Reliability-based user equilibrium in a transport network under the effects of speed limits and supply uncertainty

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Abstract:
This study investigates the system-wide traffic flow re-allocation effect of speed limits in uncertain environments. Previous studies have only considered link capacity degradation, which is only one of the factors that lead to supply uncertainty. This study examines how imposing speed limits reallocates the traffic flows in a situation of general supply uncertainty with risk-averse travelers. The effects of imposing a link-specific speed limit on link driving speed and travel time are analyzed, given the link travel time distribution before imposing the speed limit. The expected travel time and travel time standard deviation of a link with a speed limit are derived from the link travel time distribution and are both continuous, monotone, and convex functions in terms of link flow. A distribution-free, reliability-based user equilibrium with speed limits is established, in which travelers are assumed to choose routes that minimize their own travel time budget. A variational inequality formulation for the equilibrium problem is proposed and the solution properties are provided. In this study, the inefficiency of an reliability-based user equilibrium flow pattern with speed limits is defined and found to be bounded above when supply uncertainty refers to capacity degradation. The upper bound depends on the level of risk aversion of travelers, a ratio related to the design and worst-case link capacities, and the highest power of all link performance functions.

Keywords: Speed limit, supply uncertainty, distribution-free reliability-based user equilibrium, inefficiency.

1 Introduction

High-speed driving can increase the number of crashes on a road and increase the severity of injuries in a crash (see [1,2]). A speed limit is commonly imposed on a road to restrict the maximum allowable driving speed and hence prevent traffic accidents. An example in practice is the Variable Speed Limit system used in Europe and the United States (see [3,4]), which is designed to reduce the speed difference (harmonize the traffic flow) on hazardous highway segments and thus decrease the rear-end collision rate. In addition to improving road safety, speed limits also reduce high-speed–related fuel consumption and emission problems (see [5,6]).

Recent studies of optimal speed limit design (e.g., [7,8]) have mainly focused on the local effects of speed limits (i.e., on a single link). McKnight and Klein [9] and Grabowski and Morrisey [10] recognized the potential traffic flow re-allocation effect of speed limits, and Yang et al. [11] and Wang [12] investigated how imposing speed limits reallocates traffic flows in an equilibrium manner at a macroscopic network level. They suggested that imposing a speed limit alters the link travel time–flow relationship, but a traffic assignment principle still applies. Yang et al. [13] then proposed a bi-level variable speed limit design model in which a user equilibrium (UE) model with speed limits was incorporated as the lower-level model. Yang et al. [14] considered the non-obedient behavior of travelers when confronted with speed limits. These studies assume that the link travel times are deterministic. Yan et al. [15] examined the effects of speed limits on traffic equilibrium when the link travel times are random due to link capacity degradations, prompting further investigation into reliability-based user equilibrium (RUE) problems that consider travel time uncertainty and the effects of speed limits.

Travel time uncertainty is mainly caused by demand (flow) uncertainty and network supply uncertainty. The former is due to the demand fluctuation caused by, e.g., temporal factors (e.g., time of day, day of the week, or seasonal effect) and special events [16]. Relevant studies include [17,18]. Supply uncertainty (e.g., [19,20]) mainly involves link capacity variation and degradation due to factors such as adverse weather conditions, road maintenance, and traffic accidents, etc. Szeto and Wang [21] suggested that variations in the link free flow travel times due to adverse weather conditions are also a type of supply uncertainty. If travel time uncertainty is taken into consideration, the effects of imposing a link-specific speed limit on the link driving speed and travel time are more difficult to analyze, but it has practical importance for accurate evaluation of the benefits of imposing speed limits.

Travel time uncertainty also affects travelers' route choices as pointed out by the studies [22-26]. Several approaches have been proposed to model this behavior, such as the stochastic dominance approach [27,28] based on an axiomatic model of risk-averse preferences. Travelers are assumed to possess a utility function and choose routes that maximize their own utility [27,28]. Another approach [29] assumes that travelers choose routes to maximize their own probability of on-time arrival. The most common are the mean-risk type approaches, in which travelers are assumed to choose routes according to the expected travel time and the variability of travel time, and then make a tradeoff between these two factors [30,31]. Lo et al. [20] proposed the concept of travel time budget (TTB), which is the sum of the expected travel time and the travel time standard variation multiplied by a scalar. At equilibrium, a traveler minimizes his own travel time budget, which is associated with a predefined on-time arrival probability [20]. Nie and Wu [32] and Nie [33] proposed the concept of percentile travel time (PTT) for a desired on-time arrival probability, and travelers are assumed to minimize their own PPT. Chen and Zhou [34] proposed the concept of mean and excess travel time (METT), which is the sum of the travel time budget and any excess travel time. At equilibrium, travelers choose routes to minimize their METT [34]. Comparisons among the mean-risk type approaches are provided by [35], and the TTB approach is most widely adopted by scholars [36,37]. Route choices of travelers also affect the distribution of traffic flows. Therefore, it is important to consider travelers' route choice behavior in the speed limit analysis to accurately assess the benefits of imposing speed limits.

This study focuses on the effects of speed limits on link travel times and driving speeds in an environment with supply uncertainty, given the link travel time distributions before the speed limits are imposed. The mean link travel time and the link travel time standard deviation of a link with a speed limit is derived from the travel time distribution. This study extends the concept of TTB to consider the effects of speed limits and proposes a distribution-free RUE model. A variational inequality (VI) formulation for the proposed model is given, and the solution properties are examined. A definition of the inefficiency of an RUE flow pattern with speed limits is also proposed and analyzed. The main differences between our study and [15] are as follows.

- Supply uncertainty is considered to include more than capacity degradation.
- Link capacity is not required to follow a uniform distribution. In fact, the proposed approach only requires that the probability density/mass function of link travel time must be known. This approach does not exclude a case in which the link travel time distribution is derived from a capacity distribution.
- The RUE model established here does not require knowledge of the path travel time distributions (i.e., it is distribution-free), and it captures the travel time correlations between links.
- Rigorous analyses of the *mathematical* properties of the mean and standard deviation of link travel times and the solution properties of the RUE model are conducted.
- The *inefficiency* of an RUE flow pattern is defined and examined.

The study provides the following contributions.

- An analysis of the effects of a speed limit on the random link travel time given the link travel time distribution before the speed limit is imposed, which provides fundamental computation formulae for the mean link travel time and link travel time standard deviation after a speed limit is imposed.
- A distribution-free RUE model to capture both risk aversion and selfish routing behavior of travelers in a transportation network with speed limits, and consideration of the travel time correlations between links.
- An examination of the analytical and counterintuitive properties of the RUE flow pattern with speed limits.
- A definition and investigation of the inefficiency of an RUE flow pattern with speed limits.

The remainder of this paper consists of four sections. In Section 2, how the distributions of random driving speed and travel time on a link are affected by the imposition of a link speed limit is analyzed, and the mean link travel time and link travel time standard deviation are derived. In Section 3, an RUE model for a transportation network with speed limits based on the concept of TTB is formulated and the solution properties are analyzed. In Section 4, a numerical example is developed to show the system-wide benefit of imposing speed limits, and the inefficiency of an RUE flow pattern is defined and examined. A conclusion is provided in Section 5.

2 Link travel time under supply uncertainty and a speed limit

2.1 Link travel time and driving speed

Consider a transportation network $G(N, A)$, where N is the set of nodes and A is the set of links. In terms of the speed limit and source of uncertainty on each link, two assumptions are introduced.

Assumption 1. The speed–flow relationship of each link is within the normal flow regime, in which the speed decreases when the traffic flow increases. The link travel time is monotone increasing with the link flow [11].

Assumption 2. The travel demands (and hence the link flows) are deterministic. The link travel times are random variables.

