Entanglement entropy fluctuation and distribution for open systems

Jiangtao Yuan,[1] Yanxia Xing,[2,1,*] Lei Zhang,[3,1] and Jian Wang[1,†]
[1]*Department of Physics and the Center of Theoretical and Computational Physics, The University of Hong Kong, Hong Kong, China*
[2]*Department of Physics, Beijing Institute of Technology, Beijing 100081, China*
[3]*State Key Laboratory of Quantum Optics and Quantum Optics Devices, Institute of Laser Spectroscopy and Collaborative Innovation Center of Extreme Optics, Shanxi University, Taiyuan 030006, China*

(Received 14 September 2016; revised manuscript received 27 February 2017; published 4 April 2017)

The entanglement entropy generated by quantum transport, similar to any physical observable quantity, is a stochastic variable that has its distribution and can fluctuate. The fundamental question is how to define the entanglement entropy operator which allows one to discuss entanglement entropy fluctuation. By introducing the entanglement entropy operator, we develop a theoretical framework to calculate the entanglement entropy fluctuation as well as its higher order cumulants generated by electronic transport in open systems. The distribution of entanglement entropy generated by opening or closing a quantum point contact (QPC) is solved exactly. When the transmission coefficient of QPC is one-half, the entanglement entropy is maximized and fluctuationless. We also establish a general relation between the generated entanglement entropy fluctuation and charge fluctuation. We apply our theory to electronic transport through a quantum dot and study the generated entanglement entropy in the transient regime. Universal behavior is found for the cumulants of entanglement entropy at short times.

DOI: 10.1103/PhysRevB.95.155402

I. INTRODUCTION

Quantum entanglement is the fundamental feature in quantum mechanics. Due to the quantum entanglement, a local measurement may influence instantaneously the results of local measurements far away. Quantum entanglement is considered as the main resource to process quantum information [1]. In addition, entanglement is also very important for quantum cryptography and quantum computation. Quantum entanglement entropy is a measure of quantum entanglement [1]. In the context of condensed-matter physics, quantum entanglement entropy can be used to probe the topological phases and to characterize critical behavior in disordered quantum systems [2–11].

It has been shown that the entanglement at a critical point of one-dimensional (1D) quantum phase transition is related to the central charge of the associated conformal field theory making a formal connection between concepts of quantum information and condensed-matter physics [5–7]. In the presence of disorder, the average entanglement entropy was shown to play the role of the central charge [3]. Connection between entanglement entropy and topological order has been established in two dimensions which can be useful in constructing an explicit microscopic model to realize topological orders [8,9].

Recently, evidence shows that entanglement entropy alone is not enough to characterize the entanglement of quantum systems. For instance for a bipartite system it is found that the entanglement spectrum reveals much more information than the entanglement entropy for topological properties of many-body quantum states [2]. In addition, total quantum dimension of topological order can be measured by topological entropy which is a linear combination of entanglement entropy of different regions of the system [8,9]. Moreover, the entanglement spectrum (the entanglement Renyi entropies) has been studied extensively [12–18].

It is instructive to examine the analogy between the entanglement entropy and the current in quantum transport. It is known that besides the current additional information can be obtained from the shot noise and in general the *current operator* is characterized by full counting statistics (FCS) of charge transport [19]. Likewise, the entanglement entropy is a stochastic quantity that has its average and can fluctuate. The entanglement of quantum systems is fully characterized by the distribution of entanglement entropy.

The fundamental question that arises is how to describe entanglement entropy fluctuation and its distribution. In comparison with FCS, a direct way is to introduce the entanglement entropy operator.

So far most of the works on entanglement entropy focus on characterization and classification of the quantum correlations of closed many-body systems [2,3,10,11]. Less attention has been paid to the entanglement entropy of open systems where quantum noises play a dominant role [15,17,19], not to mention the entanglement entropy fluctuation. In this paper, we introduce the operator for entanglement entropy and formulate a theoretical framework of entanglement entropy fluctuation generated by quantum transport for open systems. In particular, the generating function of entanglement entropy due to the quantum transport is obtained. A general relation between FCS of charge transfer and the generated entanglement entropy fluctuation is established, suggesting an indirect way to measure entanglement entropy fluctuation. We apply our theory to a quantum point contact (QPC). In the presence of bias, eigenvalues of entanglement entropy of a QPC are obtained analytically in the long-time limit and the distribution of entanglement entropy is found to obey a modified binomial distribution. When the transmission coefficient of a QPC is one-half, all eigenvalues of the entanglement entropy operator become degenerate and the entanglement entropy is fluctuationless. For a quantum dot (QD) system in the transient regime, a universal feature has been identified for the amplitude of higher-order cumulants of entanglement entropy at short times.

*xingyx@bit.edu.cn
†jianwang@hku.hk

II. THEORETICAL FORMULATION

A. Entanglement entropy operator

We start by introducing the entanglement entropy operator. In general, the entropy operator is defined as [20]

$$\hat{S} = -k_B \ln \hat{\rho} \tag{1}$$

from which the entropy is obtained, $S = -k_B \mathrm{Tr}[\hat{\rho} \ln \hat{\rho}]$. For a bipartite system consisting of subsystems A and B, the entanglement entropy of subsystem A is given by $S_A(t) = -\mathrm{Tr}_A[\hat{\rho}_A(t) \ln \hat{\rho}_A(t)] = -\mathrm{Tr}[\hat{\rho}(t) \ln \hat{\rho}_A(t)] = -\mathrm{Tr}[\hat{\rho}(0) U^\dagger(t) \ln \hat{\rho}_A(t) U(t)]$, suggesting that the entanglement entropy operator of subsystem A is defined as

$$\hat{S}_A(t) = -U^\dagger(t) \ln \hat{\rho}_A(t) U(t), \tag{2}$$

where $U(t)$ is the evolution operator for the bipartite system, $\hat{\rho}_A(t) = \mathrm{Tr}_B[\hat{\rho}(t)]$ is the reduced density matrix of subsystem A, and $\hat{\rho}(t) = U(t)\hat{\rho}(0)U^\dagger(t)$ is the full density matrix of the system and the partial trace Tr_B is over subsystem B. The nth moment of entanglement entropy is defined as

$$\langle S_A^n \rangle = \mathrm{Tr}\{\hat{\rho}(0)[\hat{S}_A(t)]^n\} = \mathrm{Tr}_A\{\hat{\rho}_A(t)[-\ln \hat{\rho}_A(t)]^n\}. \tag{3}$$

B. Generating function of entanglement entropy

The nth cumulants of entanglement entropy can be calculated from the generating function defined as

$$Z(\lambda) = \langle e^{-i\lambda \hat{S}_A(t)} \rangle = \mathrm{Tr}_A[\hat{\rho}_A(t) e^{i\lambda \ln \hat{\rho}_A(t)}], \tag{4}$$

which is related to the distribution of entanglement entropy $p(S_A)$ via Fourier transformation

$$Z(\lambda) = \int dS_A\, p(S_A) e^{-i\lambda S_A}. \tag{5}$$

Considering an open system such as a QPC where the current can be switched on and off, which naturally defines a bipartite system with subsystems $A = L$ and $B = R$, the generating function of generated entanglement entropy of the QPC can be calculated as follows. The QPC model consists of two parts: the left lead with chemical potential μ_L and the right lead with chemical potential μ_R. The temperature of the system T is assumed to be finite.

When $t < 0$, we assume that these two leads are disconnected and the Hamiltonian $\hat{H}_0$ of the system is of the form

$$\hat{H}_0 = \sum_k [(\varepsilon_{kL} - \mu_L)\hat{c}_{kL}^\dagger \hat{c}_{kL} + (\varepsilon_{kR} - \mu_R)\hat{c}_{kR}^\dagger \hat{c}_{kR}]. \tag{6}$$

At $t = 0$, a hopping interaction $\hat{H}_T$ is suddenly added to the system connecting the two leads, therefore the Hamiltonian $\hat{H}$ is $\hat{H} = \hat{H}_0 + \hat{H}_T$. The hopping interaction $\hat{H}_T$ is given by

$$\hat{H}_T = \sum_{kk'} t_{kL,k'R} \hat{c}_{kL}^\dagger \hat{c}_{k'R} + \mathrm{H.c.} \tag{7}$$

Expanding Eq. (4) in terms of Fermionic coherent states and using Grassmann algebra to carry out the calculation, we find [21]

$$Z(\lambda) = \frac{\mathrm{Det}[Y^{i\lambda+1} + I]}{\mathrm{Det}[(Y + I)^{i\lambda+1}]}, \tag{8}$$

where $Y = M(1-M)^{-1}$ and $M = -iG_L^<(t,t)$ is the correlation matrix of the left lead with $G_L^<(t,t)$ the lesser Green's function of the left lead. The matrix element of M is defined as $M_{kk'} = \mathrm{Tr}_L[\hat{\rho}_L(t)\hat{c}_{k'L}^\dagger \hat{c}_{kL}]$. The correlation matrix M is a $N \times N$ matrix where N is the number of degrees of freedom of the left lead.

The detailed derivation of Eq. (8) is given in Appendix A. Equation (8) is the central result of this paper. The cumulant of the entanglement entropy can be obtained by taking derivative of cumulant generating function $\ln Z(\lambda)$ with respect to $-i\lambda$, from which we find the entanglement entropy [Eq. (3) when $A = L$]

$$\langle \hat{S}_L \rangle = -\mathrm{Tr}\{M \ln M + (1-M)\ln(1-M)\},$$

which agrees with the result of Ref. [19]. The entanglement entropy fluctuation is (see Appendix A)

$$\langle\langle \hat{S}_L^2 \rangle\rangle \equiv \langle (\Delta \hat{S}_L)^2 \rangle = \mathrm{Tr}\left\{(1-M)M\left[\ln\left(\frac{M}{1-M}\right)\right]^2\right\}, \tag{9}$$

where $\Delta \hat{S}_L = \hat{S}_L - \langle \hat{S}_L \rangle$. Higher-order cumulants of entanglement entropy can also be obtained. The generating function of QD systems is also derived (see Appendix B).

