Asymptotic Results for Ruin Probability in a Two-dimensional Risk Model with Stochastic Investment Returns

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Abstract

This paper considers a two-dimensional time-dependent risk model with stochastic investment returns. In the model, an insurer operates two lines of insurance business sharing a common claim number process and can invest its surplus into some risky assets. The claim number process is assumed to be a renewal counting process and the investment return is modeled by a geometric Lévy process. Furthermore, claim sizes of the two insurance businesses and their common inter-arrival times correspondingly follow a three-dimensional Sarmanov distribution. When claim-size distributions of the two insurance business are heavy tailed, we establish some uniform asymptotic estimates for the ruin probability of the model over certain time horizon.

1 Introduction

In this paper, we consider a two-dimensional renewal risk model in which an insurer operates two lines of insurance businesses sharing a common claim-number process. The common claim number process $\{N(t), t \geq 0\}$ is a renewal counting process defined by

$$N_t = \sum_{i=1}^{\infty} \mathbf{1}_{(\tau_i \leq t)}, \quad t \geq 0,$$

where $\{\tau_i, i \geq 1\}$ are the claim arrival times and $\mathbf{1}_E$ denotes the indicator function of an event E. Thus, the inter-arrival times, $\{\theta_i = \tau_i - \tau_{i-1}, i \geq 1\}$ are independent, identically distributed, and nonnegative random variables, with $\tau_0 = 0$ by convention.

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The two-dimensional surplus process of the insurer is described as

$$\begin{pmatrix} U_{1t} \\ U_{2t} \end{pmatrix} = \begin{pmatrix} x\mathrm{e}^{L(t)} \\ y\mathrm{e}^{L(t)} \end{pmatrix} + \begin{pmatrix} c_1 \int_{0-}^{t} \mathrm{e}^{L(t-s)} \mathrm{d}s \\ c_2 \int_{0-}^{t} \mathrm{e}^{L(t-s)} \mathrm{d}s \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{N_t} X_i \mathrm{e}^{L(t-\tau_i)} \\ \sum_{i=1}^{N_t} Y_i \mathrm{e}^{L(t-\tau_i)} \end{pmatrix}, \quad t \geq 0, \tag{1.1}$$

where $(x,y)^\intercal$ is the initial capital vector with $x,y > 0$, $(c_1,c_2)^\intercal$ is the premium collection rate vector with $c_1,c_2 \geq 0$, $\{(X_i,Y_i)^\intercal, i \geq 1\}$ is the sequence of claim size vectors, and $\{\mathrm{e}^{L(t)}, t \geq 0\}$ is the return process of the investment of the insurer's surplus.

In recent years, there are many papers devoted to the ruin problems of risk model (1.1) and its variants and here we only mentioned some of them closely related to this paper. Chan et al. (2003), Yuen et al. (2006), Chen et al. (2011), and Chen et al. (2012) investigated the asymptotic of ruin probabilities of risk model (1.1) without interest force, i.e. $L(t) \equiv 0$ for any $t \geq 0$. Li et al. (2007) and Zhang and Wang (2012) considered the ruin problems of a risk model similar to (1.1) but perturbed by a diffusion and without interest force. Chen et al. (2013) studied the ruin probabilities of risk model (1.1) with a constant interest force, i.e. $L(t) = rt$ for some $r > 0$ and any $t \geq 0$.

All these papers mentioned above assumed that the claim size sequences $\{X_i, i \geq 1\}$ and $\{Y_i, i \geq 1\}$ of the two lines of insurance businesses and their inter-arrival times $\{\theta_i, i \geq 1\}$ are mutually independent. Yang and Li (2014) assumed that claim size vectors $\{(X_i,Y_i)^\intercal, i \geq 1\}$ are independent and identically distributed copies of a generic random vector $(X,Y)^\intercal$ which follows a bivariate Farlie-Gumbel-Morgenstern (FGM) distribution, and investigated the asymptotic of ruin probability of risk model (1.1) with constant interest rate and stochastic premium process. This study are based on the assumption that claim sizes and their inter-arrival times are independent. Such independence assumption is proposed mainly for mathematical tractability and it is unrealistic in reality. In fact, if the deductible retained to insureds is raised, then the inter-arrival time will increase since small claims will be ruled out, while the likelihood of a large claim will increase if claim sizes are new-worse-than-used, and will decrease if claim sizes is new-better-than-used. However, there are few papers devoted to ruin problems of the two-dimensional risk model in (1.1) with dependence between claim size vectors $\{(X_i,Y_i)^\intercal, i \geq 1\}$ and claim inter-arrival times $\{\theta_i, i \geq 1\}$.

In this paper, we assume that $\{(X_i,Y_i,\theta_i)^\intercal, i \geq 1\}$ is a sequence of independent and identically distributed copies of a generic random vector $(X,Y,\theta)^\intercal$ whose distribution is given by

$$\begin{aligned} &\mathbb{P}(X \in \mathrm{d}x, Y \in \mathrm{d}y, \theta \in \mathrm{d}z) \\ &= (1 + \eta_1\varphi_1(x)\varphi_2(y) + \eta_2\varphi_1(x)\varphi_3(z) + \eta_3\varphi_2(y)\varphi_3(z))\,\mathrm{d}F(x)\mathrm{d}G(y)\mathrm{d}H(z), \end{aligned} \tag{1.2}$$

where F, G, H are the corresponding marginal distributions of X, Y, and θ, respectively. The parameters η_1, η_2, and η_3 are real numbers, and the kernels φ_1, φ_2 and φ_3 are functions satisfying

$$\mathbb{E}\varphi_1(X) = \mathbb{E}\varphi_2(Y) = \mathbb{E}\varphi_3(\theta) = 0, \tag{1.3}$$

and

$$1 + \eta_1\varphi_1(x)\varphi_2(y) + \eta_2\varphi_1(x)\varphi_3(z) + \eta_3\varphi_2(y)\varphi_3(z) \geq 0 \tag{1.4}$$

for all $x \in D_X$, $y \in D_Y$ and $z \in D_\theta$ with D_X, D_Y, and D_θ defined as

$$D_X = \{x \geq 0 : \mathbb{P}(X \in (x-\delta, x+\delta)) > 0 \text{ for all } \delta > 0\},$$

$$D_Y = \{y \geq 0 : \mathbb{P}(Y \in (y-\delta, y+\delta)) > 0 \text{ for all } \delta > 0\},$$
$$D_\theta = \{z \geq 0 : \mathbb{P}(\theta \in (z-\delta, z+\delta)) > 0 \text{ for all } \delta > 0\}.$$

For more details of multivariate Sarmanov distributions, one can refer to Lee (1996) and Kotz et al. (2000) among others.

It is worth to point out that all the studies mentioned above only considered the effect of constant interest rates. However, insurers often invest their surplus into certain portfolios consisting of some risk-free and risky assets to obtain higher risky returns. Hence, besides the dependence assumption specified in (1.2) - (1.4), we further assume that the return process $\{e^{L(t)}, t \geq 0\}$ of the investment of insurers is a geometric Lévy process, i.e. $\{L(t), t \geq 0\}$ is a standard Lévy process. This investment model is widely used in mathematical finance, see e.g. Paulsen and Gjessing (1997), Wang and Wu (2001), Heyde and Wang (2009), and Tang, et al. (2010), among others. For the general theory of Lévy processes, we refer to the monographs of Sato (1999) and Cont and Tankov (2004).

Define the finite-time ruin probability of risk model (1.1) as

$$\Psi(x,y;t) = \mathbb{P}\left(T_{\max} \leq t \mid (U_{10}, U_{20})^\intercal = (x,y)^\intercal\right),$$

where

$$T_{\max} = \inf\{t > 0 : \max\{U_{1t}, U_{2t}\} < 0\}$$

denotes the ruin time with $\inf \emptyset = \infty$ by convention.

We focus on the risk model in (1.1) with the dependence structure specified in (1.2)-(1.4) and geometric Lévy investment returns, and aim at some uniform formulas of the finite-time ruin probability over certain time regions as $(x,y)^\intercal$ tends to $(\infty,\infty)^\intercal$.

The rest of the paper consists of three sections. Section 2 introduces some frequently used notations and states the main result, Section 3 establishes some crucial lemmas, and Section 4 proves the main result of the paper.

2 Notations and Main Results

2.1 Notations

In the sequel, let $\{(X_j^*, Y_j^*, \theta_j^*)^\intercal, j \geq 1\}$ be a sequence of independent and identically distributed copies of a generic random triplet $(X^*, Y^*, \theta^*)^\intercal$ whose components X^*, Y^* and θ^* are mutually independent with distribution functions denoted by F, G, and H, respectively.

From the definition of the multivariate Sarmanov distribution specified in (1.2)-(1.4), it is easy to see that the generic claim size vector $(X,Y)^\intercal$ follows a bivariate Sarmanov distribution. Hence, by Proposition 1.1 of Yang and Wang (2013), there exist two positive constants b_1 and b_2 such that $|\varphi_1(x)| \leq b_1$ for all $x \in D_X$ and $|\varphi_2(y)| \leq b_2$ for all $y \in D_Y$. When establishing asymptotic estimate for ruin probability, we need further impose the following assumption on the two functions $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$.

H1. *The two limits* $\lim_{x\to\infty} \varphi_1(x) = d_1$ *and* $\lim_{y\to\infty} \varphi_2(y) = d_2$ *exist and the constant* $1 + \eta_1 d_1 d_2$ *is strictly positive, i.e.* $1 + \eta_1 d_1 d_2 > 0$.

This assumption, together with the definition of the multivariate Sarmanov distribution specified in (1.2)-(1.4), implies that for all $s \in D_\theta$,

$$\hat{\varphi}_3(s) := 1 + \eta_2 d_1 \varphi_3(s) \geq 0, \quad \check{\varphi}_3(s) := 1 + \eta_3 d_2 \varphi_3(s) \geq 0, \tag{2.1}$$

$$\ddot{\varphi}_3(s) := 1 + \frac{\eta_2 d_1 \varphi_3(s) + \eta_3 d_2 \varphi_3(s)}{1 + \eta_1 d_1 d_2} \geq 0. \tag{2.2}$$

Recall that by relation (1.3), $\mathbb{E}\varphi_3(\theta) = 0$. Hence, $\hat{\varphi}_3(\cdot)$, $\check{\varphi}_3(\cdot)$, and $\ddot{\varphi}_3(\cdot)$ are three well-defined probability density functions on D_θ. Define

$$\mathrm{d}\hat{H}(s) = \hat{\varphi}_3(s)\mathrm{d}H(s),\ \ \mathrm{d}\check{H}(s) = \check{\varphi}_3(s)\mathrm{d}H(s),\ \ \mathrm{d}\ddot{H}(s) = \ddot{\varphi}_3(s)\mathrm{d}H(s),\ \ s \in D_\theta, \tag{2.3}$$

and let $\hat{\theta}$, $\check{\theta}$, and $\ddot{\theta}$ be three independent random variables with distributions denoted by $\hat{H}$, $\check{H}$, and $\ddot{H}$, respectively. Furthermore, we assume that $\hat{\theta}$, $\check{\theta}$, and $\ddot{\theta}$ are independent of all the random quantities above.

With $\hat{\theta}$, $\check{\theta}$, $\ddot{\theta}$ and $\{\theta_j^*, j \geq 1\}$ specified as before, we define the following three delayed renewal counting processes as

$$\check{N}_t = \sum_{k=1}^{\infty} \mathbf{1}_{(\check{\tau}_k^* \leq t)} \quad \text{with } \check{\tau}_1^* = \check{\theta},\ \ \check{\tau}_k^* = \check{\theta} + \sum_{i=2}^{k} \theta_i^*,\ \ k \geq 2, \tag{2.4}$$

$$\ddot{N}_t = \sum_{k=1}^{\infty} \mathbf{1}_{(\ddot{\tau}_k^* \leq t)} \quad \text{with } \ddot{\tau}_1^* = \ddot{\theta},\ \ \ddot{\tau}_k^* = \ddot{\theta} + \sum_{i=2}^{k} \theta_i^*,\ \ k \geq 2, \tag{2.5}$$

$$\overline{N}_t = \sum_{i=1}^{\infty} \mathbf{1}_{(\overline{\tau}_k^* \leq t)} \quad \text{with } \overline{\tau}_1^* = \hat{\theta},\ \ \overline{\tau}_2^* = \check{\theta},\ \ \overline{\tau}_k^* = \hat{\theta} + \check{\theta} + \sum_{i=3}^{k} \theta_i^*,\ \ k \geq 3, \tag{2.6}$$

Define the renewal functions of the renewal counting processes $\{N_t, t \geq 0\}$, $\{\check{N}_t, t \geq 0\}$, $\{\ddot{N}_t, t \geq 0\}$, and $\{\overline{N}_t, t \geq 0\}$ respectively as

$$\lambda_t = \mathbb{E}N_t = \sum_{k=1}^{\infty} \mathbb{P}(\tau_k \leq t), \quad \check{\lambda}_t = \mathbb{E}\check{N}_t = \sum_{k=1}^{\infty} \mathbb{P}\left(\check{\tau}_k^* \leq t\right), \tag{2.7}$$

$$\ddot{\lambda}_t = \mathbb{E}\ddot{N}_t = \sum_{k=1}^{\infty} \mathbb{P}\left(\ddot{\tau}_k^* \leq t\right), \quad \overline{\lambda}_t = \mathbb{E}\overline{N}_t = \sum_{k=1}^{\infty} \mathbb{P}\left(\overline{\tau}_k^* \leq t\right), \quad t \geq 0. \tag{2.8}$$

When establishing asymptotic estimate of finite-time ruin probability, it is natural to restrict the region of the variable t to

$$\Lambda = \{t : 0 < \lambda_t \leq \infty\}.$$

With $\underline{t} = \inf\{t : \lambda_t > 0\} = \inf\{t : \mathbb{P}(\tau_1 \leq t) > 0\}$, it is clear that $\Lambda = [\underline{t}, \infty]$ if $\mathbb{P}(\theta = \underline{t}) > 0$, and $\Lambda = (\underline{t}, \infty]$ if $\mathbb{P}(\theta = \underline{t}) = 0$. For ease of notations, write $\Lambda_T = \Lambda \cap [0, T]$ for every fixed $T \in \Lambda$.