Two assumptions regarding supply uncertainty can be identified in the literature. Most scholars [20,35] assume that supply uncertainty refers to the link capacity degradation due to factors such as maintenance work, adverse weather conditions, and road accidents, whereas some [21] assume that supply uncertainty includes both link capacity degradation and free flow travel time variation due to adverse weather conditions. The first assumption can be regarded as a special case of the second.

Denote the random travel time before a speed limit is imposed as T^{R}. Denote the free flow travel time under the maximum allowable driving speed as $t^{\#}$. Intuitively, $\Pr\left[T^{\mathrm{R}} \geq t^{\#}\right]=1$.

The link driving speed is commonly obtained by dividing the link length by the link travel time. Thus, the driving speed, denoted as S^{R}, is

$$S^{\mathrm{R}} = L/T^{\mathrm{R}},$$

where L is the link length. The driving speed S^{R} is also a random variable, and $\Pr\left[S^{\mathrm{R}} \leq L/t^{\#}\right]=1$.

Assumption 3. The probability density/mass function (PDF/PMF) of random link travel time T^{R} depends on the link flow v.

The mean and standard deviation of the random link travel time are generally dependent on the link flow [20,36]. The PDF/PMF of a random variable is commonly dependent on its mean value and/or standard deviation, so adoption of Assumption 3 is reasonable.

Denote $f_{T^{\mathrm{R}}}(\chi', v)$ as the PDF/PMF of the random link travel time, where v is a non-negative link flow and χ' represents an outcome of travel time. The analyst can obtain the functional form of $f_{T^{\mathrm{R}}}(\chi', v)$ directly by calibration using the link flow and travel time data. Alternatively, the functional form of the link performance function can be assumed, and the functional form of $f_{T^{\mathrm{R}}}(\chi', v)$ can be derived based on the known distributions of the parameters in the link performance function (e.g., the distribution of the link capacity).

In some cases, the lower and upper supports of the travel time distribution are both finite (e.g., when the link travel time follows a triangular distribution). In other cases, the lower support equals negative infinity and/or the upper support equals positive infinity, meaning that one or both supports of the travel time distribution is/are unbounded, and even independent of the link flow (e.g., when the link travel time follows a log-normal or normal distribution). Travel times with extremely low occurrence probabilities (e.g., in a disaster scenario when the link travel time is very high) or no occurrence probabilities (e.g., when the travel time is shorter than the free flow travel time under the maximum allowable driving speed) are too trivial to be considered because they are rarely or never observed. Thus, this study assumes that *non-trivial* lower and upper bounds for the random travel time T^{R} exist, which are defined as

$$t^{\min} = \left\{x \middle| \Pr\left[T^{\mathrm{R}} \leq x\right] = \underline{\omega}\right\} \text{ and}$$

$$t^{\max} = \left\{x \middle| \Pr\left[T^{\mathrm{R}} \geq x\right] = \overline{\omega}\right\},$$

where $\bar{\varpi}$ equals a small positive number if the travel time distribution has no finite upper support and zero otherwise. Similarly, $\underline{\varpi}$ equals a small positive number if the travel time distribution has no finite lower support and zero otherwise. Intuitively, the defined lower support must be at least equal to the free flow travel time under the maximum allowable driving speed, that is, $t^{\min} \geq t^{\#}$, and the defined upper support is larger than the lower support.

If the travel time distribution has no finite upper or lower support, the value of $\bar{\varpi}$ or $\underline{\varpi}$ is decided by the analyst. For example, if the link travel time follows a normal distribution and the analyst plans to exclude the travel time instances at the two tails of the normal distribution with a total probability of 4%, then both $\bar{\varpi}$ and $\underline{\varpi}$ can be set to 0.02 (i.e., 0.04/2).

Obtaining the values of $t^{\min}$ and $t^{\max}$ requires the values of $\bar{\varpi}$ and $\underline{\varpi}$, and the PDF/PMF of T^{R}. Because the PDF/PMF of T^{R} is flow dependent, so are the above two defined supports. Denote them as $t^{\min}(v)$ and $t^{\max}(v)$, respectively.

Define two nontrivial supports for the distribution of the random driving speed S^{R}, denoted as $s^{\min}(v)$ and $s^{\min}(v)$, and equal

$$s^{\min}(v) = L \big/ t^{\max}(v) \text{ and}$$
$$s^{\max}(v) = L \big/ t^{\min}(v).$$

Assumption 4. The two supports of the random travel time, $t^{\min}(v)$ and $t^{\max}(v)$, are strictly increasing with respect to the link flow v. The supports of the random driving speed, $s^{\min}(v)$ and $s^{\max}(v)$, are strictly decreasing with respect to the link flow v.

Similarly, by collecting and calibrating link flow and travel time data, the analyst can obtain the functional forms of $t^{\min}(v)$ and $t^{\max}(v)$ directly. Other methods include assuming the functional form of a link performance function and deriving the functional forms of $t^{\min}(v)$ and $t^{\max}(v)$ based on the given distributions of parameters in the link performance function (e.g., the distribution of the link capacity).

The three functions $t^{\min}(v)$, $t^{\max}(v)$, and $f_{T^{\mathrm{R}}}(\chi', v)$ are invertible in terms of link flow.

By excluding travel times that are shorter than the nontrivial lower support and longer than the non-trivial upper support, an adjusted random link travel time is defined as below. The adjusted random link travel time, denoted by T, is

$$T = \left\{ T^{\mathrm{R}} \,\middle|\, t^{\min} \leq T^{\mathrm{R}} \leq t^{\max} \right\}.$$

Then, the probability density/mass function of T equals

$$f_T(\chi, v) = \frac{1}{1 - \left(\bar{\varpi} + \underline{\varpi}\right)} f_{T^{\mathrm{R}}}(\chi', v), \ \chi \in \left[t^{\min}, t^{\max} \right].$$

If the distribution of T has finite upper and lower supports, then $\bar{\varpi} = \underline{\varpi} = 0$, $T = T^{\mathrm{R}}$, and $f_T(\chi, v) = f_{T^{\mathrm{R}}}(\chi', v)$. The adjusted random link travel time is identical to the random link travel time before adjustment.

The adjusted link driving speed is correspondingly defined as

$$S = \left\{ S^{\mathrm{R}} \,\middle|\, s^{\min} \leq S^{\mathrm{R}} \leq s^{\max} \right\}.$$

In the following analysis, the effects of a speed limit on the adjusted random link travel time T and adjusted link driving speed S are examined.

2.2 Effects of a speed limit on link driving speed and travel time

2.2.1 General case

Following Yang et al. [11], this study assumes that the travelers drive as fast as possible while still strictly complying with the traffic regulations. Thus, this study assumes the following.

Assumption 5. For a link with a speed limit $\bar{s}$ imposed, if the *random speed* S exceeds the speed limit $\bar{s}$, the travelers drive at the speed limit. If not, their driving speeds are unaffected [15].

Based on Assumption 5, the driving speed under the speed limit $\bar{s}$, denoted as $\tilde{S}(v)$, is a random variable and equals

$$\tilde{S} = \min\left[S, \bar{s}\right]. \tag{1}$$

The travel time after the speed limit is imposed on the link, denoted as $\tilde{T}$, equals $L/\tilde{S}$. Based on Eq. (1),

$$\tilde{T} = \max\left[T, L/\bar{s}\right]. \tag{2}$$

Note that Assumption 5 and Eq. (2) are very similar to the assumptions introduced by Yan et al. [15]. However, they implicitly assume that the link travel time distribution is truncated and must have finite supports because a uniform capacity distribution is used. This study is fundamentally different because it proposes that if the link travel time distribution does not have finite lower and/or upper support, the analyst must first adjust the link random travel time by excluding some rare instances, so the distribution of the adjusted link random travel time is truncated. If and only if the random link travel time before adjustment T^{R} has positive finite supports, that is, $T = T^{\mathrm{R}}$, then Eq. (2) is equivalent to $\tilde{T} = \max\left[T^{\mathrm{R}}, L/\bar{s}\right]$, which is identical to the case in [15].