C. Entanglement entropy fluctuation and FCS

Due to the nonlocality nature of entanglement, it is very difficult to measure entanglement entropy experimentally. However, entanglement entropy could be measured indirectly through quantum noises. It has been shown that the entanglement entropy can be expanded in terms of quantum noises, i.e., cumulants of transferred charge [19]. To find the relation between FCS of charge transfer and all higher-order cumulants of entanglement entropy, we derive a relation between $Z(\lambda)$ and $\chi(\lambda)$ for a QPC system, with $\chi(\lambda)$ the generating function of FCS. For the QPC, the generating function of FCS is given by [19]

$$\chi(\lambda) = \mathrm{Det}[(1 - M + Me^{i\lambda})e^{-i\lambda Q}], \tag{10}$$

where $Q = UnP_LU^\dagger$. Here P_L is the projection operator on the left lead, n is the Fermi Dirac distribution, and U is the single-particle evolution operator. Using the spectral density $\mu(z)$ of M defined as $\mathrm{Tr}\delta(z - M)$ [19], the generating function of the entanglement entropy can be expressed as (see Appendix C)

$$\ln Z(\lambda) = \frac{2a}{\pi}\int du\left[\frac{1}{e^{2au}+1} - \frac{1}{e^{2u}+1}\right]\mathrm{Im}[\ln \chi(2iu - \pi)], \tag{11}$$

where $a = i\lambda + 1$. The relation between average entanglement entropy and FCS has been established in Ref. [19] and refined in Ref. [15]. Using Eq. (11), we express the entanglement entropy fluctuations $\langle\langle \hat{S}_L^j \rangle\rangle$ $(j > 1)$ in terms of cumulants of charge transfer. Denoting the cumulants of charge transfer as C_m so that $\ln \chi(\lambda) = \sum_{m=0}^\infty (i\lambda)^m C_m/m!$, we find (see Appendix D)

$$\ln Z(\lambda) = \sum_{m=0}^\infty \beta_m C_m/m!, \tag{12}$$

where

$$\beta_m = \frac{2a}{\pi}\int du \frac{(-2)^m}{e^{2au}+1}\mathrm{Im}\left[\left(u+i\frac{\pi}{2}\right)^m\right] \tag{13}$$

after some algebra, we have

$$\beta_{2r} = \sum_{q=0}^{r}\frac{C_{2r+1}^{2q}(-1)^{r-q+1}\pi^{2r}2^{2q}}{(2r+1)(i\lambda+1)^{2r}}(i\lambda)^{2r+1-2q}|B_{2q}| \tag{14}$$

and $\beta_{2r+1}=0$, where B_q is Bernoulli number and r and q are integers. It has been pointed out that the direct expansion of entanglement entropy in terms of ordinary cumulants of charge transfer does not converge [15]. A resummation technique using factorial cumulants [22,23] can overcome this problem [15] and the details of resummation of expansion of entanglement entropy fluctuation are given in Appendix E. For the entanglement entropy fluctuation, we find (see detailed derivation in Appendix E)

$$\langle\langle \hat{S}^2\rangle\rangle = \lim_{K\to\infty}\sum_{n=1}^{K+1}\alpha'_{2n}(K)C_{2n}, \tag{15}$$

where

$$\alpha'_{2n}(K) = \sum_{k=k_1}^{K}\sum_{k'=k'_1}^{K}\left\{\frac{-1}{kk'}[u(k+k')-u(k+k'+1)] + \sum_{r=0}^{r_1}C_{k+k'}^{r}\frac{(-1)^{1+k+k'-r}}{kk'}[u(r_k)-u(r_k+1)] + \sum_{r=0}^{r_2}(-2)C_k^r\frac{(-1)^{1+k-r}}{kk'}[u(r_k)-u(r_k+1)]\right\}, \tag{16}$$

where $u(m)=S_1(m+1,2n)/m!$ and $r_k=k+k'-r$, $k_1=\max(2n-K-2,1)$, $k'_1=\max(2n-k-2,1)$, $r_1=\min(k+k'-2n+2,k+k')$, $r_2=\min(k+k'-2n+2,k)$, and $S_1(n,k)$ is the unsigned Stirling number of the first kind. A convergence test for a QPC with a dc bias will be discussed below [see Fig. 3(b)].

III. RESULTS AND DISCUSSIONS

In this section, we use several examples to illustrate the application of our theory.

A. Distribution of entanglement entropy in a QPC

First of all, we will calculate the distribution function of entanglement entropy generated by opening or closing a QPC. In equilibrium the generating function of charge transfer is written as [17,24]

$$\chi(\lambda) = e^{-\lambda_*^2 G/4\pi^2}, \quad \sin\frac{\lambda_*}{2} = \sqrt{T}\sin\frac{\lambda}{2}, \tag{17}$$

where $G=2\ln\{[h\beta/(\pi\delta)]\sinh[\pi t/(h\beta)]\}$, t is the time duration when the QPC is open and δ is a short-time cutoff, β is the inverse temperature, T is the transmission coefficient of the QPC, λ is the counting field. When $T=1$, the FCS is Gaussian. In this case, the cumulant generating function can be obtained exactly from Eq. (11). We find $\ln Z(\lambda) = G(1-a^2)/(12a)$, where $a=i\lambda+1$ (see Appendix F for derivation). After Fourier transformation, the distribution of entanglement entropy of the left lead at zero bias is (see Appendix F for derivation)

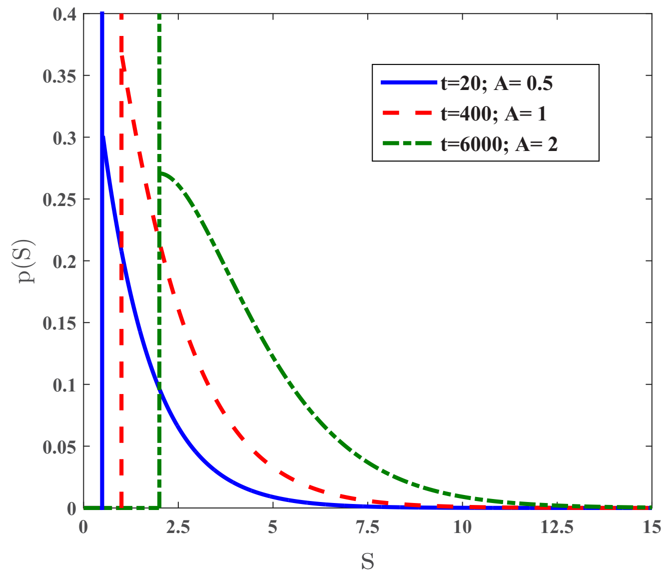

FIG. 1. Entanglement entropy distribution due to the opening of a QPC in equilibrium at different times [the vertical line represents the δ function in Eq. (18)]. Here the time t is measured in short-time cutoff δ.

$$p(S) = e^{-S}\left[\delta(S-A)+\frac{A\theta(S-A)}{x}I_1(2x)\right], \tag{18}$$

where $x=\sqrt{A(S-A)}$, $A=G/12$, and I_1 is the modified Bessel function of the first kind. In Fig. 1, the distribution of entanglement entropy is plotted at different times ($h\beta/\pi = 1$ s and $\delta = 1$ ms is used). We see that although the distribution of charge transfer is Gaussian, the generated entanglement entropy is non-Gaussian. The average entanglement entropy and its fluctuation are both equal to $2A$ which increases with time. This means that $F=1$, independent of time, where we define the "Fano" factor $F=\langle\langle\hat{S}^2\rangle\rangle/\langle\hat{S}\rangle$ which is a measure of relative fluctuation.

With a dc bias V, the generating function of charge transfer at zero temperature in the long-time limit obeys the binomial distribution [19]

$$\chi(\lambda) = [1+T(e^{i\lambda}-1)]^N. \tag{19}$$

Due to the charge quantization $N=eVt/h$ is an integer describing the number of charges detected during time t. Using Eq. (11), the generating function of $\hat{S}$ is found to be $Z(\lambda)=[T^{i\lambda+1}+(1-T)^{i\lambda+1}]^N$, from which the distribution function is obtained (see Appendix G for derivation)

$$p(S) = \sum_{m=0}^{N}p_N(m)\delta[S+\ln g_m(T)], \tag{20}$$

where $p_N(m)=C_N^m g_m(T)$ is the binomial distribution with $g_m(T)=T^m(1-T)^{N-m}$. From this expression, we see that the distribution is symmetric about $T=0.5$ and the

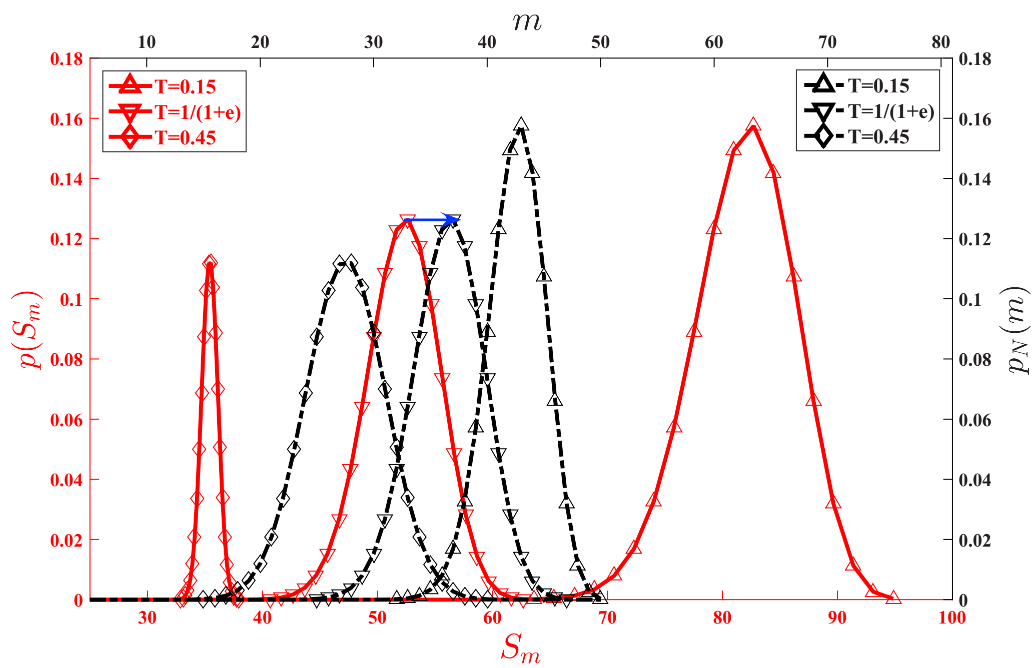

FIG. 2. Entanglement entropy distribution $p(S_m)$ of a QPC in dc bias in the long-time limit for different transmission coefficients with $N = 50$. The curves on the right (black dash-dotted lines) are binomial distribution $p_N(m)$ while the curves on the left (red solid lines) are the corresponding modified binomial distribution $p(S_m)$. The blue arrow indicates the shift between two curves when $T = 1/(1+e)$.

entanglement entropy takes $N+1$ different values $S_m \equiv -\ln g_m = -N\ln(1-T) + m\ln[(1-T)/T]$ with the probability $p_N(m)$. Therefore we can also write the envelope function of the distribution in Eq. (20) as $p_N(S_m)$ which is a modified binomial distribution (see Fig. 2).