Let (γ, σ^2, ν) be the Lévy triplet of $\{L(t), t \geq 0\}$, where $\gamma \in (-\infty, \infty)$, $\sigma \geq 0$ are two constants and ν is a Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{-\infty}^{\infty}(x^2 \wedge 1)\nu(\mathrm{d}x) < \infty$. Define the Laplace exponent of the Lévy process $\{L(t), t \geq 0\}$ as

$$\phi(s) = \log \mathbb{E}[\mathrm{e}^{-sL(1)}], \quad s \in (-\infty, \infty).$$

If $\phi(s)$ is finite, then

$$\phi(s) = -\gamma s + \frac{\sigma^2}{2}s^2 + \int_{-\infty}^{\infty} \left(\mathrm{e}^{-sx} - 1 + sx\mathbf{1}_{(|x| \leq 1)}\right) \nu(\mathrm{d}x),$$

and

$$\mathbb{E}\mathrm{e}^{-sL(t)} = \mathrm{e}^{t\phi(s)} < \infty, \quad t \geq 0;$$

see, e.g. Theorem 25.17 in Sato (1999). For our purpose, we need to impose the following boundedness assumption on the Laplace exponent $\phi(\cdot)$ of the Lévy process $\{L(t), t \geq 0\}$.

H2. *There is some constant $\eta > 0$ such that the Laplace exponent $\phi(\cdot)$ of the Lévy process $\{L(t), t \geq 0\}$ at this point is finite, i.e. $\phi(\eta) \in (-\infty, \infty)$.*

This assumption is easily satisfied by many return processes of risky investment, see e.g. Klüppelberg and Kostadinova (2008), Heyde and Wang (2009), and Guo and Wang (2013).

When investigating the asymptotic of ruin probability, we focus on the case that the marginal distributions F and G of the generic claim size vector $(X,Y)^\intercal$ of the two lines of insurance businesses are heavy-tailed. The involved classes of heavy-tailed distributions are $\mathscr{D}$, $\mathscr{L}$, and $\mathscr{R}$. By definition, a distribution B concentrated on $[0,\infty)$ is said to belong to the class $\mathscr{D}$ of dominatedly-varying-tailed distributions if $\overline{B}(x) > 0$ for all $x \geq 0$ and the relation

$$\limsup_{x\to\infty} \frac{\overline{B}(xy)}{\overline{B}(x)} < \infty$$

holds for any $0 < y < 1$; and a distribution B concentrated on $[0,\infty)$ is said to belong to the class $\mathscr{L}$ of long-tailed distributions if $\overline{B}(x) > 0$ for all $x \geq 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{B}(x-l)}{\overline{B}(x)} = 1 \tag{2.9}$$

holds for any $l \neq 0$. The intersection $\mathscr{D} \cap \mathscr{L}$ is rich enough to contain many useful heavy-tailed distributions and a famous subclass of the intersection is the class $\mathscr{R}$ of regularly-varying-tailed distributions. By definition, a distribution B concentrated on $[0,\infty)$ is said to belong to the class $\mathscr{R}_{-\alpha}$ with tail index $-\alpha$ if $\overline{B}(x) > 0$ for all $x \geq 0$ and there is some $\alpha > 0$ such that for any $y > 0$,

$$\lim_{x\to\infty} \frac{\overline{B}(xy)}{\overline{B}(x)} = y^{-\alpha}. \tag{2.10}$$

We signify the regularity property in (2.10) as $B \in \mathscr{R}_{-\alpha}$, so that $\mathscr{R}$ is the union of all $\mathscr{R}_{-\alpha}$ over the range $0 < \alpha < \infty$. For details of heavy-tailed distributions and their applications to insurance and finance, see, e.g. the monographs of Bingham, et al. (1987) and Embrechts, et al. (1997).

This work is also closely related to a significant indices of heavy-tailed distributions. For any distribution B concentrated on $(-\infty,\infty)$ and any $y > 0$, define

$$\mathbb{J}_B^+ = \inf\left\{-\frac{\log \overline{B}_*(y)}{\log y} : y > 1\right\} = -\lim_{y\to\infty} \frac{\log \overline{B}_*(y)}{\log y} \quad \text{with} \quad \overline{B}_*(y) = \liminf_{x\to\infty} \frac{\overline{B}(xy)}{\overline{B}(x)}.$$

In the terminology of Bingham, et al. (1987), $\mathbb{J}_B^+$ is called the upper Matuszewska index of the function $f(x) = (\overline{B}(x))^{-1}$, $x \geq 0$. Without any confusion, we simply call $\mathbb{J}_B^+$ the upper Matuszewska index of the distribution B. Especially, if $B \in \mathscr{D}$, then $\mathbb{J}_B^+ < \infty$; and if $B \in \mathscr{R}_{-\alpha}$ with $\alpha \geq 0$, then $\mathbb{J}_B^+ = \alpha$.

Hereafter all limit relationships are for $(x,y)^\intercal \to (\infty,\infty)^\intercal$ unless stated otherwise. For two positive functions $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ satisfying

$$l_1 = \liminf_{(x,y)^\intercal\to(\infty,\infty)^\intercal} \inf_{t\in E\neq\emptyset} \frac{a(x,y;t)}{b(x,y;t)} \leq \liminf_{(x,y)^\intercal\to(\infty,\infty)^\intercal} \sup_{t\in E\neq\emptyset} \frac{a(x,y;t)}{b(x,y;t)} = l_2,$$

we say that $a(x,y;t) \asymp b(x,y;t)$ holds uniformly for $t \in E$ if $0 < l_1 \leq l_2 < \infty$; $a(x,y;t) \lesssim b(x,y;t)$ holds uniformly for $t \in E$ if $l_2 \leq 1$; $a(x,y;t) \gtrsim b(x,y;t)$ holds uniformly for $t \in E$ if $l_1 \geq 1$; and $a(x,y;t) \sim b(x,y;t)$ holds uniformly for $t \in E$ if $l_1 = l_2 = 1$.

To our purpose, we also need the following independence assumption.

H3. *All the random sources* $\{(X_j, Y_j, \theta_j)^\intercal, j \geq 1\}$, $\{(X_j^*, Y_j^*, \theta_j^*)^\intercal, j \geq 1\}$, $\{L(t), t \geq 0\}$, *and* $(\hat{\theta}, \check{\theta}, \ddot{\theta})^\intercal$ *are mutually independent.*

2.2 Main Results

Now we are ready to state the main result of the paper.

Theorem 2.1. *Consider the two-dimensional renewal risk model introduced above. If Assumptions **H1**, **H2**, **H3** hold and claim-size distributions F and G of the two lines of insurance businesses belong to the intersection* $\mathscr{D} \cap \mathscr{L}$ *with upper Matuszewska indices* $\mathbb{J}_F^+$ *and* $\mathbb{J}_G^+$ *satisfying* $\mathbb{J}_F^+ + \mathbb{J}_G^+ < \eta$, *then the relation*

$$\Psi(x,y;t) \sim (1+\eta_1 d_1 d_2) \int_{0-}^{t} P_{x,y}(u,u) \mathrm{d}\ddot{\lambda}_u + \iint_{\Omega_{2,t}} (P_{x,y}(u,u+v) + P_{x,y}(u+v,u)) \mathrm{d}\lambda_v \mathrm{d}\overline{\lambda}_u$$
$$+ \iint_{\Omega_{2,t}} (P_{x,y}(u,u+v) + P_{x,y}(u+v,u)) \hat{\varphi}_3(u) (\mathrm{d}\check{\lambda}_v - \mathrm{d}\lambda_v) \mathrm{d}H(u) \tag{2.11}$$

holds uniformly for all $t \in \Lambda_T$ *for arbitrarily fixed* $T \in \Lambda$, *where* $\ddot{\lambda}_u$, $\overline{\lambda}_u$, $\check{\lambda}_v$ *are specified in* (2.7)-(2.8), $\hat{\varphi}_3(u)$ *is specified in* (2.1), $\Omega_{2,t} = \{(u,v)^\intercal : u,v \geq 0, u+v \leq t\}$, *and*

$$P_{x,y}(u,v) = \mathbb{P}\left(X^* \mathrm{e}^{-L(u)} > x, Y^* \mathrm{e}^{-L(v)} > y\right) \quad \textit{for any } u,v \geq 0. \tag{2.12}$$

Specifically, if F and G belong to the class $\mathscr{R}_{-\alpha}$ *for some* $0 \leq \alpha \leq \eta/2$, *then it holds uniformly for all* $t \in \Lambda_T$ *that*

$$\Psi(x,y;t) \sim (1+\eta_1 d_1 d_2) \overline{F}(x) \overline{G}(y) \int_{0-}^{t} \mathrm{e}^{u\phi(2\alpha)} \mathrm{d}\ddot{\lambda}_u + 2\overline{F}(x)\overline{G}(y) \iint_{\Omega_{2,t}} \mathrm{e}^{u\phi(2\alpha)+v\phi(\alpha)} \mathrm{d}\lambda_v \mathrm{d}\overline{\lambda}_u$$
$$+ 2\overline{F}(x)\overline{G}(y) \iint_{\Omega_{2,t}} \mathrm{e}^{u\phi(2\alpha)+v\phi(\alpha)} \hat{\varphi}_3(u) (\mathrm{d}\check{\lambda}_v - \mathrm{d}\lambda_v) \mathrm{d}H(u). \tag{2.13}$$

Remark 2.1. The results in this paper successfully captures the impact of the Sarmanov dependence of claim sizes and their common inter-arrival times on ruin asymptotic. This can be seen from relations (2.11) and (2.13), in which the Sarmanov dependence specified in (1.2)-(1.4) appears as complicate coefficients in the asymptotic formulas of finite-time ruin probability.

Remark 2.2. The Sarmanov dependence of claim sizes and their common inter-arrival times of the two lines of insurance businesses specified in (1.2)-(1.4) includes many dependence structures as its special cases. Among them, one important special case is that claim sizes of the two lines of insurance business are independent of their inter-arrival times and follows a bivariate Sarmanov distributions. Precisely, claim size vector $(X,Y)^\intercal$ is independent of its common inter-arrival time θ, i.e. $\eta_2 = \eta_3 = 0$ or $\varphi_3 \equiv 0$, and follows a bivariate Sarmanov distribution given by

$$\mathbb{P}(X \in \mathrm{d}x, Y \in \mathrm{d}y) = (1+\eta_1 \varphi_1(x) \varphi_2(y)) \mathrm{d}F(x) \mathrm{d}G(y). \tag{2.14}$$

where η_1 is a real number, and the kernels φ_1 and φ_2 are functions satisfying

$$\mathbb{E}\varphi_1(X) = \mathbb{E}\varphi_2(Y) = 0, \tag{2.15}$$

and

$$1 + \eta_1 \varphi_1(x) \varphi_2(y) \geq 0 \tag{2.16}$$

for all $x \in D_X = \{x \geq 0 : \mathbb{P}(X \in (x-\delta, x+\delta)) > 0 \text{ for all } \delta > 0\}$. Furthermore, as pointed out by Tang et al. (2011), if we let $\varphi_1(x) = 1 - 2\overline{F}(x)$ and $\varphi_2(y) = 1 - 2\overline{G}(y)$ in (2.14)-(2.16), then $(X,Y)^\intercal$ follows a bivariate FGM distribution given by

$$\mathbb{P}(X \in \mathrm{d}x, Y \in \mathrm{d}y) = \left[1 + \eta_1 \left(1 - 2\overline{F}(x)\right)\left(1 - 2\overline{G}(y)\right)\right] \mathrm{d}F(x)\mathrm{d}G(y), \quad \eta_1 \in [-1,1]. \tag{2.17}$$

For the two special cases, the results in Theorem 2.1 can be significantly improved, see Corollaries 2.1 and 2.2 below.

Note that if $\eta_2 = \eta_3 = 0$ or $\varphi_3 \equiv 0$, and $(X,Y)^\intercal$ follows a bivariate Sarmanov distribution specified in (2.14)-(2.16), then from the definitions of $\hat{H}$, $\check{H}$, and $\ddot{H}$ in (2.3), we see that $\hat{H} = \check{H} = \ddot{H} = H$. It follows from (2.4)-(2.8) that for any $t \geq 0$,

$$\check{\lambda}_t = \ddot{\lambda}_t = \overline{\lambda}_t = \lambda_t.$$

Hence, by Theorem 2.1, the following corollary holds immediately.

Corollary 2.1. *Consider the two-dimensional renewal risk model introduced above with $\eta_2 = \eta_3 = 0$ or $\varphi_3 \equiv 0$ in* (1.2)-(1.4)*. If Assumptions **H1**, **H2**, **H3** hold and claim-size distributions F and G of the two lines of insurance businesses belong to the intersection $\mathscr{D} \cap \mathscr{L}$ with $\mathbb{J}_F^+ + \mathbb{J}_G^+ < \eta$, then the relation*

$$\begin{aligned}&\Psi(x,y;t)\\&\sim (1+\eta_1 d_1 d_2)\int_{0-}^{t} P_{x,y}(u,u)\mathrm{d}\lambda_u + \iint_{\Omega_{2,t}} (P_{x,y}(u,u+v) + P_{x,y}(u+v,u))\mathrm{d}\lambda_v \mathrm{d}\lambda_u\end{aligned} \tag{2.18}$$

holds uniformly for all $t \in \Lambda_T$ for arbitrarily fixed $T \in \Lambda$ with $\Omega_{2,t} = \{(u,v)^\intercal : u, v \geq 0, u+v \leq t\}$ and $P_{x,y}(u,v)$ specified in (2.12)*. Specifically, if F and G belong to the class $\mathscr{R}_{-\alpha}$ for some $0 \leq \alpha \leq \eta/2$, then it holds uniformly for all $t \in \Lambda_T$ that*

$$\begin{aligned}&\Psi(x,y;t)\\&\sim (1+\eta_1 d_1 d_2)\overline{F}(x)\overline{G}(y)\int_{0-}^{t} \mathrm{e}^{u\phi(2\alpha)}\mathrm{d}\lambda_u + 2\overline{F}(x)\overline{G}(y)\iint_{\Omega_{2,t}} \mathrm{e}^{u\phi(2\alpha)+v\phi(\alpha)}\mathrm{d}\lambda_v \mathrm{d}\lambda_u.\end{aligned} \tag{2.19}$$

In addition, if $(X,Y)^\intercal$ follows a bivariate FGM distribution specified in (2.17) and there is some constant $\delta > 0$ such that $L(t) = \delta t$ for any $t \geq 0$, then by Corollary 2.1, the following corollary holds immediately.