The adjusted random link driving speed S is never higher than the maximum allowable driving speed $L/t^{\#}$. If the speed limit on a link is the maximum allowable driving speed or faster, the speed limit is not effective and the link behaves as if it has no speed limit. Therefore, to obtain nontrivial solutions, the following assumption is made:

Assumption 6. The link speed limit is lower than the link maximum allowable driving speed, that is, $\bar{s} < L/t^{\#}$.

For a link, given the PDF/PMF $f_T(\chi, v)$ of the adjusted random link travel time, the nontrivial supports $t^{\min}(v)$ and $t^{\max}(v)$, and the value of link speed limit $\bar{s}$ ($\bar{s} < L/t^{\#}$), the effect of the link speed limit on the random link travel time can be examined. Before the analysis, two critical flows, v^{L} and v^{U} for a link with a speed limit $\bar{s}$ ($\bar{s} < L/t^{\#}$), are defined by

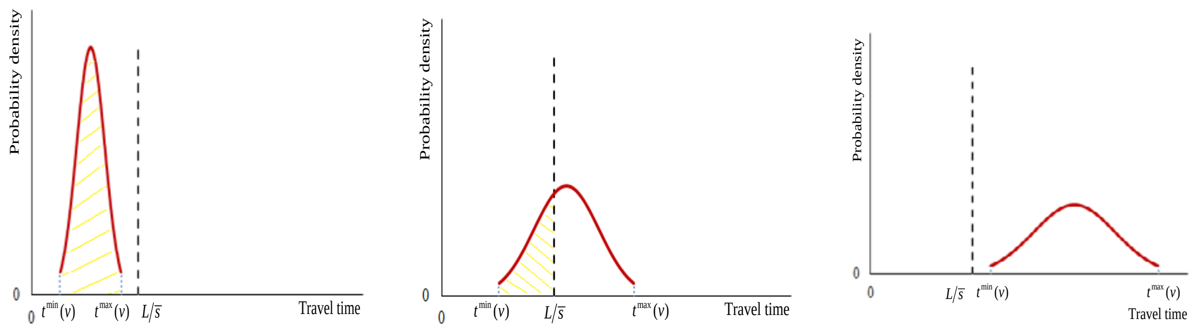
$$v^{\mathrm{L}} = t^{\max -1}(L/\bar{s}) \text{ and}$$

$$v^{\mathrm{U}} = t^{\min -1}(L/\bar{s}),$$

in which $t^{\min-1}(\cdot)$ and $t^{\max-1}(\cdot)$ are the inverse functions of $t^{\min}(\cdot)$ and $t^{\max}(\cdot)$, respectively. Based on the above definitions, the following are true: $s^{\min}(v^{\mathrm{L}})=\overline{s}$, $s^{\max}(v^{\mathrm{U}})=\overline{s}$.

Because $t^{\max}(\tilde{v})$ is larger than $t^{\min}(\tilde{v})$ for any given non-negative link flow $\tilde{v}$ and both functions are monotone increasing with respect to the link flow, the inverse function value $t^{\max-1}(x')$ is smaller than $t^{\min-1}(x')$, given any input x'. Thus, the relationship between the two critical flows is $v^{\mathrm{L}} \le v^{\mathrm{U}}$. The domain of the function $t^{\max}(v)$ is positive, so the value of the inverse function $t^{\max-1}(\cdot)$ is also positive, indicating that $v^{\mathrm{L}} > 0$.

For a link with a speed limit $\overline{s}$ ($\overline{s} < L/t^{\#}$), consider three mutually exclusive cases, as shown in Figures 1(a), (b), and (c).



(a) $v \in \left[0, v^{\mathrm{L}}\right]$ (b) $v \in \left(v^{\mathrm{L}}, v^{\mathrm{U}}\right)$ (c) $v \in \left[v^{\mathrm{U}}, +\infty\right)$

Figure 1. Probability density of the adjusted link random travel time under different link flows ($\overline{\omega} = \underline{\omega} = 0.02$)

When the link flow v is v^{L} or less (see Figure 1(a)), the following hold: $s^{\min}(v) \ge s^{\min}(v^{\mathrm{L}}) = \overline{s}$, $t^{\max}(v) \le t^{\max}(v^{\mathrm{L}}) = L/\overline{s}$. The travel time is longer than the adjusted random travel time T, meaning that the link travel time is equal to $L/\overline{s}$. The probability that the link travel time is $L/\overline{s}$ equals the shaded area in Figure 1(a), which numerically equals one. Thus, when $t^{\max}(v) \le L/\overline{s}$ holds, the travelers always drive at the speed limit and the link travel time always equals $L/\overline{s}$.

When the link flow v is between v^{L} and v^{U} (see Figure 1(b)), the following hold: $s^{\min}(v) < s^{\min}(v^{\mathrm{L}}) = \overline{s}$, $s^{\max}(v) > s^{\max}(v^{\mathrm{U}}) = \overline{s}$, and $t^{\min}(v) < L/\overline{s} < t^{\max}(v)$. Travelers occasionally drive slower than the speed limit and occasionally at the speed limit, and the link travel time occasionally equals the adjusted random travel time T or $L/\overline{s}$. The probability that travelers are driving at the speed limit and the link travel time is $L/\overline{s}$ equals the shaded area in Figure 1(b).

When the link flow v is greater than or equal to v^{U} (see Figure 1(c)), the following hold: $s^{\max}(v) \le s^{\max}(v^{\mathrm{U}}) = \overline{s}$, $t^{\min}(v) \ge t^{\min}(v^{\mathrm{U}}) = L/\overline{s}$. In all instances, the travel time is lower than the adjusted random travel time T, meaning that the speed limit does not affect the travel time. The probability that the link travel time is $L/\overline{s}$ equals the shaded area in Figure 1(c), which is zero. Thus, the travel time is not affected by the speed limit.

In conclusion, when $0 \le v \le v^{\mathrm{L}}$, travelers drive at the speed limit; when $v^{\mathrm{L}} < v < v^{\mathrm{U}}$, travelers occasionally drive at the speed limit (depending on the probability distribution of the link travel time); and when $v \ge v^{\mathrm{U}}$, travelers are not affected by the speed limit. The link flow region $\left[0, v^{\mathrm{U}}\right)$ is the *effective link flow region* for imposing the speed limit $\bar{s}$ ($\bar{s} < L/t^{\#}$) on a link in uncertain environments with supply uncertainty, in which the speed limit affects the driving speed of travelers on that link.

2.2.2 Special case in which supply uncertainty refers to capacity degradation

The link travel time distributions can be obtained by methods other than calibration. Many engineers and scholars consider link capacity degradation to be the main cause of supply uncertainty. They assume a link performance function in which the link flow and link capacity are included and treat the link capacity as a random variable with a known distribution. Thus, the travel time distribution can be derived. This approach only requires knowledge of the link capacity distribution and the link performance function.

When the flow on a link is v and the random capacity of the link is C, the travel time on that link T can be obtained by the assumed link performance function [38]:

$$T = g(v, C) = t^{\#}\left[1 + \alpha\left(v/C\right)^{\beta}\right],$$

where $t^{\#}$ is the positive link free flow travel time, and α and β are positive parameters. This function has been used in many studies, with β typically assumed to be between 2.5 and 5.0 [38]. When $\alpha = 0.15$ and $\beta = 4$, it is known as the Bureau of Public Roads (BPR) function. The parameter β is assumed to be larger than one. *In fact, the function $g(v, C)$ can be of other types of link performance functions instead of a BPR-type function.*

In $g(v, C)$, the link capacity C is a continuous random variable and follows a distribution over $\left[c^{\min}, c^{\max}\right]$, where $c^{\max}$ is the design capacity and $c^{\min}$ is the worst-case capacity. Intuitively, the worst-case capacity is the design capacity or lower, i.e., $c^{\min} \le c^{\max}$. The equal sign holds (i.e., $c^{\min} = c^{\max}$) when there is no link capacity degradation.

