Optimal reinsurance in a compound Poisson risk model with dependence

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Abstract This paper considers the problem of optimal reinsurance in a compound Poisson risk model with dependent classes of insurance business. It is assumed that the risk process in each class follows a compound Poisson process, and that all classes are correlated due to the so-called thinning-dependence structure. Under the criterion of maximizing the adjustment coefficient, methods for finding the optimal reinsurance strategies are discussed for both the expected value premium principle and the variance premium principle. Numerical examples are also provided to illustrate the impact of the model parameters on the optimal reinsurance strategies.

Keywords Adjustment coefficient · Compound Poisson risk model · Optimal reinsurance · Thinning dependence

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1 Introduction

Reinsurance is an important activity of insurance companies for transferring some of their potential risk to another party at the cost of sacrificing part of their potential profit. Because of its practical importance, the problem of optimal reinsurance has been studied extensively in the actuarial literature. Under the criterion of minimizing ruin probability or maximizing adjustment coefficient, many optimal results have been derived for the classical risk model and its diffusion approximation. For example, [2], [9] and [8] considered the problem of minimizing ruin probability; and [3]-[6] focused on constructing optimal contracts that maximize the adjustment coefficient by the martingale approach.

In the actuarial literature, insurance risk analysis in the presence of dependent risks has been one of the major topics in the past few decades. Although it might be a bit long overdue, the study of dependent risks has been extended to the problem of optimal reinsurance in recent years. One of the frequently-seen risk models with dependence assumes that correlation among different classes of insurance risks is due to common shocks. For two classes of insurance risks with common shock dependence, [1] sought the optimal excess of loss reinsurance to minimize the ruin probability based on the diffusion approximation risk model; and [7] studied the optimal proportional reinsurance problem under the variance premium principle for both the compound Poisson risk model and the associated diffusion risk model. For more than two classes of insurance business with common shock dependence, [13] considered the objective of maximizing the expected exponential utility and derived the optimal reinsurance strategy not only for the diffusion approximation risk model but also for the compound Poisson risk model.

In order to depict more realistic features of dependence, [14] first introduced the so-called risk model with thinning dependence in which claims in one class may induce claims in other classes with certain probabilities. A typical example is that a severe car accident may cause not only the loss of the damaged car but also the medical expenses of the injured driver and passengers. Mimicking the idea of [14], [11] studied the thinning relation in the discrete-time case; and [10] formulated the thinning-dependence structure in a more general framework, and investigated some basic properties of the corresponding risk process as well as the impact of the thinning dependence on ruin probability.

In the classical risk model, the risk process is given by

$$U(t) = u + ct - S(t) = u + ct - \sum_{i=1}^{N(t)} X(i), \tag{1.1}$$

where u is the initial capital, c is the premium rate, $S(t)$ is the aggregate claims process, the claim-number process $N(t)$ is a homogeneous Poisson process with intensity λ, and $\{X(i), i \geq 1\}$ is a sequence of positive, independent and identically distributed (*i.i.d.*) claim-amount random variables. It is assumed that the claim-number process $N(t)$ is independent of the claim amounts $\{X(i), i \geq 1\}$.

Suppose that an insurance company has n ($n \geq 2$) classes of business. In the setting of thinning dependence, stochastic sources that may cause a claim in at least one of the n classes are classified into m groups. It is assumed that each event in the kth group may cause a claim in the jth class with probability p_{kj} for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Thus, the total amount of claims for the jth class up to time t has the form

$$S_j(t) = \sum_{i=1}^{N_j(t)} X_j(i),$$

where $N_j(t)$ is the claim-number process of the jth class. Denote by $N^k(t)$ the number of events generated from the kth group that has occurred up to time t, and by $N_j^k(t)$ the number of claims of the jth class up to time t generated from the events in group k. Then the claim-number process of the jth class can be written as

$$N_j(t) = N_j^1(t) + N_j^2(t) + \cdots + N_j^m(t),$$

and the aggregate claims process of the entire portfolio is given by

$$S(t) = \sum_{j=1}^{n} S_j(t) = \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} X_j(i),$$

where $\{X_j(i), i \geq 1\}$ is a sequence of *i.i.d.* non-negative claim-amount random variables with common distribution F_{X_j} for class j ($j = 1, \ldots, n$).

Incorporating the thinning dependence into (1.1), we define the risk process of the jth ($j = 1, \ldots, n$) class as

$$U_j(t) = u_j + c_j t - S_j(t) = u_j + c_j t - \sum_{i=1}^{N_j(t)} X_j(i),$$

and the risk process of the company is denoted by

$$U(t) = u + ct - S(t) = u + ct - \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} X_j(i), \tag{1.2}$$

with $u = \sum_{j=1}^{n} u_j$ and $c = \sum_{j=1}^{n} c_j$. As was mentioned in [10], we need the following assumptions in order to make the analysis of $U(t)$ mathematically tractable:

(A1) The process $N^1(t), \cdots, N^m(t)$ are independent Poisson processes with intensities $\lambda_1, \cdots, \lambda_m$, respectively. For $k \neq k'$, the two vectors of claim-number processes $(N^k(t), N_1^k(t), \cdots, N_n^k(t))$ and $(N^{k'}(t), N_1^{k'}(t), \cdots, N_n^{k'}(t))$ are independent.

(A2) For each $k = 1, \cdots, m$, $N_1^k(t), \cdots, N_n^k(t)$ are conditionally independent given $N^k(t)$.

Remark 1.1. *If $m = n$ and $p_{kk} = 1$ for $k = 1, \ldots, n$, then $U(t)$ of (1.2) is the risk model of [14] with the so-called thinning-dependence structure. If $n = 2$, $m = 3$, $p_{12} = p_{21} = 0$, $p_{31} = p_{32} = 1$, $p_{11} = p_{22} = 1$, then $U(t)$ of (1.2) is the risk model with common shock for two dependent classes of business; see, for example, [7] and [12]. For $n > 2$, more general risk models with common shock can also be constructed from (1.2) by choosing the values of m and p_{kj} appropriately.*

In this paper, our aim is to find the optimal reinsurance strategy for the compound Poisson risk model with thinning dependence under the criterion of maximizing the adjustment coefficient. Section 2 presents the compound Poisson risk model with thinning dependence in the presence of proportional reinsurance. In Section 3, optimal results under both the expected value premium principle and the variance premium principle are derived. Section 4 provides some numerical examples to illustrate the impact of the model parameters on the optimal reinsurance strategies.

2 Model with proportional reinsurance

Assume that insurance company can reinsure a fraction of its claim with retention level q_j ($0 \leq q_j \leq 1$) for each risk in class j ($j = 1, 2, \ldots, n$). Let $\delta(q)$ be the reinsurance premium rate with $q = (q_1, q_2, \cdots, q_n)$. Let $U^q(t)$ be the associated risk process that describes the reserve of an insurance company at time t with reinsurance strategy q. It follows from (1.2) that

$$U^q(t) = u + (c - \delta(q))t - S^q(t), \tag{2.1}$$

where

$$S^q(t) = \sum_{j=1}^{n} S_j^q(t) = \sum_{j=1}^{n} \sum_{i=1}^{N_j(t)} q_j X_j(i).$$

Following the arguments in the proof of [10], one can show that $S^q(t)$ is still a compound Poisson risk process which can be rewritten as

$$S^q(t) = \sum_{i=1}^{N_t^X} X_i,$$

where N_t^X is a Poisson process with intensity $\lambda = \lambda_1 + \cdots + \lambda_m$, and $\{X_i, i \geq 1\}$ are *i.i.d.* with common distribution F_X having moment generating function

$$M_X(r) = \frac{1}{\lambda} \sum_{k=1}^{m} \lambda_k \prod_{j=1}^{n} (M_j(q_j r) p_{kj} + 1 - p_{kj}), \tag{2.2}$$

with $M_j(r)$ being the moment generating function of distribution F_{X_j} $(j = 1, 2, ..., n)$.

Denote the ruin time by

$$T^q = \inf\{t \geq 0 : U^q(t) < 0\},$$

and the probability of ultimate ruin by

$$\psi^q(u) = \mathrm{P}\{T^q < \infty | U^q(0) = u\}.$$

It is well known from the classical risk theory that the ruin probability has the form

$$\psi^q(u) = \frac{e^{-R^q u}}{\mathrm{E}[e^{-R^q U^q(T^q)} | T^q < \infty]},$$

where R^q is the adjustment coefficient. In general, there is no closed-form expression for the conditional expectation in the denominator. As a result, much ruin analysis has been carried out using the Lundberg inequality

$$\psi^q(u) \leq e^{-R^q u}. \tag{2.3}$$

By definition, the adjustment coefficient R^q is the positive root of

$$M_{S^q(t)-(c-\delta(q))t}(r) = e^{-r(c-\delta(q))t} e^{\lambda t [M_X(r)-1]} = 1,$$

and hence satisfies

$$(c - \delta(q))r - \lambda[M_X(r) - 1] = 0. \tag{2.4}$$

In the next section, we derive the optimal reinsurance strategy that maximizes the adjustment coefficient or minimizes the upper bound in (2.3).