In fact, S_m are eigenvalues of entanglement entropy operator of the QPC. To see this, we consider two independent electrons traversing the QPC successively; each of them has a probability T (transmission coefficient) in the right lead and a probability $1-T$ in the left lead so that the state of each electron can be viewed as a qubit with $|0\rangle$ (unoccupied state) and $|1\rangle$ (occupied state) in the L and R regions. The wave function of the system can be written as $\Psi = \psi_1 \otimes \psi_2$ where $\psi_{1/2} = t|0\rangle_L|1\rangle_R - r|1\rangle_L|0\rangle_R$ is the wave function for each electron with $|t|^2 = T$ and $|r|^2 = 1-T$ [15]. The density matrix of the system is $\rho = |\Psi\rangle\langle\Psi|$. The reduced density matrix ρ_L is found to be a 4×4 diagonal matrix with three distinct eigenvalues T^2, $T(1-T)$ (double degenerate), $(1-T)^2$. Hence the eigenvalues of entanglement entropy operator are $-\ln[T^m(1-T)^{2-m}]$ and degeneracy is C_2^m with $m = 0,1,2$. This in turn gives the probability of $p_2(m) = C_2^m T^m (1-T)^{2-m}$ which agrees with Eq. (20) for $N = 2$. The generating function is also obtained $\chi(\lambda) = \sum_m p_2(m)\exp(i\lambda m)$ which agrees with Eq. (19) for $N = 2$. This argument can easily be generalized to N electrons.

The eigenvalues S_m are equally spaced in the interval $[N\ln(1-T), N\ln T]$ with level spacing $\ln[(1-T)/T]$ which goes to zero at $T = 0.5$. At $T = 0.5$, all eigenvalues of $\hat{S}$ collapse to $N\ln 2$ and hence no fluctuation will occur. This gives the maximum entanglement entropy or the system is maximally entangled. It is interesting to note that for $T = 0.5$, the current fluctuation of the QPC is the largest. When $T = 1/(1+e)$, we have $S_m = m - N\ln(1-T)$. The corresponding distribution is a shifted binomial distribution (see Fig. 2), i.e., $p(S_m) = p_N(m-c)$ with a constant shift $c = N\ln[e/(1+e)]$.

From Eq. (20), the average entanglement entropy is found to be $\langle\hat{S}\rangle = -N[T\ln T + (1-T)\ln(1-T)]$ which has been obtained before [19] and its fluctuation is

$$\langle\langle\hat{S}^2\rangle\rangle = NT(1-T)\left|\ln\left(\frac{T}{1-T}\right)\right|^2. \tag{21}$$

As shown in Fig. 3(a), the entanglement entropy fluctuation has two maxima located at T_0 and $1-T_0$ with $T_0 \approx 0.08$ and one minimum at $T = 0.5$. In addition, $\langle\langle\hat{S}^2\rangle\rangle = 0$ at $T = 0,1$. In general, the Fano factor is a monotonic decreasing function of T in [0,0.5]. At $T \approx 0.135$ the Fano factor is $F = 1$.

In Fig. 3(b), we show the convergence test of $\langle\langle\hat{S}^2\rangle\rangle$ in terms of cumulant expansion of charge transfer after resummation. We see that the convergence rate is slower for larger fluctuation. Experimentally, the cumulants of charge transfer have been measured in the QPC with high precision [25]. For QD systems, up to the 20th order of cumulants of charge transfer have been measured experimentally [26,27]. It is conceivable that the entanglement entropy fluctuation can be measured indirectly through cumulants of charge transfer.

B. Universal behavior of entanglement entropy

Although the theory discussed above is for the QPC system, it is straightforward to treat QD systems by setting subsystem A to be QD and subsystem B to be two leads. In this case, Eq. (8) can still be used and the only difference is here $M = -iG^<(t,t)$ where $G^<(t,t)$ is the lesser Green's function of the QD (see Appendix B for derivation of generating function of QD systems). Now we consider the transient behavior of entanglement entropy generated by connecting

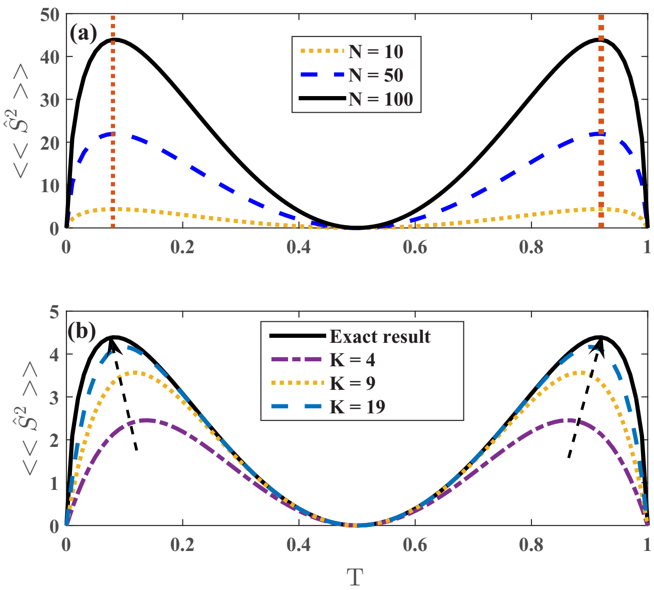

FIG. 3. Entanglement entropy fluctuation $\langle\langle \hat{S}^2 \rangle\rangle$ of the QPC. (a) $\langle\langle \hat{S}^2 \rangle\rangle$ vs transmission coefficient T for different N (number of electrons). (b) Convergence test of $\langle\langle \hat{S}^2 \rangle\rangle$ using cumulant expansion of charge transfer with resummation with K the cutoff introduced in Eq. (15) ($N = 10$).

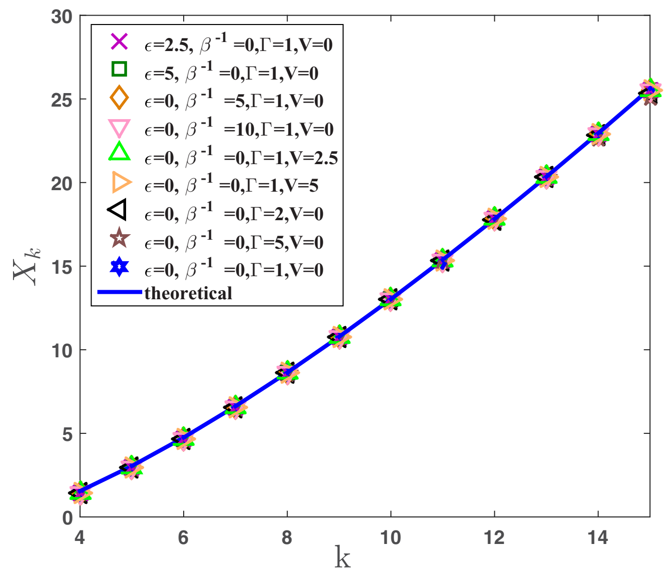

FIG. 4. Scaling function X_k vs k for different system parameters such as bias voltage V, energy level ϵ_0, linewidth function Γ, and inverse temperature β.

the QD with two leads at $t = 0$. When $t < 0$ the Hamiltonian of the system is

$$\hat{H} = \hat{H}_{\text{lead}} + \hat{H}_{\text{D}}, \tag{22}$$

where $\hat{H}_{\text{lead}} = \sum_{k\alpha} \epsilon_{k\alpha} \hat{a}^\dagger_{k\alpha} \hat{a}_{k\alpha}$ and $\hat{H}_{\text{D}} = \epsilon_0 \hat{d}^\dagger \hat{d}$. At $t = 0$, the hopping Hamiltonian $\hat{H}_{\text{T}} = \sum_{k\alpha} t_{k\alpha} \hat{a}^\dagger_{k\alpha} \hat{d} + \text{H.c.}$ is added to the system. For simplicity, we assume that at $t = 0^-$ the QD is empty and the leads are biased with $v_L = 0$ and $v_R = V$. Using the nonequilibrium Green's-function method, the time-dependent lesser Green's function $G^<$ of the QD with a single level ($\epsilon_0 = 0$) (in the wide-band limit and zero temperature) is found to be

$$G^<(t,t) = (3i/4)(1 - e^{-\Gamma t}), \tag{23}$$

where Γ is the linewidth function describing the coupling strength between QD and two leads and we have assumed that $V \gg \Gamma$. Substituting Eq. (23) into Eq. (8), we can calculate the generating function of entanglement entropy and then obtain the time-dependent behavior of higher-order cumulants $h_k(t) = \langle\langle \hat{S}(t)^k \rangle\rangle (k > 1)$. It is easy to see that all extrema of $h_k(t)$ are located at short times ($\Gamma t \ll 1$), which can be found by setting $\partial_t h_k(t) = 0$. We find (see Appendix H for derivation)

$$\langle\langle \hat{S}(t)^k \rangle\rangle_{\max} = e^{-k} k^k. \tag{24}$$

Taking the logarithm, we find a universal scaling X_k vs k that does not depend on the details of the system,

$$X_k = \ln \langle\langle \hat{S}(t)^k \rangle\rangle_{\max} \approx k \ln k, (k \gg 1). \tag{25}$$

To show this behavior is indeed universal; we have varied different system parameters such as bias voltage V, energy level ϵ_0, linewidth function Γ, and inverse temperature β. We find that at short times $G^<(t,t) = c_0 i \Gamma t$ and different system parameters only alter the coefficient c_0 which does not appear in the final scaling function X_k. Indeed, this is confirmed numerically as shown in Fig. 4 where X_k vs k is plotted for different system parameters. Here the data points were obtained using the following steps: (1) calculate $G^<(t,t)$ numerically [not using Eq. (23)]; (2) plug $G^<(t,t)$ into the expression of generating function of entanglement entropy Eq. (B3); (3) take the nth derivative with respect to $-i\lambda$ to obtain the nth cumulant of entanglement entropy; (4) find the maximum value at short times. Clearly the universal relationship among cumulants of entanglement entropy at short times is revealed. We note that a universal oscillation of cumulants of charge transfer for a QD system has been studied experimentally and analyzed theoretically [28].