The nontrivial supports of the travel time distribution T when ω is zero are

$$t^{\min}(v) = g(v, c^{\max}) \quad \text{and}$$

$$t^{\max}(v) = g\ (v, c^{\min}).$$

In the analysis, the probability density function of the link travel time must be known. If the available information is merely the link capacity distribution and the link performance function, then the PDF/PMF of the link travel time can be obtained based on the following property.

Property 1. Denote c as an outcome of link capacity and χ as an outcome of travel time. Given a link, the travel time χ and the link capacity c has the following relationship: $\chi = t(c, v)$, where $t(c, v) = t^{\#}\left(1 + \alpha\left(v/c\right)^{\beta}\right)$. Then, the travel time function $t(c, v)$ is an objective function of c. The inverse function of $t(c, v)$, denoted as $t^{-1}(\chi, v)$, exists and is differentiable. In addition, if the capacity distribution is known, then according to Proposition 3.4.1 in [39], the probability density function of the random link travel time T is

$$f_T(\chi,v) = f_C(t^{-1}(\chi,v))\left|dt^{-1}(\chi,v)/d\chi\right|.$$

Remark. In some parts of this study, a uniform distribution is used for ease of illustration. If the link capacity follows uniform distribution, the PDF of the random travel time is

$$f_T(\chi,v) = \begin{cases} \left(t^{\#}\alpha\right)^{1/\beta} v\left(\chi - t^0\right)^{-1-1/\beta} \Big/ \left(\beta\left(c^{\max} - c^{\min}\right)\right), & \chi \in \left[t^{\min}(v), t^{\max}(v)\right], \\ 0, & \text{otherwise.} \end{cases}$$

The analyst can then use the same method as in Sub-section 2.2.1 to define the three critical flow regions.

2.3 Mean and standard deviation of link travel time with consideration of a speed limit

2.3.1 General case

Without loss of generality, the network is assumed to have links with and without speed limits. Denote $\overline{A}$ as the set of links with and $\underline{A}$ as the set without speed limits. The indexes a and b, which represent links, are added to the notations as subscripts when necessary. Denote the vector of link speed limits as $\overline{\mathbf{s}}$, where $\overline{s}_a < L_a / t_a^{\#}$ for $a \in \overline{A}$ and $\overline{s}_b = +\infty$ for $b \in \underline{A}$.

For link $b \in \underline{A}$, the mean and standard deviation of random link travel time T_b, denoted by $\underset{\sim}{t}_b(v_b)$ and $\underset{\sim}{\sigma}_b(v_b)$, are calculated, respectively, by

$$\underset{\sim}{t}_b(v_b) = E\left[T_b\right] = \int_0^{+\infty} f_{T_b}(\chi, v_b)\cdot\chi d\chi \text{ and} \tag{3}$$

$$\underset{\sim}{\sigma}_b(v_b) = \sigma\left[T_b\right] = \left(\int_0^{+\infty} f_{T_b}(\chi, v_b)\cdot\chi^2 d\chi - \underset{\sim}{t}_b(v_b)^2\right)^{0.5}. \tag{4}$$

For a link $a \in \overline{A}$ with a speed limit $\overline{s}_a$ **(**$\overline{s}_a \le L_a / t_a^{\#}$**)**, the mean link travel time $\tilde{t}_a(v_a)$ is a piecewise function in terms of link flow, consisting of three pieces of functions in the domains of $\left[0, v_a^{\mathrm{L}}\right]$, $\left(v_a^{\mathrm{L}}, v_a^{\mathrm{U}}\right)$, and $\left[v_a^{\mathrm{U}}, +\infty\right)$:

$$\tilde{t}_a(v_a) = \begin{cases} L_a/\overline{s}_a, & v_a \in \left[0, v_a^{\mathrm{L}}\right], \\ h_a(v_a), & v_a \in \left(v_a^{\mathrm{L}}, v_a^{\mathrm{U}}\right), \\ \underset{\sim}{t}_a(v_a), & v_a \in \left[v_a^{\mathrm{U}}, +\infty\right), \end{cases} \tag{5}$$

in which $h_a(v_a)$ is equal to

$$h_a(v_a) = \int_0^{L_a/\overline{s}_a} \left(L_a/\overline{s}_a\right)\cdot f_{T_a}(\chi, v_a) d\chi + \int_{L_a/\overline{s}_a}^{+\infty} \chi\cdot f_{T_a}(\chi, v_a) d\chi.$$

The travel time standard deviation $\tilde{\sigma}_a(v_a)$ of link $a \in \overline{A}$ is also a piecewise function in terms of link flow, consisting of three pieces of functions in the domains of $\left[0, v_a^{\mathrm{L}}\right]$, $\left(v_a^{\mathrm{L}}, v_a^{\mathrm{U}}\right)$ and $\left[v_a^{\mathrm{U}}, +\infty\right)$:

$$\tilde{\sigma}_a(v_a) = \begin{cases} 0, & v_a \in \left[0, v_a^{\mathrm{L}}\right], \\ \left(q_a(v_a) - h_a(v_a)^2\right)^{0.5}, & v_a \in \left(v_a^{\mathrm{L}}, v_a^{\mathrm{U}}\right), \\ \underset{\sim}{\sigma}_a(v_a), & v_a \in \left[v_a^{\mathrm{U}}, +\infty\right), \end{cases} \tag{6}$$

in which $q_a(v_a)$ is

$$q_a(v_a)=\int_0^{L_a/\bar{s}_a}\left(L_a/\bar{s}_a\right)^2\cdot f_{T_a}(\chi)d\chi+\int_{L_a/\bar{s}_a}^{+\infty}\chi^2\cdot f_{T_a}(\chi)d\chi \ .$$

2.3.2 Special case in which supply uncertainty refers to capacity degradation

If the analyst considers that the supply uncertainty mainly refers to capacity degradation and that the link travel time can be captured by a BPR-type function, then for a link $b\in\underline{A}$ without speed limit, the mean and standard deviation of link travel time, according to (3)-(4), are

$$\underset{\sim}{t}_b(v_b)=t_b^{\#}+t_b^{\#}\alpha_b v_b^{\beta_b}\cdot\theta_b,\ \theta_b=\int_{c_b^{\min}}^{c_b^{\max}} f_{C_b}(c)\cdot c^{-\beta_b}\cdot dc \text{ and} \tag{7}$$

$$\underset{\sim}{\sigma}_b(v_b)=\alpha_b t_b^{\#}v_b^{\beta_b}\cdot\omega_b,\ \omega_b=\left(\int_{c_b^{\min}}^{c_b^{\max}}\left(1/c\right)^{2\beta_b}\cdot f_{C_b}(c)dc-\theta_b^{\,2}\right)^{0.5}. \tag{8}$$

For a link $a\in\bar{A}$ with a speed limit $\bar{s}_a$, the mean and standard deviation of link travel time follow the expressions (5) and (6), respectively, with $\underset{\sim}{t}_a(v_a)$ and $\underset{\sim}{\sigma}_a(v_a)$ follow Eqs. (7) and (8), and $h_a(v_a)$ and $q_a(v_a)$ are equal to

$$h_a(v_a)=\int_{c_a^{\min}}^{v_a\left(\alpha_a t_a^{\#}/\left(L_a/\bar{s}_a-t_a^{\#}\right)\right)^{1/\beta_a}} g_a(v_a,c)\cdot f_{C_a}(c)dc+\int_{v_a\left(\alpha_a t_a^{\#}/\left(L_a/\bar{s}_a-t_a^{\#}\right)\right)^{1/\beta_a}}^{c_a^{\max}}\left(L_a/\bar{s}_a\right)\cdot f_{C_a}(c)dc \text{ and} \tag{9}$$

$$q_a(v_a)=\int_{c_a^{\min}}^{v_a\left(\alpha_a t_a^{\#}/\left(L_a/\bar{s}_a-t_a^{\#}\right)\right)^{1/\beta_a}} g_a(v_a,c)^2\cdot f_{C_a}(c)dc+\int_{v_a\left(\alpha_a t_a^{\#}/\left(L_a/\bar{s}_a-t_a^{\#}\right)\right)^{1/\beta_a}}^{c_a^{\max}}\left(L_a/\bar{s}_a\right)^2\cdot f_{C_a}(c)dc. \tag{10}$$