3 Optimal results

To simplify the optimality analysis, we consider working on the model of study with two classes of business, i.e., $n = 2$, from now on. In this section, we discuss the optimal reinsurance problem for risk process (2.1) under two typical premium principles.

3.1 Expected value premium principle

Under the expected value premium principle, the reinsurance premium rate is

$$\delta(q) = (1+\eta)\sum_{j=1}^{2}\mu_j(1-q_j)\sum_{k=1}^{m}\lambda_k p_{kj}, \tag{3.1}$$

where η is the reinsurer's safety loading of two classes of insurance business. Denote the insurer's safety loading by

$$\theta = \frac{c}{\sum_{j=1}^{2}\mu_j\sum_{k=1}^{m}\lambda_k p_{kj}} - 1. \tag{3.2}$$

Plugging (2.2), (3.1) and (3.2) into (2.4), we obtain

$$\begin{aligned}&\Big[(1+\theta)\sum_{j=1}^{2}\mu_j\sum_{k=1}^{m}\lambda_k p_{kj} - (1+\eta)\sum_{j=1}^{2}\mu_j(1-q_j)\sum_{k=1}^{m}\lambda_k p_{kj}\Big]r\\ &\quad - \sum_{k=1}^{m}\lambda_k\prod_{j=1}^{2}[M_j(q_j r)p_{kj} + 1 - p_{kj}] + \lambda = 0.\end{aligned} \tag{3.3}$$

The main objective of the paper is to find the optimal reinsurance strategy $q^* = (q_1^*, q_2^*)$ that maximizes the adjustment coefficient or minimizes the upper bound for the ruin probability given in (2.3). So, we want to find the maximized adjustment coefficient R^{q^*} satisfying the following equation

$$\begin{aligned}\sup_{(q_1,q_2)}\Bigg\{&\Big[(1+\theta)\sum_{j=1}^{2}\mu_j\sum_{k=1}^{m}\lambda_k p_{kj} - (1+\eta)\sum_{j=1}^{2}\mu_j(1-q_j)\sum_{k=1}^{m}\lambda_k p_{kj}\Big]r\\ &- \sum_{k=1}^{m}\lambda_k\prod_{j=1}^{2}[M_j(q_j r)p_{kj} + 1 - p_{kj}] + \lambda\Bigg\} = 0.\end{aligned}$$

Let

$$\begin{aligned}f(q_1,q_2) =&\Big[(1+\theta)\sum_{j=1}^{2}\mu_j\sum_{k=1}^{m}\lambda_k p_{kj} - (1+\eta)\sum_{j=1}^{2}\mu_j(1-q_j)\sum_{k=1}^{m}\lambda_k p_{kj}\Big]r\\ &- \sum_{k=1}^{m}\lambda_k\prod_{j=1}^{2}[M_j(q_j r)p_{kj} + 1 - p_{kj}] + \lambda.\end{aligned}$$

Differentiating it with respect to q_1 and q_2 gives

$$\begin{cases} \dfrac{\partial f(q_1,q_2)}{\partial q_1} = (1+\eta)\mu_1 \sum\limits_{k=1}^{m} \lambda_k p_{k1} r - \sum\limits_{k=1}^{m} \lambda_k \left[r M_1'(q_1 r) p_{k1} \right] \left[M_2(q_2 r) p_{k2} + 1 - p_{k2} \right], \\ \dfrac{\partial f(q_1,q_2)}{\partial q_2} = (1+\eta)\mu_2 \sum\limits_{k=1}^{m} \lambda_k p_{k2} r - \sum\limits_{k=1}^{m} \lambda_k \left[r M_2'(q_2 r) p_{k2} \right] \left[M_1(q_1 r) p_{k1} + 1 - p_{k1} \right], \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2 f(q_1,q_2)}{\partial q_1^2} = -\sum\limits_{k=1}^{m} \lambda_k \left[r^2 M_1''(q_1 r) p_{k1} \right] \left[M_2(q_2 r) p_{k2} + 1 - p_{k2} \right] < 0, \\ \dfrac{\partial^2 f(q_1,q_2)}{\partial q_2^2} = -\sum\limits_{k=1}^{m} \lambda_k \left[r^2 M_2''(q_2 r) p_{k2} \right] \left[M_1(q_1 r) p_{k1} + 1 - p_{k1} \right] < 0, \\ \dfrac{\partial^2 f(q_1,q_2)}{\partial q_1 \partial q_2} = -\sum\limits_{k=1}^{m} \lambda_k \left[r M_1'(q_1 r) p_{k1} \right] \left[r M_2'(q_2 r) p_{k2} \right] < 0. \end{cases}$$

Lemma 3.1. *The function $f(q_1, q_2)$ is a concave function with respect to q_j for $j = 1, 2$.*

Proof. To prove that $f(q_1, q_2)$ is a concave function with respect to q_j for $j = 1, 2$, we need to show that the Hessian matrix of $f(q_1, q_2)$ is a negative definite matrix.

Denote the Hessian matrix of $f(q_1, q_2)$ by

$$\mathrm{H_E} = \begin{pmatrix} \dfrac{\partial^2 f(q_1,q_2)}{\partial q_1^2} & \dfrac{\partial^2 f(q_1,q_2)}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 f(q_1,q_2)}{\partial q_1 \partial q_2} & \dfrac{\partial^2 f(q_1,q_2)}{\partial q_2^2} \end{pmatrix} = r^2 \mathbf{A}, \tag{3.4}$$

where

$$\mathbf{A} = \begin{pmatrix} -\sum\limits_{k=1}^{m} \lambda_k M_1''(q_1 r) p_{k1} \left[M_2(q_2 r) p_{k2} + 1 - p_{k2} \right] & -\sum\limits_{k=1}^{m} \lambda_k \left[M_1'(q_1 r) p_{k1} \right] \left[M_2'(q_2 r) p_{k2} \right] \\ -\sum\limits_{k=1}^{m} \lambda_k \left[M_1'(q_1 r) p_{k1} \right] \left[M_2'(q_2 r) p_{k2} \right] & -\sum\limits_{k=1}^{m} \lambda_k M_2''(q_2 r) p_{k2} \left[M_1(q_1 r) p_{k1} + 1 - p_{k1} \right] \end{pmatrix}.$$

Because of (3.4), it is sufficient to show that $\mathbf{A}$ is negative definite. It is obvious that

$$\frac{\partial^2 f(q_1,q_2)}{\partial q_1^2} = -\sum_{k=1}^{m} \lambda_k M_1''(q_1 r) p_{k1} \left[M_2(q_2 r) p_{k2} + 1 - p_{k2} \right] < 0.$$

The remaining step is to prove

$$\begin{aligned} |\mathbf{A}| \;=\; & \sum_{k=1}^{m} \lambda_k M_1''(q_1 r) p_{k1} \left[M_2(q_2 r) p_{k2} + 1 - p_{k2} \right] \sum_{k=1}^{m} \lambda_k M_2''(q_2 r) p_{k2} \left[M_1(q_1 r) p_{k1} + 1 - p_{k1} \right] \\ & - \Big[\sum_{k=1}^{m} \lambda_k \left[M_1'(q_1 r) p_{k1} \right] \left[M_2'(q_2 r) p_{k2} \right] \Big]^2 > 0. \end{aligned} \tag{3.5}$$

