On the T^2 control chart with estimated parameters

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Abstract

Statistical monitoring of multivariate processes is becoming increasingly important in modern manufacturing environments. Typical equipment may have multiple key variables to be measured continuously. Hotelling's T^2 chart was originally applied for monitoring the mean vector of multivariate quality measurements. In practical problems, estimated parameters are needed and their use will modify the properties of control charts. The Average Run Length (ARL), an indicator of the performance of the control charts, will be larger when the estimated parameters are used. As one contribution of the paper, we provide a rigorous proof of this phenomenon which has been reported in several empirical studies. Furthermore, in order to design an efficient T^2 chart with estimated parameters, it is necessary to have a method to calculate or approximate the ARL function. An existing approach in the literature is based on extensive Monte-Carlo simulations. In this paper, we propose a novel approach by providing an analytic approximation of the ARL function in the however limited case of univariate observations.

Keywords: Average Run Length; Estimated parameters; Multivariate statistical process control; T^2 chart

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1 Introduction

Statistical monitoring of multivariate processes is becoming increasingly important in modern manufacturing environments. Typical equipment may have multiple key variables to be measured continuously. Hotelling's T^2 chart (Hotelling , 1947) was originally applied to the problem of monitoring the mean vector of multivariate quality measurements. In practice, parameters of the observation process need to be estimated first during a Phase I analysis. As a consequence, the use of such estimated parameters will modify the properties of control charts in Phase II study. The performance of multivariate charts with estimated parameters has been studied in the literature. Ryan (2011) analyzed the dependence of T^2 chart on estimated parameters. Lowry and Montgomery (1995) provided recommended sample sizes for multivariate T^2 control charts. A more detailed study on the effects of the parameter estimation on multivariate T^2 charts with χ^2 based control limits was done by Nedumaran and Pignatiello (1999). Champ *et al.* (2005) studied the T^2 charts with corrected control limits. Jensen *et al.* (2006) and Bersimis *et al.* (2006) give a thorough review of the literature concerning the effects of estimation on univariate and multivariate charts with estimated parameters.

Throughout the paper, we assume that the base vector $\mathbf{X}$ of quality measurements is distributed as the p-variate normal distribution $\mathcal{N}_p(\mu, \Sigma_0)$. The process is said *in-control* when $\mu = \mu_0$, a preassigned mean vector. When the in-control parameters μ_0 and Σ_0 are known, Hotelling's T^2 chart uses the statistics at time $k = 1, 2, \dots,$

$$T_k^2 = n \left(\overline{\mathbf{X}}_k - \mu_0\right)^{\mathrm{T}} \Sigma_0^{-1} \left(\overline{\mathbf{X}}_k - \mu_0\right), \tag{1.1}$$

where $\overline{\mathbf{X}}_k$ is the mean of the k-th sample with sample size n. The chart gives an out-of-control signal at time i when T_i is above a control limit h.

A widely used method to measure the performance of a control chart is through the average run length (ARL), which is the expected number of the plotted chart statistics before a signal is observed. When μ_0 and Σ_0 are unknown, we replace them by some sample estimates for them. Sample estimators $\hat{\mu}_0$ and $\hat{\Sigma}_0$ are based on data acquired prior to the on-going inspection process. Assume that there are m such independent random samples $\{\mathbf{X}_{i,1}, \mathbf{X}_{i,2}, \dots, \mathbf{X}_{i,n}\}$ of size n ($i = 1, 2, \dots, m$) from an in-control process. Then the sample estimators $\hat{\mu}_0$ and $\hat{\Sigma}_0$ are:

$$\hat{\mu}_0 = \overline{\overline{\mathbf{X}}} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{X}_{i,j},$$

$$\hat{\Sigma}_0 = \overline{\mathbf{S}} = \frac{1}{m(n-1)} \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\mathbf{X}_{i,j} - \overline{\mathbf{X}}_i\right) \left(\mathbf{X}_{i,j} - \overline{\mathbf{X}}_i\right)^{\mathrm{T}}.$$

The T^2 statistics are then replaced with

$$T_k^2 = n\left(\overline{\mathbf{X}}_k - \hat{\mu}_0\right)^{\mathrm{T}} \hat{\Sigma}_0^{-1} \left(\overline{\mathbf{X}}_k - \hat{\mu}_0\right). \quad (1.2)$$

The whole procedure of T^2 chart is then divided into two phases. In Phase One, the chart is used to determine whether the m independent random samples each with size n is generated from an in-control process and then the parameter estimators can be obtained through these samples. In Phase Two, these estimated parameters are directly used in testing whether new sample points deviate from the in-control status based on some control limits. Sullivan and Woodall (1996) and Kim *et al.* (2003) have discussed the problem of adapting control charts for the preliminary analysis of multivariate observations and also recommend a method for preliminary analysis of multivariate observations that does not require any simulation for the control limit for Phase One. For Phase Two, the main task is to determine a control limit h which keeps the in-control ARL at a given level (e.g. 200 or 500). The out-of-control ARL are also to be estimated and they should be kept as small as possible. If μ_0 and Σ_0 are known, each statistic T_k^2 in (1.1) $(k = 1, 2, \ldots)$ follow a χ^2 distribution independently with p degree of freedom and the RL of a chart is a geometric random variable. For example, the multivariate Shewart control chart has an upper control limit $h = \chi^2_{p,1-\alpha}$ with an in-control $ARL = 1/\alpha$. Several authors provided an analysis on $\{T_k^2\}$ when sample estimators $\hat{\mu}_0$ and $\hat{\Sigma}_0$ are used as in (1.2). For example, Wierda (1994) showed that $\left[(mn - m - p + 1)/p(m+1)(n-1)\right] T_k^2$ is distributed as a Fisher distribution $F_{p,mn-m-p+1}$. In Phase Two, if the subsequent statistics T_k^2 were independent at different time point $k = 1, 2, \ldots$, the control limit h^* corresponding to an in-control $ARL = 1/\alpha$ would be

$$h^* = \frac{p(m+1)(n-1)}{m(n-1)+1-p} F_{1-\alpha,p,mn-m-p+1}, \quad \text{and} \quad \mathrm{P}\left(T_1^2 > h^*\right) = \alpha, \quad (1.3)$$

where $F_{1-\alpha,p,mn-m-p+1}$ is the α-th upper quantile of $F_{p,mn-m-p+1}$. However, as estimators are used in place of known parameters, the T_k^2's become dependent so that with the control limit h^* specified above, the actual in-control ARL indeed differs from the target value α^{-1}.