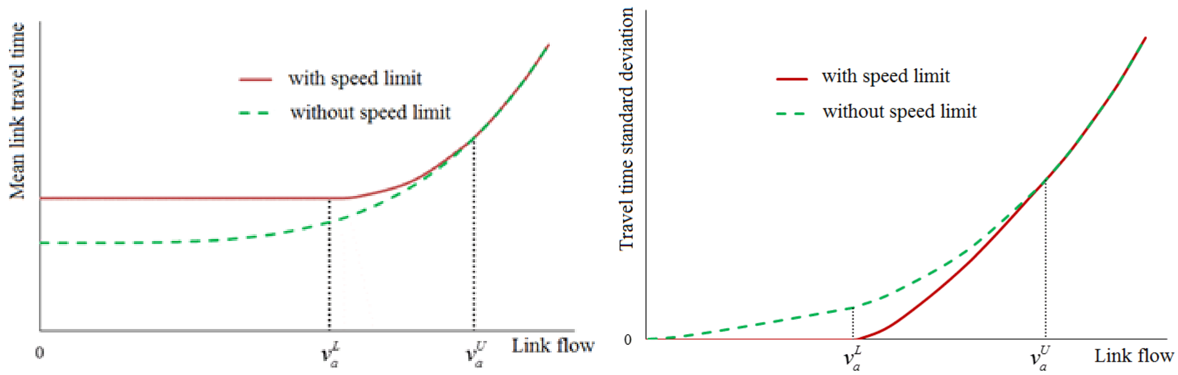
If the random link capacity follows a uniform distribution, the mean and standard deviation of link travel time follow the expressions provided in the study by Yan et al. [15].

2.3.3 Properties of the mean and standard deviation of link travel time with the consideration of a speed limit

The mathematical properties of the link travel time, such as the continuity, monotonicity, and convexity of the mean and standard deviation, are extremely important and must be examined. Yan et al. [15] only provide formulations for the mean and variance of link travel time based on an assumed link performance function and uniform capacity distribution, but do not fully consider the mathematical properties. The mathematical properties are critical because they directly affect the solution properties of the to-be-defined equilibrium, such as the existence of solutions and uniqueness of travel times at equilibrium.

In the following, the mathematical properties of the mean and standard deviation of link travel time in the special case in which the supply uncertainty refers to capacity degradation and the link performance function is the BPR-type function are provided.

Property 2. If the supply uncertainty refers to link capacity degradation and the link travel time is captured by a BPR-type function, the expected travel time of a link without any speed limit is a *continuous*, *strictly increasing*, and *strictly convex* function of its link flow. The expected travel time of a link with a speed limit is a *continuous*, *non-decreasing*, and *convex* function of link flow.



(a) (b)

Figure 2. Mean and standard deviation of travel time of a link with and without a speed limit

Property 3. If the supply uncertainty refers to link capacity degradation and the link travel time is captured by a BPR-type function, the travel time standard deviation of a link without any speed limit is a *continuous*, *strictly increasin*g, and *strictly convex* function of its link flow. The travel time standard deviation of a link with a speed limit is a *continuous*, *non-decreasing*, and *convex* function of its link flow.

Based on Figure 2, it is concluded that imposing a speed limit on a link may increase its mean link travel time but may reduce its link travel time standard deviation.

Noting that Property 2 and Property 3 hold not only in the special case in which the supply uncertainty refers to capacity degradation, but also hold when the supply uncertainty refers to more than capacity degradation and the link travel time before imposing a speed limit follows other types of distributions (e.g., a normal distribution with a mean and standard deviation that both increase monotonically with respect to link flow).

3 Reliability-based user equilibrium assignment model with speed limits

Denote RS as the set of origin-destination (O-D) pairs. The travel demand d_{rs} for O-D pair $rs \in RS$ is assumed to be fixed and given, and $\mathbf{d}$ is the vector of travel demands. Let P and P_{rs} be the set of all feasible paths in the network and for O-D pair rs, respectively. Denote f_p as the flow on path p. Denote also $\mathbf{f} = (f_p)_{p\in P}$ and $\mathbf{v} = (v_a)_{a\in A}$ as the path and link flow vectors, respectively. The feasible path flow set Ω_f and the link flow set Ω_v are defined as follows:

$$\Omega_f = \left\{ \mathbf{f} \,\middle|\, f_p \geq 0, \forall p \in P; \sum_{p\in P_{rs}} f_p = d_{rs}, \forall rs \in RS \right\} \text{ and } \Omega_v = \left\{ \mathbf{v} \,\middle|\, v_a = \sum_{p\in P} f_p \delta_p^a, \forall \mathbf{f} \in \Omega_f, \forall a \in A \right\},$$

where δ_p^a is an indicator variable that equals one if link a is on path p, and zero otherwise. Denote $\mathbf{t} = (t_a)_{a\in A}$ as the mean link travel time vector, and $\boldsymbol{\sigma}$ as the link travel time covariance matrix.

3.1 Travel time budgets

To model the travelers' selfish routing behavior, a classic traffic assignment principle, namely the UE principle as proposed by Wardrop [40], is commonly adopted. The UE principle assumes that all travelers are selfish and choose their routes to minimize their own travel time

[40]. At equilibrium, the travel times of all used routes are not longer than those of the unused routes for the same O-D pair [40].

In transport networks with supply uncertainty, travelers' route choices are also affected by travel time variations. Empirical studies [22,23] suggest that some travelers are risk-averse and consider both the travel time and its variation when making route choices. The TTB approach [20,37,41] is frequently used to capture the risk aversion and selfish routing behavior of travelers because it only requires the mean and standard deviation of path travel times, a parameter related to the level of risk aversion of travelers, and the probability of being late. The TTB approach does not require the distributions of the path travel time, which is a compound random variable consisting of random link travel times. *Obtaining a path travel time distribution from link travel time distributions is difficult, because the link travel time distributions may not be identical, and the link travel times are commonly correlated. Imposing speed limits on links alter the link travel time distributions, making it more difficult to acquire the path travel time distributions.*

The TTB of a path is defined as

Path TTB = Mean path travel time + Safety margin.

The mean path travel time consists of the mean link travel times of links on that path. Based on Eqs. (3) and (5), the mean travel time on link $a \in A$, $t_a(v_a)$, can be expressed by

$$t_a(v_a) = \begin{cases} \underset{\sim}{t}_a(v_a), & \text{if } a \in \underline{A}, \\ \tilde{t}_a(v_a), & \text{if } a \in \bar{A}. \end{cases} \tag{11}$$

The mean travel time on path *p*, denoted as q_p, equals

$$q_p = \sum_{a \in A} t_a(v_a) \delta_p^a, \ \forall p \in P. \tag{12}$$

The safety margin refers to the extra time set aside by travelers for the trip to avoid arriving late. Common measures of the safety margin include travel time standard deviation and travel time variance, usually multiplied by a non-negative scalar, which reflects the level of risk aversion of travelers. This study uses the weighted travel time standard deviation as the safety margin because it has the same unit as the mean travel time.