Corollary 2.2. *Consider the risk model* (1.1) *in which $\{(X,Y)^\intercal, (X_i,Y_i)^\intercal, i \geq 1\}$, independent of $\{\theta_i, i \geq 1\}$, is a sequence of independent and identically distributed random vectors following a common FGM distribution specified in* (2.17) *with $\eta_1 \in (-1,1]$. If there is some constant $\delta > 0$ such that $L(t) = \delta t$ for any $t \geq 0$ and claim size distributions F and G of the two lines of insurance businesses belong to the intersection $\mathscr{D} \cap \mathscr{L}$, then the relation*

$$\begin{aligned}\Psi(x,y;t) \sim{}& (1+\eta_1)\int_{0-}^{t} \overline{F}\left(x\mathrm{e}^{\delta u}\right)\overline{G}\left(y\mathrm{e}^{\delta u}\right)\mathrm{d}\lambda_u\\&+ \iint_{\Omega_{2,t}} \left\{\overline{F}\left(x\mathrm{e}^{\delta(u+v)}\right)\overline{G}\left(y\mathrm{e}^{\delta u}\right) + \overline{F}\left(x\mathrm{e}^{\delta u}\right)\overline{G}\left(y\mathrm{e}^{\delta(u+v)}\right)\right\}\mathrm{d}\lambda_v \mathrm{d}\lambda_u\end{aligned} \tag{2.20}$$

holds uniformly for all $t \in \Lambda_T$ for arbitrarily fixed $T \in \Lambda$ with $\Omega_{2,t} = \{(u,v)^\intercal : u, v \geq 0, u+v \leq t\}$ and $P_{x,y}(u,v)$ specified in (2.12)*. Specifically, if F and G belong to the class $\mathscr{R}_{-\alpha}$ for some $\alpha \geq 0$, then it holds uniformly for all $t \in \Lambda_T$ that*

$$\Psi(x,y;t) \sim (1+\eta_1)\overline{F}(x)\overline{G}(y)\int_{0-}^{t} \mathrm{e}^{2\delta u}\mathrm{d}\lambda_u + 2\overline{F}(x)\overline{G}(y)\iint_{\Omega_{2,t}} \mathrm{e}^{\alpha(2u+v)}\mathrm{d}\lambda_v \mathrm{d}\lambda_u. \tag{2.21}$$

Remark 2.3. Yang and Li (2014) studied a similar problem to that in Corollary 2.2. Hence, our work extends theirs in the following three directions: (i) they considered the case of constant interest force while ours considered the case of risky investment; (ii) they used a bivariate FGM dependence specified in (2.17) to model claim size vector $(X,Y)^\intercal$ and assumed that $(X,Y)^\intercal$ is independent of its inter-arrival times θ, while we used a more general three-dimensional Sarmanov distribution to model the dependence of the random vector $(X,Y,\theta)^\intercal$ of claim sizes and their common inter-arrival times of the two lines of insurances businesses; (iii) their results hold for a fixed time t while ours are equipped with local uniformity in time t, which greatly enhances the theoretical and applied interests of the results.

3 Lemmas

To prove Theorem 2.1, we need to recall some well-known results in the literature and establish some crucial lemmas. For any distribution $B \in \mathscr{D}$ and $p > \mathbb{J}_B^+$, Proposition 2.2.1 of Bingham, et al. (1987) shows that there are positive constants C_p and D_p such that

$$\frac{\overline{B}(y)}{\overline{B}(x)} \le C_p \left(\frac{x}{y}\right)^p \tag{3.1}$$

holds uniformly for all $x \ge y \ge D_p$. Fixing the variable y in (3.1) leads to

$$x^{-p} = o(\overline{B}(x)) \quad \text{for any } p > \mathbb{J}_B^+. \tag{3.2}$$

Lemma 3.1. *Let Z and W be two independent and nonnegative random variables with Z distributed by B. If $B \in \mathscr{D}$, then for arbitrarily fixed $\delta > 0$ and $p > \mathbb{J}_B^+$, there is a positive constant C without relation to W and δ such that for all large x,*

$$\mathbb{P}(ZW > \delta x \mid W) \le C\overline{B}(x)\{\delta^{-p}W^p + \mathbf{1}_{(W<\delta)}\}. \tag{3.3}$$

Proof. See Lemma 3.2 of Heyde and Wang (2009). □

Lemma 3.2. *For any distribution $B \in \mathscr{L}$, there is a function $l(\cdot) : (0,\infty) \mapsto (0,\infty)$ satisfying*
(a) $l(x) < x/2$ for all $x > 0$,
(b) $l(x) \to \infty$ as $x \to \infty$,
(c) $l(x)$ is slowly varying at infinity,
such that, for every $c_0, c_1 \ge 0$, the relation

$$\overline{B}(xy \pm c_0 l(x)y \pm c_1 y) \sim \overline{B}(xy)$$

holds uniformly for all $y \in [a,b]$ for arbitrarily fixed $0 < a < b < \infty$.

Proof. By Lemma 4.1 of Li, et al. (2010), there is a slowly varying function $l(\cdot) : (0,\infty) \mapsto (0,\infty)$ satisfying (a)-(c) such that for every $d_0, d_1 \ge 0$,

$$\overline{B}(x \pm d_0 l(x) \pm d_1) \sim \overline{B}(x \pm d_0 l(x)) \sim \overline{B}(x),$$

where at the first step we used the definition of $\mathscr{L}$. Since $l(\cdot)$ is slowly varying at infinity, by Theorem 1.2.1 of Bingham, et al. (1987), the relation

$$l(xy) \sim l(x)$$

holds uniformly for all $y \in [a,b]$ for arbitrarily fixed $0 < a \le b < \infty$. By the two relations, we can conclude the proof. □

In the following, for notational convenience, for every $k \geq 1$ and arbitrarily fixed $t \geq 0$, denote $\vec{\theta}_k = (\theta_1, \ldots, \theta_k)^\intercal$ and $\vec{s}_k = (s_1, \ldots, s_k)^\intercal$, write $t_k = \sum_{i=1}^k s_i$ and define

$$\Omega_{k,t} = \{\vec{s}_k : s_1, \ldots, s_k \geq 0, t_k \leq t\}. \tag{3.4}$$

Let $(\widetilde{X}^*, \widetilde{Y}^*, \widetilde{\theta}^*)^\intercal$ be a random vector consisting independent components and independent of all the random quantities in Assumption **H3**. The components $\widetilde{X}^*$, $\widetilde{Y}^*$, and $\widetilde{\theta}^*$ have distributions $\widetilde{F}$, $\widetilde{G}$, and $\widetilde{H}$ which are defined as

$$\mathrm{d}\widetilde{F} = \left(1 - \frac{\varphi_1}{b_1}\right)\mathrm{d}F,\ \mathrm{d}\widetilde{G} = \left(1 - \frac{\varphi_2}{b_2}\right)\mathrm{d}G,\ \mathrm{d}\widetilde{H} = \left(1 - \frac{\varphi_3}{b_3}\right)\mathrm{d}H, \tag{3.5}$$

where φ_i, $i = 1,2,3$, are the three functions specified in (1.2). Let $\{(\widetilde{X}_j^*, \widetilde{Y}_j^*, \widetilde{\theta}_j^*)^\intercal, j \geq 1\}$ be a sequence of independent and identically distributed copies of the random vector $(\widetilde{X}^*, \widetilde{Y}^*, \widetilde{\theta}^*)^\intercal$.

Lemma 3.3. *Let $l(\cdot)$ be the function specified in Lemma 3.2 and for arbitrarily fixed $\tilde{c}_0$, $\tilde{c}_1$, $\bar{c}_0$, $\bar{c}_1 \geq 0$, write $x' = x - \tilde{c}_0 l(x) - \tilde{c}_1$ and $y' = y - \bar{c}_0 l(y) - \bar{c}_1$. Under the conditions of Theorem 2.1, for arbitrarily fixed $1 \leq i, j \leq k < \infty$ and $t \in \Lambda$, it holds uniformly for all $\vec{s}_k \in \Omega_{k,t}$ that*

$$\mathbb{P}\left(X^* \mathrm{e}^{-L(t_i)} > x', Y^* \mathrm{e}^{-L(t_j)} > y'\right) \sim P_{x,y}(t_i, t_j) \asymp \overline{F}(x)\overline{G}(y), \tag{3.6}$$

$$\mathbb{P}\left(\widetilde{X}^* \mathrm{e}^{-L(t_i)} > x', Y^* \mathrm{e}^{-L(t_j)} > y'\right) \sim \left(1 - \frac{d_1}{b_1}\right) P_{x,y}(t_i, t_j), \tag{3.7}$$

$$\mathbb{P}\left(X^* \mathrm{e}^{-L(t_i)} > x', \widetilde{Y}^* \mathrm{e}^{-L(t_j)} > y'\right) \sim \left(1 - \frac{d_2}{b_2}\right) P_{x,y}(t_i, t_j), \tag{3.8}$$

$$\mathbb{P}\left(\widetilde{X}^* \mathrm{e}^{-L(t_i)} > x', \widetilde{Y}^* \mathrm{e}^{-L(t_j)} > y'\right) \sim \left(1 - \frac{d_1}{b_1}\right)\left(1 - \frac{d_2}{b_2}\right) P_{x,y}(t_i, t_j) \tag{3.9}$$

with $P_{x,y}(t_i, t_j)$ specified in (2.12). Specifically, if $F \in \mathscr{R}_{-\alpha}$ and $G \in \mathscr{R}_{-\alpha}$ for some $0 \leq \alpha \leq \eta/2$, then it holds uniformly for all $\vec{s}_k \in \Omega_{k,t}$ that

$$P_{x,y}(t_i, t_j) \sim \overline{F}(x)\overline{G}(y)\mathrm{e}^{(t_i \wedge t_j)\phi(2\alpha)}\mathrm{e}^{(t_i \vee t_j - t_i \wedge t_j)\phi(\alpha)}. \tag{3.10}$$

Proof. We first prove the uniformity of the second relation in (3.6). Clearly, for all $\vec{s}_k \in \Omega_{k,t}$,

$$0 < \underline{w}_t := \mathrm{e}^{-\sup_{0 \leq s \leq t} L(s)} \leq \mathrm{e}^{-L(t_n)} \leq \mathrm{e}^{-\inf_{0 \leq s \leq t} L(s)} := \overline{w}_t, \quad n = i, j. \tag{3.11}$$

The lemma of Willekens (1987) asserts that for all $t > 0$ and all $u > u_0 > 0$,

$$\mathbb{P}\left(-\inf_{0 \leq s \leq t} L(s) > u\right)\mathbb{P}\left(-\sup_{0 \leq s \leq t} L(s) > -u_0\right) \leq \mathbb{P}\left(-L(t) > u - u_0\right).$$

This, together with the fact $\mathbb{E}\mathrm{e}^{-\nu L(t)} \leq \mathbb{E}^{\nu/\eta}\mathrm{e}^{-\eta L(t)} = \mathrm{e}^{t\nu\phi(\eta)/\eta} < \infty$ for any $0 \leq \nu \leq \eta$, implies that $\mathbb{E}\overline{w}_t^\nu < \infty$. Arbitrarily choose $p > \mathbb{J}_F^+$ and $q > \mathbb{J}_G^+$ such that $p + q \leq \eta$. By Lemma 3.1 and inequality (3.11), we can derive that, for all large x, y and $\vec{s}_k \in \Omega_{k,t}$

$$\begin{aligned}
P_{x,y}(t_i, t_j) &\leq \mathbb{P}\left(X^* \overline{w}_t > x, Y^* \overline{w}_t > y\right) \\
&= \mathbb{E}\left\{\mathbb{P}\left(X^* \overline{w}_t > x \mid \overline{w}_t\right)\mathbb{P}\left(Y^* \overline{w}_t > y \mid \overline{w}_t\right)\right\} \\
&\leq C_p C_q \overline{F}(x)\overline{G}(y)\mathbb{E}\left\{\left(\overline{w}_t^p + \mathbf{1}_{(\overline{w}_t < 1)}\right)\left(\overline{w}_t^q + \mathbf{1}_{(\overline{w}_t < 1)}\right)\right\} \\
&\leq C_p C_q \overline{F}(x)\overline{G}(y)\left\{\mathbb{E}\overline{w}_t^{p+q} + \mathbb{E}\overline{w}_t^p + \mathbb{E}\overline{w}_t^q + 1\right\}
\end{aligned}$$

$$\leq C\overline{F}(x)\overline{G}(y).$$

Arbitrarily choose $a \in (0,1)$ such that $\mathbb{P}(\underline{w}_t \geq a) > 1/2$. Hence, by inequality (3.1), we can derive that, for all large x, y and $\vec{s}_k \in \Omega_{k,t}$,

$$\begin{aligned} P_{x,y}(t_i,t_j) &\geq \mathbb{P}\left(X^*\underline{w}_t > x, Y^*\underline{w}_t > y, \underline{w}_t \geq a\right) \\ &\geq \overline{F}(x/a)\overline{G}(y/a)\mathbb{P}(\underline{w}_t \geq a) \\ &\geq 2^{-1}C_p^{-1}C_q^{-1}\overline{F}(x)\overline{G}(y). \end{aligned}$$

By the two estimates above, we obtain the uniformity of the second relation in (3.6).

Next, we prove the uniformity of the first relation in (3.6). We only prove it for the case $i > j$ since the proofs of the uniformity of the first relation in (3.6) for the cases $i < j$ and $i = j$ are similar or simpler. For arbitrarily fixed $\varepsilon > 0$, choose some small $a \in (0,1)$ and large $b \in (1,\infty)$ such that

$$\mathbb{E}\left\{\left(\overline{w}_t^{p+q} + \overline{w}_t^{p} + \overline{w}_t^{q} + 1\right)\left(\mathbf{1}_{(\overline{w}_t > b)} + \mathbf{1}_{(\underline{w}_t < a)}\right)\right\} < \varepsilon. \tag{3.12}$$

For the fixed a and b, write

$$E_1 = \{\mathrm{e}^{-L(t_j)} < a\},\ E_2 = \{\mathrm{e}^{-L(t_j)} > b\},\ E_3 = \{a \leq \mathrm{e}^{-L(t_j)} \leq b\}, \tag{3.13}$$

and

$$E_1^* = \{\mathrm{e}^{-[L(t_i)-L(t_j)]} < a\},\ E_2^* = \{\mathrm{e}^{-[L(t_i)-L(t_j)]} > b\},\ E_3^* = \{a \leq \mathrm{e}^{-[L(t_i)-L(t_j)]} \leq b\}. \tag{3.14}$$

With the notation E_m, $m = 1,2,3$, specified in (3.13), we can derive that

$$\begin{aligned} \mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)} > x', Y^*\mathrm{e}^{-L(t_j)} > y'\right) &\leq \sum_{m=1}^{3} \mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)} > x', Y^*\mathrm{e}^{-L(t_j)} > y', E_m\right) \\ &:= I_1(x',y';\vec{s}_k) + I_2(x',y';\vec{s}_k) + I_3(x',y';\vec{s}_k). \end{aligned} \tag{3.15}$$

Using the notation E_m^*, $m = 1,2,3$, in (3.14), we can further split $I_3(x',y';\vec{s}_k)$ into three parts.

$$\begin{aligned} I_3(x',y';\vec{s}_k) &\leq \sum_{m=1}^{3} \mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)} > x', Y^*\mathrm{e}^{-L(t_j)} > y', E_3, E_m^*\right) \\ &:= I_{31}(x',y';\vec{s}_k) + I_{32}(x',y';\vec{s}_k) + I_{33}(x',y';\vec{s}_k). \end{aligned} \tag{3.16}$$