Champ *et al.* (2005) took the dependence among the T_k^2 into consideration and offered corrected control limit h using intensive simulations. More precisely, using the sample estimators $\hat{\mu}_0$ and $\hat{\Sigma}_0$, the current statistic T_k^2 of (1.2) with inspected sample mean $\bar{\mathbf{X}}^k$ computed from the sample $\mathbf{X}^k = \{\mathbf{X}_{k,1}, \mathbf{X}_{k,2}, \ldots, \mathbf{X}_{k,n}\}$ can be written as:

$$T_k^2 = m(n-1)\left(\mathbf{Z}_k + \sqrt{n}\delta - \frac{1}{\sqrt{m}}\mathbf{Z}_0\right)^{\mathrm{T}} \mathbf{W}_0^{-1} \left(\mathbf{Z}_k + \sqrt{n}\delta - \frac{1}{\sqrt{m}}\mathbf{Z}_0\right), \quad (1.4)$$

$\mathbf{Z}_0 = \sqrt{mn}\mathbf{P}_0^{-1}\left(\overline{\overline{\mathbf{X}}} - \mu_0\right) \sim \mathrm{N}_p(\mathbf{0}, \mathrm{I})$, $\mathbf{Z}_k = \sqrt{n}\mathbf{P}_0^{-1}\left(\overline{\mathbf{X}^k} - \mu_0\right) \sim \mathrm{N}_p(\mathbf{0}, \mathrm{I})$, $\delta = \mathbf{P}_0^{-1}(\mu - \mu_0)$, $\mathbf{W}_0 = m(n-1)\mathbf{P}_0^{-1}\overline{\mathbf{S}}\left(\mathbf{P}_0^{-1}\right)^{\mathrm{T}} \sim \mathrm{Wishart}_p(\mathrm{I},\ m(n-1))$, and the matrix $\mathbf{P}_0$ is defined by the factorization $\Sigma_0 = \mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}$. Moreover, $\mathbf{Z}_0$, $\mathbf{Z}_k$, and $\mathbf{W}_0$ are independent in this representation. If we further use an orthogonal transformation $\mathbf{B}$ such that $\mathbf{B}\delta = (d, 0, \ldots, 0)^{\mathrm{T}}$, we have

$$T_k^2 = m\,(n-1)\left(\mathbf{Z}_k^* + \sqrt{n}d\mathbf{e} - \frac{1}{\sqrt{m}}\mathbf{Z}_0^*\right)^{\mathrm{T}} \mathbf{W}_0^{*-1}\left(\mathbf{Z}_k^* + \sqrt{n}d\mathbf{e} - \frac{1}{\sqrt{m}}\mathbf{Z}_0^*\right), \qquad (1.5)$$

where $\mathbf{e}$ is the p $\times$ 1 vector $(1, 0, \ldots, 0)^{\mathrm{T}}$, $\mathbf{Z}_0^* = \mathbf{B}\mathbf{Z}_0 \sim \mathrm{N}_p(\mathbf{0}, \mathrm{I})$, $\mathbf{Z}_k^* = \mathbf{B}\mathbf{Z}_k \sim \mathrm{N}_p(\mathbf{0}, \mathrm{I})$, $\mathbf{W}_0^* = \mathbf{B}\mathbf{W}_0\mathbf{B}^{\mathrm{T}} \sim \mathrm{Wishart}_p(\mathrm{I}, mn - m)$ and they are independent. Notice that $d^2 = \|\delta\|^2 = (\mu - \mu_0)^{\mathrm{T}}\,\Sigma_0^{-1}(\mu - \mu_0)$ is the non-central parameter. Two important features of the T^2 chart with estimated parameters emerge from this representation. First, the distribution of each T_k^2 depends on the unknown parameters μ_0 and Σ_0 through d only; second, even though the samples, say $k = 1, 2 \ldots$, in Phase II are independent, the statistics $\{T_k^2\}_{k\geqslant 1}$ are dependent since they share the same random variables $\mathbf{Z}_0^*$, $\{\mathbf{Z}_k^*\}_{k\geqslant 1}$ and $\mathbf{W}_0^*$ which are functions of the estimated parameters. However, the $\{T_k^2\}_{k\geqslant 1}$ still have the same marginal distribution.

Despite the interesting representation (1.5) , the joint distribution of $\{T_k^2\}_{k\geqslant 1}$ is far from being known. This makes it a particular challenge to determine a control limit h for a given in-control ARL. One attempt consists of keeping the same control limit h as if the parameters were known and not estimated, namely the traditional limit h^*. However, it has been reported in the literature that this seemingly natural approach inflates the actual ARL_0 (in-control) which becomes larger than the target ARL_0, see e.g. Bersimis *et al.* (2006). Moreover, Champ *et al.* (2005) provided detailed simulations to verify this phenomenon. As first main contribution of the paper, we will provide a formal proof of the phenomenon.

Furthermore, the design of an efficient T^2 chart with estimated parameters requires an appropriate method to calculate or approximate the ARL function. Champ *et al.* (2005) proposes an effective approach where the ARL function is tabulated using extensive Monte-Carlo simulation. Typically, given a target ARL_0, say 200 with m samples of block length n in Phase I, the corrected UCL $\bar{h}$ can be found by large size simulations and binary adjustments of h. Tables in Champ *et al.* (2005) report the corrected UCL $\bar{h}$ for $ARL_0 = 200$ and a wide range of parameters: dimension $2 \leqslant p \leqslant 10$, sample size $30 \leqslant m \leqslant 100$ and block length $3 \leqslant n \leqslant 15$.

Although this Monte-Carlo approach is valuable and precious, one may search for other approximation methods for several reasons. First, since the random ARL function has a quite large variance, such simulation procedures may suffer from numerical instability. Second, as for all tabulating approaches, a practitioner may face a problem where the parameters p, m and n are not documented in the established tables. A method based on analytic approximation relaxes the restriction on m and n and provides a more convenient alternative to designing a T^2 chart. As second main contribution of this paper, we establish such an analytic approximation for the ARL function. Due to the complexity of the problem, we limit ourselves to the univariate case $p = 1$ and we establish a numerical method for the determination of a control limit h when an arbitrary in-control ARL is given.

2 Comparison between target and actual values of ARL when using the traditional control limit

For a given upper control limit h, the ARL function of the T^2 chart is

$$\begin{aligned} ARL\,(h) &= \sum_{k=0}^{\infty} k\mathrm{P}\,(\text{Run length} = k) = \sum_{k=0}^{\infty} \mathrm{P}\,(\text{Run length} > k) \\ &= \sum_{k=0}^{\infty} \mathrm{P}\left(T_1^2 \leqslant h,\ T_2^2 \leqslant h, \ldots, T_{k-1}^2 \leqslant h,\ T_k^2 \leqslant h\right). \end{aligned} \tag{2.1}$$

Meanwhile, the traditional control limit h^* is derived by assuming the independence between the T_k^2's, which is

$$ARL_0 \backsimeq \sum_{k=0}^{\infty} \left\{\mathrm{P}\left(T_1^2 \leqslant h^*\right)\right\}^k = \frac{1}{\mathrm{P}\,(T_1^2 > h^*)} = \frac{1}{\alpha}, \tag{2.2}$$

where h^* equals to the α-th upper quantile of T_1^2 as given in (1.3). For example, if the target ARL_0 is 200, one finds $\alpha = \frac{1}{200}$ and the value of h^* is derived.