C. Entanglement entropy and Renyi entropies

Now we discuss the relation between entanglement entropy fluctuations and Renyi entropies which are defined as ($n > 1$) [12,15]

$$S_n = \frac{1}{1-n} \ln \left[\text{Tr}\left(\hat{\rho}_A^n\right)\right]. \tag{26}$$

Clearly, S_1 is the ordinary entanglement entropy. From Eq. (20), Renyi entropies S_n can be calculated analytically for the QPC since we have all the eigenvalues of the reduced density matrix. We find $S_n = N \ln[T^n + (1-T)^n]/(1-n)$. Comparing with the expression of generating function of entanglement entropy $Z(\lambda)$, we arrive at

$$S_n = \frac{1}{1-n} \ln Z[i(1-n)]. \tag{27}$$

In fact, this is a general relation valid for noninteger n and any noninteracting systems. This can be derived using the relation [17] $\ln \text{Tr}[\hat{\rho}^\nu] = \text{Tr} \ln[M^\nu + (1-M)^\nu]$ where M is defined after Eq. (8). In terms of cumulants of entanglement entropy, we find

$$S_n = \sum_{m=1}^{\infty} \frac{(1-n)^{m-1}}{m!} \langle\langle \hat{S}^m \rangle\rangle. \tag{28}$$

We see that Renyi entropies and cumulants of entanglement entropy are different representations capable of characterizing the entanglement entropy operator while the cumulant of entanglement entropy is a direct measure of fluctuations of entanglement entropy.

For a general bipartite system with or without interaction, the reduced density matrix ρ_A can be found using Schmidt decomposition $\rho_A = \sum_i p_i^2 |i\rangle_A \langle i|_A$, from which the nth moment of entanglement entropy is obtained,

$$\langle \hat{S}_L^n \rangle = (-1)^n \sum_i p_i^{2n} \left(\ln p_i^2 \right)^n, \tag{29}$$

and the nth cumulants can be calculated similarly.

IV. SUMMARY

To summarize, we have introduced the entanglement entropy operator which enables us to discuss entanglement entropy fluctuation and distribution of entanglement entropy. A theoretical framework for calculating entanglement entropy fluctuation and distribution of entanglement entropy has been developed for open systems. A general relation between entanglement entropy fluctuation generated by quantum transport and FCS of charge transfer has been established, making entanglement entropy fluctuation a measurable quantity. We have applied our theory to QPC and QD systems. In the transient regime, the maximum amplitude of cumulants of entanglement entropy of the QD show universal scaling at short times, independent of system parameters. Although our theory presented here is for noninteracting systems, it can be generalized to interacting systems.

ACKNOWLEDGMENTS

This work was financially supported by GRF (Grant No. 17311116), the University Grant Council (Contract No. AoE/P-04/08) of the Government of HKSAR, the National Natural Science Foundation of China (Grants No. 11374246 and No. 11674024), and the grant of Trans-Century Training Program Foundation for the Talents by the State Education Commission (Grant No. NCET-13-0048).

APPENDIX A: GENERATING FUNCTION OF ENTANGLEMENT ENTROPY FOR THE QPC SYSTEM

In this appendix, we derive the expression for the generating function of the entanglement entropy for a QPC system. Initially ($t = 0^-$) the system can be described by the equilibrium state density matrix defined as

$$\hat{\rho}(0) = \frac{1}{Z} e^{-\beta \hat{H}_0}, \tag{A1}$$

where $\beta = 1/k_B T$ and $Z \equiv \mathrm{Tr}[e^{-\beta \hat{H}_0}]$. After $t = 0$, the time dependent evolution of the full density matrix is determined by Liouville's equation (here we set $\hbar = 1$),

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t) = e^{-i\hat{H}t}\hat{\rho}(0)e^{i\hat{H}t}. \tag{A2}$$

In order to evaluate the full density matrix $\hat{\rho}(t)$ and the reduced density matrix $\hat{\rho}_L(t)$ occurring later on, it is convenient and powerful to use the techniques of Fermionic coherent states and Grassmann algebras. In the following, we just briefly summarize some important properties of Grassmann algebras and list some useful formulas.

Denoting ξ_i and ξ_j as Grassmann variables, the following anticommutative properties hold: $\xi_i \xi_j = -\xi_j \xi_i$ and ${\xi_i}^2 = {\xi_j}^2 = 0$. In terms of the Grassmann variables, the many-body Fermionic coherent state can be defined as

$$|\vec{\xi}\rangle = |\xi_1 \xi_2 \cdots \xi_N\rangle = \exp\left(-\sum_{i=1}^{N} \xi_i \hat{c}_i^\dagger \right) |0\rangle. \tag{A3}$$

One can easily check that $|\vec{\xi}\rangle$ is the eigenstate of the Fermionic annihilation operator $\hat{c}_i$, i.e.,

$$\hat{c}_i |\vec{\xi}\rangle = \xi_i |\vec{\xi}\rangle. \tag{A4}$$

By taking the Hermitian conjugation, we have

$$\langle \vec{\xi} | \hat{c}_i^\dagger = \langle \vec{\xi} | \xi_i^*. \tag{A5}$$

The following formulas of the Grassmann algebras are useful to carry out calculations:

$$\langle \vec{\xi} | \exp\left(\sum_{ij} \hat{c}_i^\dagger A_{ij} \hat{c}_j \right) |\vec{\xi}'\rangle = \exp\left[\sum_{ij} \xi_i^* (e^A)_{ij} \xi_j^{'} \right], \tag{A6}$$

$$\mathrm{Tr}[A] = \int D[\vec{\xi}^*, \vec{\xi}] e^{-\sum_i \xi_i^* \xi_i} \langle -\vec{\xi} | A | \vec{\xi} \rangle, \tag{A7}$$

$$\mathrm{Det}[A] = \int D[\vec{\xi}^*, \vec{\xi}] e^{-\sum_{ij} \xi_i^* A_{ij} \xi_j}, \tag{A8}$$

$$\hat{1} = \int D[\vec{\xi}^*, \vec{\xi}] e^{-\sum_i \xi_i^* \xi_i} |\vec{\xi}\rangle\langle\vec{\xi}|, \tag{A9}$$

$$\langle \vec{\xi}' | \vec{\xi} \rangle = e^{\sum_i \xi_i^{'*} \xi_i}, \tag{A10}$$

where we have used the notation $D[\vec{\xi}^*, \vec{\xi}] \equiv \prod_i d\xi_i^* d\xi_i$.

For a QPC system, without loss of generality, we focus on the left lead and the reduced density matrix $\hat{\rho}_L(t)$ can be obtained by tracing out the degrees of freedom of the right lead of the full density matrix $\hat{\rho}(t)$,

$$\hat{\rho}_L(t) = \mathrm{Tr}_\mathrm{R}[\hat{\rho}(t)]. \tag{A11}$$

The entanglement entropy of the left lead is defined as

$$S_L(t) = -\mathrm{Tr}_L[\hat{\rho}_L(t) \ln \hat{\rho}_L(t)]. \tag{A12}$$

Combining Eq. (A2) with Eq. (A11), we have

$$\hat{\rho}_L(t) = \mathrm{Tr}_\mathrm{R}[e^{-i(\hat{H}_0 + \hat{H}_T)t} \hat{\rho}(0) e^{i(\hat{H}_0 + \hat{H}_T)t}], \tag{A13}$$

and meanwhile we can write the Fermionic coherent state as

$$\langle \vec{\xi}, \vec{\eta} | = \exp\left[-\sum_k \xi_{kL} \hat{c}_{kL}^\dagger - \sum_k \eta_{kR} \hat{c}_{kR}^\dagger \right] |0\rangle, \tag{A14}$$

where ξ_{kL} (η_{kR}) is associated with the degrees of freedom of the left (right) lead. Thus the matrix element $\langle\vec{\xi}|\hat{\rho}_L(t)|\vec{\xi}'\rangle$ of the reduced density matrix $\hat{\rho}_L(t)$ can be calculated in the Fermionic coherent state representation,

$$\begin{aligned}\langle\vec{\xi}|\hat{\rho}_L(t)|\vec{\xi}'\rangle &= \langle\vec{\xi}|\mathrm{Tr_R}[\hat{\rho}(t)]|\vec{\xi}'\rangle = \int D[\vec{\eta}^*,\vec{\eta}]e^{-\sum_i \eta_i^*\eta_i}\langle\vec{\xi},-\vec{\eta}|\hat{\rho}(t)|\vec{\xi}',\vec{\eta}\rangle\\ &= \int D[\vec{\eta}^*,\vec{\eta}]e^{-\sum_i \eta_i^*\eta_i}\langle\vec{\xi},-\vec{\eta}|e^{-i(\hat{H}_0+\hat{H}_T)t}\hat{\rho}(0)e^{i(\hat{H}_0+\hat{H}_T)t}|\vec{\xi}',\vec{\eta}\rangle,\end{aligned} \tag{A15}$$

where Eq. (A7) has been used. By inserting Eq. (A9) into Eq. (A15), we obtain

$$\begin{aligned}\langle\vec{\xi}|\hat{\rho}_L(t)|\vec{\xi}'\rangle = \frac{1}{Z}\int & D[\vec{\eta}^*,\vec{\eta}]D[\vec{\psi}_L^*,\vec{\psi}_L]D[\vec{\psi}_R^*,\vec{\psi}_R]D[\vec{\lambda}_L^*,\vec{\lambda}_L]D[\vec{\lambda}_R^*,\vec{\lambda}_R]e^{-\sum_i\eta_i^*\eta_i}e^{-\sum_i\psi_{Li}^*\psi_{Li}}e^{-\sum_i\psi_{Ri}^*\psi_{Ri}}e^{-\sum_i\lambda_{Li}^*\lambda_{Li}}e^{-\sum_i\lambda_{Ri}^*\lambda_{Ri}}\\ &\times\langle\vec{\xi},-\vec{\eta}|e^{-i(\hat{H}_0+\hat{H}_T)t}|\vec{\psi}_L,\vec{\psi}_R\rangle\langle\vec{\psi}_L,\vec{\psi}_R|e^{-\beta\hat{H}_0}|\vec{\lambda}_L,\vec{\lambda}_R\rangle\langle\vec{\lambda}_L,\vec{\lambda}_R|e^{i(\hat{H}_0+\hat{H}_T)t}|\vec{\xi}',\vec{\eta}\rangle,\end{aligned} \tag{A16}$$

where quantities like $|\vec{\Psi}_L\rangle$ and $|\vec{\lambda}_L\rangle$ are intermediate Grassmann variables. Since all terms like $e^{-\beta\hat{H}_0}$, $e^{i(\hat{H}_0+\hat{H}_T)t}$, and $e^{-i(\hat{H}_0+\hat{H}_T)t}$ are quadratic functions, after carrying out the Gaussian integral we immediately conclude that the final result of Eq. (A16) is also a quadratic function which can be formally written as