Since $0 < p_{kj} < 1$, the left-hand side of (3.5) becomes

$$\begin{aligned}
&\sum_{k=1}^{m} \lambda_k M_1''(q_1 r) p_{k1} \left[M_2(q_2 r) p_{k2} + 1 - p_{k2}\right] \sum_{k=1}^{m} \lambda_k M_2''(q_2 r) p_{k2} \left[M_1(q_1 r) p_{k1} + 1 - p_{k1}\right] \\
&\quad - \Big[\sum_{k=1}^{m} \lambda_k \left(M_1'(q_1 r) p_{k1}\right) \left(M_2'(q_2 r) p_{k2}\right)\Big]^2 \\
>& \sum_{k=1}^{m} \lambda_k p_{k1} p_{k2} M_1''(q_1 r) M_2(q_2 r) \sum_{k=1}^{m} \lambda_k p_{k1} p_{k2} M_2''(q_2 r) M_1(q_1 r) \\
&\quad - \Big[\sum_{k=1}^{m} \lambda_k \left(M_1'(q_1 r) p_{k1}\right) \left(M_2'(q_2 r) p_{k2}\right)\Big]^2 \\
\geq& \left(\sum_{k=1}^{m} \sqrt{\lambda_k p_{k1} p_{k2} M_1''(q_1 r) M_2(q_2 r)} \sqrt{\lambda_k p_{k1} p_{k2} M_2''(q_2 r) M_1(q_1 r)}\right)^2 \\
&\quad - \Big[\sum_{k=1}^{m} \lambda_k \left(M_1'(q_1 r) p_{k1}\right) \left(M_2'(q_2 r) p_{k2}\right)\Big]^2 \\
=& \Big(\sum_{k=1}^{m} \lambda_k p_{k1} p_{k2}\Big)^2 \Big[M_1''(q_1 r) M_2(q_2 r) M_2''(q_2 r) M_1(q_1 r) - [M_1'(q_1 r) M_2'(q_2 r)]^2\Big] \\
=& \Big(\sum_{k=1}^{m} \lambda_k p_{k1} p_{k2}\Big)^2 \bigg(\mathrm{E}\left[X_1^2 e^{q_1 r X_1}\right] \mathrm{E}\left[e^{q_1 r X_1}\right] \mathrm{E}\left[X_2^2 e^{q_2 r X_2}\right] \mathrm{E}\left[e^{q_2 r X_2}\right] \\
&\quad - \Big(\mathrm{E}\left[X_1 e^{q_1 r X_1}\right] \mathrm{E}\left[X_2 e^{q_2 r X_2}\right]\Big)^2\bigg) \\
\geq& \Big(\sum_{k=1}^{m} \lambda_k p_{k1} p_{k2}\Big)^2 \bigg[\Big(\mathrm{E}\Big[X_1 e^{\frac{1}{2} q_1 r X_1} e^{\frac{1}{2} q_1 r X_1}\Big]\Big)^2 \Big(\mathrm{E}\Big[X_2 e^{\frac{1}{2} q_2 r X_2} e^{\frac{1}{2} q_2 r X_2}\Big]\Big)^2 \\
&\quad - \Big(\mathrm{E}\left[X_1 e^{q_1 r X_1}\right] \mathrm{E}\left[X_2 e^{q_2 r X_2}\right]\Big)^2\bigg] \\
=& 0,
\end{aligned}$$

where the second inequality and the last one are due to Hölder's inequality. □

From Lemma 3.1, we see that (q_1, q_2) maximizing $f(q_1, q_2)$ satisfies the following equations

$$\begin{cases} (1+\eta)\mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} r - \sum_{k=1}^{m} \lambda_k \left[r M_1'(q_1 r) p_{k1}\right] \left[M_2(q_2 r) p_{k2} + 1 - p_{k2}\right] = 0, \\ (1+\eta)\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2} r - \sum_{k=1}^{m} \lambda_k \left[r M_2'(q_2 r) p_{k2}\right] \left[M_1(q_1 r) p_{k1} + 1 - p_{k1}\right] = 0. \end{cases} \tag{3.6}$$

For notational convenience, we set $a = q_1 r$ and $b = q_2 r$. Equation (3.6) becomes

$$\begin{cases} (1+\eta)\mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} - \sum_{k=1}^{m} \lambda_k \left[M_1'(a) p_{k1}\right] \left[M_2(b) p_{k2} + 1 - p_{k2}\right] = 0, \\ (1+\eta)\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2} - \sum_{k=1}^{m} \lambda_k \left[M_2'(b) p_{k2}\right] \left[M_1(a) p_{k1} + 1 - p_{k1}\right] = 0. \end{cases} \tag{3.7}$$

For checking the existence and uniqueness of the solution to (3.7), we need the following lemmas.

Lemma 3.2. *There is a unique positive solution to each of the following equations*

$$(1+\eta)\mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} = \sum_{k=1}^{m} \lambda_k M_1'(a) p_{k1}, \tag{3.8}$$

and

$$(1+\eta)\mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} = \sum_{k=1}^{m} \lambda_k \mu_1 p_{k1} \left[M_2(b) p_{k_2} + 1 - p_{k2} \right]. \tag{3.9}$$

Proof. We first discuss equation (3.8). From equation (3.8), we have

$$M_1'(a) = \mu_1(1+\eta).$$

Since $M_1'(a)$ is an increasing function with $M_1'(0) = \mu_1$, equation (3.8) must have a unique positive solution a_1 given by

$$a_1 = M_1'^{-1}(\mu_1(1+\eta)).$$

For equation (3.9), let

$$g_0(b) = \sum_{k=1}^{m} \lambda_k \mu_1 p_{k1} \left[M_2(b) p_{k_2} + 1 - p_{k2} \right].$$

It is easy to check that

$$\begin{aligned} g_0(0) &= \sum_{k=1}^{m} \lambda_k \mu_1 p_{k1}, \\ g_0'(b) &= \sum_{k=1}^{m} \lambda_k \mu_1 p_{k1} \mathrm{E}(X_2 e^{bX_2}) > 0, \\ g_0''(b) &= \sum_{k=1}^{m} \lambda_k \mu_1 p_{k1} \mathrm{E}(X_2^2 e^{bX_2}) > 0, \end{aligned}$$

which imply that $g_0(b)$ is an increasing convex function. Furthermore, the left-hand side of equation (3.9) is a constant which is larger than $g_0(0)$, and the right-hand side of (3.9) tends to ∞ as $b \to \infty$. Therefore, equation (3.9) has a unique positive root denoted by b_1. □

Lemma 3.3. *There is a unique positive solution to each of the following equations*

$$(1+\eta)\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2} = \sum_{k=1}^{m} \lambda_k \mu_2 p_{k2} \left[M_1(a) p_{k_1} + 1 - p_{k1} \right], \tag{3.10}$$

and

$$(1+\eta)\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2} = \sum_{k=1}^{m} \lambda_k M_2'(b) p_{k2}. \tag{3.11}$$

Proof. Similar to the proof of Lemma 3.2, one can show that equations (3.10) and (3.11) have unique positive roots denoted by a_2 and b_2, respectively. □

The next lemma states the existence and uniqueness of the solution to (3.7).

Lemma 3.4. *Let a_1, b_1, a_2 and b_2 be the unique positive roots of equation* (3.8),(3.9),(3.10) *and* (3.11), *respectively. If*

$$\begin{cases} a_1 > a_2, \\ b_1 < b_2, \end{cases}$$

or

$$\begin{cases} a_1 < a_2, \\ b_1 > b_2 \end{cases}$$

hold, then equation (3.7) *has a unique positive solution (t_1, t_2).*

Proof. Let

$$G_1(a,b) = (1+\eta)\mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} - \sum_{k=1}^{m} \lambda_k \left[M_1'(a)p_{k1}\right]\left[M_2(b)p_{k2} + 1 - p_{k2}\right],$$

and

$$G_2(a,b) = (1+\eta)\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2} - \sum_{k=1}^{m} \lambda_k \left[M_2'(b)p_{k2}\right]\left[M_1(a)p_{k1} + 1 - p_{k1}\right].$$

Assume that $G_1(a,b) = 0$ with $a = f_1(b)$ and $G_2(a,b) = 0$ with $a = f_2(b)$. Differentiating both sides of $G_1(a,b) = 0$ with respect to b yields

$$\sum_{k=1}^{m} \lambda_k \left[f_1'(b) M_1''(a) p_{k1} \left[M_2(b)p_{k2} + 1 - p_{k2}\right] + M_1'(a) M_2'(b) p_{k1} \right] = 0,$$

which in turn gives

$$f_1'(b) = -\frac{\sum_{k=1}^{m} \lambda_k M_1'(a) M_2'(b) p_{k1}}{\sum_{k=1}^{m} \lambda_k f_1'(b) M_1''(a) p_{k1} [M_2(b)p_{k2} + 1 - p_{k2}]} < 0.$$