The path travel time standard deviation consists of the travel time variances of links on that path and the travel time covariances between two links on that path. According to Eqs. (4) and (6), the link travel time standard deviation, $\sigma_a(v_a)$, $\forall a \in A$ is expressed by

$$\sigma_a(v_a) = \begin{cases} \underset{\sim}{\sigma}_a(v_a), & \text{if } a \in \underline{A}, \\ \tilde{\sigma}_a(v_a), & \text{if } a \in \bar{A}. \end{cases} \tag{13}$$

The path travel time standard deviation, denoted as ς_p, equals

$$\varsigma_p = \sqrt{\sum_{a \in A} \left(\sigma_a(v_a)\right)^2 \delta_p^a + \sum_{a \in A} \sum_{b \in A, b \neq a} \sigma_{ab} \cdot \delta_p^a \delta_p^b}, \ \forall p \in P, \tag{14}$$

in which σ_{ab} is the travel time covariance between any two distinct links a and b, and σ_{ab} is assumed to be non-decreasing with respect to the total path flows that pass through both links (i.e., $\sum_{p \in P} f_p \delta_p^a \delta_p^b$). By the Cauchy-Schwarz inequality, the travel time covariance is bounded above: $|\sigma_{ab}| \leq \sigma_a \cdot \sigma_b$.

According to the definition of TTB and Eqs. (11)-(14), the travel time budget on path *p*, denoted as b_p, is expressed mathematically as

$$b_p = q_p + \lambda \cdot \varsigma_p,\ \forall p \in P,$$

where the non-negative parameter λ represents the level of risk aversion of travelers. The λ value reflects how travelers make trade-offs between the mean travel time and the travel time variation [16]. A larger λ indicates that travelers are more risk-averse and more concerned with travel time variations, and vice versa [21]. In the case of $\lambda = 0$, travelers are risk neutral and ignore the travel time variations [21] .

The λ value is associated with the chance constraint, which guarantees the path travel time reliability or equals the probability of on-time arrival [17]. For example, if the path travel time follows a normal distribution, and λ equals 1.65, then the probability of a traveler arriving at his destination on time is 95%, and the probability that his path travel time is less than the path travel time budget is 95% [20,37]. In our study, the path travel time distribution is not required to be known explicitly. The λ value can be calibrated directly based on travel time data. The chance constraint is implicit here.

To facilitate the study, the level of risk aversion is assumed to be the same for all travelers. Nevertheless, this assumption can be easily relaxed by considering multiple classes of travelers, each of which has their own λ value to represent the level of risk aversion of travelers in that user class.

3.2 Equilibrium conditions

This study assumes that travelers acquire information on the mean path travel time and its standard deviation when speed limits are imposed by their past experiences, choose routes to minimize their own path travel time budget, and eventually settle into a long-term habitual equilibrium pattern [37]. An equilibrium is reached only if the travel time budgets of all used routes are not higher than those of the unused routes for the same O-D pair [37]. This equilibrium is referred to as RUE [37]. Given an RUE instance $\left(G, \mathbf{d}, \mathbf{t}, \boldsymbol{\sigma}, \overline{\mathbf{s}}\right)$, the path flow pattern at RUE, denoted as $\mathbf{f}^* = \left(f_p^*\right)_{p \in P}$, satisfies the following conditions:

$$f_p^*\left(b_p^* - \pi_{rs}\right) = 0,\ \forall p \in P_{rs},\ \forall rs \in RS \quad \text{and}$$

$$\left(b_p^* - \pi_{rs}\right) \geq 0,\ \forall p \in P_{rs},\ \forall rs \in RS,$$

in which π_{rs} represents the minimum path travel time budget between O-D pair *rs* and b_p^* denotes the travel time budget for $\mathbf{f}^*$.

The superscript * is used to denote variables that are functions of $\mathbf{f}^*$ and can also apply to variables other than travel time budget, such as t_a^*, q_p^*, and ς_p^*. The superscript in the path flow vector can be replaced by other notions (instead of *) to denote other flow patterns, and variables with the same superscript are functions of that flow pattern.

3.3 Variational inequality formulation and its properties

The proposed RUE model cannot be solved by conventional convex minimization techniques because the path travel time budgets are nonaddictive. Even a UE model with nonaddictive path travel costs requires special solution methods (Meng et al. [42]). Nevertheless, following [37], an optimal solution to the RUE model can be obtained by solving the following VI problem.

Proposition 1. The RUE model with speed limits can be obtained by solving the following VI problem: to determine $\mathbf{f}^* \in \Omega_f$ such that

$$\sum_{p\in P}\left(\tilde{f}_p - f_p^*\right)\cdot b_p^* \geq 0,\ \forall\tilde{\mathbf{f}} = \left(\tilde{f}_p\right)_{p\in P} \in \Omega_f\,. \tag{15}$$

Proof. This proof follows that provided in [43] by replacing the path travel time with a path TTB.

The mapping function in the above VI is indeed monotone, as stated below.

Property 4. The mappings $\mathbf{b}:\Omega_f \to \mathbb{R}^{\mathrm{card}(P)}$, $\mathbf{q}:\Omega_f \to \mathbb{R}^{\mathrm{card}(P)}$, $\boldsymbol{\varsigma}:\Omega_f \to \mathbb{R}^{\mathrm{card}(P)}$, where $\mathbf{b}=(b_p)_{p\in P}$, $\mathbf{q}=(q_p)_{p\in P}$, $\boldsymbol{\varsigma}=(\varsigma_p)_{p\in P}$, and $\mathbb{R}^n$ refers to a *n*-dimensional real number space, are monotone in terms of path flows $\mathbf{f}=(f_p)_{p\in P}$.

The feasible set of path flows (i.e., Ω_f) is closed and convex (see the statement (13.i) in [44]). The set is also bounded because the flow on a path between an O-D pair must not be greater than the O-D demand. It is also clear that $\mathbf{b}$ is also continuous with respect to $\mathbf{f}$. Thus, a solution to the VI problem (15) exists (see statement (8) in [44]). This, along with the monotone results, is useful because existing path-based projection algorithms that guarantee convergence under the assumption that monotone mapping can be used to solve the VI problem (15).

The studies [11,45] suggested that the classical UE models with speed limits may not have unique path flow solutions. Although multiple RUE flow patterns with speed limits may exist, the properties of the RUE flow patterns can be summarized, as stated below.

Proposition 2. At RUE, the mean link travel time vector $\mathbf{t}^*=\left(t_a^*\right)_{a\in A}$, the mean path travel time vector $\mathbf{q}^*=\left(q_p^*\right)_{p\in P}$, the path travel time standard deviation vector $\boldsymbol{\varsigma}^*=\left(\varsigma_p^*\right)_{p\in P}$, and the minimum path travel time budget vector $\mathbf{b}^*=(b_p^*)_{p\in P}$ are unique.

Proof. Assume that there exists another RUE flow pattern $\overline{\mathbf{f}}^*$ and the associated variables are $\overline{\mathbf{b}}^*$, $\overline{\mathbf{q}}^*$, $\overline{\boldsymbol{\varsigma}}^*$, and $\overline{\mathbf{t}}^*$. Based on Proposition 1 and Property 4, $\left(\mathbf{f}^*-\overline{\mathbf{f}}^*\right)^{\mathrm{T}}\left(\overline{\mathbf{b}}^*-\mathbf{b}^*\right)=0$ or $\left(\mathbf{f}^*-\overline{\mathbf{f}}^*\right)^{\mathrm{T}}\left(\overline{\mathbf{q}}^*-\mathbf{q}^*\right)+\left(\mathbf{f}^*-\overline{\mathbf{f}}^*\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\varsigma}}^*-\boldsymbol{\varsigma}^*\right)=0$. The monotonicity of $\mathbf{q}$ and $\boldsymbol{\varsigma}$ means that the first and the second terms of the left side of the equation are equal to zero. The first term is equivalent to $\sum_{a\in A}\left(v_a^*-\overline{v}_a^*\right)\cdot\left(\overline{t}_a^*-t_a^*\right)=0$, by which it is deduced that $\overline{t}_a^*{=}t_a^*$ for each link. A similar argument holds for the path travel time standard deviation and hence for the minimum path travel time budget. ■