However, the statistics T_k^2's are indeed dependent. There will be therefore a bias between the actual ARL and the target ARLusing the independence approximation (2.2) above. Actually, it has been observed for years that using estimated parameters leads to an in-control ARL that is higher than the expected one.

To the best of our knowledge, this paper is the first to give a formal proof of this phenomenon.

Theorem 1. *For the T^2 chart in* (1.5) *using estimated parameters $\hat{\mu}_0$ and $\hat{\Sigma}_0$, we have $ARL\,(h) \geqslant \frac{1}{P(T_1^2 > h)}$ for all $h > 0$. In particular, if the traditional control limit h^* in* (1.3) *is used, then the actual in-control ARL is always larger than the target ARL. i.e. $ARL_0 \geqslant \frac{1}{\alpha}$.*

Proof. By (2.1),

$$ARL\,(h) = \sum_{k=0}^{\infty} \mathrm{P}\left(T_1^2 \leqslant h, \ldots, T_k^2 \leqslant h\right) = \sum_{k=0}^{\infty} \mathrm{E}\,\left\{\mathbf{1}_{\left(T_1^2 \leqslant h\right)}, \ldots, \mathbf{1}_{\left(T_k^2 \leqslant h\right)}\right\}. \tag{2.3}$$

Here $\mathbf{1}_A$ denotes the indicator function of an event A. Note that in the representation (1.5), the $\{\mathbf{Z}_k^*\}_{k \geqslant 1}$ are independent of $\mathbf{Z}_0^*$ and $\mathbf{W}_0^*$. As a result, conditionally to $\mathbf{Z}_0^*$ and $\mathbf{W}_0^*$, the statistics $\mathbf{T}_k^2$'s are independent. Therefore, by the conditioning method,

$$\begin{aligned} \mathrm{E}\,\left\{\mathbf{1}_{\left(T_1^2 \leqslant h\right)}, \ldots, \mathbf{1}_{\left(T_k^2 \leqslant h\right)}\right\} &= \mathrm{E}\,\left\{\mathrm{E}\left(\mathbf{1}_{\left(T_1^2 \leqslant h\right)}, \ldots, \mathbf{1}_{\left(T_k^2 \leqslant h\right)} \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right)\right\} \\ &= \mathrm{E}\,\left[\left\{\mathrm{E}\left(\mathbf{1}_{\left(T_1^2 \leqslant h\right)} \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right)\right\}^k\right]. \end{aligned} \tag{2.4}$$

After applying the Jensen's inequality $\mathrm{E}\left[X^k\right] \geqslant \{\mathrm{E}[X]\}^k$ to the nonnegative variables $X = \mathrm{E}\left(\mathbf{1}_{\left(T_1^2 \leqslant h\right)} \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right)$ and observing that $\mathrm{E}[X] = \mathrm{E}\left[\mathbf{1}_{\left(T_1^2 \leqslant h\right)}\right] = \mathrm{P}\left(T_1^2 \leqslant h\right)$, we have

$$\mathrm{E}\left\{\mathbf{1}_{\left(T_1^2 \leqslant h\right)}, \ldots, \mathbf{1}_{\left(T_k^2 \leqslant h\right)}\right\} \geqslant \left\{\mathrm{E}\left[\mathrm{E}\left(\mathbf{1}_{\left(T_1^2 \leqslant h\right)} \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right)\right]\right\}^k = \left\{\mathrm{P}\left(T_1^2 \leqslant h\right)\right\}^k.$$

Hence,

$$ARL(h) \geqslant \sum_{k=0}^{\infty} \left\{\mathrm{P}\left(T_1^2 \leqslant h\right)\right\}^k = \frac{1}{\mathrm{P}\left(T_1^2 > h\right)}. \tag{2.5}$$

In particular, for the in-control ARL using traditional control limit as UCL,

$$ARL(h^*) \geqslant \frac{1}{\mathrm{P}_0\left(T_1^2 > h^*\right)} = \frac{1}{\alpha}.$$

□

Theorem 1 formally establishes the fact reported in the literature (Bersimis *et al.* , 2006) that using the traditional limit h^* in the presence of estimated parameters will inflates the in-control ARL. Moreover, we observe that Equation (2.5) shows that the ARL(h) function is increasing in h, so that the needed UCL $\bar{h}$ for a given target in-control ARL must be smaller than the traditional limit h^*. As for the out-of-control ARL, again by the monotonicity of the ARL function, the wrong use of h^* instead of the needed UCL $\bar{h}$ will also increases the underlying out-of-control ARL.

3 Analytic approximations for the in-control ARL in the univariate case

As explained in the introduction, we develop an approximation method for the in-control ARL function with univariate observations. To begin with, we give an exact analytic expression of the in-control ARL.

Proposition 1. *When $p = 1$,*

$$ARL_0 = E\left(\frac{1}{\overline{\Phi}\left(b_0\sqrt{h} + a_0\right) + \overline{\Phi}\left(b_0\sqrt{h} - a_0\right)}\right), \tag{3.1}$$

where $a_0 = \mathbf{Z}_0^*/\sqrt{m}$, $b_0 = \sqrt{\mathbf{W}_0^*}/\sqrt{m(n-1)}$, $\overline{\Phi}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$ and $\mathbf{Z}_0^*$, $\mathbf{W}_0^*$ are the variables defined in (1.5) .

Proof. First, we rewrite (1.5) when $p = 1$ and assume all Phase II observation $\{T_k^2\}_{k=1,2,\ldots}$ are in-control ($d = 0$):

$$T_k^2 = \frac{m(n-1)}{\mathbf{W}_0^*}\left(\mathbf{Z}_k^* - \frac{1}{\sqrt{m}}\mathbf{Z}_0^*\right)^2 = \left(\frac{\mathbf{Z}_k^* - a_0}{b_0}\right)^2, \tag{3.2}$$

where $a_0 = \mathbf{Z}_0^*/\sqrt{m}$, $b_0 = \sqrt{\mathbf{W}_0^*}/\sqrt{m(n-1)}$. Here $\mathbf{Z}_0^* \sim \mathrm{N}(\mathbf{0},1)$, $\mathbf{W}_0^* \sim \chi^2_{m(n-1)}$. Particularly, $T_1^2 = \left(\frac{\mathbf{Z}_1^* - a_0}{b_0}\right)^2$.

