A note on optimal insurance risk control with multiple reinsurers

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December 13, 2016

Abstract

This note revisits the problem in Meng et al. [15] [H. Meng, T.K. Siu, H. Yang, Optimal insurance risk control with multiple reinsurers. J. Comput. Appl. Math. 306 (2016), 40-52] where an optimal insurance risk control problem was discussed in a diffusion approximation model with multiple reinsurers adopting variance premium principles. It was shown in Meng et al. [15] that under a certain technical condition, a combined proportional reinsurance treaty is an optimal form in a class of plausible reinsurance treaties. From both theoretical and practical perspectives, an interesting question may be whether the combined proportional reinsurance treaty is still an optimal form in a quite considerably larger class of plausible reinsurance treaties. This note addresses this question and shows that a combined proportional reinsurance treaty is still an optimal form.

Keywords: Optimal reinsurance treaties; Enlargement of reinsurance space; Variance premium principles; Multiple reinsurers; Lagrangian function
MSC2000: Primary 93E20; Secondary 60J25; 91B30

1 Introduction

Since the classical work by Borch [4] and Arrow [1], many aspects of generalizations have been studied in the literature. One direction of these generalizations is to consider multiple (re)-insurance parties, see, for example, Asimit et al. [2], Chi and Meng [6], Cong and Tan [7] and Boonen et al. [3]. For some related discussions and relevant literature, see Cheung and Lo [5] and Boonen et al. [3]. Some recent works on generalizations to dynamic modeling settings and multiple re-insurers include, for example, Meng [12], Meng et al. [14] and Meng et al. [15]. See Meng et al. [14] for some related discussions and relevant literature.

The purpose of this paper is to further generalize the recent work of Meng et al. [15] by enlarging the space of reinsurance treaties with a view to presenting a scientific inquiry on optimal reinsurance policies in a more general and flexible modeling environment. Using

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the Lagrangian function method, we establish the optimality result that the combined proportional reinsurance treaty is still an optimal form in a quite considerably larger class of plausible reinsurance treaties than the one considered in Meng et al. [15]. The organization of this paper is as follows. The next section presents the model formulation. The main result, namely the optimality result, is presented in Section 3.

2 Model formulation

In Meng et al. [15], we discussed an optimal risk control problem in a diffusion approximation model with the multiple reinsurers adopting variance premium principles. That is, the surplus process of the insurance company without dividends satisfies

$$dX(t) = \upsilon(g_0(t,Z), g_1(t,Z), \cdots, g_m(t,Z))dt + \sigma(g_0(t,Z))dB(t), \quad (2.1)$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion on a given complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$,

$$\upsilon(g_0(t,Z), g_1(t,Z), \cdots, g_m(t,Z)) = \lambda \left(\theta_0 \sigma^2 - \sum_{j=1}^{m} \theta_j \mathbb{E}[(g_j(t,Z))^2] \right), \quad (2.2)$$

and

$$\sigma(g_0(t,Z)) = \sqrt{\lambda \mathbb{E}[(g_0(t,Z))^2]}, \quad (2.3)$$

where $\mathbb{E}[\cdot]$ is the expectation taken under $\mathbb{P}$; Z is a nonnegative random variable; $\sigma^2 = \mathbb{E}[Z^2]$ and $\lambda, \theta_j > 0$, $j = 0, 1, \cdots, m$; the expectations in (2.2) and (2.3) are taken only for the random variable Z under the measure $\mathbb{P}$. In Meng et al.[15], a reinsurance strategy $(g_0, g_1, \cdots, g_m)$ is called an admissible policy if it satisfies the following four conditions:

(1) for each $z \in \Re^+$ and each $j = 0, 1, \cdots, m$, $\{g_j(t,z); t \geq 0\}$ is $\{\mathcal{F}_t\}_{t\geq 0}$-predictable;

(2) for each $(t, \omega) \in [0, \infty) \times \Omega$, $g_j(t, z, \omega)$ is Borel-measurable in z;

(3) $g_j(t,z) \geq 0$, $\sum_{j=0}^{m} g_j(t,z) = z$;

(4) There is at least a pair (k, l) such that

$$\sqrt{\mathbb{E}\left[\left(\sum_{i\geq 0, i\neq k} g_i(t,Z)\right)^2\right]} - \sum_{i\geq 0, i\neq k,l} \sqrt{\mathbb{E}[(g_i(t,Z))^2]} \geq 0,$$

where $1 \leq k, l \leq m$, $k \neq l$.