By Lemma 3.1, Lemma 3.2 and (3.12), we have, for all large x, y and $\vec{s}_k \in \Omega_{k,t}$,

$$\begin{aligned} I_1(x',y';\vec{s}_k) + I_{31}(x',y';\vec{s}_k) &\leq 2\mathbb{P}\left(X^*\overline{w}_t > x', Y^*\overline{w}_t > y', \underline{w}_t < a\right) \\ &= 2\mathbb{E}\left\{\mathbf{1}_{(\underline{w}_t < a)}\mathbb{P}\left(X^*\overline{w}_t > x' \mid \underline{w}_t, \overline{w}_t\right)\mathbb{P}\left(Y^*\overline{w}_t > y' \mid \underline{w}_t, \overline{w}_t\right)\right\} \\ &\leq C\overline{F}(x')\overline{G}(y')\mathbb{E}\left\{\left(\overline{w}_t^{p} + \mathbf{1}_{(\overline{w}_t < 1)}\right)\left(\overline{w}_t^{q} + \mathbf{1}_{(\overline{w}_t < 1)}\right)\mathbf{1}_{(\underline{w}_t < a)}\right\} \\ &\leq C\overline{F}(x)\overline{G}(y)\mathbb{E}\left\{\left(\overline{w}_t^{p+q} + \overline{w}_t^{p} + \overline{w}_t^{q} + 1\right)\mathbf{1}_{(\underline{w}_t < a)}\right\} \\ &\leq C\varepsilon\overline{F}(x)\overline{G}(y). \end{aligned} \tag{3.17}$$

Similarly, we have, for all large x, y and $\vec{s}_k \in \Omega_{k,t}$,

$$I_2(x',y';\vec{s}_k) + I_{32}(x',y';\vec{s}_k) \leq 2\mathbb{P}\left(X^*\overline{w}_t > x', Y^*\overline{w}_t > y', \overline{w}_t > b\right)$$

$$
\begin{aligned}
&= 2\mathbb{E}\left\{\mathbf{1}_{(\overline{w}_t>b)}\mathbb{P}\left(X^*\overline{w}_t > x' \mid \overline{w}_t\right)\mathbb{P}\left(Y^*\overline{w}_t > y' \mid \overline{w}_t\right)\right\}\\
&\le C\overline{F}(x')\overline{G}(y')\mathbb{E}\left\{\left(\overline{w}_t^p+\mathbf{1}_{(\overline{w}_t<1)}\right)\left(\overline{w}_t^q+\mathbf{1}_{(\overline{w}_t<1)}\right)\mathbf{1}_{(\overline{w}_t>b)}\right\}\\
&\le C\overline{F}(x)\overline{G}(y)\mathbb{E}\left\{\left(\overline{w}_t^{p+q}+\overline{w}_t^p+\overline{w}_t^q+1\right)\mathbf{1}_{(\overline{w}_t>b)}\right\}\\
&\le C\varepsilon\overline{F}(x)\overline{G}(y), \qquad (3.18)
\end{aligned}
$$

For $I_{33}(x',y';\vec{s}_k)$, by Lemma 3.2, it holds uniformly for all $\vec{s}_k \in \Omega_{k,t}$ that

$$
\begin{aligned}
I_{33}(x',y';\vec{s}_k) &= \int_a^b\int_a^b \overline{F}\left(\frac{x'}{uv}\right)\overline{G}\left(\frac{y'}{u}\right)\mathbb{P}\left(\mathrm{e}^{-L(t_j)}\in \mathrm{d}u\right)\mathbb{P}\left(\mathrm{e}^{-[L(t_i)-L(t_j)]}\in \mathrm{d}v\right)\\
&\sim \int_a^b\int_a^b \overline{F}\left(\frac{x}{uv}\right)\overline{G}\left(\frac{y}{u}\right)\mathbb{P}\left(\mathrm{e}^{-L(t_j)}\in \mathrm{d}u\right)\mathbb{P}\left(\mathrm{e}^{-[L(t_i)-L(t_j)]}\in \mathrm{d}v\right) \qquad (3.19)\\
&= \mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x, Y^*\mathrm{e}^{-L(t_j)}>y, E_3, E_3^*\right) = I_{33}(x,y;\vec{s}_k) \qquad (3.20)\\
&\le \mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x, Y^*\mathrm{e}^{-L(t_j)}>y\right) = P_{x,y}(t_i,t_j). \qquad (3.21)
\end{aligned}
$$

By (3.15)-(3.18), (3.21) and the second relation in (3.6), we obtain the upper-bound version of the first relation in (3.6).

We turn to prove the lower-bound version of the first relation in (3.6). In fact, with E_m, E_m^*, $m = 1, 2, 3$, specified in (3.13)-(3.14) and $I_m(\cdot,\cdot;\vec{s}_k)$, $I_{3m}(\cdot,\cdot;\vec{s}_k)$ defined in (3.15)-(3.16), by relation (3.20), it holds uniformly for all $\vec{s}_k \in \Omega_{k,t}$ that

$$
\begin{aligned}
\mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x', Y^*\mathrm{e}^{-L(t_j)}>y'\right) &\ge I_{33}(x',y';\vec{s}_k) \sim I_{33}(x,y;\vec{s}_k)\\
&= \mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x, Y^*\mathrm{e}^{-L(t_j)}>y, E_3, E_3^*\right)\\
&\ge \mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x, Y^*\mathrm{e}^{-L(t_j)}>y\right)\\
&\quad - \sum_{m=1}^{2}\mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x, Y^*\mathrm{e}^{-L(t_j)}>y, E_m\right)\\
&\quad - \sum_{m=1}^{2}\mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x, Y^*\mathrm{e}^{-L(t_j)}>y, E_3, E_m^*\right)\\
&= P_{x,y}(t_i,t_j) - \sum_{m=1}^{2}\left(I_m(x,y;\vec{s}_k)+I_{3m}(x,y;\vec{s}_k)\right). \qquad (3.22)
\end{aligned}
$$

Following the derivation of (3.17)-(3.18) with some obvious modifications, we can obtain that, for all large x,y and all $\vec{s}_k \in \Omega_{k,t}$,

$$
\sum_{m=1}^{2}\left(I_m(x,y;\vec{s}_k)+I_{3m}(x,y;\vec{s}_k)\right) \le C\varepsilon\overline{F}(x)\overline{G}(y). \qquad (3.23)
$$

By (3.22)-(3.23) and the uniformity of the second relation in (3.6), we obtain the lower-bound version of the first relation (3.6).

Next, we prove the uniformity of relations (3.7), (3.8), and (3.9). Recall that $\widetilde{X}^*$ and $\widetilde{Y}^*$ are independent random variables with distributions $\widetilde{F}$ and $\widetilde{G}$, respectively. By (3.5), it is easy to verify that

$$
\overline{\widetilde{F}}(x) = \int_x^\infty\left(1-\frac{\varphi_1(u)}{b_1}\right)\mathrm{d}F(u) \sim \left(1-\frac{d_1}{b_1}\right)\overline{F}(x) \quad \text{as } x\to\infty, \qquad (3.24)
$$

$$\overline{\overline{G}}(y)=\int_y^\infty\left(1-\frac{\varphi_2(v)}{b_2}\right)\mathrm{d}G(v)\sim\left(1-\frac{d_2}{b_2}\right)\overline{G}(y)\quad\text{as } y\to\infty. \tag{3.25}$$

By going along the same lines as in the proof of the uniformity of the first relation in (3.6) and using relations (3.24) and (3.25) when necessary, we can obtain the uniformity of relations (3.7), (3.8), and (3.9).

It remains to prove the uniformity of relation (3.10) when F and G belong to the class $\mathscr{R}_{-\alpha}$ for some $0\le 2\alpha\le\eta$. Without loss of generality, we assume $i>j$. In this case, the uniformity of relation (3.10) reduces to a concise expression. Namely, uniformly for all $\vec{s}_k\in\Omega_{k,t}$,

$$P_{x,y}(t_i,t_j)\sim\overline{F}(x)\overline{G}(y)\mathrm{e}^{t_j\phi(2\alpha)}\mathrm{e}^{(t_i-t_j)\phi(\alpha)}. \tag{3.26}$$

In fact, Theorem 1.5.2 of Bingham, et al. (1987) shows that, for any distribution $B\in\mathscr{R}_{-\alpha}$ with $0<\alpha<\infty$,

$$\lim_{x\to\infty}\sup_{y\in[a,b]}\left|\frac{\overline{B}(xy)}{\overline{B}(x)}-y^{-\alpha}\right|=0. \tag{3.27}$$

Now following the derivations in (3.12)-(3.23) and further applying relation (3.27) to (3.19), we can obtain the uniformity of relation (3.26) and conclude the proof. □

Lemma 3.4. *Let $l(\cdot)$ be the function specified in Lemma 3.2 and for arbitrarily fixed $\tilde{c}_0$, $\tilde{c}_1$, $\bar{c}_0$ and $\bar{c}_1\ge 0$, write $x'=x-\tilde{c}_0l(x)-\tilde{c}_1$ and $y'=y-\bar{c}_0l(y)-\bar{c}_1$. Under the conditions of Theorem 2.1, for arbitrarily fixed $1\le i\ne j\le k<\infty$ and $t\in\Lambda$, the relations*

$$\mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)}>x',Y_i\mathrm{e}^{-L(t_i)}>y'\mid\theta_i=s_i\right)\sim(1+\eta_1d_1d_2)\ddot{\varphi}_3(s_i)P_{x,y}(t_i,t_i), \tag{3.28}$$

$$\mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)}>x',Y_j\mathrm{e}^{-L(t_j)}>y'\mid\theta_i=s_i,\theta_j=s_j\right)\sim\hat{\varphi}_3(s_i)\check{\varphi}_3(s_j)P_{x,y}(t_i,t_j) \tag{3.29}$$

hold uniformly for all $\vec{s}_k\in\Omega_{k,t}$, where $\hat{\varphi}_3(\cdot)$, $\check{\varphi}_3(\cdot)$, and $\ddot{\varphi}_3(\cdot)$ are specified in (2.1)-(2.3)*, and $P_{x,y}(\cdot,\cdot)$ is specified in* (2.12)*.*

Proof. First, we prove the uniformity of relation (3.28). Applying the decomposition

$$\begin{aligned}
&1+\eta_1\varphi_1\varphi_2+\eta_2\varphi_1\varphi_3+\eta_3\varphi_2\varphi_3\\
&=(1+\eta_1b_1b_2+\eta_2b_1b_3+\eta_3b_2b_3)-(\eta_1b_1b_2+\eta_2b_1b_3)\left(1-\frac{\varphi_1}{b_1}\right)\\
&\quad-(\eta_1b_1b_2+\eta_3b_2b_3)\left(1-\frac{\varphi_2}{b_2}\right)-(\eta_2b_1b_3+\eta_3b_2b_3)\left(1-\frac{\varphi_3}{b_3}\right)\\
&\quad+\eta_1b_1b_2\left(1-\frac{\varphi_1}{b_1}\right)\left(1-\frac{\varphi_2}{b_2}\right)+\eta_2b_1b_3\left(1-\frac{\varphi_1}{b_1}\right)\left(1-\frac{\varphi_3}{b_3}\right)\\
&\quad+\eta_3b_2b_3\left(1-\frac{\varphi_2}{b_2}\right)\left(1-\frac{\varphi_3}{b_3}\right)
\end{aligned} \tag{3.30}$$

to the probability $\mathbb{P}(X_i\mathrm{e}^{-L(t_i)}>x',Y_i\mathrm{e}^{-L(t_i)}>y',\theta_i=s_i)$ leads to

$$\begin{aligned}
&\mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)}>x',Y_i\mathrm{e}^{-L(t_i)}>y',\theta_i=s_i\right)\\
&=(1+\eta_1b_1b_2+\eta_2b_1b_3+\eta_3b_2b_3)\mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x',Y^*\mathrm{e}^{-L(t_i)}>y',\theta^*=s_i\right)
\end{aligned}$$

$$
\begin{aligned}
&-(\eta_1 b_1 b_2+\eta_2 b_1 b_3)\mathbb{P}\left(\widetilde{X}^*\mathrm{e}^{-L(t_i)}>x',Y^*\mathrm{e}^{-L(t_i)}>y',\theta^*=s_i\right)\\
&-(\eta_1 b_1 b_2+\eta_3 b_2 b_3)\mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x',\widetilde{Y}^*\mathrm{e}^{-L(t_i)}>y',\theta^*=s_i\right)\\
&-(\eta_2 b_1 b_3+\eta_3 b_2 b_3)\mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x',Y^*\mathrm{e}^{-L(t_i)}>y',\widetilde{\theta}^*=s_i\right)\\
&+\eta_1 b_1 b_2\mathbb{P}\left(\widetilde{X}^*\mathrm{e}^{-L(t_i)}>x',\widetilde{Y}^*\mathrm{e}^{-L(t_i)}>y',\theta^*=s_i\right)\\
&+\eta_2 b_1 b_3\mathbb{P}\left(\widetilde{X}^*\mathrm{e}^{-L(t_i)}>x',Y^*\mathrm{e}^{-L(t_i)}>y',\widetilde{\theta}^*=s_i\right)\\
&+\eta_3 b_2 b_3\mathbb{P}\left(X^*\mathrm{e}^{-L(t_i)}>x',\widetilde{Y}^*\mathrm{e}^{-L(t_i)}>y',\widetilde{\theta}^*=s_i\right). \qquad (3.31)
\end{aligned}
$$

Denote the seven probability terms on the right-hand side of equality (3.31) by $J_l(x',y';\vec{s}_k)$, $l=1,2,\ldots,7$, respectively. By Lemma 3.3, it holds uniformly for all $\vec{s}_k\in\Omega_{k,t}$ that

$$
\begin{aligned}
\frac{J_1(x',y';\vec{s}_k)}{\mathbb{P}(\theta_i=s_i)}&\sim P_{x,y}(t_i,t_i), &(3.32)\\
\frac{J_2(x',y';\vec{s}_k)}{\mathbb{P}(\theta_i=s_i)}&\sim\left(1-\frac{d_1}{b_1}\right)P_{x,y}(t_i,t_i), &(3.33)\\
\frac{J_3(x',y';\vec{s}_k)}{\mathbb{P}(\theta_i=s_i)}&\sim\left(1-\frac{d_2}{b_2}\right)P_{x,y}(t_i,t_i) &(3.34)\\
\frac{J_5(x',y';\vec{s}_k)}{\mathbb{P}(\theta_i=s_i)}&\sim\left(1-\frac{d_1}{b_1}\right)\left(1-\frac{d_2}{b_2}\right)P_{x,y}(t_i,t_i). &(3.35)
\end{aligned}
$$

Noting that for any $z\geq 0$,

$$
\mathbb{P}\left(\widetilde{\theta}^*\in\mathrm{d}z\right)=\left(1-\frac{\varphi_3(z)}{b_3}\right)\mathbb{P}\left(\theta^*\in\mathrm{d}z\right),
$$

by Lemma 3.3, it holds uniformly for all $\vec{s}_k\in\Omega_{k,t}$ that

$$
\begin{aligned}
\frac{J_4(x',y';\vec{s}_k)}{\mathbb{P}(\theta_i=s_i)}&\sim\left(1-\frac{\varphi_3(s_i)}{b_3}\right)P_{x,y}(t_i,t_i), &(3.36)\\
\frac{J_6(x',y';\vec{s}_k)}{\mathbb{P}(\theta_i=s_i)}&\sim\left(1-\frac{d_1}{b_1}\right)\left(1-\frac{\varphi_3(s_i)}{b_3}\right)P_{x,y}(t_i,t_i), &(3.37)\\
\frac{J_7(x',y';\vec{s}_k)}{\mathbb{P}(\theta_i=s_i)}&\sim\left(1-\frac{d_2}{b_2}\right)\left(1-\frac{\varphi_3(s_i)}{b_3}\right)P_{x,y}(t_i,t_i). &(3.38)
\end{aligned}
$$

Plugging (3.32)-(3.38) into (3.31) and performing a slightly tedious calculation give the uniformity of relation (3.28).