Moreover, it follows from Lemma 3.2 that the equations $G_1(0,b) = 0$ and $G_1(a,0) = 0$ have unique positive solutions a_1 and b_1, respectively. Therefore, the function $f_1(b)$ is decreasing with

$$\begin{cases} f_1(0) = a_1 > 0, \\ f_1^{-1}(0) = b_1 > 0. \end{cases}$$

Along the same lines, one can show that $f_2'(b) < 0$. This implies that $f_2(b)$ is also a decreasing function with

$$\begin{cases} f_2(0) = a_2 > 0, \\ f_2^{-1}(0) = b_2 > 0. \end{cases}$$

As a result, if the following inequalities

$$\begin{cases} f_1(0) > f_2(0), \\ f_1^{-1}(0) < f_2^{-1}(0), \end{cases}$$

or

$$\begin{cases} f_1(0) < f_2(0), \\ f_1^{-1}(0) > f_2^{-1}(0) \end{cases}$$

hold, the functions $f_1(b)$ and $f_2(b)$ have at least one point of intersection at some $t_2 > 0$. By mimicking the proof of Lemma 3.2 of [7], one can show that $t_{11} = t_{12}$ and $t_{21} = t_{22}$ if both (t_{11}, t_{21}) and (t_{12}, t_{22}) are solutions to (3.7). So, equation (3.7) has a unique positive root (t_1, t_2) with $t_1 = f_1(t_2) = f_2(t_2)$. □

Let (t_1, t_2) be the solution to (3.7). It follows from the relationship between (3.6) and (3.7) that $t_1 = q_1 r$ and $t_2 = q_2 r$. Plugging $r = t_1/q_1$ into (3.3) and $r = t_2/q_2$ into (3.3) separately, we obtain the following two equations

$$\begin{cases} \left[(1+\theta)\sum_{j=1}^{2}\mu_j\sum_{k=1}^{m}\lambda_k p_{kj} - (1+\eta)\sum_{j=1}^{2}\mu_j(1-q_j)\sum_{k=1}^{m}\lambda_k p_{kj}\right]\dfrac{t_1}{q_1} \\ \qquad -\sum_{k=1}^{m}\lambda_k\prod_{j=1}^{2}[M_j(t_j)p_{kj} + 1 - p_{kj}] + \lambda = 0, \\ \left[(1+\theta)\sum_{j=1}^{2}\mu_j\sum_{k=1}^{m}\lambda_k p_{kj} - (1+\eta)\sum_{j=1}^{2}\mu_j(1-q_j)\sum_{k=1}^{m}\lambda_k p_{kj}\right]\dfrac{t_2}{q_2} \\ \qquad -\sum_{k=1}^{m}\lambda_k\prod_{j=1}^{2}[M_j(t_j)p_{kj} + 1 - p_{kj}] + \lambda = 0, \end{cases}$$

which imply that

$$\begin{cases} q_1 = \dfrac{(\theta-\eta)t_1 \sum\limits_{j=1}^{2} \mu_j \sum\limits_{k=1}^{m} \lambda_k p_{kj}}{\sum\limits_{k=1}^{m} \lambda_k \prod\limits_{j=1}^{2} \left[M_j(t_j)p_{kj} + 1 - p_{kj}\right] - \lambda - (1+\eta) \sum\limits_{j=1}^{2} \mu_j t_j \sum\limits_{k=1}^{m} \lambda_k p_{kj}}, \\ q_2 = \dfrac{(\theta-\eta)t_2 \sum\limits_{j=1}^{2} \mu_j \sum\limits_{k=1}^{m} \lambda_k p_{kj}}{\sum\limits_{k=1}^{m} \lambda_k \prod\limits_{j=1}^{2} \left[M_j(t_j)p_{kj} + 1 - p_{kj}\right] - \lambda - (1+\eta) \sum\limits_{j=1}^{2} \mu_j t_j \sum\limits_{k=1}^{m} \lambda_k p_{kj}}. \end{cases} \tag{3.12}$$

We next discuss the optimal reinsurance strategy for each line. Note that retention levels should lie between $[0, 1]$. When q_1 or q_2 is outside the $[0, 1]$ interval, we set it to be 0 (if it is negative) or 1 (if it is greater 1), and recalculate the retention level for the other line. When both retention levels fall outside the interval, we regard the boundary values as the optimal reinsurance strategies. The details are illustrated as follows.

For $q_1 < 0$ and $0 \le q_2 \le 1$, the optimal reinsurance strategy for the first line is set to be 0, i.e., $q_1^* = 0$. It means that the insurance company should buy reinsurance for all of its risks in the first line. In this case, we need to recalculate the value of q_2. To do so, one can plug $q_1^* = 0$ into equation (3.3) and let $t_2' = q_2' r$. This results in

$$(\theta-\eta) \sum_{j=1}^{2} \mu_j \sum_{k=1}^{m} \lambda_k p_{kj} \frac{t_2'}{q_2'} = \sum_{k=1}^{m} \lambda_k \left[M_2(t_2')p_{k2} + 1 - p_{k2}\right] - (1+\eta)\mu_2 t_2' \sum_{k=1}^{m} \lambda_k p_{k2} - \lambda,$$

which gives

$$q_2' = \frac{t_2'(\theta-\eta) \sum\limits_{j=1}^{2} \mu_j \sum\limits_{k=1}^{m} \lambda_k p_{kj}}{\left[M_2(t_2') - (1+\eta)\mu_2 t_2' - 1\right] \sum\limits_{k=1}^{m} \lambda_k p_{k2}}. \tag{3.13}$$

To figure out the value of t_2', we directly plug $q_1^* = 0$ into equation (3.3) and take the first derivative on both sides with respect to q_2'. So, we have

$$M_2'(t_2') = (1+\eta)\mu_2,$$

which yields

$$t_2' = M_2'^{-1}\left[(1+\eta)\mu_2\right].$$

Plugging it into (3.13), we obtain

$$q_2' = \frac{M_2'^{-1}\left[(1+\eta)\mu_2\right](\theta-\eta) \sum\limits_{j=1}^{2} \mu_j \sum\limits_{k=1}^{m} \lambda_k p_{kj}}{\Big[(1+\eta)\mu_2 - (1+\eta)\mu_2 M_2'^{-1}\left[(1+\eta)\mu_2\right]\Big] \sum\limits_{k=1}^{m} \lambda_k p_{k2}}.$$

Therefore, the optimal reinsurance strategy (q_1^*, q_2^*) for the insurance company has the form $(0, (q_2')_+ \wedge 1)$ with $(q_2')_+ = \max(q_2', 0)$.

For $q_1 < 0$ and $q_2 \notin [0, 1]$, the optimal reinsurance strategy (q_1^*, q_2^*) should be set as $(0, 0)$ for $q_2 < 0$ and $(0, 1)$ for $q_2 > 1$.

For $q_1 > 1$ and $0 \le q_2 \le 1$, we take $q_1^* = 1$ as the optimal reinsurance strategy for the first line. Similar to the derivation of (3.13), we obtain the following revised value of q_2

$$q_2'' = \frac{t_2''\Big[(1+\theta)\sum_{j=1}^{2}\mu_j\sum_{k=1}^{m}\lambda_k p_{kj} - (1+\eta)\mu_2\sum_{k=1}^{m}\lambda_k p_{k2}\Big]}{\sum_{k=1}^{m}\lambda_k\prod_{j=1}^{2}\Big[M_j(t_j'')p_{kj} + 1 - p_{kj}\Big] - \lambda - (1+\eta)\sum_{j=1}^{2}\mu_j t_j''\sum_{k=1}^{m}\lambda_k p_{kj}}, \tag{3.14}$$

where (t_1'', t_2'') is the solution to (3.7), which can be alternatively derived by plugging $q_1^* = 1$ into equation (3.6) and letting $t_j'' = q_j r,\ j = 1, 2$. As a result, the optimal reinsurance strategy for the second line is $q_2^* = (q_2'')_+ \wedge 1$.

For $q_1 > 1$ and $q_2 \notin [0, 1]$, the optimal reinsurance strategy $(q_1^*,\ q_2^*)$ will be $(1, 0)$ for $q_2 < 0$ and $(1, 1)$ for $q_2 > 1$.

For $q_1, q_2 \in [0, 1]$, the optimal reinsurance strategy $(q_1^*,\ q_2^*)$ for the insurance company is just (q_1, q_2).