We write $\mathcal{A}_{1-4}$ for the space of all reinsurance strategies satisfying the above four conditions. Meng et al. [15] showed that within the class of reinsurance policies that satisfy $\mathcal{A}_{1-4}$, a combined proportional reinsurance treaty is an optimal form. Indeed, the condition (4) is purely technical and was used in Meng et al. [15] to prove the optimality result of reinsurance strategy. As illustrated in Example 2 in Section 2 of Meng et al. [15], this purely technical condition rules out some reinsurance strategies. From both theoretical and practical perspectives, an interesting question may be what is an optimal form of reinsurance strategies if this purely technical condition is relaxed. From the practical perspective, the relaxation of the purely technical condition would enhance the applicability and practicality of the theoretical results in Meng et al. [15]. From the theoretical point of view, the relaxation of the purely technical condition would strengthen the universality of the optimal form of reinsurance strategies as a combined proportional reinsurance treaty by considering a (possibly) considerably larger space of reinsurance strategies. We shall discuss this interesting problem in this note and show that a combined proportional reinsurance strategy is still an optimal form even when the condition (4) is dropped.

3 Optimal reinsurance form

This section presents the main result of this note. The key mathematical tool used to establish the optimality result is the Lagrangian method. Note that the Lagrangian method was used in Meng et al. [15], Section 4, to derive an explicit solution to the optimal dividend problem within the class of combined proportional reinsurance strategies. Indeed, a considerable amount of effort has been given to discussing optimal forms of reinsurance treaties in continuous-time insurance risk models. Some examples of these works are Meng and Zhang [8], Meng and Siu [9], [10], [11], Meng [12], and Meng et al. [13], [14], among others. For interested audience, please see Meng et al. [15] and the relevant references therein for related discussions. Here we use $\mathcal{A}_{1-3}$ for denoting the space of all reinsurance strategies only satisfying the conditions (1)-(3) in section 1. In this section, we shall show that in the wider class $\mathcal{A}_{1-3}$ of reinsurance strategies, a combined proportional reinsurance treaty is still an optimal form.

Firstly, for any fixed reals $z \geq 0, \gamma > 0$, we adopt the Lagrangian function method to construct the following function:

$$f(y_0, y_1, \cdots, y_m) = -\sum_{j=1}^{m} \theta_j y_j^2 - \gamma y_0^2 + \frac{2z}{\sum_{j=1}^{m} \frac{1}{\theta_j} + \frac{1}{\gamma}} \left(\sum_{j=0}^{m} y_j - z \right), \tag{3.1}$$

where $(y_0, y_1, \cdots, y_m) \in \Re^{m+1}$.

It can be seen that (3.1) can be rewritten as:

$$\begin{aligned} f(y_0, y_1, \cdots, y_m) &= -\sum_{j=1}^{m} \theta_j (y_j - \bar{y}_j(\gamma, z))^2 - \gamma (y_0 - \bar{y}_0(\gamma, z))^2 \\ &\quad - \frac{2z^2}{\sum_{j=1}^{m} \frac{1}{\theta_j} + \frac{1}{\gamma}} + \sum_{j=1}^{m} \theta_j (\bar{y}_j(\gamma, z))^2 + \gamma (\bar{y}_0(\gamma, z))^2, \end{aligned} \tag{3.2}$$

where $(\bar{y}_0(\gamma, z), \bar{y}_1(\gamma, z), \cdots, \bar{y}_m(\gamma, z))$ is

$$\bar{y}_0(\gamma, z) = \frac{\frac{1}{\gamma}}{\sum_{j=1}^{m} \frac{1}{\theta_j} + \frac{1}{\gamma}} z, \tag{3.3}$$

$$\bar{y}_j(\gamma, z) = \frac{\frac{1}{\theta_j}}{\sum_{j=1}^{m} \frac{1}{\theta_j} + \frac{1}{\gamma}} z, \; j = 1, \cdots, m. \tag{3.4}$$