Next, we prove the uniformity of relation (3.29). Note that (X_i,θ_i) and (Y_j,θ_j) are independent random vectors for every fixed $i\neq j$. Applying the decomposition

$$
\begin{aligned}
1+\eta_2\varphi_1\varphi_3=(1+\eta_2 b_1 b_3)-\eta_2 b_1 b_3&\left(1-\frac{\varphi_1}{b_1}\right)\\
-\eta_2 b_1 b_3\left(1-\frac{\varphi_3}{b_3}\right)&+\eta_2 b_1 b_3\left(1-\frac{\varphi_1}{b_1}\right)\left(1-\frac{\varphi_3}{b_3}\right) \qquad (3.39)
\end{aligned}
$$

to the probability $\mathbb{P}(X_i\mathrm{e}^{-L(t_i)}>x',Y_j\mathrm{e}^{-L(t_j)}>y',\theta_i=s_i,\theta_j=s_j)$ leads to

$$
\mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)}>x',Y_j\mathrm{e}^{-L(t_j)}>y',\theta_i=s_i,\theta_j=s_j\right)
$$

$$
\begin{aligned}
&= (1+\eta_2 b_1 b_3)\mathbb{P}\left(X_i^* \mathrm{e}^{-L(t_i)} > x', Y_j \mathrm{e}^{-L(t_j)} > y', \theta_i^* = s_i, \theta_j = s_j\right) \\
&\quad - \eta_2 b_1 b_3 \mathbb{P}\left(\widetilde{X}_i^* \mathrm{e}^{-L(t_i)} > x', Y_j \mathrm{e}^{-L(t_j)} > y', \theta_i^* = s_i, \theta_j = s_j\right) \\
&\quad - \eta_2 b_1 b_3 \mathbb{P}\left(X_i^* \mathrm{e}^{-L(t_i)} > x', Y_j \mathrm{e}^{-L(t_j)} > y', \widetilde{\theta}_i^* = s_i, \theta_j = s_j\right) \\
&\quad + \eta_2 b_1 b_3 \mathbb{P}\left(\widetilde{X}_i^* \mathrm{e}^{-L(t_i)} > x', Y_j \mathrm{e}^{-L(t_j)} > y', \widetilde{\theta}_i^* = s_i, \theta_j = s_j\right) \\
&:= (1+\eta_2 b_1 b_3) J_1'(x',y';\vec{s}_k) - \eta_2 b_1 b_3 \left\{ J_2'(x',y';\vec{s}_k) + J_3'(x',y';\vec{s}_k) - J_4'(x',y';\vec{s}_k) \right\}. \qquad (3.40)
\end{aligned}
$$

Furthermore, applying the decomposition

$$
\begin{aligned}
1+\eta_3 \varphi_2 \varphi_3 &= (1+\eta_3 b_2 b_3) - \eta_3 b_2 b_3 \left(1 - \frac{\varphi_2}{b_2}\right) \\
&\quad - \eta_3 b_2 b_3 \left(1 - \frac{\varphi_3}{b_3}\right) + \eta_3 b_2 b_3 \left(1 - \frac{\varphi_2}{b_2}\right)\left(1 - \frac{\varphi_3}{b_3}\right), \qquad (3.41)
\end{aligned}
$$

to $J_1'(x',y';\vec{s}_k)$ in (3.40) leads to

$$
\begin{aligned}
&J_1'(x',y';\vec{s}_k) \\
&= (1+\eta_3 b_2 b_3)\mathbb{P}\left(X_i^* \mathrm{e}^{-L(t_i)} > x', Y_j^* \mathrm{e}^{-L(t_j)} > y', \theta_i^* = s_i, \theta_j^* = s_j\right) \\
&\quad - \eta_3 b_2 b_3 \mathbb{P}\left(X_i^* \mathrm{e}^{-L(t_i)} > x', \widetilde{Y}_j^* \mathrm{e}^{-L(t_j)} > y', \theta_i^* = s_i, \theta_j^* = s_j\right) \\
&\quad - \eta_3 b_2 b_3 \mathbb{P}\left(X_i^* \mathrm{e}^{-L(t_i)} > x', Y_j^* \mathrm{e}^{-L(t_j)} > y', \theta_i^* = s_i, \widetilde{\theta}_j^* = s_j\right) \\
&\quad + \eta_3 b_2 b_3 \mathbb{P}\left(X_i^* \mathrm{e}^{-L(t_i)} > x', \widetilde{Y}_j^* \mathrm{e}^{-L(t_j)} > y', \theta_i^* = s_i, \widetilde{\theta}_j^* = s_j\right) \\
&:= (1+\eta_3 b_2 b_3) J_{11}'(x',y';\vec{s}_k) - \eta_3 b_2 b_3 \left\{ J_{12}'(x',y';\vec{s}_k) + J_{13}'(x',y';\vec{s}_k) - J_{14}'(x',y';\vec{s}_k) \right\}. \qquad (3.42)
\end{aligned}
$$

By Lemma 3.3, it holds uniformly for all $\vec{s}_k \in \Omega_{k,t}$ that

$$
\frac{J_{11}'(x',y';\vec{s}_k)}{\mathbb{P}\left(\theta_i = s_i, \theta_j = s_j\right)} \sim P_{x,y}(t_i,t_j), \qquad (3.43)
$$

$$
\frac{J_{12}'(x',y';\vec{s}_k)}{\mathbb{P}\left(\theta_i = s_i, \theta_j = s_j\right)} \sim \left(1 - \frac{d_2}{b_2}\right) P_{x,y}(t_i,t_j), \qquad (3.44)
$$

$$
\frac{J_{13}'(x',y';\vec{s}_k)}{\mathbb{P}\left(\theta_i = s_i, \theta_j = s_j\right)} \sim \left(1 - \frac{\varphi_3(s_j)}{b_3}\right) P_{x,y}(t_i,t_j), \qquad (3.45)
$$

$$
\frac{J_{14}'(x',y';\vec{s}_k)}{\mathbb{P}\left(\theta_i = s_i, \theta_j = s_j\right)} \sim \left(1 - \frac{d_2}{b_2}\right)\left(1 - \frac{\varphi_3(s_j)}{b_3}\right) P_{x,y}(t_i,t_j). \qquad (3.46)
$$

Plugging (3.43)-(3.46) into (3.42) gives that, uniformly for all $\vec{s}_k \in \Omega_{k,t}$,

$$
\frac{J_1'(x',y';\vec{s}_k)}{\mathbb{P}\left(\theta_i = s_i, \theta_j = s_j\right)} \sim \left(1 + \eta_3 d_2 \varphi_3(s_j)\right) P_{x,y}(t_i,t_j) = \check{\varphi}_3(s_j) P_{x,y}(t_i,t_j). \qquad (3.47)
$$

Similar to the derivations in (3.41)-(3.47), we obtain, uniformly for all $\vec{s}_k \in \Omega_{k,t}$,

$$
\frac{J_2'(x',y';\vec{s}_k)}{\mathbb{P}\left(\theta_i = s_i, \theta_j = s_j\right)} \sim \left(1 - \frac{d_1}{b_1}\right) \check{\varphi}_3(s_j) P_{x,y}(t_i,t_j), \qquad (3.48)
$$

$$\frac{J_3'(x',y';\vec{s}_k)}{\mathbb{P}\left(\theta_i=s_i,\theta_j=s_j\right)}\sim\left(1-\frac{\varphi_3(s_i)}{b_3}\right)\check{\varphi}_3(s_j)P_{x,y}(t_i,t_j), \tag{3.49}$$

$$\frac{J_4'(x',y';\vec{s}_k)}{\mathbb{P}\left(\theta_i=s_i,\theta_j=s_j\right)}\sim\left(1-\frac{d_1}{b_1}\right)\left(1-\frac{\varphi_3(s_i)}{b_3}\right)\check{\varphi}_3(s_j)P_{x,y}(t_i,t_j). \tag{3.50}$$

Plugging (3.47)-(3.50) into (3.40) gives the uniformity of relation (3.29). This ends the proof. □

Following the method used in the proof of Lemmas 3.3-3.4 with some obvious modifications, we can obtain the following lemma.

Lemma 3.5. *Let Θ be a nonnegative random variable independent of $(X,Y,\theta)^\intercal$ and $(X^*,Y^*,\theta^*)^\intercal$. Under the conditions of Theorem 2.1, if there is some constant $\eta_1>\mathbb{J}_F^++\mathbb{J}_G^+$ such that $\mathbb{E}\Theta^{\eta_1}<\infty$, then for arbitrarily fixed $i\geq 1$ and $T\in\Lambda$, it holds uniformly for all $u\in\Lambda_T$ that*

$$\mathbb{P}\left(X_i\Theta>x\mid\theta_i=u\right)\sim\hat{\varphi}_3(u)\mathbb{P}\left(X^*\Theta>x\right)\quad \text{as } x\to\infty,$$
$$\mathbb{P}\left(Y_i\Theta>y\mid\theta_i=u\right)\sim\check{\varphi}_3(u)\mathbb{P}\left(Y^*\Theta>y\right)\quad \text{as } y\to\infty,$$
$$\mathbb{P}\left(X_i\Theta>x,Y_i\Theta>y\mid\theta_i=u\right)\sim(1+\eta_1d_1d_2)\ddot{\varphi}_3(u)\mathbb{P}\left(X^*\Theta>x,Y^*\Theta>y\right),$$

and for every $1\leq i\neq j<\infty$, it holds uniformly for all $(u,v)\in\Omega_{2,T}$ that

$$\mathbb{P}\left(X_i\Theta>x,Y_j\Theta>y\mid\theta_i=u,\theta_j=v\right)\sim\hat{\varphi}_3(u)\check{\varphi}_3(v)\mathbb{P}\left(X^*\Theta>x,Y^*\Theta>y\right),$$

where $\hat{\varphi}_3(\cdot)$, $\check{\varphi}_3(\cdot)$, and $\ddot{\varphi}_3(\cdot)$ are defined in (2.1)-(2.3).

For any $t\geq 0$, define

$$Z_t=\int_{0-}^t \mathrm{e}^{-L(s)}\mathrm{d}s. \tag{3.51}$$

Lemma 3.6. *Under the conditions of Theorem 2.1, for arbitrarily fixed $k\geq 1$, $T\in\Lambda$, and $\tilde{c}_1,\tilde{c}_2\geq 0$, it holds uniformly for all $t\in\Lambda_T$ that*

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^k X_i\mathrm{e}^{-L(\tau_i)}-\tilde{c}_1Z_t>x,\sum_{j=1}^k Y_j\mathrm{e}^{-L(\tau_j)}-\tilde{c}_2Z_t>y,N_t=k\right)\\
&\sim(1+\eta_1d_1d_2)\sum_{i=1}^k\mathbb{P}\left(X^*\mathrm{e}^{-L(\ddot{\tau}_i^*)}>x,Y^*\mathrm{e}^{-L(\ddot{\tau}_i^*)}>y,\ddot{N}_t=k\right)\\
&\quad+\sum_{1\leq i\neq j\leq k}\mathbb{P}\left(X^*\mathrm{e}^{-L(\overline{\tau}_i^*)}>x,Y^*\mathrm{e}^{-L(\overline{\tau}_j^*)}>y,\overline{N}_t=k\right)
\end{aligned} \tag{3.52}$$

with $\{\ddot{\tau}_j^*,\overline{\tau}_j^*,j\geq 1\}$, $\{\ddot{N}_t,\overline{N}_t,t\geq 1\}$ specified in (2.5)-(2.6).

Proof. We only prove the uniformity of relation (3.52) when $k\geq 2$. The proof of the uniformity of relation (3.52) when $k=1$ is similar but simpler. Trivially, for any $t\in\Lambda_T$ and $k\geq 2$,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^k X_i\mathrm{e}^{-L(\tau_i)}-\tilde{c}_1Z_t>x,\sum_{j=1}^k Y_j\mathrm{e}^{-L(\tau_j)}-\tilde{c}_2Z_t>y,N_t=k\right)\\
&=\int\cdots\int_{\Omega_{k,t}}\mathbb{P}\left(\sum_{i=1}^k X_i\mathrm{e}^{-L(t_i)}-\tilde{c}_1Z_t>x,\sum_{j=1}^k Y_j\mathrm{e}^{-L(t_j)}-\tilde{c}_2Z_t>y\,\middle|\,\vec{\theta}_k=\vec{s}_k\right)\overline{H}(t-t_k)\prod_{l=1}^k\mathrm{d}H(s_l)
\end{aligned}$$

$$:= \int \cdots \int_{\Omega_{k,t}} K(x,y;\vec{s}_k,t)\overline{H}(t-t_k)\prod_{l=1}^{k}\mathrm{d}H(s_l). \tag{3.53}$$

