^{2\upsilon}}f(M(z(\beta_1,\beta_2))^{1/\upsilon})\\ &\quad= \frac{\upsilon^2}{M^{2\upsilon}}f\left(M\left(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}|y(\sigma,\tau)|^{\upsilon}d\sigma d\tau\right)^{1/\upsilon}\right).\end{aligned}$$

This completes the proof. ☐

Funding: The research of Zhao Changjian was supported by the National Natural Science Foundation of China (11371334), and the research of Wing Sum Cheung was partially supported by the HKU URC grant.

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Received September 29, 2012; accepted March 13, 2014.