Obviously, $(\bar{y}_0(\gamma, z), \bar{y}_1(\gamma, z), \cdots, \bar{y}_m(\gamma, z))$ is the maximum point of $f(y_0, y_1, \cdots, y_m)$ over all points $(y_0, y_1, \cdots, y_m) \in \Re^{m+1}$ and it satisfies

$$\sum_{j=0}^{m} \bar{y}_j(\gamma, z) = z. \tag{3.5}$$

In particular, for each $\tilde{y}_j \geq 0$ satisfying $\sum_{j=0}^m \tilde{y}_j = z$, we have

$$f(\bar{y}_0(\gamma, z), \bar{y}_1(\gamma, z), \cdots, \bar{y}_m(\gamma, z)) \quad \geq \quad f(\tilde{y}_0, \tilde{y}_1, \cdots, \tilde{y}_m), \tag{3.6}$$

that is

$$-\sum_{j=1}^m \theta_j (\bar{y}_j(\gamma, z))^2 - \gamma(\bar{y}_0(\gamma, z))^2 \geq -\sum_{j=1}^m \theta_j \tilde{y}_j^2 - \gamma \tilde{y}_0^2. \tag{3.7}$$

Similarly, it can be shown that $(\bar{y}_1(z), \cdots, \bar{y}_m(z))$ is the maximum point of $f_0(y_1, \cdots, y_m)$ over all points $(y_1, \cdots, y_m) \in \Re^m$ and it satisfies the condition that

$$\sum_{j=1}^m \bar{y}_j(z) \quad = \quad z, \tag{3.8}$$

where

$$f_0(y_1, \cdots, y_m) \quad = \quad -\sum_{j=1}^m \theta_j y_j^2 + \frac{2z}{\sum_{j=1}^m \frac{1}{\theta_j}} \left(\sum_{j=1}^m y_j - z \right), \tag{3.9}$$

and

$$\bar{y}_j(z) \quad = \quad \frac{\frac{1}{\theta_j}}{\sum_{j=1}^m \frac{1}{\theta_j}} z, \; j = 1, \cdots, m. \tag{3.10}$$

Thus for each $\tilde{y}_j \geq 0$ satisfying $\sum_{j=1}^m \tilde{y}_j = z$, we have

$$f_0(\bar{y}_1(z), \cdots, \bar{y}_m(z)) \quad \geq \quad f_0(\tilde{y}_1, \cdots, \tilde{y}_m), \tag{3.11}$$

that is

$$-\sum_{j=1}^m \theta_j (\bar{y}_j(z))^2 \geq -\sum_{j=1}^m \theta_j \tilde{y}_j^2. \tag{3.12}$$

We now put z as the nonnegative random variable Z and fix a reinsurance strategy $\{\tilde{g}_j(t, Z)\}_{j=0}^m \in \mathcal{A}_{1-3}$.

Noting $\mathbb{E}[(\tilde{g}_0(t, Z))^2] \in [0, \sigma^2]$, three cases are considered: $\mathbb{E}[(\tilde{g}_0(t, Z))^2] \in (0, \sigma^2)$, $\mathbb{E}[(\tilde{g}_0(t, Z))^2] = 0$ or $\mathbb{E}[(\tilde{g}_0(t, Z))^2] = \sigma^2$. The last two cases are degenerate cases.

When $\mathbb{E}[(\tilde{g}_0(t, Z))^2] \in (0, \sigma^2)$ and noting

$$\mathbb{E}[(\bar{y}_0(\gamma, Z))^2] = \left(\frac{\frac{1}{\gamma}}{\sum_{j=1}^m \frac{1}{\theta_j} + \frac{1}{\gamma}} \right)^2 \sigma^2, \tag{3.13}$$

we can choose a predictable process $\{\gamma_t\}$ such that

$$\mathbb{E}[(\bar{y}_0(\gamma_t, Z))^2] = \mathbb{E}[(\tilde{g}_0(t, Z))^2]. \tag{3.14}$$

Note that the expectations are taken only for the random variable Z under the measure $\mathbb{P}$.