Write $x' = x - l(x)$ and $y' = y - l(y)$ with the function $l(\cdot)$ specified as in Lemma 3.2. Trivially,

$$\begin{aligned}
&K(x,y;\vec{s}_k,t)\\
&\le \mathbb{P}\left(\sum_{i=1}^{k} X_i\mathrm{e}^{-L(t_i)} > x, \sum_{j=1}^{k} Y_j\mathrm{e}^{-L(t_j)} > y \,\Big|\, \vec{\theta}_k = \vec{s}_k\right)\\
&\le \mathbb{P}\left(\bigcup_{i=1}^{k}\{X_i\mathrm{e}^{-L(t_i)} > x'\}, \bigcup_{j=1}^{k}\{Y_j\mathrm{e}^{-L(t_j)} > y'\} \,\Big|\, \vec{\theta}_k = \vec{s}_k\right)\\
&\quad + \mathbb{P}\left(\bigcup_{i=1}^{k}\{X_i\mathrm{e}^{-L(t_i)} > x'\}, \sum_{j=1}^{k} Y_j\mathrm{e}^{-L(t_j)} > y, \bigcap_{n=1}^{k}\{Y_n\mathrm{e}^{-L(t_n)} \le y'\} \,\Big|\, \vec{\theta}_k = \vec{s}_k\right)\\
&\quad + \mathbb{P}\left(\sum_{i=1}^{k} X_i\mathrm{e}^{-L(t_i)} > x, \bigcap_{m=1}^{k}\{X_m\mathrm{e}^{-L(t_m)} \le x'\}, \bigcup_{j=1}^{k}\{Y_j\mathrm{e}^{-L(t_j)} > y' \,\Big|\, \vec{\theta}_k = \vec{s}_k\right)\\
&\quad + \mathbb{P}\left(\sum_{i=1}^{k} X_i\mathrm{e}^{-L(t_i)} > x, \sum_{j=1}^{k} Y_j\mathrm{e}^{-L(t_j)} > y, \bigcap_{1\le m,n\le k}\{X_m\mathrm{e}^{-L(t_m)} \le x', Y_n\mathrm{e}^{-L(t_n)} \le y'\} \,\Big|\, \vec{\theta}_k = \vec{s}_k\right)\\
&:= K_1(x,y;\vec{s}_k) + K_2(x,y;\vec{s}_k) + K_3(x,y;\vec{s}_k) + K_4(x,y;\vec{s}_k).
\end{aligned} \tag{3.54}$$

For $K_1(x,y;\vec{s}_k)$, by Lemma 3.4, it holds uniformly for all $\vec{s}_k \in \Omega_{k,T}$ that

$$\begin{aligned}
K_1(x,y;\vec{s}_k) &\le \sum_{i=1}^{k}\sum_{j=1}^{k} \mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)} > x', Y_j\mathrm{e}^{-L(t_j)} > y' \,\Big|\, \vec{\theta}_k = \vec{s}_k\right)\\
&\sim (1+\eta_1 d_1 d_2)\sum_{i=1}^{k} \ddot{\varphi}_3(s_i)P_{x,y}(t_i,t_i) + \sum_{1\le i\ne j\le k} \hat{\varphi}_3(s_i)\check{\varphi}_3(s_j)P_{x,y}(t_i,t_j)\\
&:= \widetilde{P}_{x,y}(\vec{s}_k).
\end{aligned} \tag{3.55}$$

Take $p > \mathbb{J}_F^+$ and $q > \mathbb{J}_G^+$ such that $p+q \le \eta$. Recall that $\mathbb{E}\overline{w}_T^{v} < \infty$ for any $0 \le v \le \eta$. Hence, for arbitrarily fixed $\varepsilon > 0$, choose some $b > 0$ large enough such that

$$\mathbb{E}\left\{\mathbf{1}_{(\overline{w}_T > b)}(\overline{w}_T^{p}+1)(\overline{w}_T^{q}+1)\right\} < \varepsilon. \tag{3.56}$$

For the fixed b, by (3.11), we have, with $k_1 = k^{-1}$ and $k_2 = (k-1)^{-1}$,

$$\begin{aligned}
&K_2(x,y;\vec{s}_k)\\
&\le \sum_{1\le i,j,n\le k, j\ne n} \mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)} > x', Y_j\mathrm{e}^{-L(t_j)} > k_2 l(y), Y_n\mathrm{e}^{-L(t_n)} > k_1 y \mid \vec{\theta}_k = \vec{s}_k\right)\\
&\le \sum_{1\le i,j,n\le k, j\ne n} \mathbb{P}\left(X_i\overline{w}_T > x', Y_j\overline{w}_T > k_2 l(y), Y_n\overline{w}_T > k_1 y \mid \vec{\theta}_k = \vec{s}_k\right)\\
&\le \sum_{1\le i,j,n\le k, j\ne n} \mathbb{P}\left(X_i > x'/b, Y_j > k_2 l(y)/b, Y_n > k_1 y/b \mid \vec{\theta}_k = \vec{s}_k\right)\\
&\quad + \sum_{1\le i,j,n\le k, j\ne n} \mathbb{P}\left(X_i\overline{w}_T > x', Y_j\overline{w}_T > k_2 l(y), Y_n\overline{w}_T > k_1 y, \overline{w}_T > b \mid \vec{\theta}_k = \vec{s}_k\right)
\end{aligned}$$

$$:= K_{21}(x,y;\vec{s}_k) + K_{22}(x,y;\vec{s}_k). \tag{3.57}$$

By Lemma 3.5, we can derive that, for all large x,y and $\vec{s}_k \in \Omega_{k,T}$,

$$\begin{aligned}
& K_{21}(x,y;\vec{s}_k) \\
& \le \sum_{1\le i=j\ne n\le k} \mathbb{P}\left(X_i > x'/b, Y_i > k_2 l(y)/b \mid \theta_i = s_i\right) \mathbb{P}\left(Y_n > k_1 y/b \mid \theta_n = s_n\right) \\
& \quad + \sum_{1\le i=n\ne j\le k} \mathbb{P}\left(X_i > x'/b, Y_i > k_1 y/b \mid \theta_i = s_i\right) \mathbb{P}\left(Y_j > k_2 l(y)/b \mid \theta_j = s_j\right) \\
& \quad + \sum_{1\le i\ne j\ne n\le k} \mathbb{P}\left(X_i > x'/b \mid \theta_i = s_i\right) \mathbb{P}\left(Y_j > k_2 l(y)/b \mid \theta_j = s_j\right) \mathbb{P}\left(Y_n > k_1 y/b \mid \theta_j = s_j\right) \\
& \le C\overline{F}(x'/b)\overline{G}(k_2 l(y)/b)\overline{G}(k_1 y/b) \sum_{1\le i\ne n\le k} (1+\eta_1 d_1 d_2)\ddot{\varphi}_3(s_i)\check{\varphi}_3(s_n) \\
& \quad + C\overline{F}(x'/b)\overline{G}(k_1 y/b)\overline{G}(k_2 l(y)/b) \sum_{1\le i\ne j\le k} (1+\eta_1 d_1 d_2)\ddot{\varphi}_3(s_i)\check{\varphi}_3(s_j) \\
& \quad + C\overline{F}(x'/b)\overline{G}(k_2 l(y)/b)\overline{G}(k_1 y/b) \sum_{1\le i\ne j\ne n\le k} \hat{\varphi}_3(s_i)\check{\varphi}_3(s_j)\check{\varphi}_3(s_n) \\
& \le C\varepsilon\overline{F}(x)\overline{G}(y)\sum_{i=1}^{k}(1+\eta_1 d_1 d_2)\ddot{\varphi}_3(s_i) + C\varepsilon\overline{F}(x)\overline{G}(y)\sum_{1\le i\ne j\le k}\hat{\varphi}_3(s_i)\check{\varphi}_3(s_j) \\
& \le C\varepsilon\widetilde{P}_{x,y}(\vec{s}_k),
\end{aligned} \tag{3.58}$$

where at the third step we used inequality (3.1), Lemma 3.2, and the boundedness of $\check{\varphi}_3(\cdot)$, while at the last step we used relation (3.6). Note that by (3.1), Lemma 3.3(i), and (3.56), we have, for all large x,y,

$$\begin{aligned}
& \mathbb{P}\left(X^*\overline{w}_T > x', Y^*\overline{w}_T > k_1 y, \overline{w}_T > b\right) \\
& = \mathbb{E}\left\{\mathbf{1}_{(\overline{w}_T > b)}\mathbb{P}(X^*\overline{w}_T > x' \mid \overline{w}_T)\mathbb{P}(Y^*\overline{w}_T > k_1 y \mid \overline{w}_T)\right\} \\
& \le C\overline{F}(x')\overline{G}(k_1 y)\mathbb{E}\left\{\mathbf{1}_{(\overline{w}_T > b)}(\overline{w}_T^p + 1)(\overline{w}_T^q + 1)\right\} \\
& \le C\varepsilon\overline{F}(x)\overline{G}(y).
\end{aligned} \tag{3.59}$$

Hence, by Lemma 3.5 and relation (3.6) in Lemma 3.3, we can derive that, for all large x,y,

$$\begin{aligned}
K_{22}(x,y;\vec{s}_k) & \le C\sum_{1\le i=n\le k}\mathbb{P}\left(X_i\overline{w}_T > x', Y_i\overline{w}_T > k_1 y, \overline{w}_T > b \mid \theta_i = s_i\right) \\
& \quad + C\sum_{1\le i\ne n\le k}\mathbb{P}\left(X_i\overline{w}_T > x', Y_n\overline{w}_T > k_1 y, \overline{w}_T > b \mid \theta_i = s_i, \theta_n = s_n\right) \\
& \le C\sum_{1\le i=n\le k}(1+\eta_1 d_1 d_2)\ddot{\varphi}_3(s_i)\mathbb{P}\left(X^*\overline{w}_T > x', Y^*\overline{w}_T > k_1 y, \overline{w}_T > b\right) \\
& \quad + C\sum_{1\le i\ne n\le k}\hat{\varphi}_3(s_i)\check{\varphi}_3(s_n)\mathbb{P}\left(X^*\overline{w}_T > x', Y^*\overline{w}_T > k_1 y, \overline{w}_T > b\right) \\
& \le C\varepsilon\overline{F}(x)\overline{G}(y)\sum_{i=1}^{k}(1+\eta_1 d_1 d_2)\ddot{\varphi}_3(s_i) + C\varepsilon\overline{F}(x)\overline{G}(y)\sum_{1\le i\ne n\le k}\hat{\varphi}_3(s_i)\check{\varphi}_3(s_n) \\
& \le C\varepsilon\widetilde{P}_{x,y}(\vec{s}_k).
\end{aligned} \tag{3.60}$$

It follows from (3.57)-(3.60) that, for all large x,y and $\vec{s}_k \in \Omega_{k,T}$,

$$K_2(x,y;\vec{s}_k) \le C\varepsilon\widetilde{P}_{x,y}(\vec{s}_k). \tag{3.61}$$

A similar derivation to that in (3.57)-(3.61) gives that, for all large x,y and $\vec{s}_k \in \Omega_{k,T}$,

$$K_3(x,y;\vec{s}_k) \le C\varepsilon \widetilde{P}_{x,y}(\vec{s}_k). \tag{3.62}$$

Let b be the positive constant specified in (3.56). We have,

$$\begin{aligned}
&K_4(x,y;\vec{s}_k)\\
&\le \sum_{\substack{1\le i\neq m\le k,\\ 1\le j\neq n\le k}} \mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)} > k_2 l(x), X_m\mathrm{e}^{-L(t_m)} > k_1 x, Y_j\mathrm{e}^{-L(t_j)} > k_2 l(y), Y_n\mathrm{e}^{-L(t_n)} > k_1 y \mid \vec{\theta}_k = \vec{s}_k\right)\\
&\le \sum_{\substack{1\le i\neq m\le k,\\ 1\le j\neq n\le k}} \mathbb{P}\left(X_i\overline{w}_T > k_2 l(x), X_m\overline{w}_T > k_1 x, Y_j\overline{w}_T > k_2 l(y), Y_n\overline{w}_T > k_1 y \mid \vec{\theta}_k = \vec{s}_k\right)\\
&\le \sum_{\substack{1\le i\neq m\le k,\\ 1\le j\neq n\le k}} \mathbb{P}\left(X_i > k_2 l(x)/b, X_m > k_1 x/b, Y_j > k_2 l(y)/b, Y_n > k_1 y/b \mid \vec{\theta}_k = \vec{s}_k\right)\\
&+ \sum_{\substack{1\le i\neq m\le k,\\ 1\le j\neq n\le k}} \mathbb{P}\left(X_i\overline{w}_T > k_2 l(x), X_m\overline{w}_T > k_1 x, Y_j\overline{w}_T > k_2 l(y), Y_n\overline{w}_T > k_1 y, \overline{w}_T > b \mid \vec{\theta}_k = \vec{s}_k\right)\\
&:= K_{41}(x,y;\vec{s}_k) + K_{42}(x,y;\vec{s}_k).
\end{aligned} \tag{3.63}$$

For $K_{41}(x,y;\vec{s}_k)$, similar to the derivation in (3.58), we can obtain, for all large x,y and $\vec{s}_k \in \Omega_{k,T}$,

$$\begin{aligned}
&K_{41}(x,y;\vec{s}_k)\\
&\le \sum_{1\le i\neq m=n\neq j\le k} \mathbb{P}\left(X_i > k_2 l(x)/b \mid \theta_i = s_i\right)\mathbb{P}\left(X_m > k_1 x/b, Y_m > k_1 y/b \mid \theta_m = s_m\right)\\
&+ \sum_{\substack{1\le i\neq m, j\neq n\le k,\\ m\neq n, i=n}} \mathbb{P}\left(X_m > k_1 x/b \mid \theta_m = s_m\right)\mathbb{P}\left(X_n > k_2 l(x)/b, Y_n > k_1 y/b \mid \theta_n = s_n\right)\\
&+ \sum_{\substack{1\le i\neq m, j\neq n\le k,\\ m\neq n, i\neq n}} \mathbb{P}\left(X_i > k_2 l(x)/b \mid \theta_i = s_i\right)\mathbb{P}\left(X_m > k_1 x/b \mid \theta_m = s_m\right)\mathbb{P}\left(Y_n > k_1 y/b \mid \theta_n = s_n\right)\\
&\le C\varepsilon \widetilde{P}_{x,y}(\vec{s}_k).
\end{aligned} \tag{3.64}$$

For $K_{42}(x,y;\vec{s}_k)$, similar to the derivation in (3.59)-(3.60), we have, for all large x,y and $\vec{s}_k \in \Omega_{k,T}$,

$$\begin{aligned}
K_{42}(x,y;\vec{s}_k) &\le C\sum_{1\le m=n\le k} \mathbb{P}\left(X_m\overline{w}_T > k_1 x, Y_m\overline{w}_T > k_1 y, \overline{w}_T > b \mid \theta_m = s_m\right)\\
&\quad + C\sum_{1\le m\neq n\le k} \mathbb{P}\left(X_m\overline{w}_T > k_1 x, Y_n\overline{w}_T > k_1 y, \overline{w}_T > b \mid \theta_m = s_m, \theta_n = s_n\right)\\
&\le C\varepsilon \widetilde{P}_{x,y}(\vec{s}_k).
\end{aligned} \tag{3.65}$$

By (3.63)-(3.65), we have, for all large x,y and $\vec{s}_k \in \Omega_{k,T}$,

$$K_4(x,y;\vec{s}_k) \le C\varepsilon \widetilde{P}_{x,y}(\vec{s}_k). \tag{3.66}$$

Combining (3.53)-(3.55), (3.61)-(3.62), and (3.66) gives that, uniformly for all $t \in \Lambda_T$,

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i\mathrm{e}^{-L(\tau_i)} > x, \sum_{j=1}^{k} Y_j\mathrm{e}^{-L(\tau_j)} > y, N_t = k\right)$$

$$
\begin{aligned}
&\lesssim \int_{\Omega_{k,t}}\cdots\int \widetilde{P}_{x,y}(\vec{s}_k)\overline{H}(t-t_k)\prod_{l=1}^{k}\mathrm{d}H(s_l)\\
&=(1+\eta_1 d_1 d_2)\sum_{i=1}^{k}\mathbb{P}\left(X^*\mathrm{e}^{-L(\ddot{\tau}_i^*)}>x, Y^*\mathrm{e}^{-L(\ddot{\tau}_i^*)}>y, \ddot{N}_t=k\right)\\
&\quad+\sum_{1\le i\ne j\le k}\mathbb{P}\left(X^*\mathrm{e}^{-L(\overline{\tau}_i^*)}>x, Y^*\mathrm{e}^{-L(\overline{\tau}_j^*)}>y, \overline{N}_t=k\right).
\end{aligned}
$$

This completes the proof of the upper-bound version of relation (3.52).