$$\langle\vec{\xi}|\hat{\rho}_L(t)|\vec{\xi}'\rangle = \frac{1}{Z'}\exp\left[\sum_{kk'}\xi_k^*Y_{kk'}(t)\xi_{k'}'\right], \tag{A17}$$

where Z' is a normalization factor. Here we emphasize that the remaining Grassmann variables ξ_k and $\xi_{k'}^*$ are associated with the degrees of freedom of the left lead and the dimension of the matrix $Y_{ij}(t)$ is the same as the number of degrees of freedom of the left lead. Furthermore the normalization condition requires $\mathrm{Tr_L}[\hat{\rho}_L(t)] = 1$ which determines Z'. Since

$$\begin{aligned}\mathrm{Tr_L}[\hat{\rho}_L(t)] &= \int D[\vec{\xi}^*,\vec{\xi}]e^{-\sum_i\xi_i^*\xi_i}\langle-\vec{\xi}|\hat{\rho}_L(t)|\vec{\xi}\rangle\\ &= \frac{1}{Z'}\int D[\vec{\xi}^*,\vec{\xi}]e^{-\sum_i\xi_i^*\xi_i}e^{(-\sum_{kk'}\xi_k^*Y_{kk'}(t)\xi_{k'})}\\ &= \frac{1}{Z'}\mathrm{Det}[Y(t)+I]\end{aligned} \tag{A18}$$

from which we have $Z' = \mathrm{Det}[Y(t)+I]$ and finally the Eq. (A16) can be rewritten as

$$\langle\vec{\xi}|\hat{\rho}_L(t)|\vec{\xi}'\rangle = \frac{1}{\mathrm{Det}[Y(t)+I]}e^{\sum_{kk'}\xi_k^*Y_{kk'}(t)\xi_{k'}'}. \tag{A19}$$

To further relate the matrix $Y(t)$ to the correlation function matrix $M_{kk'}(t)$ defined as $M_{kk'}(t) = \mathrm{Tr_L}[\hat{\rho}_L(t)\hat{c}_{k'L}^\dagger\hat{c}_{kL}]$, we can find

$$\begin{aligned}M_{kk'}(t) &= \int D[\vec{\xi}^*,\vec{\xi}]D[\vec{\eta}^*,\vec{\eta}]e^{-\sum_i\xi_i^*\xi_i}e^{-\sum_i\eta_i^*\eta_i}\langle-\vec{\eta}|\hat{\rho}_L(t)|\vec{\xi}\rangle\langle\vec{\xi}|\hat{c}_{k'L}^\dagger\hat{c}_{kL}|\vec{\eta}\rangle\\ &= \frac{1}{Z'}\int D[\vec{\xi}^*,\vec{\xi}]D[\vec{\eta}^*,\vec{\eta}]\xi_{k'}^*\eta_k e^{-\sum_i\xi_i^*\xi_i}e^{-\sum_i\eta_i^*\eta_i}e^{-\sum_{ij}\eta_i^*Y_{ij}(t)\xi_j}e^{\sum_i\xi_i^*\eta_i}\\ &= \frac{\mathrm{Det}[Y(t)+I]}{Z'}\left(\frac{Y(t)}{Y(t)+I}\right)_{kk'} = \left(\frac{Y(t)}{Y(t)+I}\right)_{kk'}.\end{aligned} \tag{A20}$$

In the above derivation, we have used Eqs. (A7) and (A9) in the first line and Eq. (A19) in the second line. For the third line we have used Eq. (A8) and Wick's theorem of Grassmann variables,

$$\frac{\int D[\vec{\psi}^*,\vec{\psi}]\psi_i\psi_j^*e^{-\sum_{ij}\psi_i^*A_{ij}\psi_j}}{\int D[\vec{\psi}^*,\vec{\psi}]e^{-\sum_{ij}\psi_i^*A_{ij}\psi_j}} = (A^{-1})_{ij}. \tag{A21}$$

Note that the definition of correlation matrix $M(t)$ is related to the lesser Green's function of the left lead $G_L^<(t,t)$ by the relation $M(t) = -iG_L^<(t,t)$. Since Eq. (A17) holds for arbitrary $\vec{\xi}$ and $\vec{\xi}'$, using Eq. (A6), we know that

$$\hat{\rho}_L(t) = \frac{1}{\mathrm{Det}[Y(t)+I]}e^{[\sum_{kk'}\hat{c}_k^\dagger\ln Y(t)_{kk'}\hat{c}_{k'}]}, \tag{A22}$$

from Eq. (A22) we have

$$\ln\hat{\rho}_L(t) = \ln\frac{1}{\mathrm{Det}[Y(t)+I]} + \sum_{kk'}\hat{c}_k^\dagger\ln Y(t)_{kk'}\hat{c}_{k'}. \tag{A23}$$

With the above mathematical preparations we can calculate the statistical distribution function of entanglement entropy. Similar to the FCS of charge transfer, the moment generating function (MGF) $Z(\lambda)$ and the cumulant generating function (CGF) $\ln Z(\lambda)$ of entanglement entropy are defined as

$$Z(\lambda) = \langle e^{-i\lambda\hat{S}_L(t)}\rangle = \sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\langle\hat{S}_L^j(t)\rangle, \tag{A24}$$

$$\ln Z(\lambda) = \langle\langle e^{-i\lambda\hat{S}_L(t)}\rangle\rangle = \sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\langle\langle\hat{S}_L^j(t)\rangle\rangle, \tag{A25}$$

where $\langle \hat{S}_L^j(t)\rangle$ is the jth moment and $\langle\langle \hat{S}_L^j(t)\rangle\rangle$ is the jth cumulant. By means of Grassmann algebras we derive the MGF $Z(\lambda)$ as follows:

$$\begin{aligned} Z(\lambda) &= \langle e^{-i\lambda \hat{S}_L(t)}\rangle = \mathrm{Tr}[\hat{\rho}(0)\hat{U}^\dagger(t)e^{i\lambda\ln\hat{\rho}_L(t)}\hat{U}(t)] \\ &= \mathrm{Tr_L}[\hat{\rho}_L(t)e^{i\lambda\ln\hat{\rho}_L(t)}] \\ &= \int D[\vec{\xi}^*,\vec{\xi}]D[\vec{\eta}^*,\vec{\eta}]e^{-\sum_i \xi_i^*\xi_i}e^{-\sum_i \eta_i^*\eta_i}\langle -\vec{\xi}|\hat{\rho}_L(t)|\vec{\eta}\rangle\langle\vec{\eta}|e^{i\lambda\ln\hat{\rho}_L(t)}|\vec{\xi}\rangle \\ &= \frac{1}{Z'}e^{-i\lambda\ln Z'}\int D[\vec{\xi}^*,\vec{\xi}]e^{-\sum_i\xi_i^*\xi_i}\int D[\vec{\eta}^*,\vec{\eta}]e^{-\sum_i\eta_i^*\eta_i}e^{-\sum_{ij}\xi_i^*Y_{ij}\eta_j}e^{\sum_{ij}\eta_i^*Y_{ij}^{i\lambda}\xi_j} \\ &= \frac{1}{Z'}e^{-i\lambda\ln Z'}\int D[\vec{\xi}^*,\vec{\xi}]e^{-\sum_i\xi_i^*\xi_i}e^{-\sum_{ij}\xi_i^*Y_{ij}^{i\lambda+1}\xi_j} \\ &= \frac{\mathrm{Det}[I+Y^{i\lambda+1}]}{\mathrm{Det}[I+Y]}e^{-i\lambda\ln[\mathrm{Det}[I+Y]]}. \end{aligned} \tag{A26}$$

In the second line we have used Eqs. (A7) and (A9) and in the third line we have used Eqs. (A6), (A19), and (A23). In the last line we have used Eq. (A8). Thus the CGF is

$$\ln Z(\lambda) = \ln(\mathrm{Det}[Y^{i\lambda+1}+I]) - (1+i\lambda)\ln(\mathrm{Det}[Y+I]). \tag{A27}$$

Now we can calculate the average entanglement entropy and its higher-order moments (or cumulants) according to the MGF (or the CGF). With the help of Jacobi's formula $\frac{\partial}{\partial t}\mathrm{Det}[A(t)] = \mathrm{Tr}[\mathrm{adj}(A)\frac{\partial A(t)}{\partial t}]$ and the relation $\mathrm{adj}(A) = \mathrm{Det}[A]A^{-1}$, we have

$$\begin{aligned} \langle\hat{S}_L(t)\rangle &= -\frac{\partial}{\partial(i\lambda)}Z(\lambda)\Big|_{\lambda=0} \\ &= -\mathrm{Tr}\left[\frac{Y}{Y+I}\ln Y\right] + \mathrm{Tr}[\ln(Y+I)]. \end{aligned} \tag{A28}$$

For the second-order moment $\langle\hat{S}_L^2(t)\rangle$,

$$\begin{aligned} \langle\hat{S}_L^2(t)\rangle &= \frac{\partial^2}{\partial(i\lambda)^2}Z(\lambda)\Big|_{\lambda=0} \\ &= \mathrm{Tr}\left[\frac{Y}{(Y+I)^2}(\ln Y)^2\right] \\ &\quad + \left\{\mathrm{Tr}\left[\frac{Y}{Y+I}\ln Y\right] - \mathrm{Tr}\ln(Y+I)\right\}^2. \end{aligned} \tag{A29}$$

Using Eqs. (A20) and (A28) we obtain

$$\langle\langle\hat{S}_L\rangle\rangle = \langle\hat{S}_L\rangle = -\mathrm{Tr}[M\ln M + (1-M)\ln(1-M)] \tag{A30}$$

and

$$\begin{aligned} \langle\langle\hat{S}_L^2\rangle\rangle &= \langle\hat{S}_L^2\rangle - \langle\hat{S}_L\rangle^2 \\ &= \mathrm{Tr}\left[\frac{Y}{(Y+I)^2}(\ln Y)^2\right] \\ &= \mathrm{Tr}\left[\ln\left(\frac{M}{1-M}\right)M\ln\left(\frac{M}{1-M}\right)(1-M)\right]. \end{aligned} \tag{A31}$$

Although Eqs. (A26) and (A27) are formal solutions, we can use y_j, the eigenspectrum of Y (or M), to express it, for instance, $\mathrm{Det}[Y^{i\lambda+1}+1] = \prod_j(y_j^{i\lambda+1}+1)$.