Letting

$$\bar{g}_j(t,Z) = \bar{y}_j(\gamma_t, Z),\ j = 0, 1, \cdots, m, \tag{3.15}$$

we have, with (3.7),

$$-\sum_{j=1}^{m} \theta_j (\bar{g}_j(t,Z))^2 - \gamma_t (\bar{g}_0(t,Z))^2 \geq -\sum_{j=1}^{m} \theta_j (\tilde{g}_j(t,Z))^2 - \gamma_t (\tilde{g}_0(t,Z))^2. \tag{3.16}$$

Taking the expectation for the random variable Z under $\mathbb{P}$ on both sides of (3.16) gives:

$$-\sum_{j=1}^{m} \theta_j \mathbb{E}[(\bar{g}_j(t,Z))^2] \geq -\sum_{j=1}^{m} \theta_j \mathbb{E}[(\tilde{g}_j(t,Z))^2], \tag{3.17}$$

which results in

$$\upsilon(\bar{g}_0(t,Z), \bar{g}_1(t,Z), \cdots, \bar{g}_m(t,Z)) \geq \upsilon(\tilde{g}_0(t,Z), \tilde{g}_1(t,Z), \cdots, \tilde{g}_m(t,Z)). \tag{3.18}$$

When $\mathbb{E}[(\tilde{g}_0(t,Z))^2] = 0$, (i.e., the insurance company cedes all of the losses to the m reinsurance companies), setting

$$\bar{g}_0(t,Z) = 0,\ \bar{g}_j(t,Z) = \bar{y}_j(Z),\ j = 1, \cdots, m, \tag{3.19}$$

and using (3.12) give:

$$-\sum_{j=1}^{m} \theta_j (\bar{g}_j(t,Z))^2 \geq -\sum_{j=1}^{m} \theta_j (\tilde{g}_j(t,Z))^2. \tag{3.20}$$

Taking the expectation for the random variable Z under $\mathbb{P}$ then gives:

$$\upsilon(\bar{g}_0(t,Z), \bar{g}_1(t,Z), \cdots, \bar{g}_m(t,Z)) \geq \upsilon(\tilde{g}_0(t,Z), \tilde{g}_1(t,Z), \cdots, \tilde{g}_m(t,Z)). \tag{3.21}$$

When $\mathbb{E}[(\tilde{g}_0(t,Z))^2] = \sigma^2$, (i.e., the insurance company retains all of the losses), letting

$$\bar{g}_0(t,Z) = Z,\ \bar{g}_j(t,Z) = 0,\ j = 1, \cdots, m, \tag{3.22}$$

gives:

$$\upsilon(\bar{g}_0(t,Z), \bar{g}_1(t,Z), \cdots, \bar{g}_m(t,Z)) = \upsilon(\tilde{g}_0(t,Z), \tilde{g}_1(t,Z), \cdots, \tilde{g}_m(t,Z)). \tag{3.23}$$

Consequently, we have the following optimality result.

Theorem 3.1. *For any fixed reinsurance strategy* $(\tilde{g}_0, \tilde{g}_1 \cdots, \tilde{g}_m) \in \mathcal{A}_{1-3}$*, there exists a combined proportional reinsurance treaty* $(\bar{g}_0, \bar{g}_1 \cdots, \bar{g}_m) \in \mathcal{A}_{1-3}$ *such that*

$$\begin{aligned} \upsilon(\bar{g}_0(t,Z), \bar{g}_1(t,Z), \cdots, \bar{g}_m(t,Z)) &\geq \upsilon(\tilde{g}_0(t,Z), \tilde{g}_1(t,Z), \cdots, \tilde{g}_m(t,Z)), \\ \sigma(\bar{g}_0(t,Z)) &= \sigma(\tilde{g}_0(t,Z)). \end{aligned}$$

This theorem shows that, under general case (i.e., without the condition (4)), a combined proportional reinsurance treaty is still an optimal form.

Remark 3.1. *In J. Comput. Appl. Math. 306 (2016), 40-52, we considered an optimal risk control problem in a diffusion approximation model with multiple reinsurers and dividends payout. In the present paper, we only focus on an optimal risk control problem without dividend payments. Dividend payouts may not affect the optimal form of reinsurance treaty. In particular, the main result presented in Theorem 3.1 may hold since at an intuitive level, the dividends strategy may be kept fixed and another reinsurance strategy may be chosen to enlarge the surplus of the insurance company.*

Acknowledgements H. Meng's research is supported by the Nature Science Foundation of China (11271385,11471171) and the Program for Innovation Research in Central University of Finance and Economics. H. Yang's research was supported by Research Grants Council of the Hong Kong Special Administrative Region (project No. HKU 17330816).

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