It remains to prove the lower-bound version of relation (3.52). Let $b>0$ be the constant specified in (3.56) and take $D>0$ such that $D/T>b$. With $\tilde{x}=x+\tilde{c}_1D$, $\tilde{y}=y+\tilde{c}_2D$, we have

$$
\begin{aligned}
K(x,y;\vec{s}_k,t) &\ge \sum_{i=1}^{k}\sum_{j=1}^{k}\mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)}>\tilde{x}, Y_j\mathrm{e}^{-L(t_j)}>\tilde{y}\mid \vec{\theta}_k=\vec{s}_k\right)\\
&\quad-\sum_{i=1}^{k}\sum_{j=1}^{k}\mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)}>\tilde{x}, Y_j\mathrm{e}^{-L(t_j)}>\tilde{y}, Z_t>D\mid \vec{\theta}_k=\vec{s}_k\right)\\
&\quad-\sum_{1\le i,j,m\le k, i\ne m}\mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)}>\tilde{x}, X_m\mathrm{e}^{-L(t_m)}>\tilde{x}, Y_j\mathrm{e}^{-L(t_j)}>\tilde{y}\mid \vec{\theta}_k=\vec{s}_k\right)\\
&\quad-\sum_{1\le i,j,n\le k, j\ne n}\mathbb{P}\left(X_i\mathrm{e}^{-L(t_i)}>\tilde{x}, Y_j\mathrm{e}^{-L(t_j)}>\tilde{y}, Y_n\mathrm{e}^{-L(t_n)}>\tilde{y}\mid \vec{\theta}_k=\vec{s}_k\right)\\
&:= K_1'(\tilde{x},\tilde{y};\vec{s}_k)-K_2'(\tilde{x},\tilde{y};\vec{s}_k,t)-K_3'(\tilde{x},\tilde{y};\vec{s}_k)-K_4'(\tilde{x},\tilde{y};\vec{s}_k).
\end{aligned} \tag{3.67}
$$

By Lemma 3.4, it holds uniformly for all $\vec{s}_k\in\Omega_{k,T}$ that

$$
\begin{aligned}
K_1'(\tilde{x},\tilde{y};\vec{s}_k) &\sim (1+\eta_1 d_1 d_2)\sum_{1\le i\le k}\ddot{\varphi}_3(s_i)P_{x,y}(t_i,t_i)+\sum_{1\le i\ne j\le k}\hat{\varphi}_3(s_i)\check{\varphi}_3(s_j)P_{x,y}(t_i,t_j)\\
&=\widetilde{P}_{x,y}(\vec{s}_k).
\end{aligned} \tag{3.68}
$$

Noting that $Z_T\le \overline{w}_T\cdot T$ and $D/T>b$, we have, for all $\vec{s}_k\in\Omega_{k,T}$ and $t\in\Lambda_T$,

$$
\begin{aligned}
K_2'(\tilde{x},\tilde{y};\vec{s}_k,t) &\le \sum_{i=1}^{k}\sum_{j=1}^{k}\mathbb{P}\left(X_i\overline{w}_T>\tilde{x}, Y_j\overline{w}_T>\tilde{y}, \overline{w}_T>D/T\mid \vec{\theta}_k=\vec{s}_k\right)\\
&\le \sum_{i=1}^{k}\sum_{j=1}^{k}\mathbb{P}\left(X_i\overline{w}_T>\tilde{x}, Y_j\overline{w}_T>\tilde{y}, \overline{w}_T>b\mid \vec{\theta}_k=\vec{s}_k\right)\\
&\le C\varepsilon\overline{F}(\tilde{x})\overline{G}(\tilde{y})\left\{(1+\eta_1 d_1 d_2)\sum_{i=1}^{k}\ddot{\varphi}_3(s_i)+\sum_{1\le i\ne j\le k}\hat{\varphi}_3(s_i)\check{\varphi}_3(s_j)\right\}\\
&\le C\varepsilon\widetilde{P}_{x,y}(\vec{s}_k),
\end{aligned} \tag{3.69}
$$

where at the third step we used a similar derivation to that in (3.59)-(3.60), while at the last step we used Lemma 3.2 and relation (3.6) in Lemma 3.3. Following the derivation in (3.56)-(3.61) with some obvious modifications, we have, for all large x,y and $\vec{s}_k\in\Omega_{k,T}$,

$$
K_3'(\tilde{x},\tilde{y};\vec{s}_k)+K_4'(\tilde{x},\tilde{y};\vec{s}_k)\le C\varepsilon\overline{F}(\tilde{x})\overline{G}(\tilde{y})\left\{(1+\eta_1 d_1 d_2)\sum_{1\le i\le k}\ddot{\varphi}_3(s_i)+\sum_{1\le i\ne j\le k}\hat{\varphi}_3(s_i)\check{\varphi}_3(s_j)\right\}
$$

$$\leq C\varepsilon \widetilde{P}_{x,y}(\vec{s}_k). \tag{3.70}$$

Combining (3.67)-(3.69) gives the lower-bound version of relation (3.52). This ends the proof. □

Lemma 3.7. *Under the conditions of Theorem 2.1, for arbitrarily fixed $T \in \Lambda$,*

$$\lim_{m\to\infty} \lim_{(x,y)^\intercal\to(\infty,\infty)^\intercal} \sup_{t\in\Lambda_T} \frac{\sum_{k=m}^{\infty} \mathbb{P}\left(\sum_{i=1}^{k} X_i \mathrm{e}^{-L(\tau_i)} > x, \sum_{j=1}^{k} Y_j \mathrm{e}^{-L(\tau_j)} > y, N_t = k\right)}{(1+\eta_1 d_1 d_2)\ddot{P} + \overline{P}} = 0, \tag{3.71}$$

$$\lim_{m\to\infty} \lim_{(x,y)^\intercal\to(\infty,\infty)^\intercal} \sup_{t\in\Lambda_T} \frac{\sum_{k=m}^{\infty} \sum_{i=1}^{k} \mathbb{P}\left(X^* \mathrm{e}^{-L(\ddot{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\ddot{\tau}_i^*)} > y, \ddot{N}_t = k\right)}{\ddot{P}} = 0, \tag{3.72}$$

$$\lim_{m\to\infty} \lim_{(x,y)^\intercal\to(\infty,\infty)^\intercal} \sup_{t\in\Lambda_T} \frac{\sum_{k=m}^{\infty} \sum_{1\leq i\neq j\leq k} \mathbb{P}\left(X^* \mathrm{e}^{-L(\overline{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\overline{\tau}_j^*)} > y, \overline{N}_t = k\right)}{\overline{P}} = 0, \tag{3.73}$$

where $\{\ddot{\tau}_j^*, j \geq 1\}$, $\{\overline{\tau}_j^*, j \geq 1\}$ *are specified in* (2.5)-(2.6)*,*

$$\ddot{P} = \mathbb{P}\left(X_1^* \mathrm{e}^{-L(\ddot{\tau}_1^*)} > x, Y^* \mathrm{e}^{-L(\ddot{\tau}_1^*)} > y, \ddot{\tau}_1^* \leq t\right),$$

$$\overline{P} = \mathbb{P}\left(X^* \mathrm{e}^{-L(\overline{\tau}_1^*)} > x, Y^* \mathrm{e}^{-L(\overline{\tau}_2^*)} > y, \overline{\tau}_2^* \leq t\right) + \mathbb{P}\left(X^* \mathrm{e}^{-L(\overline{\tau}_2^*)} > x, Y^* \mathrm{e}^{-L(\overline{\tau}_1^*)} > y, \overline{\tau}_2^* \leq t\right).$$

Proof. With $P_{x,y}(\cdot,\cdot)$ specified in (2.12) and $\ddot{\tau}_1^*$ in (2.5), by relation (3.6) in Lemma 3.3, it holds uniformly for all $t \in \Lambda_T$ that

$$\ddot{P} = \int_{0-}^{t} P_{x,y}(u,u)\ddot{\varphi}_3(u)\mathrm{d}H(u) \asymp \overline{F}(x)\overline{G}(y) \int_{0-}^{t} \ddot{\varphi}_3(u)\mathrm{d}H(u). \tag{3.74}$$

With $\overline{\tau}_1^*$ and $\overline{\tau}_2^*$ specified in (2.6), by relation (3.6) in Lemma 3.3, it holds uniformly for all $t \in \Lambda_T$ that

$$\begin{aligned}\overline{P} &= \iint_{\Omega_{2,t}} \hat{\varphi}_3(u)\check{\varphi}_3(v)(P_{x,y}(u,u+v) + P_{x,y}(u+v,u))\mathrm{d}H(u)\mathrm{d}H(v) \\ &\asymp \overline{F}(x)\overline{G}(y) \iint_{\Omega_{2,t}} \hat{\varphi}_3(u)\check{\varphi}_3(v)\mathrm{d}H(u)\mathrm{d}H(v).\end{aligned} \tag{3.75}$$

Denote the summand in the numerator of (3.71) by P_k. For all $t \in \Lambda_T$ and any $k \geq 2$,

$$\begin{aligned}P_k &\leq \sum_{1\leq i,j\leq k} \mathbb{P}\left(X_i \mathrm{e}^{-L(\tau_i)} > \frac{x}{k}, Y_j \mathrm{e}^{-L(\tau_j)} > \frac{y}{k}, N_t = k\right) \\ &= \sum_{i=1}^{k} \int\cdots\int_{\Omega_{k,t}} \mathbb{P}\left(X_i \mathrm{e}^{-L(t_i)} > \frac{x}{k}, Y_i \mathrm{e}^{-L(t_i)} > \frac{y}{k} \mid \theta_i = s_i\right) \overline{H}(t-t_k) \prod_{l=1}^{k} \mathrm{d}H(s_l) \\ &\quad + \sum_{1\leq i\neq j\leq k} \int\cdots\int_{\Omega_{k,t}} \mathbb{P}\left(X_i \mathrm{e}^{-L(t_i)} > \frac{x}{k}, Y_j \mathrm{e}^{-L(t_j)} > \frac{y}{k} \mid \theta_i = s_i, \theta_j = s_j\right) \overline{H}(t-t_k) \prod_{l=1}^{k} \mathrm{d}H(s_l) \\ &:= P_{k1} + P_{k2}.\end{aligned} \tag{3.76}$$

Take $p > \mathbb{J}_F^+$ and $q > \mathbb{J}_G^+$ such that $p+q \leq \eta$. By relation (3.28) in Lemma 3.4 and relation (3.6) in Lemma 3.3, we can derive that, for all large x, y and all $t \in \Lambda_T$,

$$P_{k1} \sim \sum_{i=1}^{k} \int\cdots\int_{\Omega_{k,t}} (1+\eta_1 d_1 d_2)\ddot{\varphi}_3(s_i) P_{x/k,y/k}(t_i,t_i) \overline{H}(t-t_k) \prod_{l=1}^{k} \mathrm{d}H(s_l)$$

$$\begin{aligned}
&\asymp k\overline{F}\left(\frac{x}{k}\right)\overline{G}\left(\frac{x}{k}\right)\int\cdots\int_{\Omega_{k,t}}(1+\eta_1 d_1 d_2)\ddot{\varphi}_3(s_1)\overline{H}(t-t_k)\prod_{l=1}^{k}\mathrm{d}H(s_l)\\
&\le C_p C_q k^{p+q+1}\overline{F}(x)\overline{G}(y)\int_{0-}^{t}(1+\eta_1 d_1 d_2)\ddot{\varphi}_3(u)\mathbb{P}(N_{t-u}=k-1)\mathrm{d}H(u)\\
&\le Ck^{p+q+1}\overline{F}(x)\overline{G}(y)\mathbb{P}(N_T\ge k-1)\int_{0-}^{t}(1+\eta_1 d_1 d_2)\ddot{\varphi}_3(u)\mathrm{d}H(u),
\end{aligned}\tag{3.77}$$

where at the second step we interchanged s_i and s_1 for every $1\le i\le k$, while at the third step we used inequality (3.1). By relation (3.29) in Lemma 3.4 and relation (3.6) in Lemma 3.3, we have, for all large x,y and all $t\in\Lambda_T$,

$$\begin{aligned}
P_{k2}&\sim\sum_{1\le i\ne j\le k}\int\cdots\int_{\Omega_{k,t}}\hat{\varphi}_3(s_i)\check{\varphi}_3(s_j)P_{x/k,y/k}(t_i,t_j)\overline{H}(t-t_k)\prod_{l=1}^{k}\mathrm{d}H(s_l)\\
&\asymp k(k-1)\overline{F}\left(\frac{x}{k}\right)\overline{G}\left(\frac{x}{k}\right)\int\cdots\int_{\Omega_{k,t}}\hat{\varphi}_3(s_1)\check{\varphi}_3(s_2)\overline{H}(t-t_k)\prod_{l=1}^{k}\mathrm{d}H(s_l)\\
&\le Ck^{p+q+2}\overline{F}(x)\overline{G}(y)\iint_{\Omega_{2,t}}\hat{\varphi}_3(u)\check{\varphi}_3(v)\mathbb{P}(N_{t-u-v}=k-2)\,\mathrm{d}H(u)\mathrm{d}H(v)\\
&\le Ck^{p+q+2}\overline{F}(x)\overline{G}(y)\mathbb{P}(N_T\ge k-2)\iint_{\Omega_{2,t}}\hat{\varphi}_3(u)\check{\varphi}_3(v)\mathrm{d}H(u)\mathrm{d}H(v),
\end{aligned}\tag{3.78}$$

where at the second step we used the interchanges of $s_i\leftrightharpoons s_1$ and $s_j\leftrightharpoons s_2$ for every $1\le i\ne j\le k$. By (3.74)-(3.78), we obtain, for all large x,y and all $t\in\Lambda_T$,

$$\begin{aligned}
&\frac{\sum_{k=m}^{\infty}\mathbb{P}\left(\sum_{i=1}^{k}X_i\mathrm{e}^{-L(\tau_i)}>x,\sum_{j=1}^{k}Y_j\mathrm{e}^{-L(\tau_j)}>y,N_t=k\right)}{(1+\eta_1 d_1 d_2)\ddot{P}+\overline{P}}\\
&\le C\sum_{k=m}^{\infty}k^{\eta+2}\mathbb{P}(N_T\ge k-2)\le C\sum_{k=m-2}^{\infty}k^{\eta+3}\mathbb{P}(N_T=k)=C\mathbb{E}\{N_T^{\eta+3}\mathbf{1}_{(N_T\ge m-2)}\}.
\end{aligned}$$

By Lemma 3.2 in Hao and Tang (2008), there exists some $\hbar>0$ such that $\mathbb{E}\mathrm{e}^{\hbar N_T}<\infty$. Hence, the last expectation above tends to 0 as $m\to\infty$. Thus, relation (3.71) holds.