4 Analysis of reliability-based user equilibrium with speed limits

4.1 Counterintuitive example

Example 1. Consider the three-link network in Figure 3. The travel demand from node A to node C is $d_{AC}=5$ and that from node B to node C is $d_{BC}=3$. The supply uncertainty refers to capacity degradation. The capacities follow uniform distributions and the design capacities are $c_1^{\max}=6$; $c_2^{\max}=6$; and $c_3^{\max}=5$. The worst-case capacities are $c_1^{\min}=4$; $c_2^{\min}=4$; and $c_3^{\min}=3$. The unit for the demand and capacities is thousands of passenger-car-units per hour. The free flow travel times are $t_1^{\#}=5$; $t_2^{\#}=2$; and $t_3^{\#}=1$. The time unit is hours. The coefficients in the BPR functions are $\alpha_1=0.4$; $\alpha_2=\alpha_3=0.15$; $\beta_a=4$, $\forall a\in A$. The link length of Link 2 is $L_2=180$ miles. The speed limit on Link 2 (if imposed) equals 60 miles/hour. Set $\lambda=2$.

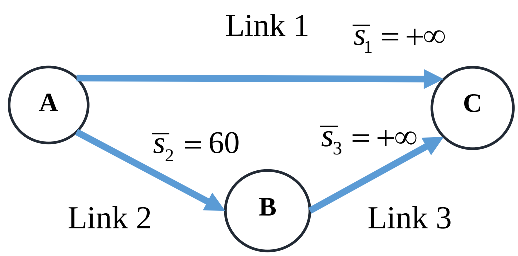


Figure 3. Three-link network

For O-D pair AC, there are two routes. Path 1 consists of Link 1, and Path 2 consists of Links 2 and 3. For O-D pair BC, the only route (Path 3) consists of Link 3. The link flows are $v_1=f_1$; $v_2=f_2$; and $v_3=f_2+f_3$. The travel time budgets for the three paths are $b_1=t_1(f_1)+\lambda\cdot\sigma_1(f_1)$; $b_2=t_2(f_2)+t_3(f_2+f_3)+\lambda\cdot\varsigma_2$; and $b_3=t_3(f_2+f_3)+\lambda\cdot\sigma_3(f_2+f_3)$. The optimal flows, mean travel times, travel time variations, and minimum travel time budget at RUE without and with the speed limit are summarized in Table 1.

Table 1. RUE solutions for Example 1

	$\underset{\sim}{f}_1^*$	$\underset{\sim}{f}_2^*$	$\underset{\sim}{f}_3^*$	$\underset{\sim}{v}_3^*$ $(\underset{\sim}{f}_2^*+\underset{\sim}{f}_3^*)$	$\underset{\sim}{\pi}_{AC}$	$\underset{\sim}{\pi}_{BC}$	$\sum_{p\in P}\underset{\sim}{f}_p^*\underset{\sim}{b}_p^*$
Without speed limit	1.69	3.31	3	6.31	5.59	3.52	38.51
	$\tilde{f}_1^*$	$\tilde{f}_2^*$	$\tilde{f}_3^*$	$\tilde{v}_3^*$ $(\tilde{f}_2^*+\tilde{f}_3^*)$	$\tilde{\pi}_{AC}$	$\tilde{\pi}_{BC}$	$\sum_{p\in P}\tilde{f}_p^*\tilde{b}_p^*$
With speed limit	2.21	2.79	3	5.79	5.78	2.79	37.27

The speed limit (if imposed) is on Link 2, which implies that it is on Path 2 for O-D pair AC. The minimum travel time budget for O-D pair AC with the speed limit is larger than that without (see Table 1), which is within our expectation. Interestingly, the minimum travel time budget for O-D pair BC with the speed limit is smaller than that without. Also, the sum of the travel time budgets of all travelers (see the right-most column of the table) is smaller when Link 2 has a speed limit imposed, that is, 37.27 < 38.51. The results suggest that *imposing a speed limit on one path for one O-D pair can reduce the minimum travel time budget for another O-D pair, and reduce the system-wide travel time budget.*

The explanation of the above phenomenon is as follows. When Link 2 has a speed limit imposed on it, the travel time budget of Path 2 increases. Fewer travelers between O-D pair AC choose Path 2 at RUE and more travelers between this O-D pair therefore choose Path 1. The path flow $\tilde{f}_2^*$ at RUE with the speed limit is smaller than that at RUE without the speed limit (i.e., $\underset{\sim}{f}_2^*$). The path flows $\underset{\sim}{f}_3^*$ and $\tilde{f}_3^*$ both equal d_{BC} because there is only one path (Path 3) for O-D pair BC. Thus, the flow on Link 3 at RUE (i.e., $\tilde{f}_2^* + \tilde{f}_3^*$) with the speed limit is smaller than that without speed limit (i.e., $\underset{\sim}{f}_2^* + \underset{\sim}{f}_3^*$). The minimum travel time budget is reduced for O-D pair BC, which equals $t_3(f_2 + f_3) + \lambda \cdot \sigma_3(f_2 + f_3)$ and is a monotone function of the flow on link 3. In summary, imposing a speed limit on Link 2 diverts the travelers for O-D pair AC from Link 2 to Link 1, leading to a reduction of the traffic flows on both Links 2 and 3. The minimum travel time budget for O-D pair BC is thus reduced. The sum of the travel time budgets of all travelers (i.e., $\sum_{p \in P} \tilde{f}_p^* \tilde{b}_p^*$) is also reduced, which is apparently a benefit of imposing speed limits.

4.2 Inefficiency of an RUE flow pattern with speed limits

In the final subsection, an important indicator is examined, which is the inefficiency of an RUE flow pattern that reveals how much the system performance degrades due to the selfish routing behavior of travelers. The inefficiency of an RUE flow pattern for a transport network with stochastic capacities have been examined by [37,46]. However, [37] assumed that there are no speed limits in a transport network. [46] mainly focused on how the inefficiency of an RUE flow pattern is affected by the amount of travel time information given to the travelers. [46] also assumed that travelers are risk-neutral and do not consider travel time variations. With the consideration of speed limits and risk aversion which affect the travelers' route choice decisions, the inefficiency of an RUE flow pattern is unclear and worthy of examination. Unlike the numerical results presented in Section 4.1, the results in this sub-section are analytical results.

The analysis in the section is based on an RUE instance $(G, \mathbf{d}, \mathbf{t}, \boldsymbol{\sigma}, \overline{\mathbf{s}})$ in Section 3. The RUE problem generalizes UE problem, which can be regarded as a type of non-cooperative game in which each traveler aims to find a route to minimize his disutility (i.e., path travel time). A UE flow pattern is known to lead to system efficiency loss compared with a system optimal flow pattern. In the UE problem, a common system performance measure from the perspective of economics is the total system travel time. A natural system performance measure for RUE problem is the *total travel time budget*, defined as $\sum_{p \in P} f_p b_p$.