Finally, when $q_1 \in [0, 1]$ and $q_2 \notin [0, 1]$, we set the optimal reinsurance strategy for the second line as $q_2^* = 0$ for $q_2 < 0$ and $q_2^* = 1$ for $q_2 > 1$. In these cases, the value of q_1 needs to be recalculated. Steps similar to the previous recalculations can be applied for finding the revised values of q_1 denoted by q_1' for $q_2^* = 0$ and q_1'' for $q_2^* = 1$. The optimal reinsurance strategy $(q_1^*,\ q_2^*)$

is then given by $((q_1')_+ \wedge 1, 0)$ and $((q_1'')_+ \wedge 1, 1)$, respectively. In summary, we have

$$(q_1^*, q_2^*) = \begin{cases} (0,0), & q_1 < 0,\ q_2 < 0, \\ (0, (q_2')_+ \wedge 1), & q_1 < 0,\ 0 \le q_2 \le 1, \\ (0,1), & q_1 < 0,\ q_2 > 1, \\ ((q_1')_+ \wedge 1, 0), & 0 \le q_1 \le 1,\ q_2 < 0, \\ (q_1, q_2), & 0 \le q_1 \le 1,\ 0 \le q_2 \le 1, \\ ((q_1'')_+ \wedge 1, 1), & 0 \le q_1 \le 1,\ q_2 > 1, \\ (1,0), & q_1 > 1,\ q_2 < 0, \\ (1, (q_2'')_+ \wedge 1), & q_1 > 1,\ 0 \le q_2 \le 1, \\ (1,1), & q_1 > 1,\ q_2 > 1. \end{cases}$$

In addition to the discussion on q_1 and q_2, we briefly describe how explicit expressions for the adjustment coefficient can be obtained. Consider the case of $q_j^* = q_j \in [0,1]$ for $j = 1,2$. Plugging $q_j^* = q_j$ into equation (3.3) and setting $t_j = q_j r$ for $j = 1,2$ lead to

$$R^{q^*} = \frac{\sum_{k=1}^{m} \lambda_k \prod_{j=1}^{2} [M_j(t_j) p_{kj} + 1 - p_{kj}] - \lambda - (1+\eta) \sum_{j=1}^{2} \mu_j t_j \sum_{k=1}^{m} \lambda_k p_{kj}}{(\theta - \eta) \sum_{j=1}^{2} \mu_j \sum_{k=1}^{m} \lambda_k p_{kj}}.$$

When $q_1^* = 0$, one can use the same technique to obtain the following adjustment coefficient

$$R^{q^*} = \begin{cases} 0, & \text{if } (q_1^*, q_2^*) = (0,0), \\ \dfrac{[M_2(t_2') - 1] \sum_{k=1}^{m} \lambda_k p_{k2} - (1+\eta) \mu_2 t_2' \sum_{k=1}^{m} \lambda_k p_{k2}}{(\theta - \eta) \sum_{j=1}^{2} \mu_j \sum_{k=1}^{m} \lambda_k p_{kj}}, & \text{if } (q_1^*, q_2^*) = (0, q_2'), \\ \dfrac{[M_2(t_2') - 1] \sum_{k=1}^{m} \lambda_k p_{k2}}{(\theta - \eta) \sum_{j=1}^{2} \mu_j \sum_{k=1}^{m} \lambda_k p_{kj} + (1+\eta)\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2}}, & \text{if } (q_1^*, q_2^*) = (0,1). \end{cases}$$

Note that the optimal reinsurance strategy $(q_1^*, q_2^*) = (0,0)$ gives $R^{q^*} = 0$ due to (3.3). Along the same lines, one can derive closed-form expressions for the maximized adjustment coefficient for other cases.

3.2 Variance premium principle

Under the variance premium principle, the reinsurance premium rate is

$$\delta(q) = \sum_{j=1}^{2} \mu_j(1-q_j) \sum_{k=1}^{m} \lambda_k p_{kj} + \Lambda_1 \Big[\sum_{j=1}^{2} (1-q_j)^2(\mu_j^2+\sigma_j^2) \sum_{k=1}^{m} \lambda_k p_{kj} + \sum_{l=1}^{m} \sum_{k=1}^{2} \sum_{j\neq k}^{2} \mu_j \mu_k (1-q_j)(1-q_k) \lambda_l p_{lj} p_{lk} \Big],$$

where Λ_1 is the reinsurer's safety loading of the two classes of insurance business. Denote the insurer's safety loading by Λ which is given by

$$\Lambda = \frac{c - \sum_{j=1}^{2} \mu_j \sum_{k=1}^{m} \lambda_k p_{kj}}{\sum_{j=1}^{2} (\mu_j^2+\sigma_j^2) \sum_{k=1}^{m} \lambda_k p_{kj} + \sum_{l=1}^{m} \sum_{k=1}^{2} \sum_{j\neq k}^{2} \mu_j \mu_k \lambda_l p_{lj} p_{lk}}.$$

Similar to the case under the expected value premium principle, the adjustment coefficient must satisfy equation (2.4), which turns out to be

$$\begin{aligned} &\Big\{ \Lambda \Big(\sum_{j=1}^{2} (\mu_j^2+\sigma_j^2) \sum_{k=1}^{m} \lambda_k p_{kj} + \sum_{l=1}^{m} \sum_{k=1}^{2} \sum_{j\neq k}^{2} \mu_j \mu_k \lambda_l p_{lj} p_{lk} \Big) + \sum_{j=1}^{2} \mu_j \sum_{k=1}^{m} \lambda_k p_{kj} \\ &\quad - \sum_{j=1}^{2} \mu_j(1-q_j) \sum_{k=1}^{m} \lambda_k p_{kj} - \Lambda_1 \Big[\sum_{j=1}^{2} (1-q_j)^2(\mu_j^2+\sigma_j^2) \sum_{k=1}^{m} \lambda_k p_{kj} \\ &\quad + \sum_{l=1}^{m} \sum_{k=1}^{2} \sum_{j\neq k}^{2} \mu_j \mu_k (1-q_j)(1-q_k) \lambda_l p_{lj} p_{lk} \Big] \Big\} r \\ &\quad - \sum_{k=1}^{m} \lambda_k \prod_{j=1}^{2} [M_j(q_j r) p_{kj} + 1 - p_{kj}] + \lambda = 0. \end{aligned} \tag{3.15}$$

To obtain the maximized adjustment coefficient R^{q^*}, we need to find the solution to the following equation

$$\begin{aligned} \sup_{(q_1,q_2)} &\Big\{ \Big\{ \Lambda \Big(\sum_{j=1}^{2} (\mu_j^2+\sigma_j^2) \sum_{k=1}^{m} \lambda_k p_{kj} + \sum_{l=1}^{m} \sum_{k=1}^{2} \sum_{j\neq k}^{2} \mu_j \mu_k \lambda_l p_{lj} p_{lk} \Big) + \sum_{j=1}^{2} \mu_j \sum_{k=1}^{m} \lambda_k p_{kj} \\ &- \sum_{j=1}^{2} \mu_j(1-q_j) \sum_{k=1}^{m} \lambda_k p_{kj} - \Lambda_1 \Big[\sum_{j=1}^{2} (1-q_j)^2(\mu_j^2+\sigma_j^2) \sum_{k=1}^{m} \lambda_k p_{kj} \\ &+ \sum_{l=1}^{m} \sum_{k=1}^{2} \sum_{j\neq k}^{2} \mu_j \mu_k (1-q_j)(1-q_k) \lambda_l p_{lj} p_{lk} \Big] \Big\} r - \sum_{k=1}^{m} \lambda_k \prod_{j=1}^{2} [M_j(q_j r) p_{kj} + 1 - p_{kj}] + \lambda \Big\} = 0. \end{aligned}$$

Let

$$
\begin{aligned}
f(q_1,q_2) \quad = \quad & \Big\{\Lambda\Big(\sum_{j=1}^{2}(\mu_j^2+\sigma_j^2)\sum_{k=1}^{m}\lambda_k p_{kj}+\sum_{l=1}^{m}\sum_{k=1}^{2}\sum_{j\neq k}^{2}\mu_j\mu_k\lambda_l p_{lj}p_{lk}\Big) \\
& +\sum_{j=1}^{2}\mu_j\sum_{k=1}^{m}\lambda_k p_{kj}-\sum_{j=1}^{2}\mu_j(1-q_j)\sum_{k=1}^{m}\lambda_k p_{kj} \\
& -\Lambda_1\Big[\sum_{j=1}^{2}(1-q_j)^2(\mu_j^2+\sigma_j^2)\sum_{k=1}^{m}\lambda_k p_{kj} \\
& +\sum_{l=1}^{m}\sum_{k=1}^{2}\sum_{j\neq k}^{2}\mu_j\mu_k(1-q_j)(1-q_k)\lambda_l p_{lj}p_{lk}\Big]\Big\}r \\
& -\sum_{k=1}^{m}\lambda_k\prod_{j=1}^{2}\left[M_j(q_jr)p_{kj}+1-p_{kj}\right]+\lambda. \qquad (3.16)
\end{aligned}
$$