Denote the conditional probability $\mathrm{P}^* = \mathrm{P}\left(T_1^2 > h \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right) \in (0,1)$. Then based on the distribution of T_1^2 above, we have

$$\begin{aligned}
\mathrm{P}^* &= \mathrm{P}\left(T_1 < -\sqrt{h} \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right) + \mathrm{P}\left(T_1 > \sqrt{h} \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right) \\
&= \mathrm{P}\left(\mathbf{Z}_1^* < -b_0\sqrt{h} + a_0 \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right) + \mathrm{P}\left(\mathbf{Z}_k^* > b_0\sqrt{h} + a_0 \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right) \\
&= \overline{\Phi}\left(b_0\sqrt{h} + a_0\right) + \overline{\Phi}\left(b_0\sqrt{h} - a_0\right). \qquad (3.3)
\end{aligned}$$

According to (2.3), (2.4) and (3.3) above,

$$\begin{aligned}
ARL_0(h) &= \sum_{k=0}^{\infty} \mathrm{E}\left[\left\{\mathrm{E}\left(\mathbf{1}_{\left(T_1^2 \leqslant h\right)} \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right)\right\}\right]^k \\
&= \sum_{k=0}^{\infty} \mathrm{E}\left[\left\{\mathrm{P}\left(T_1^2 \leqslant h \mid \mathbf{Z}_0^*, \mathbf{W}_0^*\right)\right\}\right]^k \\
&= \sum_{k=0}^{\infty} \mathrm{E}\left[1 - \mathrm{P}^*\right]^k = \mathrm{E}\left(\frac{1}{\mathrm{P}^*}\right) \\
&= \mathrm{E}\left(\frac{1}{\overline{\Phi}\left(b_0\sqrt{h} + a_0\right) + \overline{\Phi}\left(b_0\sqrt{h} - a_0\right)}\right)
\end{aligned}$$

□

We have obtained an analytic expression of ARL_0, which depends on the conditional probabilities $G(a_0, b_0, h) = \left(\overline{\Phi}\left(b_0\sqrt{h} + a_0\right) + \overline{\Phi}\left(b_0\sqrt{h} - a_0\right)\right)^{-1}$. We seek for an approximation of $G(a_0, b_0, h)$ in order to find an effective approximation of the ARL function (3.1) .

There is a traditional approximation to evaluate the complementary cumulative distribution function $\overline{\Phi}(x)$ (see Abramowitz and Stegun (1965)):

$$
\begin{aligned}
\overline{\Phi}(x) &\backsimeq \frac{Z(x)}{x}\left\{1-\frac{1}{x^2}+\frac{1\cdot 3}{x^4}-\cdots+\frac{(-1)^{N_1}\cdot 1\cdot 3\cdots(2N_1-1)}{x^{2N_1}}\right\} \\
&= Z(x)\sum_{i=0}^{N_1}\frac{(-1)^i\cdot 1\cdot 3\cdots(2i-1)}{x^{2i+1}}, \qquad x\neq 0,
\end{aligned} \tag{3.4}
$$

where $Z(x) = (2\pi)^{-1/2}e^{-x^2/2}$ is the standard normal density function and $N_1 \geq 1$ determines the degree of the expansion (here for $i = 0$, the product $1 \cdot 3 \cdots (2i-1)$ is empty so that its value equals to 1 by convention). The main idea is then to approximate the denominator of the G function with (3.4).

Let $y = b_0\sqrt{h} = \sqrt{\frac{\mathbf{W_0^*}h}{m(n-1)}}$ and $z = a_0 = \frac{\mathbf{Z_0^*}}{\sqrt{m}}$ where $\mathrm{Z}_0^* \sim \mathrm{N}(\mathbf{0}, 1)$, $\mathrm{W}_0^* \sim \chi^2_{m(n-1)}$. We find that z is small with high probability since $\mathrm{E}(z) = 0$ and $\mathrm{var}(z) = \frac{1}{m}$, where m is usually large. Expanding $\overline{\Phi}$ around y, we can rewrite G as

$$
G(a_0, b_0, h) = \frac{1}{2\overline{\Phi}(y) + m(z,y)} = \frac{1}{2\overline{\Phi}(y)} - l(z,y), \tag{3.5}
$$

where $m(z,y)$ and $l(z,y)$ are the errors associated to these approximations which are small compared with $\frac{1}{2\overline{\Phi}(y)}$. Therefore, the procedure of approximating $G(a_0, b_0, h)$ can be decomposed into two steps. First, we estimate the main part $\frac{1}{2\overline{\Phi}(y)}$. Second, we find an estimation for $l(z,y)$, which will helpfully lead to an approximation of G and finally of the ARL function in (3.1) .

3.1 Approximation of $\frac{1}{2\overline{\Phi}(y)}$

We rewrite (3.4) as

$$
\overline{\Phi}(y) \backsimeq \frac{Z(y)}{f_1(y)}, \qquad \text{with} \quad \frac{1}{f_1(y)} = \sum_{i=0}^{N_1}\frac{(-1)^i\cdot 1\cdot 3\cdots(2i-1)}{y^{2i+1}}, \tag{3.6}
$$

where $y \neq 0$ and $Z(y) \neq 0$. Then

$$
\frac{1}{2\overline{\Phi}(y)} \backsimeq \frac{1}{2Z(y)}\cdot f_1(y). \tag{3.7}
$$

Next, we expend the rational function $f_1(y)$ as $\sum_{i=0}^{N_2} c_i y^{1-2i}$, a sum of negative powers in y, where N_2 is the order of the expansion to be fixed according to a desired accuracy. To be clear, we list below two examples of $f_1(y)$ ($N_1 = 2$ or 3) and the associated expansions using $N_2 = 3$ or 5.

	$f_1(y)$	$N_2 = 3$	$N_2 = 5$
$N_1 = 2$	$\left(y^{-1} - y^{-3} + 3y^{-5}\right)^{-1}$	$y + y^{-1} - 2y^{-3} - 5y^{-5}$	$y + y^{-1} - 2y^{-3} - 5y^{-5} + y^{-7} + 16y^{-9}$
$N_1 = 3$	$\left(y^{-1} - y^{-3} + 3y^{-5} - 15y^{-7}\right)^{-1}$	$y + y^{-1} - 2y^{-3} + 10y^{-5}$	$y + y^{-1} - 2y^{-3} + 10y^{-5} + 31y^{-7} - 29y^{-9}$

Finally, we obtain the target approximation as follows

$$\frac{1}{2\overline{\Phi}\left(y\right)} \simeq \frac{1}{2Z\left(y\right)}\left(\sum_{i=0}^{N_2} c_i y^{1-2i}\right), \tag{3.8}$$

and its accuracy will be determined by an appropriate choice of $N_1 \geqslant 0$ and $N_2 \geqslant 1$.