APPENDIX B: GENERATING FUNCTION OF ENTANGLEMENT ENTROPY FOR THE QD SYSTEM

The derivation for the generating function of entanglement entropy for a QD system is similar to that for the QPC system. In this case we focus on the QD and both leads are treated as environments. For simplicity we consider a single level QD. When $t < 0$, the Hamiltonian of the system is $\hat{H}_0 = \hat{H}_E + \hat{H}_D$, where $\hat{H}_E = \sum_k[(\varepsilon_{kL}-\mu_L)\hat{c}_{kL}^\dagger\hat{c}_{kL} + (\varepsilon_{kR}-\mu_R)\hat{c}_{kR}^\dagger\hat{c}_{kR}]$ is the Hamiltonian of the isolated leads and $\hat{H}_D = \epsilon_0\hat{d}^\dagger\hat{d}$ is the Hamiltonian of the isolated QD. At $t = 0$, a hopping interaction $\hat{H}_T$ is suddenly added to the system connecting the QD and the leads, therefore the Hamiltonian $\hat{H}$ is $\hat{H} = \hat{H}_0 + \hat{H}_T$. The hopping interaction $\hat{H}_T$ is $\hat{H}_T = \sum_{k\alpha} t_{k\alpha}\hat{c}_{k\alpha}^\dagger\hat{d} + \mathrm{H.c.}$ The initial density matrix of the whole system is assumed to be

$$\hat{\rho}(0) = \hat{\rho}_E \otimes \hat{\rho}_D, \tag{B1}$$

where $\hat{\rho}_E = \frac{1}{Z}e^{-\beta\hat{H}_E}$ with $Z = \mathrm{Tr}[e^{-\beta\hat{H}_E}]$ and $\hat{\rho}_D$ is the initial density matrix for the QD. Since the isolated QD is noninteracting the density matrix can be written in some kind of quadratic thermal density matrix form. For example, the initial density matrix has the form $\hat{\rho}_D = \frac{1}{Z_1}e^{-\beta_{\mathrm{eff}}\hat{H}_D}$ where $Z_1 = \mathrm{Tr}[e^{-\beta_{\mathrm{eff}}\hat{H}_D}]$. This assertion guarantees that all the derivations discussed in Appendix A are still valid for the QD system. The entanglement entropy of the QD system is defined as

$$S_D(t) = -\mathrm{Tr}_D[\hat{\rho}_D(t)\ln\hat{\rho}_D(t)], \tag{B2}$$

where $\hat{\rho}_D(t) = -\mathrm{Tr_E}[\hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)] = -\mathrm{Tr_E}[e^{-i\hat{H}t}\hat{\rho}(0)e^{i\hat{H}t}]$. As we have pointed out, from the theoretical point of view, the dynamics of the reduced system, the QD, is similar to that of the left lead discussed above. The only difference is that for now we should use the correlation matrix of the QD instead of that of the left lead. Therefore the generating function of entanglement entropy of the QD system is

$$\ln Z_D(\lambda) = \ln\left(\mathrm{Det}\left[Y_D^{i\lambda+1}+I\right]\right) - (1+i\lambda)\ln(\mathrm{Det}[Y_D+I]). \tag{B3}$$

Here we emphasize that the subscript D stands for the corresponding matrix of the QD which should be distinguished from Eq. (A27). The entanglement entropy of the QD can be expressed as

$$\begin{aligned} S_D(t) &= -\mathrm{Tr_D}\{M_D(t)\ln M_D(t) + [1-M_D(t)]\ln[1-M_D(t)]\} \\ &= -\mathrm{Tr_D}\{-iG_D^<(t,t)\ln[-iG_D^<(t,t)] + [1+iG_D^<(t,t)]\ln[1+iG_D^<(t,t)]\}, \end{aligned} \tag{B4}$$

where $M_D(t)$ is the correlation matrix of the QD and $G_D^<(t,t)$ is the lesser Green's function of the QD and the relation $M_D(t) = -iG_D^<(t,t)$ has been used.

APPENDIX C: DERIVATION OF EQ. (11)

Introducing the spectrum density $\mu(z)$ of the correlation matrix M as done by Klich and Levitov [19] which is defined as

$$\mu(z) = \frac{1}{\pi}\mathrm{Im}[\partial_z \ln\chi(z-i0^+)], \tag{C1}$$

where $\ln\chi(z)$ is the CGF of charge transfer which can be expanded as $\ln\chi(z) = \ln\chi[\xi(z)] = \sum_m \frac{[i\xi(z)]^m}{m!}C_m$ in terms of the cumulants C_m and $z(\xi)$ satisfies the relation $\ln(\frac{1}{z}-1) = i\xi + i\pi$. Then using Eqs. (A20) and (A27), we follow exactly the same steps as in Ref. [19] and find

$$\ln Z(\lambda) = \int_0^1 dz\mu(z)\ln\left[\left(\frac{z}{1-z}\right)^{(i\lambda+1)} + 1\right] - (i\lambda+1)\int_0^1 dz\mu(z)\ln\left[\frac{z}{1-z}+1\right]. \tag{C2}$$

Substituting Eq. (C1) into Eq. (C2) and integrating by parts, we have

$$\begin{aligned} \ln Z(\lambda) &= -\int_0^1 dz\frac{1}{\pi}\mathrm{Im}[\ln\chi(z_-)]\partial_z\ln\left[\left(\frac{z}{1-z}\right)^{(i\lambda+1)}+1\right] + (i\lambda+1)\int_0^1 dz\frac{1}{\pi}\mathrm{Im}[\ln\chi(z_-)]\partial_z\ln\left[\frac{z}{1-z}+1\right] \\ &= -\int_0^1 dz\frac{1}{\pi}\mathrm{Im}\{\ln\chi[\xi(z)]\}\partial_z\ln\left[\left(\frac{z}{1-z}\right)^{(i\lambda+1)}+1\right] + (i\lambda+1)\int_0^1 dz\frac{1}{\pi}\mathrm{Im}\{\ln\chi[\xi(z)]\}\partial_z\ln\left[\frac{z}{1-z}+1\right] \\ &= \int du\frac{2}{\pi}\mathrm{Im}\{\ln\chi[\xi(u)]\}(i\lambda+1)g(u) \\ &= \int du\frac{2}{\pi}\mathrm{Im}[\ln\chi(2iu-\pi)](i\lambda+1)g(u), \end{aligned} \tag{C3}$$

where $z_- = z - i0^+$ and $g(u) = \frac{e^{2u}}{e^{2u}+1} - \frac{e^{2u(i\lambda+1)}}{e^{2u(i\lambda+1)}+1}$. In the third line of this equation we have changed the variable, $u = \frac{1}{2}\ln(\frac{z}{1-z})$ and the new variable $\xi(u)$ satisfies $i\xi = -2u - i\pi$. Thus we have constructed a relation between the CGF of the entanglement entropy and that of charge transfer. This relation means that once we know the CGF of charge transfer, in principle we can obtain the CGF of the entanglement entropy.

APPENDIX D: DERIVATION OF EQS. (12)–(14)

When calculating the jth cumulant $\langle\langle \hat{S}_L^j(t)\rangle\rangle$ for $j>1$, we can neglect the second term which is proportional to $i\lambda+1$ in Eq. (C2). Thus the effective cumulant generating function of entanglement entropy is

$$\begin{aligned} \ln Z(\lambda) &= \int du\frac{1}{\pi}\mathrm{Im}\{\ln\chi[\xi(u)]\}\frac{2(i\lambda+1)}{e^{2u(i\lambda+1)}+1} \\ &= \sum_{m=1}^{\infty}\frac{\beta_m}{m!}C_m, \end{aligned} \tag{D1}$$

where we have used the relation $\ln\chi(\xi) = \sum_m (i\xi)^m\frac{C_m}{m}!$ in the above derivation. In Eq. (C3), the coefficient β_m is given by

$$\beta_m = \int du\frac{(-2)^m}{\pi}\mathrm{Im}\left[\left(u+\frac{i\pi}{2}\right)^m\right]\frac{2(i\lambda+1)}{e^{2u(i\lambda+1)}+1}. \tag{D2}$$

In Eq. (D2), after integration by parts we have

$$\begin{aligned} \frac{\pi\beta_m}{(-2)^m} &= \int du\mathrm{Im}\left[\left(u+\frac{i\pi}{2}\right)^m\right]\frac{(i\lambda+1)}{e^{2u(i\lambda+1)}+1} \\ &= \int du\mathrm{Im}\left[\frac{\left(u+\frac{i\pi}{2}\right)^{(m+1)}}{m+1}\right]\frac{2(i\lambda+1)^2}{[e^{u(i\lambda+1)}+e^{-u(i\lambda+1)}]^2} \\ &= \mathrm{Im}\left[\int du\frac{u^{m+1}}{m+1}\frac{-2(\alpha+1)^2}{[e^{u(\alpha+1)}e^{-i\alpha(\pi/2)} - e^{-u(\alpha+1)}e^{i\alpha(\pi/2)}]^2}\right] \\ &= \mathrm{Im}\left[\int du\frac{u^{m+1}}{m+1}\frac{-2(\alpha+1)^2}{\sinh^2\left(u\alpha+u-i\alpha\frac{\pi}{2}\right)}\right], \end{aligned} \tag{D3}$$

where we have set $i\lambda = \alpha$ and will treat α as a real quantity from now on until Eq. (D8). We have also changed u to $u - \frac{i\pi}{2}$ in Eq. (D3). Changing the variable again from $u\alpha + u - i\alpha\frac{\pi}{2}$ to u, we obtain from Eq. (D3)

$$\frac{\pi\beta_m}{(-2)^m} = \mathrm{Im}\left[\int du\frac{\left(u+\frac{i\pi\alpha}{2}\right)^{m+1}}{m+1}\frac{-1}{2(\alpha+1)^m\sinh^2 u}\right] \tag{D4}$$

from which we find

$$\beta_m = -\frac{(-2)^m}{\pi(m+1)(\alpha+1)^m}\mathrm{Im}\left[\int du\frac{\left(u+\frac{i\pi\alpha}{2}\right)^{m+1}}{\sinh^2 u}\right]. \tag{D5}$$

To carry out the above integral, we expand $(u+\frac{i\pi\alpha}{2})^{m+1}$ according to the binomial theorem

$$\left(u+\frac{i\pi\alpha}{2}\right)^{m+1}=\sum_{p=0}^{m+1}C_{m+1}^{p}u^{p}\left(\frac{i\pi\alpha}{2}\right)^{m+1-p}, \quad \text{(D6)}$$

where $C_m^n=\frac{m!}{n!(m-n)!}$ is the binomial coefficient. We then obtain

$$\beta_m=\sum_{q=0}^{(m+1)/2}\frac{(-2)^{m+1}C_{m+1}^{2q}\pi^{2q}}{\pi(m+1)(\alpha+1)^m}\mathrm{Im}\left[\left(\frac{i\pi\alpha}{2}\right)^{m+1-2q}\right]|B_{2q}|. \quad \text{(D7)}$$

In deriving Eq. (D7), we have used the integral $\int_0^\infty\frac{u^{2m}du}{\sinh^2 u}=\pi^{2m}|B_{2m}|$ and here B_n are Bernoulli numbers. For a specific q, $m+1-2q$ must be odd number to give nonzero β_m. Thus $m=2r$ is even number and $[\frac{m+1}{2}]=r$. Finally, we obtain

$$\beta_{m=2r}=\sum_{q=0}^{r}\pi^{2r}2^{2q}\frac{(-1)^{r-q+1}C_{2r+1}^{2q}}{(2r+1)(\alpha+1)^{2r}}|B_{2q}|\alpha^{2r+1-2q} \quad \text{(D8)}$$

and

$$\beta_{m=2r+1}=0. \quad \text{(D9)}$$

Now we can conclude that all higher-order cumulants of entanglement entropy can be expanded in terms of even order cumulants of charge transfer C_m which can be experimentally measured. This insight proposed by Klich and Levitov is now generalized into higher-order cumulants of entanglement entropy.