Differentiating $f(q_1,q_2)$ with respect to q_1 and q_2, we obtain

$$
\begin{cases}
\dfrac{\partial f(q_1,q_2)}{\partial q_1} = & r\mu_1\sum\limits_{k=1}^{m}\lambda_k p_{k1}-2r\Lambda_1 q_1(\mu_1^2+\sigma_1^2)\sum\limits_{k=1}^{m}\lambda_k p_{k1} \\
& +2r\Lambda_1(\mu_1^2+\sigma_1^2)\sum\limits_{k=1}^{m}\lambda_k p_{k1}+2r\Lambda_1\mu_1\mu_2(1-q_2)\sum\limits_{l=1}^{m}\lambda_l p_{l1}p_{l2} \\
& -\sum\limits_{k=1}^{m}\lambda_k\left[rM_1'(q_1r)p_{k1}\right]\left[M_2(q_2r)p_{k2}+1-p_{k2}\right], \\
\dfrac{\partial f(q_1,q_2)}{\partial q_2} = & r\mu_2\sum\limits_{k=1}^{m}\lambda_k p_{k2}-2r\Lambda_1 q_2(\mu_2^2+\sigma_2^2)\sum\limits_{k=1}^{m}\lambda_k p_{k2} \\
& +2r\Lambda_1(\mu_2^2+\sigma_2^2)\sum\limits_{k=1}^{m}\lambda_k p_{k2}+2r\Lambda_1\mu_1\mu_2(1-q_1)\sum\limits_{l=1}^{m}\lambda_l p_{l1}p_{l2} \\
& -\sum\limits_{k=1}^{m}\lambda_k\left[rM_2'(q_2r)p_{k2}\right]\left[M_1(q_1r)p_{k1}+1-p_{k1}\right],
\end{cases}
$$

and

$$
\begin{cases}
\dfrac{\partial^2 f(q_1,q_2)}{\partial q_1^2} = -2r\Lambda_1(\mu_1^2+\sigma_1^2)\sum\limits_{k=1}^{m}\lambda_k p_{k1}-\sum\limits_{k=1}^{m}\lambda_k\left[r^2M_1''(q_1r)p_{k1}\right]\left[M_2(q_2r)p_{k2}+1-p_{k2}\right]<0, \\
\dfrac{\partial^2 f(q_1,q_2)}{\partial q_2^2} = -2r\Lambda_1(\mu_2^2+\sigma_2^2)\sum\limits_{k=1}^{m}\lambda_k p_{k2}-\sum\limits_{k=1}^{m}\lambda_k\left[r^2M_2''(q_2r)p_{k2}\right]\left[M_1(q_1r)p_{k1}+1-p_{k1}\right]<0, \\
\dfrac{\partial^2 f(q_1,q_2)}{\partial q_1\partial q_2} = -2r\Lambda_1\mu_1\mu_2\sum\limits_{l=1}^{m}\lambda_l p_{l1}p_{l2}-\sum\limits_{k=1}^{m}\lambda_k\left[rM_1'(q_1r)p_{k1}\right]\left[rM_2'(q_2r)p_{k2}\right]<0.
\end{cases}
$$

Lemma 3.5. *The function $f(q_1,q_2)$ defined in* (3.16) *is a concave function with respect to q_1 and q_2.*

Proof. Similar to the proof of Lemma 3.1, it is sufficient to prove that the Hessian matrix of $f(q_1,q_2)$ is negative definite. Denote by $\mathrm{H_V}$ the Hessian matrix of $f(q_1,q_2)$ under the variance premium principle. It can be shown that

$$
\mathrm{H_V} = \begin{pmatrix} \dfrac{\partial^2 f(q_1,q_2)}{\partial q_1^2} & \dfrac{\partial^2 f(q_1,q_2)}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 f(q_1,q_2)}{\partial q_1 \partial q_2} & \dfrac{\partial^2 f(q_1,q_2)}{\partial q_2^2} \end{pmatrix} = r^2\mathbf{A} + 2r\Lambda_1\mathbf{B},
$$

where $\mathbf{A}$ is defined in (3.4) and

$$
\mathbf{B} = \begin{pmatrix} -(\mu_1^2+\sigma_1^2)\sum_{k=1}^m \lambda_k p_{k1} & -\mu_1\mu_2 \sum_{l=1}^m \lambda_l p_{l1} p_{l2} \\ -\mu_1\mu_2 \sum_{l=1}^m \lambda_l p_{l1} p_{l2} & -(\mu_2^2+\sigma_2^2)\sum_{k=1}^m \lambda_k p_{k2} \end{pmatrix}.
$$

Since the proof of $|\mathbf{A}| > 0$ is given in the proof of Lemma 3.1, we only need to show that $|\mathbf{B}| > 0$, i.e.,

$$
(\mu_1^2+\sigma_1^2)\sum_{k=1}^m \lambda_k p_{k1} (\mu_2^2+\sigma_2^2)\sum_{k=1}^m \lambda_k p_{k2} - \Big(\mu_1\mu_2 \sum_{l=1}^m \lambda_l p_{l1} p_{l2}\Big)^2 > 0. \tag{3.17}
$$

By Hölder's inequality, the left-hand side of (3.17) becomes

$$
\begin{aligned}
&(\mu_1^2+\sigma_1^2)\sum_{k=1}^m \lambda_k p_{k1} (\mu_2^2+\sigma_2^2)\sum_{k=1}^m \lambda_k p_{k2} - \Big(\mu_1\mu_2 \sum_{l=1}^m \lambda_l p_{l1} p_{l2}\Big)^2 \\
\geq&(\mu_1^2+\sigma_1^2)(\mu_2^2+\sigma_2^2)\Big(\sum_{k=1}^m \lambda_k \sqrt{p_{k1}p_{k2}}\Big)^2 - \Big(\mu_1\mu_2 \sum_{l=1}^m \lambda_l p_{l1} p_{l2}\Big)^2 \\
>&0,
\end{aligned}
$$

where the last inequality is obtained using

$$
\begin{cases} \mu_j^2 + \sigma_j^2 > \mu_j^2, & \text{for } j = 1, 2, \\ \sum_{k=1}^m \lambda_k \sqrt{p_{k1}p_{k2}} > \sum_{l=1}^m \lambda_l p_{l1} p_{l2}, & \text{for } k = l \in \{1, \dots, m\}. \end{cases}
$$

Thus, $\mathbf{B}$ is a negative definite matrix, and hence $\mathrm{H_V}$ is also negative definite.

□

Consequently, (q_1, q_2) which maximizes $f(q_1, q_2)$ satisfies the following equations

$$
\begin{cases}
r\mu_1 \sum_{k=1}^m \lambda_k p_{k1} - 2r\Lambda_1 q_1(\mu_1^2+\sigma_1^2)\sum_{k=1}^m \lambda_k p_{k1} + 2r\Lambda_1(\mu_1^2+\sigma_1^2)\sum_{k=1}^m \lambda_k p_{k1} \\
\qquad +2r\Lambda_1\mu_1\mu_2(1-q_2)\sum_{l=1}^m \lambda_l p_{l1}p_{l2} - \sum_{k=1}^m \lambda_k [rM_1'(q_1 r)p_{k1}][M_2(q_2 r)p_{k2} + 1 - p_{k2}] = 0, \\
r\mu_2 \sum_{k=1}^m \lambda_k p_{k2} - 2r\Lambda_1 q_2(\mu_2^2+\sigma_2^2)\sum_{k=1}^m \lambda_k p_{k2} + 2r\Lambda_1(\mu_2^2+\sigma_2^2)\sum_{k=1}^m \lambda_k p_{k2} \\
\qquad +2r\Lambda_1\mu_1\mu_2(1-q_1)\sum_{l=1}^m \lambda_l p_{l1}p_{l2} - \sum_{k=1}^m \lambda_k [rM_2'(q_2 r)p_{k2}][M_1(q_1 r)p_{k1} + 1 - p_{k1}] = 0,
\end{cases} \tag{3.18}
$$

which can be simplified as

$$
\begin{cases}
\mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1(1-q_1)(\mu_1^2+\sigma_1^2) \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1\mu_1\mu_2(1-q_2) \sum_{l=1}^{m} \lambda_l p_{l1} p_{l2} \\
\quad - \sum_{k=1}^{m} \lambda_k [M_1'(q_1 r) p_{k1}][M_2(q_2 r) p_{k2} + 1 - p_{k2}] = 0, \\
\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2} + 2\Lambda_1(1-q_2)(\mu_2^2+\sigma_2^2) \sum_{k=1}^{m} \lambda_k p_{k2} + 2\Lambda_1\mu_1\mu_2(1-q_1) \sum_{l=1}^{m} \lambda_l p_{l1} p_{l2} \\
\quad - \sum_{k=1}^{m} \lambda_k [M_2'(q_2 r) p_{k2}][M_1(q_1 r) p_{k1} + 1 - p_{k1}] = 0.
\end{cases} \tag{3.19}
$$

Let $t_1 = q_1 r$ and $t_2 = q_2 r$. Then, (3.19) can be rewritten as

$$
\begin{cases}
\mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1(1-\frac{t_1}{r})(\mu_1^2+\sigma_1^2) \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1\mu_1\mu_2(1-\frac{t_2}{r}) \sum_{l=1}^{m} \lambda_l p_{l1} p_{l2} \\
\quad - \sum_{k=1}^{m} \lambda_k [M_1'(t_1) p_{k1}][M_2(t_2) p_{k2} + 1 - p_{k2}] = 0, \\
\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2} + 2\Lambda_1(1-\frac{t_2}{r})(\mu_2^2+\sigma_2^2) \sum_{k=1}^{m} \lambda_k p_{k2} + 2\Lambda_1\mu_1\mu_2(1-\frac{t_1}{r}) \sum_{l=1}^{m} \lambda_l p_{l1} p_{l2} \\
\quad - \sum_{k=1}^{m} \lambda_k [M_2'(t_2) p_{k2}][M_1(t_1) p_{k1} + 1 - p_{k1}] = 0.
\end{cases} \tag{3.20}
$$

The following lemmas are useful for proving the existence and uniqueness of the solution to (3.20).