3.2 Approximation of $l\left(z,y\right)$

The remainder $l\left(z,y\right)$ is the actual difference between $\frac{1}{\overline{\Phi}(y+z)+\overline{\Phi}(y-z)}$ and $\frac{1}{2\overline{\Phi}(y)}$. First, we estimate $\overline{\Phi}\left(y+z\right)$ and $\overline{\Phi}\left(y-z\right)$ by rational functions in x. According to (3.4),

$$\begin{aligned}\overline{\Phi}\left(y+z\right) &\backsimeq Z\left(y+z\right)\sum_{i=0}^{N_1}\frac{\left(-1\right)^i 1\cdot 3\ldots\left(2i-1\right)}{\left(y+z\right)^{2i+1}}\\ &= Z\left(y\right)\exp\left(-\frac{2yz+z^2}{2}\right)\sum_{i=0}^{N_1}\frac{\left(-1\right)^i 1\cdot 3\ldots\left(2i-1\right)}{\left(y+z\right)^{2i+1}}.\end{aligned}$$

Next, we expand the term $\exp\left(-\frac{2yz+z^2}{2}\right)\sum_{i=0}^{N_1}\frac{(-1)^i 1\cdot 3\ldots(2i-1)}{(y+z)^{2i+1}}$ in a power series of z according to z's property (z is small) by keeping the four first powers as $e_0(y) + e_1\left(y\right)z + e_2\left(y\right)z^2 + e_3\left(y\right)z^3$. More powers in z may be used but we find that the approximation is accurate enough with four terms. Then $\overline{\Phi}\left(y+z\right)$ can be written as

$$\overline{\Phi}\left(y+z\right) = Z\left(y\right)\left(e_0(y) + e_1\left(y\right)z + e_2\left(y\right)z^2 + e_3\left(y\right)z^3\right) + o\left(z^3\right), \tag{3.9}$$

where each $e_i\left(y\right)$ $\left(i = 0,1,2,3\right)$ is a power function in y.

Similarly, $\overline{\Phi}\left(y-z\right)$ can be written as

$$\overline{\Phi}\left(y-z\right) = Z\left(y\right)\left(e_0(y) - e_1\left(y\right)z + e_2\left(y\right)z^2 - e_3\left(y\right)z^3\right) + o\left(z^3\right). \tag{3.10}$$

According to (3.9) and (3.10) and by assuming $z = 0$, we find $e_0(y) = \frac{\overline{\Phi}(y)}{Z(y)} \backsimeq \frac{Z(y)}{f_1(y)}\cdot\frac{1}{Z(y)} =$

$\frac{1}{f_1(y)}$. Thus with the notation $f_2(y) = (f_1(y))^2 e_1(y)$, $l(z,y)$ can be estimated as:

$$\begin{aligned} l(z,y) &= \frac{1}{2\overline{\Phi}(y)} - \frac{1}{\overline{\Phi}(y+z) + \overline{\Phi}(y-z)} \\ &\approx \frac{f_1(y)}{2Z(y)} \left(1 - \frac{1}{1 + f_1(y)\, e_2(y)\, z^2 + o(z^3)}\right) \\ &= \frac{f_1(y)}{2Z(y)} \sum_{i=1}^{\infty} (-1)^{i+1} \left(f_1(y)\, e_2(y)\, z^2 + o\left(z^3\right)\right)^i \\ &= \frac{z^2}{2Z(y)} \cdot f_2(y) + o\left(z^3\right). \end{aligned} \tag{3.11}$$

Furthermore, by a combination of the expansions $e_2(y)$ and $f_1(y)$, we get $f_2(y)$ of form $\sum_{i=0}^{N_2} d_i y^{(3-2i)}$. We list several examples of $f_2(y)$ and their expansions with different values of N_1 and N_2 ($N_1 = 2$ or 3, $N_2 = 3$ or 5).

	$N_1 = 2$	$N_1 = 3$
$e_2(y)$	$\left(\frac{1}{2}y + \frac{15}{2}y^{-5} + 45y^{-7}\right)$	$\left(\frac{1}{2}y - \frac{105}{2}y^{-7} - 420y^{-9}\right)$
$f_2(y)$	$\left(\frac{1}{2}y + \frac{15}{2}y^{-5} + 45y^{-7}\right)\left(y^{-1} - y^{-3} + 3y^{-5}\right)^{-2}$	$\left(\frac{1}{2}y - \frac{105}{2}y^{-7} - 420y^{-9}\right)\left(y^{-1} - y^{-3} + 3y^{-5} - 15y^{-7}\right)^{-2}$
$N_2 = 3$	$\frac{1}{2}y^3 + y - \frac{3}{2}y^{-1} + \frac{1}{2}y^{-3} + 58y^{-5}$	$\frac{1}{2}y^3 + y - \frac{3}{2}y^{-1} + 8y^{-3} - \frac{19}{2}y^{-5}$
$N_2 = 5$	$\frac{1}{2}y^3 + y - \frac{3}{2}y^{-1} + \frac{1}{2}y^{-3} + 58y^{-5} + \frac{189}{2}y^{-7} - \frac{401}{2}y^{-9}$	$\frac{1}{2}y^3 + y - \frac{3}{2}y^{-1} + 8y^{-3} - \frac{19}{2}y^{-5} - 543y^{-7} - \frac{1391}{2}y^{-9}$

The final approximation for $l(z,y)$ is

$$l(z,y) \approx \frac{z^2}{2Z(y)} \left(\sum_{i=0}^{N_2} d_i y^{(3-2i)}\right). \tag{3.12}$$

By combining (3.8) and (3.12) together with (3.5), finally, we get the following approximation

$$G(a_0, b_0, h) \backsimeq \frac{1}{2Z(y)} \left(\sum_{i=0}^{N_2} c_i y^{1-2i}\right) - \frac{z^2}{2Z(y)} \left(\sum_{i=0}^{N_2} d_i y^{(3-2i)}\right), \tag{3.13}$$

where $y = \sqrt{\frac{\mathbf{W_0^*} h}{m(n-1)}}$, $z = \frac{\mathbf{Z}_0^*}{\sqrt{m}}$ and $\mathbf{Z}_0^* \sim \mathrm{N}(\mathbf{0}, 1)$ $\mathbf{W}_0^* \sim \chi^2_{m(n-1)}$.