APPENDIX E: RESUMMATION OF $\langle\langle\hat{S}^2\rangle\rangle$

The entanglement entropy fluctuation is written as

$$\langle\langle\hat{S}^2\rangle\rangle=\mathrm{Tr}\left[\ln\left(\frac{1-M}{M}\right)M\ln\left(\frac{1-M}{M}\right)(1-M)\right]. \quad \text{(E1)}$$

Expanding $\ln M$ in terms of $1-M$ and $\ln(1-M)$ in terms of M, we have

$$\langle\langle\hat{S}^2\rangle\rangle=\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}\frac{A_{kk'}}{kk'}, \quad \text{(E2)}$$

where $A_{kk'}=\mathrm{Tr}\{(M^{k+k'+1}-M^{k+k'+2})+[M(1-M)^{k+k'}-M^2(1-M)^{k+k'}]-2[(1-M)^kM^{k'+1}-(1-M)^kM^{k'+2}]\}$. Unlike the ordinary MGF or CGF, we should use the factorial generating function (FGF) $\chi_f(z)$ [15,22] defined as

$$\chi_f(z)=\sum_n(z+1)^nP(n), \quad \text{(E3)}$$

where $P(n)$ is the probability for transferring n particles during the measuring time. The factorial cumulant generating function (FCGF) is

$$\ln\chi_f(z)=\sum_{n=1}^{\infty}\frac{z^n}{n!}F_n, \quad \text{(E4)}$$

where F_n are the factorial cumulants. Comparing with the ordinary CGF

$$\ln\chi(\lambda)=\sum_{n=1}^{\infty}\frac{(i\lambda)^n}{n!}C_n. \quad \text{(E5)}$$

It is easy to see in the following relation:

$$\chi_f(z)=\chi[-i\ln(z+1)]. \quad \text{(E6)}$$

Therefore the FCGF can be rewritten as [15,19]

$$\begin{aligned}\ln\chi_f(z)&=\ln\chi[-i\ln(z+1)]\\&=\mathrm{Tr}[\ln(1+zM)-\ln(z+1)Q],\end{aligned} \quad \text{(E7)}$$

where Q is defined in Ref. [15]. We can directly calculate the factorial cumulant F_k according to the FCGF,

$$\begin{aligned}F_k&=\partial_z^k\ln\chi_f(z)\big|_{z=0}\\&=(-1)^{k-1}(k-1)!\mathrm{Tr}[M^k-Q],\end{aligned} \quad \text{(E8)}$$

then we have

$$\mathrm{Tr}[M^k]=\mathrm{Tr}[Q]+\frac{F_k}{(-1)^{k-1}(k-1)!}. \quad \text{(E9)}$$

Substituting Eq. (E9) into Eq. (E2), we obtain

$$\begin{aligned}A_{kk'}=&\frac{F_{k+k'+1}}{(-1)^{k+k'}(k+k')!}-\frac{F_{k+k'+2}}{(-1)^{k+k'+1}(k+k'+1)!}\\&+\sum_{r=0}^{k+k'}C_{k+k'}^{r}\left[\frac{F_{k+k'-r+1}}{(k+k'-r)!}+\frac{F_{k+k'-r+2}}{(k+k'-r+1)!}\right]\\&-2(-1)^{k'}\sum_{r=0}^{k}C_k^r\left[\frac{F_{k+k'-r+1}}{(k+k'-r)!}+\frac{F_{k+k'-r+2}}{(k+k'-r+1)!}\right].\end{aligned} \quad \text{(E10)}$$

Using the relation $F_n=\sum_{k=1}^{n}S_1(n,k)(-1)^{n-k}C_k$ [22] to relate the factorial cumulants to the ordinary cumulants, we obtain

$$\begin{aligned}A_{kk'}=&\sum_{n=1}^{k+k'+1}\frac{(-1)^{1-n}S_1(k+k'+1,n)}{(k+k')!}C_n\\&-\sum_{n=1}^{k+k'+2}\frac{(-1)^{1-n}S_1(k+k'+2,n)}{(k+k'+1)!}C_n\\&+\sum_{n=1}^{r_k+1}\sum_{r=0}^{k+k'}C_{k+k'}^{r}\frac{(-1)^{1-n+r_k}S_1(r_k+1,n)}{(r_k)!}C_n\\&+\sum_{n=1}^{r_k+2}\sum_{r=0}^{k+k'}C_{k+k'}^{r}\frac{(-1)^{2-n+r_k}S_1(r_k+2,n)}{(r_k+1)!}C_n\\&-2(-1)^{k'}\sum_{n=1}^{r_k+1}\sum_{r=0}^{k}C_k^r\frac{(-1)^{1-n+r_k}S_1(r_k+1,n)}{(r_k)!}C_n\\&-2(-1)^{k'}\sum_{n=1}^{r_k+2}\sum_{r=0}^{k}C_k^r\frac{(-1)^{2-n+r_k}S_1(r_k+2,n)}{(r_k+1)!}C_n,\end{aligned} \quad \text{(E11)}$$

where $r_k=k+k'-r$ and $S_1(n,k)$ is the unsigned Stirling number of the first kind. Replacing the sum of following infinite series by taking the limit of the sum of a finite series,

$$\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} = \lim_{K\to\infty}\sum_{k=1}^{K}\sum_{k'=1}^{K}, \tag{E12}$$

and then interchanging the order of summation we have

$$\sum_{k=1}^{K}\sum_{k'=1}^{K}\sum_{n=1}^{k+k'+2} = \sum_{n=1}^{2K+2}\sum_{k=k_1}^{K}\sum_{k'=k_1'}^{K}, \tag{E13}$$

$$\sum_{k=1}^{K}\sum_{k'=1}^{K}\sum_{n=1}^{k+k'-r+2}\sum_{r=0}^{k+k'} = \sum_{n=1}^{2K+2}\sum_{k=k_1}^{K}\sum_{k'=k_1'}^{K}\sum_{r=0}^{r_1}, \tag{E14}$$

$$\sum_{k=1}^{K}\sum_{k'=1}^{K}\sum_{n=1}^{k+k'-r+2}\sum_{r=0}^{k} = \sum_{n=1}^{2K+2}\sum_{k=k_1}^{K}\sum_{k'=k_1'}^{K}\sum_{r=0}^{r_2}, \tag{E15}$$

where $k_1 = \max(n-K-2,1)$, $k_1' = \max(n-k-2,1)$, $r_1 = \min(k+k'-n+2,k+k')$, and $r_2 = \min(k+k'-n+2,k)$;

$$\langle\langle \hat{S}^2\rangle\rangle = \lim_{K\to\infty}\sum_{n=1}^{2K+2}\alpha_n'(K)C_n, \tag{E16}$$

where

$$\begin{aligned}\alpha_n'(K) = \sum_{k=k_1}^{K}\sum_{k'=k_1'}^{K}\Bigg\{&\frac{(-1)^{1-n}}{kk'}[u(k+k')-u(k+k'+1)]\\ &+\sum_{r=0}^{r_1}C_{k+k'}^{r}\frac{(-1)^{1-n+k+k'-r}}{kk'}[u(r_k)-u(r_k+1)]\\ &+\sum_{r=0}^{r_2}(-2)C_k^r\frac{(-1)^{1-n+k-r}}{kk'}[u(r_k)-u(r_k+1)]\Bigg\}\end{aligned} \tag{E17}$$

with $u(m) = S_1(m+1,n)/m!$ and $r_k = k+k'-r$. We point out that $\alpha_{2p+1}'(K) = 0 (p = 0,1,\ldots)$ numerically which agrees with the theoretical result.

APPENDIX F: DERIVATION OF EQ. (18)

Considering a full transmission ($T = 1$) QPC system at zero temperature and zero bias, the MGF of charge transfer $\chi(\xi)$ is Gaussian given by

$$\ln\chi(\xi) = -\frac{\xi^2 G}{4\pi^2}, \tag{F1}$$

where $G = 2\pi^2 C_2$ due to Gaussian distribution and C_2 is the second-order cumulant of charge transfer. As discussed above, once we know the CGF of charge transfer we can use it to calculate that of entanglement entropy. According to the relation $\xi = 2iu - \pi$ we find from Eq. (F1) that

$$\ln\chi[\xi(u)] = -\frac{(2iu-\pi)^2 G}{4\pi^2} \tag{F2}$$

with

$$\mathrm{Im}\{\ln\chi[\xi(u)]\} = \frac{Gu}{\pi}. \tag{F3}$$

Substituting Eq. (F3) into Eq. (C3) we obtain

$$\ln Z(\lambda) = \frac{2aG}{\pi^2}\int u\,du\left[\frac{e^{2u}}{e^{2u}+1} - \frac{e^{2au}}{e^{2au}+1}\right], \tag{F4}$$

where we have set $i\lambda + 1 = a$. Carrying out the integral in Eq. (F4), we obtain the CGF of the entanglement entropy for QPC as

$$\ln Z(\lambda) = \frac{G}{12a}(1-a^2). \tag{F5}$$

Setting $A = \frac{G}{12}$, the MGF of the entanglement entropy is

$$Z(\lambda) = e^{-Ai\lambda(i\lambda+2)/(i\lambda+1)}. \tag{F6}$$

Having the MGF we can calculate the distribution function $p(s)$ of the entanglement entropy as follows:

$$p(s) = \int\frac{d\lambda}{2\pi}e^{A[1/(i\lambda+1)-(i\lambda+1)]}e^{i\lambda s}. \tag{F7}$$

In order to do this integral we expand $e^{A/(i\lambda+1)}$ in power series,

$$e^{A/(i\lambda+1)} = \sum_{k=0}^{\infty}\frac{A^k}{k!(i\lambda+1)^k}. \tag{F8}$$