Lemma 3.6. *There is a unique positive solution to each of the following equations*

$$
\begin{aligned}
& \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1(1-\frac{t_1}{r})(\mu_1^2+\sigma_1^2) \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1\mu_1\mu_2 \sum_{l=1}^{m} \lambda_l p_{l1} p_{l2} \\
= & \sum_{k=1}^{m} \lambda_k M_1'(t_1) p_{k1},
\end{aligned} \tag{3.21}
$$

and

$$
\begin{aligned}
& \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1(\mu_1^2+\sigma_1^2) \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1\mu_1\mu_2(1-\frac{t_2}{r}) \sum_{l=1}^{m} \lambda_l p_{l1} p_{l2} \\
= & \sum_{k=1}^{m} \lambda_k \mu_1 p_{k1} [M_2(t_2) p_{k2} + 1 - p_{k2}].
\end{aligned} \tag{3.22}
$$

Proof. We first consider equation (3.21). Let

$$
\begin{cases}
g_1(t_1) = \sum_{k=1}^{m} \lambda_k M_1'(t_1) p_{k1}, \\
g_2(t_1) = \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1(1-\frac{t_1}{r})(\mu_1^2+\sigma_1^2) \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1\mu_1\mu_2 \sum_{l=1}^{m} \lambda_l p_{l1} p_{l2}.
\end{cases}
$$

Then, we have

$$
\begin{aligned}
g_1(0) =& \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1}, \\
g_1'(t_1) =& \sum_{k=1}^{m} \lambda_k p_{k1} \mathrm{E}(X_1^2 e^{t_1 X_1}) > 0, \\
g_1''(t_1) =& \sum_{k=1}^{m} \lambda_k p_{k1} \mathrm{E}(X_1^3 e^{t_1 X_1}) > 0.
\end{aligned}
$$

These imply that $g_1(t_1)$ is an increasing convex function. On the other hand, $g_2(t_1)$ is a decreasing linear function with respect to t_1 with

$$
g_2(0) = \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1(\mu_1^2 + \sigma_1^2) \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1 \mu_1 \mu_2 \sum_{l=1}^{m} \lambda_l p_{l1} p_{l2} > g_1(0).
$$

It is not difficult to see that $g_1(t_1)$ and $g_2(t_1)$ have a unique point of intersection at some $\bar{t}_1 > 0$. That is, equation (3.21) has a unique positive solution.

We now consider equation (3.22). Let

$$
\begin{cases}
g_3(t_2) = \sum_{k=1}^{m} \lambda_k \mu_1 p_{k1} [M_2(t_2) p_{k2} + 1 - p_{k2}], \\
g_4(t_2) = \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1(\mu_1^2 + \sigma_1^2) \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1 \mu_1 \mu_2 (1 - \dfrac{t_2}{r}) \displaystyle\sum_{l=1}^{m} \lambda_l p_{l1} p_{l2}.
\end{cases}
$$

It is easy to see that

$$
\begin{aligned}
g_3(0) =& \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1}, \\
g_3'(t_2) =& \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} p_{k2} \mathrm{E}(X_2 e^{t_2 X_2}) > 0, \\
g_3''(t_2) =& \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} p_{k2} \mathrm{E}(X_2^2 e^{t_2 X_2}) > 0.
\end{aligned}
$$

So, one can conclude that $g_3(t_2)$ is an increasing convex function. Also, it is easily seen that $g_4(t_2)$ is a decreasing linear function with respect to t_2 with

$$
g_4(0) = \mu_1 \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1(\mu_1^2 + \sigma_1^2) \sum_{k=1}^{m} \lambda_k p_{k1} + 2\Lambda_1 \mu_1 \mu_2 \sum_{l=1}^{m} \lambda_l p_{l1} p_{l2} > g_3(0).
$$

As a consequence, $g_3(t_2)$ and $g_4(t_2)$ have a unique point of intersection at some $\bar{t}_2 > 0$. That is, equation (3.22) has a unique positive solution.

□

Lemma 3.7. *There is a unique positive solution to each of the following equations*

$$\begin{aligned}
&\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2} + 2\Lambda_1(\mu_2^2+\sigma_2^2)\sum_{k=1}^{m}\lambda_k p_{k1} + 2\Lambda_1\mu_1\mu_2(1-\frac{t_1}{r})\sum_{l=1}^{m}\lambda_l p_{l1}p_{l2} \\
= &\sum_{k=1}^{m}\lambda_k\mu_2 p_{k2}[M_1(t_1)p_{k1}+1-p_{k1}],
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
&\mu_2 \sum_{k=1}^{m} \lambda_k p_{k2} + 2\Lambda_1(1-\frac{t_2}{r})(\mu_2^2+\sigma_2^2)\sum_{k=1}^{m}\lambda_k p_{k2} + 2\Lambda_1\mu_1\mu_2\sum_{l=1}^{m}\lambda_l p_{l1}p_{l2} \\
= &\sum_{k=1}^{m}\lambda_k M_2'(t_2)p_{k2}.
\end{aligned} \tag{3.24}$$

Proof. Similar to the proof of Lemma 3.6, one can show that equations (3.23) and (3.24) have a unique positive root $\tilde{t}_1$ and $\tilde{t}_2$, respectively. □

The next lemma states the existence and uniqueness of the solution to equation (3.20).

Lemma 3.8. *Let $\bar{t}_1$, $\bar{t}_2$, $\tilde{t}_1$ and $\tilde{t}_2$ be the unique positive roots of equations* (3.21),(3.22),(3.23) *and* (3.24)*, respectively. If*

$$\begin{cases} \bar{t}_1 > \tilde{t}_1, \\ \bar{t}_2 < \tilde{t}_2, \end{cases}$$

or

$$\begin{cases} \bar{t}_1 < \tilde{t}_1, \\ \bar{t}_2 > \tilde{t}_2 \end{cases}$$

hold, then equation (3.20) *has a unique positive solution* (t_1, t_2)*.*

Proof. The proof is similar to that of Lemma 3.4. □

Put $t_1 = t_1(r)$ and $t_2 = t_2(r)$. Plugging these into (3.15) yields

$$\begin{aligned}
&\Big[(\Lambda-\Lambda_1)\sum_{j=1}^{2}(\mu_j^2+\sigma_j^2)\sum_{k=1}^{m}\lambda_k p_{kj} + 2(\Lambda-\Lambda_1)\mu_1\mu_2\sum_{k=1}^{m}\lambda_k p_{k1}p_{k2}\Big]r^2 \\
&+\Big[\sum_{j=1}^{2}\mu_j t_j(r)\sum_{k=1}^{m}\lambda_k p_{kj} + 2\Lambda_1\sum_{j=1}^{2}t_j(r)(\mu_j^2+\sigma_j^2)\sum_{k=1}^{m}\lambda_k p_{kj} \\
&+2\Lambda_1\mu_1\mu_2\Big(t_1(r)+t_2(r)\Big)\sum_{k=1}^{m}\lambda_k p_{k1}p_{k2} - \sum_{k=1}^{m}\lambda_k\prod_{j=1}^{2}[M_j(t_j(r))p_{kj}+1-p_{kj}] \\
&+\lambda\Big]r - \Lambda_1\sum_{j=1}^{2}t_j(r)^2(\mu_j^2+\sigma_j^2)\sum_{k=1}^{m}\lambda_k p_{kj} - 2\mu_1\mu_2\Lambda_1 t_1(r)t_2(r)\sum_{k=1}^{m}\lambda_k p_{k1}p_{k2} = 0.
\end{aligned} \tag{3.25}$$

The maximized adjustment coefficient R^{q^*} can be obtained by solving (3.25). The optimal strategies for both lines of business are then given by

$$\begin{cases} q_1^* = \dfrac{t_1(R^{q^*})}{R^{q^*}}, \\ q_2^* = \dfrac{t_2(R^{q^*})}{R^{q^*}}. \end{cases} \tag{3.26}$$

Remark 3.1. *If reasonable parameter values are used (for example,* Λ *and* Λ_1 *are close to each other), one should be able to obtain the solution to equation* (3.25) *numerically. A few examples are presented in the next section. Also, many previous results have shown that retention levels under the variance premium principle fall inside the interval* $[0, 1]$. *See, for example, [7] and [13].*

4 Numerical examples

In this section, we carry out a few numerical studies using the results obtained in the previous sections for two classes of insurance business and two groups of stochastic sources, i.e., $n = m = 2$.