Proposition 2. *Based on the approximation* (3.13), *an analytic approximation for* ARL_0 *formula is given by*

$$\widehat{ARL_0}(h) = \frac{\sqrt{2\pi}}{2} \sum_{i=0}^{N_2} \left(c_i - \frac{d_i}{m}\right) g_i(h), \tag{3.14}$$

where

$$g_i(h)=\frac{\Gamma\left(\frac{mn-m}{2}-\frac{2i-3}{2}\right)\left(1-\frac{h}{mn-m}\right)^{\left(\frac{2i-3}{2}-\frac{mn-m}{2}\right)}}{2^{\frac{2i-3}{2}}\Gamma\left(\frac{mn-m}{2}\right)\left(\frac{h}{mn-m}\right)^{\frac{2i-3}{2}}}.$$

Proof. By (3.13), (3.1) and $\mathrm{E}\left(z^2\right)=\mathrm{E}\left[\frac{\left(\mathbf{Z}_0^*\right)^2}{m}\right]=\frac{1}{m}$, we have

$$\begin{aligned}\widehat{ARL_0}(h)=&\mathrm{E}\left[\frac{1}{2Z(y)}\left(\sum_{i=0}^{N_2}c_iy^{(3-2i)}\right)-\frac{z^2}{2Z(y)}\left(\sum_{i=0}^{N_2}d_iy^{(3-2i)}\right)\right]\\ =&\frac{\sqrt{2\pi}}{2}\sum_{i=0}^{N_2}\left(c_i-\frac{d_i}{m}\right)\mathrm{E}\left[\exp\left(\frac{y^2}{2}\right)y^{(3-2i)}\right].\end{aligned}$$

Let $g_i(h)=\mathrm{E}\left[\exp\left(\frac{y^2}{2}\right)y^{(3-2i)}\right]$. Since $\mathrm{W}_0^*\sim\chi^2_{m(n-1)}$ and $y=b_0\sqrt{h}=\sqrt{\frac{\mathbf{W}_0^*h}{m(n-1)}}$ is bounded with high probability, for each i $(0\leqslant i\leqslant N_2)$, we have

$$\begin{aligned}g_i(h)&=\mathrm{E}\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{y^2}{2}\right)^k y^{(3-2i)}\right]\\ &=\sum_{k=0}^{\infty}\frac{1}{2^k k!}\mathrm{E}\left[y^{(3-2i+2k)}\right]\\ &=\sum_{k=0}^{\infty}\frac{1}{2^k k!}\left(\frac{h}{mn-m}\right)^{\frac{3-2i+2k}{2}}\mathrm{E}\left[\mathrm{W}_0^{*\frac{(3-2i+2k)}{2}}\right]\\ &=\sum_{k=0}^{\infty}\frac{1}{2^k k!}\left(\frac{h}{mn-m}\right)^{\frac{3-2i+2k}{2}}\cdot 2^{\frac{(3-2i+2k)}{2}}\frac{\Gamma\left(k+\frac{3}{2}-i+\frac{mn-m}{2}\right)}{\Gamma\left(\frac{mn-m}{2}\right)}\\ &=\frac{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{h}{mn-m}\right)^k\Gamma\left(k+\frac{mn-m}{2}-\frac{2i-3}{2}\right)}{2^{\frac{2i-3}{2}}\Gamma\left(\frac{mn-m}{2}\right)\left(\frac{h}{mn-m}\right)^{\frac{2i-3}{2}}}.\end{aligned}\tag{3.15}$$

Here the condition $N_2<\frac{m(n-1)+3}{2}$ must be satisfied for the definition of the Gamma function.

Now, we calculate the numerator of (3.15) with the notation $a = \frac{h}{mn-m}$, $b = \frac{mn-m}{2} - \frac{2i-3}{2}$.

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{a^k}{k!} \Gamma(k+b) &= \sum_{k=0}^{\infty} \frac{a^k}{k!} \int_0^{\infty} \exp(-t)\, t^{k+b-1} dt \\ &= \int_0^{\infty} \left[\sum_{k=0}^{\infty} \frac{(at)^k}{k!} \right] \exp(-t)\, t^{b-1} dt \\ &= \int_0^{\infty} \exp\left[-(1-a)\,t\right] t^{b-1} dt \\ &= \Gamma(b)\,(1-a)^{-b}, \end{aligned} \tag{3.16}$$

i.e. $\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{h}{mn-m}\right)^k \Gamma\left(k + \frac{mn-m}{2} - \frac{2i-3}{2}\right) = \Gamma\left(\frac{mn-m}{2} - \frac{2i-3}{2}\right) \left(1 - \frac{h}{mn-m}\right)^{\left(\frac{2i-3}{2} - \frac{mn-m}{2}\right)}$.

Substituting (3.16) into (3.15) , we get

$$g_i(h) = \frac{\Gamma\left(\frac{mn-m}{2} - \frac{2i-3}{2}\right)\left(1 - \frac{h}{mn-m}\right)^{\left(\frac{2i-3}{2} - \frac{mn-m}{2}\right)}}{2^{\frac{2i-3}{2}} \Gamma\left(\frac{mn-m}{2}\right) \left(\frac{h}{mn-m}\right)^{\frac{2i-3}{2}}},$$

and finally we have $\widehat{ARL_0}(h) = \frac{\sqrt{2\pi}}{2} \sum_{i=0}^{N_2} \left(c_i - \frac{d_i}{m}\right) g_i(h)$. □

4 The approximation orders N_1 and N_2

Our approximation formula (3.14) depends on two parameters, namely, the orders N_1 and N_2 introduced in the expansions. These parameters determine the coefficients $\{c_i\}$ and $\{d_i\}$ and finally the accuracy of the approximation (3.14). Below we give some indications on the choice of these parameters.

First, we decide the value of N_1. The criterion is to choose N_1 such that the approximation (3.6) of $\overline{\Phi}(y)$ is accurate enough.

According to Abramowitz and Stegun (1965), the difference between $\overline{\Phi}(y)$ and the approximation (3.6)

$$\frac{Z(y)}{f_1(y)} = Z(y) \sum_{i=0}^{N_1} \frac{(-1)^i\, 1 \cdot 3 \ldots (2i-1)}{y^{2i+1}}$$

has the form

$$R_{N_1}(y) \quad = \overline{\Phi}(y) - \frac{Z(y)}{f_1(y)} = \quad (-1)^{N_1+1} \cdot 1 \cdot 3 \cdot \cdots \cdot (2N_1+1) \int_y^{\infty} \frac{Z(t)}{t^{2N_1+2}} dt$$

whose absolute value is a decreasing function of y ($y > 0$) when N_1 is given. So if we choose an N_1 to ensure $|R_{N_1}(y_0)| < 10^{-3}$ at y_0, then this also holds for any $y > y_{0,}$. Since in our application, $y = \sqrt{\frac{\mathbf{W}_0^* h}{m(n-1)}}$ where $\mathbf{W}_0^* \sim \chi^2_{m(n-1)}$, we can see that y has the magnitude of $\sqrt{h}$, which usually ranges from 2 to 10 (usually the h is ranging from 4 to 100). So we compare $\overline{\Phi}(y)$ with the approximation $\frac{Z(y)}{f_1(y)} = Z(y)\sum_{i=0}^{N_1} \frac{(-1)^i 1\cdot 3 \ldots (2i-1)}{y^{2i+1}}$ at the point $y_0 = 2$ in order to find an appropriate N_1.