It remains to prove relations (3.72)-(3.73). In fact, by going along the same lines of the proof of relation (3.71) with some obvious modifications, we can verify relations (3.72)-(3.73) and conclude the proof. □

4 Proof of Theorem 2.1

First we prove the upper-bound version of relation (2.11). Trivially, for arbitrarily fixed $t\in\Lambda_T$,

$$\begin{aligned}
\Psi(x,y;t)&=\mathbb{P}\left(\bigcup_{0\le s\le t}\left\{\mathrm{e}^{-L(s)}U_{1s}<0,\mathrm{e}^{-L(s)}U_{2s}<0\right\}\,\Big|\,U_{10}=x,U_{20}=y\right)\\
&=\mathbb{P}\left(\bigcup_{0<s\le t}\left\{\sum_{i=1}^{N_s}X_i\mathrm{e}^{-L(\tau_i)}-c_1Z_s>x,\sum_{j=1}^{N_s}Y_j\mathrm{e}^{-L(\tau_j)}-c_2Z_s>y\right\}\right)
\end{aligned}\tag{4.1}$$

$$
\begin{aligned}
&\le \mathbb{P}\left(\sum_{i=1}^{N_t} X_i \mathrm{e}^{-L(\tau_i)} > x, \sum_{j=1}^{N_t} Y_j \mathrm{e}^{-L(\tau_j)} > y\right) \\
&= \left(\sum_{k=1}^{m} + \sum_{k=m+1}^{\infty}\right) \mathbb{P}\left(\sum_{i=1}^{k} X_i \mathrm{e}^{-L(\tau_i)} > x, \sum_{j=1}^{k} Y_j \mathrm{e}^{-L(\tau_j)} > y, N_t = k\right) \\
&:= M_1(x,y;t) + M_2(x,y;t). \qquad (4.2)
\end{aligned}
$$

By the uniformity of relation (3.52) in Lemma 3.6, it holds uniformly for all $t \in \Lambda_T$ that

$$
\begin{aligned}
M_1(x,y;t) &\sim (1+\eta_1 d_1 d_2) \sum_{k=1}^{m} \sum_{i=1}^{k} \mathbb{P}\left(X^* \mathrm{e}^{-L(\ddot{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\ddot{\tau}_i^*)} > y, \ddot{N}_t = k\right) \\
&\quad + \sum_{k=1}^{m} \left(\sum_{i=1}^{k} \sum_{j=i+1}^{k} + \sum_{j=1}^{k} \sum_{i=j+1}^{k}\right) \mathbb{P}\left(X^* \mathrm{e}^{-L(\overline{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\overline{\tau}_j^*)} > y, \overline{N}_t = k\right) \\
&:= (1+\eta_1 d_1 d_2) M_{11}(x,y;t) + M_{12}(x,y;t) + M_{13}(x,y;t). \qquad (4.3)
\end{aligned}
$$

Interchanging the order of the sums of i and k gives that

$$
\begin{aligned}
M_{11}(x,y;t) &\le \sum_{k=1}^{\infty} \sum_{i=1}^{k} \mathbb{P}\left(X^* \mathrm{e}^{-L(\ddot{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\ddot{\tau}_i^*)} > y, \ddot{N}_t = k\right) \\
&= \sum_{i=1}^{\infty} \mathbb{P}\left(X^* \mathrm{e}^{-L(\ddot{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\ddot{\tau}_i^*)} > y, \ddot{\tau}_i^* \le t\right) \\
&:= M'_{11}(x,y;t). \qquad (4.4)
\end{aligned}
$$

Interchanging the order of the sum of (i,j) and k leads to

$$
\begin{aligned}
M_{12}(x,y;t) &\le \sum_{k=1}^{\infty} \sum_{i=1}^{k} \sum_{j=i+1}^{k} \mathbb{P}\left(X^* \mathrm{e}^{-L(\overline{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\overline{\tau}_j^*)} > y, \overline{N}_t = k\right) \\
&= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \mathbb{P}\left(X^* \mathrm{e}^{-L(\overline{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\overline{\tau}_j^*)} > y, \overline{\tau}_j^* \le t\right) \\
&:= M'_{12}(x,y;t). \qquad (4.5)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
M_{13}(x,y;t) &\le \sum_{k=1}^{\infty} \sum_{j=1}^{k} \sum_{i=j+1}^{k} \mathbb{P}\left(X^* \mathrm{e}^{-L(\overline{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\overline{\tau}_j^*)} > y, \overline{N}_t = k\right) \\
&= \sum_{j=1}^{\infty} \sum_{i=j+1}^{\infty} \mathbb{P}\left(X^* \mathrm{e}^{-L(\overline{\tau}_i^*)} > x, Y^* \mathrm{e}^{-L(\overline{\tau}_j^*)} > y, \overline{\tau}_i^* \le t\right) \\
&:= M'_{13}(x,y;t). \qquad (4.6)
\end{aligned}
$$

By (4.3)-(4.6), it holds uniformly for all $t \in \Lambda_T$ that

$$
M_1(x,y;t) \lesssim (1+\eta_1 d_1 d_2) M'_{11}(x,y;t) + M'_{12}(x,y;t) + M'_{13}(x,y;t). \qquad (4.7)
$$

Note that with $\ddot{P}$ and $\overline{P}$ specified as in Lemma 3.7,

$$
(1+\eta_1 d_1 d_2) M'_{11}(x,y;t) + M'_{12}(x,y;t) + M'_{13}(x,y;t) \ge (1+\eta_1 d_1 d_2)\ddot{P} + \overline{P}. \qquad (4.8)
$$

Hence, by (4.8) and relation (3.71) in Lemma 3.7,

$$\lim_{m\to\infty}\lim_{(x,y)^\intercal\to(\infty,\infty)^\intercal}\sup_{t\in\Lambda_T}\frac{M_2(x,y;t)}{(1+\eta_1d_1d_2)M'_{11}(x,y;t)+M'_{12}(x,y;t)+M'_{13}(x,y;t)}=0. \tag{4.9}$$

By (4.2), (4.7), and (4.9), it holds uniformly for all $t\in\Lambda_T$ that

$$\Psi(x,y;t)\lesssim(1+\eta_1d_1d_2)M'_{11}(x,y;t)+M'_{12}(x,y;t)+M'_{13}(x,y;t). \tag{4.10}$$

Recalling the definition of $M'_{11}(x,y;t)$ in (4.4) with $\{\ddot{\tau}_i^*,i\ge1\}$ specified in (2.5), we have

$$M'_{11}(x,y;t)=\sum_{i=1}^{\infty}\int_{0-}^{t}P_{x,y}(u,u)\mathbb{P}(\ddot{\tau}_i^*\in\mathrm{d}u)=\int_{0-}^{t}P_{x,y}(u,u)\mathrm{d}\ddot{\lambda}_u, \tag{4.11}$$

where $\{\ddot{\lambda}_t,t\ge0\}$ is the renewal function of $\{\ddot{N}_t,t\ge0\}$ and specified in (2.8). From the definition of $M'_{12}(x,y;t)$ in (4.5) with $\{\overline{\tau}_i^*,i\ge1\}$ specified in (2.6), it is easy to see that

$$\begin{aligned}
&M'_{12}(x,y;t)\\
&=\sum_{j=2}^{\infty}\iint_{\Omega_{2,t}}\mathbb{P}\left(X^*\mathrm{e}^{-L(u)}>x,Y^*\mathrm{e}^{-L(u+v)}>y\right)\mathbb{P}(\overline{\tau}_j^*-\overline{\tau}_1^*\in\mathrm{d}v)\mathbb{P}(\overline{\tau}_1^*\in\mathrm{d}u)\\
&\quad+\sum_{i=2}^{\infty}\sum_{j=i+1}^{\infty}\iint_{\Omega_{2,t}}\mathbb{P}\left(X^*\mathrm{e}^{-L(u)}>x,Y^*\mathrm{e}^{-L(u+v)}>y\right)\mathbb{P}(\overline{\tau}_j^*-\overline{\tau}_i^*\in\mathrm{d}v)\mathbb{P}(\overline{\tau}_i^*\in\mathrm{d}u)\\
&=\iint_{\Omega_{2,t}}P_{x,y}(u,u+v)\hat{\varphi}_3(u)(\mathrm{d}\check{\lambda}_v-\mathrm{d}\lambda_v)\mathrm{d}H(u)+\iint_{\Omega_{2,t}}P_{x,y}(u,u+v)\mathrm{d}\lambda_v\mathrm{d}\overline{\lambda}_u
\end{aligned} \tag{4.12}$$

with $\{\check{\lambda}_t,t\ge0\}$ and $\{\overline{\lambda}_t,t\ge0\}$ specified in (2.7)-(2.8). Similarly, by the definition of $M'_{13}(x,y;t)$ in (4.6) with $\{\overline{\tau}_i^*,i\ge1\}$ specified in (2.6), we can obtain

$$M'_{13}(x,y;t)=\iint_{\Omega_{2,t}}P_{x,y}(u+v,u)\hat{\varphi}_3(u)(\mathrm{d}\check{\lambda}_v-\mathrm{d}\lambda_v)\mathrm{d}H(u)+\iint_{\Omega_{2,t}}P_{x,y}(u+v,u)\mathrm{d}\lambda_v\mathrm{d}\overline{\lambda}_u. \tag{4.13}$$

By (4.10)-(4.13), we obtain the upper-bound version of relation (2.11).

Next, we prove the lower-bound version of relation (2.11). In fact, by (4.1) and Lemma 3.6, it holds uniformly for all $t\in\Lambda_T$ that

$$\begin{aligned}
\Psi(x,y;t)&\ge\mathbb{P}\left(\sum_{i=1}^{N_t}X_i\mathrm{e}^{-L(\tau_i)}-c_1Z_t>x,\sum_{j=1}^{N_t}Y_j\mathrm{e}^{-L(\tau_j)}-c_2Z_t>y\right)\\
&\ge\sum_{k=1}^{m}\mathbb{P}\left(\sum_{i=1}^{k}X_i\mathrm{e}^{-L(\tau_i)}-c_1Z_t>x,\sum_{j=1}^{k}Y_j\mathrm{e}^{-L(\tau_j)}-c_2Z_t>y,N_t=k\right)\\
&\sim(1+\eta_1d_1d_2)\sum_{k=1}^{m}\sum_{i=1}^{k}\mathbb{P}\left(X^*\mathrm{e}^{-L(\ddot{\tau}_i^*)}>x,Y^*\mathrm{e}^{-L(\ddot{\tau}_j^*)}>y,\ddot{N}_t=k\right)\\
&\quad+\sum_{k=1}^{m}\left(\sum_{i=1}^{k}\sum_{j=i+1}^{k}+\sum_{j=1}^{k}\sum_{i=j+1}^{k}\right)\mathbb{P}\left(X^*\mathrm{e}^{-L(\overline{\tau}_i^*)}>x,Y^*\mathrm{e}^{-L(\overline{\tau}_j^*)}>y,\overline{N}_t=k\right)\\
&=(1+\eta_1d_1d_2)M'_{11}(x,y;t)+M'_{12}(x,y;t)+M'_{13}(x,y;t)\\
&\quad-(1+\eta_1d_1d_2)\sum_{k=m+1}^{\infty}\sum_{i=1}^{k}\mathbb{P}\left(X^*\mathrm{e}^{-L(\ddot{\tau}_i^*)}>x,Y^*\mathrm{e}^{-L(\ddot{\tau}_i^*)}>y,\ddot{N}_t=k\right)
\end{aligned}$$

$$-\sum_{k=m+1}^{\infty}\sum_{1\le i\ne j\le k}\mathbb{P}\left(X^*\mathrm{e}^{-L(\overline{\tau}_i^*)}>x,Y^*\mathrm{e}^{-L(\overline{\tau}_j^*)}>y,\overline{N}_t=k\right). \tag{4.14}$$

with $M'_{1l}(x,y;t)$, $l=1,2,3$, specified in (4.4)-(4.6), respectively. For the last two terms in (4.14), by (4.8) and relations (3.72)-(3.73) in Lemma 3.7, we obtain

$$\lim_{m\to\infty}\lim_{(x,y)^\intercal\to(\infty,\infty)^\intercal}\sup_{t\in\Lambda_T}\frac{\sum_{k=m+1}^{\infty}\sum_{i=1}^{k}\mathbb{P}\left(X^*\mathrm{e}^{-L(\ddot{\tau}_i^*)}>x,Y^*\mathrm{e}^{-L(\ddot{\tau}_i^*)}>y,\ddot{N}_t=k\right)}{(1+\eta_1d_1d_2)M'_{11}(x,y;t)+M'_{12}(x,y;t)+M'_{13}(x,y;t)}=0, \tag{4.15}$$

and

$$\lim_{m\to\infty}\lim_{(x,y)^\intercal\to(\infty,\infty)^\intercal}\sup_{t\in\Lambda_T}\frac{\sum_{k=m+1}^{\infty}\sum_{1\le i\ne j\le k}\mathbb{P}\left(X^*\mathrm{e}^{-L(\overline{\tau}_i^*)}>x,Y^*\mathrm{e}^{-L(\overline{\tau}_j^*)}>y,\overline{N}_t=k\right)}{(1+\eta_1d_1d_2)M'_{11}(x,y;t)+M'_{12}(x,y;t)+M'_{13}(x,y;t)}=0. \tag{4.16}$$

By (4.14)-(4.16), it holds uniformly for all $t\in\Lambda_T$ that

$$\Psi(x,y;t)\gtrsim(1+\eta_1d_1d_2)M'_{11}(x,y;t)+M'_{12}(x,y;t)+M'_{13}(x,y;t).$$

This, together with equalities (4.11)-(4.13), gives the lower-bound version of relation (2.11).

Finally, we prove the local uniformity of relation (2.13). In fact, following the method used in the proof of the uniformity of relation (2.11) and applying relation (3.10) in Lemma 3.3 to (4.11)-(4.13), we can obtain the local uniformity of relation (2.13) and conclude the proof.

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