Substituting Eq. (F8) into Eq. (F7) we have

$$\begin{aligned}p(s) &= \int\frac{d\lambda}{2\pi}\sum_{k=0}^{\infty}\frac{A^k}{k!(i\lambda+1)^k}e^{-A(i\lambda+1)}e^{i\lambda s}\\ &= e^{-s}\delta(s-A) + \int\frac{d\lambda}{2\pi}\sum_{k=1}^{\infty}e^{-A}\frac{A^k}{k!(i\lambda+1)^k}e^{-i\lambda(A-s)}.\end{aligned} \tag{F9}$$

For the second term, the residue of $\frac{1}{(i\lambda+1)^k}e^{-i\lambda(A-s)}$ is given by

$$\begin{aligned}&\mathrm{Res}\left[\frac{1}{(i\lambda+1)^k}e^{-i\lambda(A-s)}\right]\\ &= \lim_{\lambda\to i}\frac{1}{(k-1)!}\frac{d^{k-1}}{d\lambda^{k-1}}\left[(\lambda-i)^k\frac{1}{(k-1)!}\frac{d^{k-1}}{d\lambda^{k-1}}\right]\\ &= \frac{e^{A-s}}{i(k-1)!}(s-A)^{k-1}.\end{aligned} \tag{F10}$$

Then choosing a proper contour and using the residue theorem on the complex plane, we have

$$\int\frac{d\lambda}{2\pi}\frac{e^{-A}A^k}{k!(i\lambda+1)^k}e^{-i\lambda(A-s)} = \theta(s-A)\frac{e^{-s}A^k(s-A)^{k-1}}{k!(k-1)!}. \tag{F11}$$

Combining Eq. (F9) with Eq. (F11) finally we find

$$\begin{aligned}p(s) &= e^{-s}\delta(s-A) + \sum_{k=1}^{\infty}\theta(s-A)\frac{e^{-s}A^k(s-A)^{k-1}}{k!(k-1)!}\\ &= e^{-s}\delta(s-A) + \frac{e^{-s}\theta(s-A)A}{i\sqrt{A(s-A)}}J_1(2i\sqrt{A(s-A)}).\end{aligned} \tag{F12}$$

In the second line of this equation, we have used the expression

$$J_n(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k+n}, \tag{F13}$$

where $J_n(z)$ $(n = 0,1,2,\ldots)$ are Bessel functions of the first kind.

APPENDIX G: DERIVATION OF EQ. (20)

Considering a QPC system with finite bias V in the long-time limit, the process of charge transfer is binomial and the corresponding CGF of charge transfer is

$$\ln \chi(\xi) = \frac{eVt}{h} \ln[1 + T(e^{i\xi} - 1)], \tag{G1}$$

where t is the measuring time and T is the transmission coefficient. In this case the spectrum density defined in Eq. (C1) is easy to find,

$$\mu(z) = \frac{eVt}{h}\delta(z - T). \tag{G2}$$

Substituting into Eq. (C2) and using Eq. (G2), we have

$$\begin{aligned}\ln Z(\lambda) &= \int_0^1 dz\mu(z) \ln\left[\left(\frac{z}{1-z}\right)^{i\lambda+1} + 1\right] \\ &\quad - (i\lambda + 1)\int_0^1 dz\mu(z) \ln\left[\frac{z}{1-z} + 1\right] \\ &= \frac{eVt}{h} \ln[T^{i\lambda+1} + (1 - T)^{i\lambda+1}].\end{aligned} \tag{G3}$$

Thus the MGF of the entanglement entropy is

$$Z(\lambda) = [T^{i\lambda+1} + (1 - T)^{i\lambda+1}]^{eVt/h}. \tag{G4}$$

Similarly to Eq. (F9) we can calculate the distribution function $p(s)$,

$$\begin{aligned}p(s) &= \int \frac{d\lambda}{2\pi} e^{i\lambda s}[T^{i\lambda+1} + (1 - T)^{i\lambda+1}]^{eVt/h} \\ &= \int \frac{d\lambda}{2\pi} e^{i\lambda s} \sum_{k=0}^{\infty} C_N^k T^{(i\lambda+1)(N-k)}(1 - T)^{(i\lambda+1)k} \\ &= \sum_{k=0}^{N} C_N^k T^{N-k}(1 - T)^k \int \frac{d\lambda}{2\pi} e^{i\lambda s} e^{i\lambda(N-k)\ln T} e^{i\lambda k \ln(1-T)} \\ &= \sum_{k=0}^{N} C_N^k T^{N-k}(1 - T)^k \\ &\quad \times \delta[s + (N - k)\ln T + k\ln(1 - T)],\end{aligned} \tag{G5}$$

where we have set $N = \frac{eVt}{h}$ which is the number of charge transfer and C_N^k is the binomial coefficient.

APPENDIX H: UNIVERSAL BEHAVIOR

Considering a single level QD with no bias, using the time-dependent nonequilibrium Green's-function method, we find in the wide-band limit

$$\begin{aligned}G_{\mathrm{D}}^{<}(t,t) &= \int \frac{d\omega}{2\pi}\{\bar{G}_{\mathrm{D}}^{<}(\omega)[1 + e^{-\Gamma t} - 2\cos(\omega - \epsilon_0)te^{-(\Gamma/2)t}]\},\end{aligned} \tag{H1}$$

where $\bar{G}_{\mathrm{D}}^{<}(\omega)$ is the steady-state lesser Green's function of the QD and Γ is the linewidth. Thus the correlation matrix $M_{\mathrm{D}}(t)$ can be directly related to $G_{\mathrm{D}}^{<}(t,t)$ by

$$M_{\mathrm{D}}(t) = -iG_{\mathrm{D}}^{<}(t,t). \tag{H2}$$

When $\Gamma t \ll 1$, the short-time behavior of $M_{\mathrm{D}}(t)$ is found to be

$$M_{\mathrm{D}}(t) = c_0\Gamma t + o(t^2) \tag{H3}$$

and

$$Y_{\mathrm{D}}(t) = \frac{M_{\mathrm{D}}(t)}{1 - M_{\mathrm{D}}(t)} = c_0\Gamma t + o(t^2), \tag{H4}$$

where c_0 is a constant which depends on system parameters such as Γ, V, ϵ_0, and β. Substituting Eq. (H4) into Eq. (B3), it is easy to find

$$\langle\langle \hat{S}_D^k \rangle\rangle \simeq (-1)^k Y_{\mathrm{D}}^{i\lambda+1}\big|_{\lambda=0}(\ln Y_{\mathrm{D}})^k \simeq (-1)^k c_0\Gamma t[\ln(c_0\Gamma t)]^k. \tag{H5}$$

Now we look for the extrema of $\langle\langle \hat{S}_D^k \rangle\rangle$ by means of

$$\frac{\partial}{\partial(\Gamma t)}\langle\langle \hat{S}_D^k \rangle\rangle = 0; \tag{H6}$$

we have

$$c_0\Gamma t = e^{-k} \tag{H7}$$

thus

$$\langle\langle \hat{S}_D^k \rangle\rangle_{\max} = e^{-k}k^k. \tag{H8}$$

Finally we obtain

$$\ln\left[\langle\langle \hat{S}_D^k \rangle\rangle_{\max}\right] = -k + k\ln k \simeq k\ln k \tag{H9}$$

for large k. This equality is universal to characterize the relation among maximum amplitudes of higher-order cumulants of entanglement entropy for the QD system in the transient regime.

[1] C. H. Bennett and D. P. DiVincenzo, Nature (London) **404**, 247 (2000).
[2] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).
[3] G. Refael and J. E. Moore, Phys. Rev. Lett. **93**, 260602 (2004).
[4] X. Jia, A. R. Subramaniam, I. A. Gruzberg, and S. Chakravarty, Phys. Rev. B **77**, 014208 (2008).
[5] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
[6] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B **424**, 443 (1994).
[7] P. Calabrese and J. Cardy, J. Stat. Mech.: Theory Exp. (2004) P06002.

[8] M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[9] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[10] A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B **83**, 075102 (2011).
[11] L. Fidkowski and A. Kitaev, Phys. Rev. B **81**, 134509 (2010).
[12] M. B. Hastings, I. González, A. B. Kallin, and R. G. Melko, Phys. Rev. Lett. **104**, 157201 (2010).
[13] Y. Zhang, T. Grover, and A. Vishwanath, Phys. Rev. Lett. **107**, 067202 (2011).
[14] M. Fagotti, P. Calabrese, and J. E. Moore, Phys. Rev. B **83**, 045110 (2011).
[15] H. F. Song, S. Rachel, C. Flindt, I. Klich, N. Laflorencie, and K. Le Hur, Phys. Rev. B **85**, 035409 (2012).
[16] M. H. Ansari and Y. V. Nazarov, Phys. Rev. B **91**, 174307 (2015).
[17] K. H. Thomas and C. Flindt, Phys. Rev. B **91**, 125406 (2015).
[18] Y. Utsumi, Phys. Rev. B **92**, 165312 (2015).
[19] I. Klich and L. Levitov, Phys. Rev. Lett. **102**, 100502 (2009).
[20] D. N. Zubarev, *Nonequilibrium Statistical Thermodynamics* (Springer, New York, 1995).
[21] Assuming the eigenvalue spectrum of M is m_i, we have $Z(\lambda) = [\prod_j (y_j^{i\lambda+1} + 1)/(y_j + 1)^{i\lambda+1}]$ with $y_j = m_j/(1 - m_j)$.
[22] D. Kambly, C. Flindt, and M. Büttiker, Phys. Rev. B **83**, 075432 (2011).
[23] H. F. Song, C. Flindt, S. Rachel, I. Klich, and K. Le Hur, Phys. Rev. B **83**, 161408 (2011).
[24] L. S. Levitov, H. Lee, and G. B. Lesovik, J. Math. Phys. **37**, 4845 (1996).
[25] G. Gershon, Y. Bomze, E. V. Sukhorukov, and M. Reznikov, Phys. Rev. Lett. **101**, 016803 (2008).
[26] C. Fricke, F. Hohls, N. Sethubalasubramanian, L. Fricke, and R. J. Haug, Appl. Phys. Lett. **96**, 202103 (2010).
[27] C. Fricke, F. Hohls, C. Flindt, and R. J. Haug, Phys. E (Amsterdam, Neth.) **42**, 848 (2010).
[28] C. Flindt, C. Fricke, F. Hohls, T. Novotny, K. Netocny, T. Brandes, and R. J. Haug, Proc. Natl. Acad. Sci. USA **106**, 10116 (2009).