Table 1: Comparison between $R_{N_1}(y_0)$ and $\frac{Z(y_0)}{f_1(y_0)}$ when $y_0 = 2$ (accurate to five decimal places)

N_1	1	2	3	4	5	6
$R_{N_1}(y_0)$	0.00250	-0.00256	0.00377	-0.00730	0.01761	-0.05090

According to Table 1, we can see that the absolute values of $R_{N_1}(y_0)$ are larger than 10^{-3} with alternate signs ($y_0 = 2$), which are not negligible. In order to reduce the error $R_{N_1}(y_0)$, we decide to average the approximation $\frac{Z(y)}{f_1(y)}$ ($N_1 = 1,\ 2,\ 3,\ 4$) by taking advantage of the $R_{N_1}(y)$'s signs. Equivalently, we average (3.14) when $N_1 = 1,\ 2,\ 3,\ 4$ and $y = 2$. Moreover, the averaged approximation (3.4) will perform better when $y \geqslant 2$, since $R_{N_1}(y)$ is smaller for larger y. As a result, we choose several $N_1 = 1,\ 2,\ 3,\ 4$ rather than choosing a single N_1 because of $R_{N_1}(y_0)$'s non-negligible values.

The choice of N_2 directly affects the deviation of $f_1(y) = \left(\sum_{i=0}^{N_1} \frac{(-1)^i 1\cdot 3 \ldots (2i-1)}{y^{2i+1}}\right)^{-1}$ in (3.7) from its Taylor expansions $\sum_{i=0}^{N_2} c_i y^{1-2i}$ and the deviation of $f_2(y) = (f_1(y))^2 e_2(y)$ in (3.11) from its Taylor expansions $\sum_{i=0}^{N_2} d_i y^{(3-2i)}$. The values of $f_1(y)$ and $f_2(y) = (f_1(y))^2 e_2(y)$ converge to their Taylor expansions when $y \geqslant 2$. Therefore, we should select large N_2 in order to approximate $f_1(y)$ and $f_2(y)$ precisely by using their Taylor expansions. Then the final approximation can also be more accurate by sharing the same coefficients $\{c_i\}$ and $\{d_i\}$. We choose the smallest N_2 ensuring $\left|f_1(y) - \sum_{i=0}^{N_2} c_i y^{1-2i}\right| < 10^{-4}$ and $\left|f_2(y) - \sum_{i=0}^{N_2} d_i y^{(3-2i)}\right| < 10^{-4}$. By numerical computation, we obtain several N_2's corresponding to different values of y for $1 \leqslant N_1 \leqslant 4$ and they are given in Table 2.

Table 2: Minimum values of N_2 for different values of y when $1 \leqslant N_1 \leqslant 4$ ensuring the error bound $\left|f_1\left(y\right) - \sum_{i=0}^{N_2} c_i y^{1-2i}\right| < 10^{-4}$ (top), and ensuring the error bound $\left|f_2\left(y\right) - \sum_{i=0}^{N_2} d_i y^{(3-2i)}\right| < 10^{-4}$ (bottom).

		y	2	2.5	3	3.5	4	4.5
$N_1 = 1$	$f_1\left(y\right) = \frac{1}{\left(\frac{1}{y} - \frac{1}{y^3}\right)}$	N_2	7	5	3	3	3	3
$N_1 = 2$	$f_1\left(y\right) = \frac{1}{\left(\frac{1}{y} - \frac{1}{y^3} + \frac{3}{y^5}\right)}$	N_2	11	7	5	5	4	3
$N_1 = 3$	$f_1\left(y\right) = \frac{1}{\left(\frac{1}{y} - \frac{1}{y^3} + \frac{3}{y^5} - \frac{15}{y^7}\right)}$	N_2	17	10	7	5	4	4
$N_1 = 4$	$f_1\left(y\right) = \frac{1}{\left(\frac{1}{y} - \frac{1}{y^3} + \frac{3}{y^5} - \frac{15}{y^7} + \frac{105}{y^9}\right)}$	N_2	36	13	9	6	5	5

		y	2	2.5	3	3.5	4	4.5
$N_1 = 1$	$f_2\left(y\right) = \frac{\left(\frac{1}{2}y - \frac{3}{2}y^{-3} - 6y^{-5}\right)}{\left(\frac{1}{y} - \frac{1}{y^3}\right)}$	N_2	10	7	6	5	5	4
$N_1 = 2$	$f_2\left(y\right) = \frac{\left(\frac{1}{2}y + \frac{15}{2}y^{-5} + 45y^{-7}\right)}{\left(\frac{1}{y} - \frac{1}{y^3} + \frac{3}{y^5}\right)^2}$	N_2	13	10	8	6	6	5
$N_1 = 3$	$f_1\left(y\right) = \frac{\left(\frac{1}{2}y - \frac{105}{2}y^{-7} - 420y^{-9}\right)}{\left(\frac{1}{y} - \frac{1}{y^3} + \frac{3}{y^5} - \frac{15}{y^7}\right)^2}$	N_2	28	14	10	8	7	5
$N_1 = 4$	$f_1\left(y\right) = \frac{\left(\frac{1}{2}y - \frac{945}{2}y^{-9} - 4725y^{-11}\right)}{\left(\frac{1}{y} - \frac{1}{y^3} + \frac{3}{y^5} - \frac{15}{y^7} + \frac{105}{y^9}\right)^2}$	N_2	65	18	11	9	6	6

From Table 2, we find that for a given N_1, the value of N_2 decreases when y increases to ensure an approximation error less than 10^{-4} for both $f_1(y)$ and $f_2(y)$, that is, the appropriate value of N_2 for $y = 2$ will remain appropriate for all $y \geqslant 2$. According to Table 2 and Table 3, we know that the optimal choice of N_2 when $y = 2$ is $N_2 = 65$ to satisfy all the circumstances $1 \leqslant N_1 \leqslant 4$.