For illustration purpose, it is assumed that the claim sizes for both lines, X_1 and X_2, are exponentially distributed with mean μ_1 and μ_2, respectively.

Example 1. We set $\theta = 0.3$, $\eta = 0.4$, $p_{11} = p_{22} = 1$, $p_{12} = p_{21} = 0.5$, $\mu_1 = \mu_2 = 1$, $\Lambda = 0.3$, $\Lambda_1 = 0.4$, $\sigma_1 = \sigma_2 = 1$ and $\lambda_2 = 2$. The results are shown in Tables 1 and 2.

Table 1: Effect of λ_1 on optimal reinsurance strategies under expected value premium principle

λ_1	1	2	3	4	5	6	7	8	9	10
q_1^*	0.436458	0.463596	0.477171	0.485302	0.490711	0.494566	0.497452	0.499692	0.501481	0.502942
q_2^*	0.485302	0.463596	0.448081	0.436458	0.427431	0.420219	0.414326	0.409421	0.405274	0.401722

Table 2: Effect of λ_1 on optimal reinsurance strategies under variance premium principle

λ_1	1	2	3	4	5	6	7	8	9	10
q_1^*	0.209473	0.212291	0.213776	0.214701	0.215334	0.215796	0.216149	0.216427	0.216652	0.216838
q_2^*	0.214701	0.212291	0.210659	0.209473	0.208569	0.207856	0.207280	0.206804	0.206403	0.206062

It is exhibited in Tables 1 and 2 that as λ_1 increases, the optimal retention level for the first line increases while the optimal retention level for the second line decreases. This implies that when the intensity of the claim number for the first line becomes larger, the insurer would like to retain a greater share of each claim in the first line but a smaller share of each claim in the second line. A possible explanation is that a greater λ_1 yields a greater reinsurance premium which in turn leads to a higher reinsurance cost. $\square$

Example 2. Let $\theta = 0.3$, $\eta = 0.4$, $\Lambda = 0.3$, $\Lambda_1 = 0.4$, $p_{11} = p_{22} = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$ and $p_{21} = 0.5$. The results are shown in Tables 3 and 4.

Table 3: Effect of p_{12} on optimal reinsurance strategies under expected value premium principle

p_{12}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
q_1^*	0.458931	0.454358	0.449084	0.443114	0.436458	0.429122	0.421114	0.412442	0.403116
q_2^*	0.466809	0.471109	0.475648	0.480390	0.485302	0.490356	0.495530	0.500802	0.506155

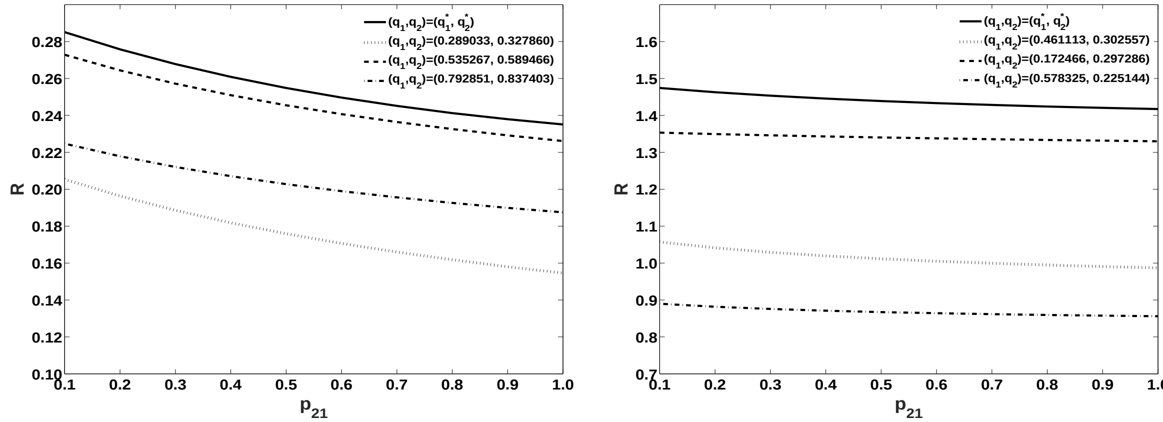
Table 4: Effect of p_{12} on optimal reinsurance strategies under variance premium principle

p_{12}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
q_1^*	0.212259	0.211611	0.210928	0.210215	0.209473	0.208705	0.207912	0.207097	0.206262
q_2^*	0.213187	0.213533	0.213904	0.214294	0.214701	0.215120	0.215550	0.215989	0.216435

Tables 3 and 4 give the values of q_1^* and q_2^* when p_{12} changes. Given other things being equal, as p_{12} moves from 0.1 to 0.9, the optimal retention level for the first line decreases while the optimal retention level for the second line increases. Note that p_{12} represents the probability that the stochastic source from the first group causes claims in the second class of insurance business. A greater value of p_{12} may yield a larger number of claims in the second line. Therefore, the results imply that the greater the probability of having claims in the second line caused by the first group of stochastic sources is, the less share of each claim in the first line but the larger share of each claim in the second line the insurance company would like to retain possibly due to the reason mentioned in Example 1. $\square$

Along the same lines, one can perform similar numerical analysis to assess the effects of λ_2 and p_{21} on the optimal reinsurance strategies. It is expected that when the value of λ_2 gets larger, the optimal retention level for the first line becomes smaller while the optimal retention level for the second line becomes bigger, and that a greater value of p_{21} yields a higher optimal retention level for the first line but a lower optimal retention level for the second line.

Example 3. Let $\theta = 0.3$, $\eta = 0.4$, $\Lambda = 0.3$, $\Lambda_1 = 0.4$, $p_{11} = p_{22} = 1$, $p_{12} = 0.5$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = \mu_2 = 1$ and $\sigma_1 = \sigma_2 = 1$. The results are shown in Figure 1.



(a) Expected value premium principle

(b) Variance premium principle

Figure 1: Optimal reinsurance strategies maximize the adjustment coefficient

Under the expected value premium principle, the optimal reinsurance strategies are $q_1^* = 0.436458$ and $q_2^* = 0.485302$, and under the variance premium principle, the optimal reinsurance strategies are $q_1^* = 0.209473$ and $q_2^* = 0.214701$. For comparison, we arbitrarily choose another three groups of reinsurance strategies. Based on these four reinsurance strategies, we plot R against p_{21} under the expected value premium principle in Figure 1(a) and under the variance premium principle in Figure 1(b). We see from both figures that the curve on the top refers to the case where optimal reinsurance strategies are applied. These are consistent with the results obtained in the previous section, i.e., when the optimal reinsurance strategies are applied, the adjustment coefficient attains its maximum under both the expected value premium principle and the variance premium principle.

□

5 Concluding remarks

This paper examines the problem of optimal proportional reinsurance in a risk model with correlated classes of insurance business. It is assumed that the claim-number processes among classes possess the thinning-dependence structure. For this risk model, we derive the optimal reinsurance strategies with the objective of maximizing the adjustment coefficient for two commonly-used premium principles. Under the expected value premium principle, we are able to obtain explicit expressions for q_1^* and q_2^*. To restrict these values to the unit interval, we propose a method to recalculate the optimal values. On the other hand, under the variance premium principle, the associated equations for finding the optimal values are more complicated than those under the expected value premium principle, and hence explicit expressions for the optimal reinsurance strategies cannot be derived. Finally, we carry out a few numerical examples to illustrate the impact of the model parameters on the optimal reinsurance strategies.

It is expected that our method can still be applied to derive the optimal strategy for the thinning model with more than two lines of insurance business. Undoubtedly, the derivation of the main results becomes much more challenging in this case. For further research, one may incorporate some investment features into the model of study, and then investigate the corresponding optimal investment and reinsurance problem with thinning dependence. Another interesting topic is to consider a similar optimal problem with another objective of optimization such as maximizing the expected utility of terminal wealth.

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