Summarizing the discussion above, we have made our decision to approximate ARL_0 by choosing $1 \leqslant N_1 \leqslant 4$, $N_2 = 65$ and averaging this four cases as follows:

$$\widehat{ARL_0}(h) \quad = \quad \frac{1}{4}\sum_{i=1}^{4}\widehat{ARL_0}(h \mid N_1 = i,\ N_2 = 65) \tag{4.1}$$

where $\widehat{ARL_0}(h \mid N_1 = i,\ N_2 = 65)$ $(i = 1,\ 2,\ 3,\ 4)$ are from (3.14) . This is the final approximation used for the remaining of the paper.

5 Monte-Carlo evaluation of the proposed analytic ARL approximation

The previous section lays the groundwork that allows us to approximate the values of ARL_0 of the T^2 chart with estimated parameters. Here we focus on practical implementation of the estimations.

According to the analytic approximation (4.1) , we can obtain a series of control limits with different m and n by setting the right side of the equation (4.1) a given value, for example, $ARL_0 = 200$ or 500. Then by programming, we can compute the corresponding control limit h and thus the control chart criterion can be confirmed with given m and n when $p = 1$.

According to (3.14) and (4.1), we first compute and list each $h(N_1 = i,\ N_2 = 65)$ ensuring $\widehat{ARL_0}(h \mid N_1 = i,\ N_2 = 65) = 200$ and 500 $(i = 1,\ 2,\ 3,\ 4)$ when $m = 50,\ 70,\ 100$ with size of subgroup $n = 5,\ 10,\ 15$. The computation is based on Monte Carlo. Then for each group, we average four values of $h(N_1 = i,\ N_2 = 65)$ $(i = 1,\ 2,\ 3,\ 4)$ to get final control limit $\bar{h}$ that we propose to use in practice for given m and n. The results are given in Tables 3 and 4 for ARL_0 of value 200 and 500, respectively. In the last column of these tables, the values of the corresponding traditional UCL h^* are given for comparison. Recall that given m, n and ARL_0, h^* is determined in (1.3). These tables show in particular that the actual UCL h, hence our approximated value $\overline{h}$, are smaller than the traditional h^* as predicted by Theorem 1, Although these differences seem not that big, the corresponding ARL values can be indeed very different due to the large variation of the ARL values.

After computing all the values of control limit $\bar{h}$ for various combination of m and n, we next check its accuracy by simulations. The simulation method is based on (1.5) following Champ *et al.* (2005). For each case with specific m, n and ARL_0, the number of independent repetitions is 10000.

Table 3: Control limits for each $N_1 = 1,\ 2,\ 3,\ 4$ ensuring $ARL_0 = 200$ using m subgroups of size n when $N_2 = 65$ and their averages.

m	n	$h\,(N_1 = 1)$	$h\,(N_1 = 2)$	$h\,(N_1 = 3)$	$h\,(N_1 = 4)$	$\bar{h}$	h^*
50	5	7.8122	7.9149	7.8469	7.9115	7.8714	8.2183
	10	7.9035	8.0093	7.9398	8.0052	7.9645	8.1169
	15	7.9299	8.0366	7.9665	8.0323	7.9913	8.0882
70	5	7.8109	7.9121	7.8462	7.9078	7.8693	8.1202
	10	7.8761	7.9796	7.9125	7.9748	7.9358	8.0486
	15	7.8950	7.9991	7.9316	7.9941	7.9550	8.0283
100	5	7.8111	7.9113	7.8468	7.9062	7.8689	8.0473
	10	7.8569	7.9585	7.8933	7.9532	7.9154	7.9976
	15	7.8700	7.9721	7.9066	7.9666	7.9289	7.9835

Table 4: Control limits for each $N_1 = 1,\ 2,\ 3,\ 4$ ensuring $ARL_0 = 500$ using m subgroups of size n when $N_2 = 65$ and their averages.

m	n	$h\,(N_1 = 1)$	$h\,(N_1 = 2)$	$h\,(N_1 = 3)$	$h\,(N_1 = 4)$	$\bar{h}$	h^*
50	5	9.4637	9.5318	9.4949	9.5238	9.5036	10.0023
	10	9.5970	9.6680	9.6296	9.6594	9.6385	9.8557
	15	9.6356	9.7074	9.6687	9.6987	9.6776	9.8143
70	5	9.4719	9.5395	9.5034	9.5312	9.5115	9.8708
	10	9.5673	9.6369	9.5998	9.6282	9.6081	9.7675
	15	9.5948	9.6650	9.6276	9.6562	9.6359	9.7383
100	5	9.4799	9.5471	9.5115	9.5386	9.5193	9.7734
	10	9.5467	9.6154	9.5790	9.6065	9.5869	9.7018
	15	9.5660	9.6350	9.5985	9.6261	9.6064	9.6815

Table 5: Empirical ARL_0 performance of the approximate control limit $\bar{h}$ with target $ARL_0 = 200$

m	n	$\bar{h}$	Empirical ARL ($s.e.$)
50	5	7.8714	202.2(243.9)
	10	7.9645	200.8(216.7)
	15	7.9913	198.5(213.0)
70	5	7.8693	200.4(225.5)
	10	7.9358	199.3(213.9)
	15	7.9550	198.3(209.8)
100	5	7.8689	205.1(220.0)
	10	7.9154	196.0(207.1)
	15	7.9289	200.2(203.6)

Table 6: Empirical ARL_0 performance of the approximate control limit $\bar{h}$ with target $ARL_0 = 500$

m	n	$\bar{h}$	Empirical ARL_0 ($s.e.$)
50	5	9.5036	510.0(691.0)
	10	9.6385	503.8(570.2)
	15	9.6776	510.5(557.1)
70	5	9.5115	506.0(612.0)
	10	9.6081	499.4(556.3)
	15	9.6359	504.4(538.7)
100	5	9.5193	505(576)
	10	9.5869	506.4(531.3)
	15	9.6064	505.7(529.4)

Table 5 and Table 6 give empirical ARL_0 values for several combinations of m and n when $p = 1$ using the control limits $\bar{h}$ for target $ARL_0 = 200$ and 500, respectively. We observe that in both Tables, the empirical values of the ARL_0 from the approximation UCL $\bar{h}$ are very close to the target one within an error of 2% at most. One may also observe quite large standard errors of the empirical ARL_0 which are however common in such Monte-Carlo experiments.

6 Concluding remarks

When estimated parameters from a Phase I study are used in the design of a Phase II control chart, it has been widely observed in the literature that the properties of the chart are modified. As a theoretical contribution of the paper, we have provided a rigorous proof of this phenomenon in the paper. Next, in addition to existing Monte-Carlo based determination method of control limits, we have proposed an analytic method based on an approximation of the *ARL* function. However, due to the complexity of the approximation problem, our solution is limited to univariate observations. How to extend such approximation procedure to more general multivariate observations remains an important question to explore in the future.

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