The total travel time budget at RUE is not smaller than that at the reliability-based system optimum (RSO), which assumes that all travelers are cooperative and choose routes to minimize the total travel time budget. The *RSO flow pattern* is the flow pattern that minimizes the total travel time budget, denoted as $\mathbf{f}^0 = \left(f_p^0\right)_{p \in P}$. Mathematically, it equals

$$\mathbf{f}^0 = \arg\left[\min_{\mathbf{f} \in \Omega_f} \sum_{p \in P} f_p b_p\right]. \tag{16}$$

At RUE, the total travel time budget equals $\sum_{p\in P} f_p^* b_p^*$. At RSO, the total travel time budget equals $\sum_{p\in P} f_p^0 b_p^0$. The *inefficiency of an RUE flow pattern* of instance $\left(G,\mathbf{d},\mathbf{t},\boldsymbol{\sigma},\overline{\mathbf{s}}\right)$, denoted as $\rho_\lambda\left(G,\mathbf{d},\mathbf{t},\boldsymbol{\sigma},\overline{\mathbf{s}}\right)$, is then defined as follows:

$$\rho_\lambda\left(G,\mathbf{d},\mathbf{t},\boldsymbol{\sigma},\overline{\mathbf{s}}\right)=\sum_{p\in P} f_p^* b_p^* \Big/ \sum_{p\in P} f_p^0 b_p^0 . \tag{17}$$

The total travel time budget at RSO must exist and be unique, because the objective function of the minimization problem in the square brackets in Eq. (16) is convex and the minimum must be unique. The total travel time budget at RUE can be alternatively expressed by $\sum_{rs\in RS} d_{rs}\pi_{rs}$, because 1) all travelers on their paths between the same O-D pair $rs\in RS$ have exactly the same travel time budget, which equals the minimum travel time budget π_{rs}; and 2) the sum of flows on the used paths for the same O-D pair equals the demand for that O-D pair. According to Proposition 2, the minimum path travel time budget of each O-D pair at RUE is unique. Thus, $\sum_{rs\in RS} d_{rs}\pi_{rs}$ is unique and the total travel time budget at RUE is also unique. The denominator and numerator on the right side of Eq. (17) are unique. Thus, the inefficiency is unique.

The RUE problem introduced in Section 3 can be used in transportation planning, by which the system manager can predict the traffic flow distribution in a transport network. The inefficiency of an RUE flow pattern, on the other hand, is an economic evaluation index, by which the system manager can quickly determine the relative reduction of system performance induced by the selfish routing behavior of travelers. Commonly, given an RUE instance, the system manager can acquire the RUE flow pattern and the value of $\rho_\lambda\left(G,\mathbf{d},\mathbf{t},\boldsymbol{\sigma},\overline{\mathbf{s}}\right)$ simultaneously. However, the inefficiency of an RUE flow pattern may be individually assessed by parties who do not have full information of the RUE instance (i.e., G, $\mathbf{d}$, $\mathbf{t}$, $\boldsymbol{\sigma}$, and $\overline{\mathbf{s}}$). These parties are concerned with evaluating or estimating the inefficiency of an RUE flow pattern with less information. In the following, an analytical upper bound of the inefficiency of an RUE flow pattern, which only depends on few parameters, is presented.

Proposition 3. The inefficiency of an RUE flow pattern of the instance $\left(G,\mathbf{d},\mathbf{t},\boldsymbol{\sigma},\overline{\mathbf{s}}\right)$ is bounded above, as:

$$\rho_\lambda\left(G,\mathbf{d},\mathbf{t},\boldsymbol{\sigma},\overline{\mathbf{s}}\right)\le\left(1+\lambda\varepsilon_{\max}\right)\big/\left(1-\mu'\right),$$

in which $\varepsilon_{\max}=\max\limits_{a\in A}\theta_a\big/\omega_a$, $\beta_{\max}=\max\limits_{a\in A}\beta_a$, and $\mu'=\beta_{\max}\left(1+\beta_{\max}\right)^{-1/\beta_{\max}-1}$.

Proof. See Appendix A.

The above results are based on the case in which supply uncertainty refers to capacity degradation and the link performance function is the BPR function.

The upper bound of the inefficiency of an RUE flow pattern depends on the level of risk aversion of travelers λ, the largest coefficient $\varepsilon_{\max}$ ($\varepsilon_{\max}$ further depends on the design and worst-case link capacities), and the largest power of all link performance functions $\beta_{\max}$. The upper bound is independent of the network topology, the travel demands, and the speed limits.

The upper bound of the inefficiency of an RUE flow pattern is increasing with respect to λ, $\varepsilon_{\max}$, and $\beta_{\max}$, which can be directly deduced from the formulation of the upper bound.

With the result in Proposition 3, any interested parties can quickly estimate the inefficiency of an RUE flow pattern, as long as they acquire the values of λ, $\varepsilon_{\max}$, and $\beta_{\max}$. In the following, an illustration example is presented.

Example 2. The urban transport planning center of a city in China is preparing the annual traffic assessment report for the city. One of the indexes in the report is the inefficiency of the RUE flow pattern during peak hours. The analysts realize that their record of the RUE instance (i.e., the network topology, the demand pattern, link travel time functions, travel time standard deviations, and the speed limits) is not updated, so that they are not able to calculate the exact value of the inefficiency of the RUE flow pattern. Instead of using tremendous resources and manpower to update the information of the RUE instance, the analysts decide to use the analytical upper bound of the inefficiency of the RUE flow pattern as an alternative. The analysts only need to: 1) conduct on-line or off-line surveys to obtain the level of risk aversion of the travelers (e.g., λ=1.65) ; 2) check those roads which are constantly under adverse environments that lead to capacity degradation and find out $\varepsilon_{\max}$ (e.g., $\varepsilon_{\max}$=0.10); and 3) check the local road design manual to find out $\beta_{\max}$ (e.g., $\beta_{\max}$=3). Based on the values of these parameters, the analysts can quickly obtain an estimate of the inefficiency of the RUE flow pattern, which is less than 2.2 (i.e., $\left(1+\lambda\varepsilon_{\max}\right)/\left(1-\mu'\right)$). The interpretation is that the system performance when the travelers choose routes selfishly is not higher than 2.2 times of its theoretical minimum value.

5 Conclusion

In this study, how link speed limits reallocate the traffic flows under a general supply uncertainty situation and with risk-averse travelers is analyzed. How the random travel time and speed on a link are altered by its link-specific speed limit and how to derive the mean link travel time and link travel time standard deviation after a speed limit is imposed are analyzed, given the PDF/PMF of the (adjusted) random link travel time before the speed limit is imposed on the link. The case in which supply uncertainty refers to capacity degradation is discussed in detail. The fundamental formulae of the mean and standard deviation of link travel time under the effects of a speed limit are provided. A distribution-free RUE model is established to capture both risk aversion and selfish routing behavior of travelers in a transportation network with speed limits, and a VI is proposed to solve the RUE model. The key results of the study are summarized below.

- The mean link travel time and link travel time standard deviation with a speed limit are shown to be continuous, monotone, and convex functions in terms of link flow.
- The mean link travel times, the mean path travel times, the path travel time standard deviations, and the minimum travel time budgets are proven to be unique at RUE.
- Imposing a speed limit on the path of one O-D pair can reduce the minimum travel time budget of another O-D pair at RUE and the total path travel time budget of all travelers.
- The inefficiency in the case in which supply uncertainty refers to capacity degradation is proven to be bounded above, which depends on the level of risk aversion of travelers, a ratio related to the design and worst-case link capacities, and the highest power of all BPR type functions used.

Our proposed model can be used by other researchers to establish bi-level network design problems that involve speed limits and uncertain environments. The above key results guarantee that the lower-level RUE model can always be solved using conventional solution methods.

This study examines the effects of speed limits on the link travel times in peak hours, when the travel demands can be regarded as deterministic. A future research possibility would be to further capture stochastic demands, in addition to supply uncertainty. The system managers could then follow these steps (see Lam et al. [36]): 1) express all of the link travel times before speed limits are imposed as functions of link flows and link capacities; 2) define the distributions of link capacities and the distributions of demand; 3) enumerate all possible paths for each O-D pair and assign each path a to-be-determined variable that presents the proportion of the path flow at equilibrium and the demand for this O-D pair; 4) derive the distributions of path flows based on the distributions of demands; and 5) derive the distributions of link flows based on the distributions of path flows. From these five steps, the mean and standard deviation of the link travel time after a speed limit is imposed are derived, using the distribution of link flow, the distribution of link capacity, the speed limit, and the rule described by $\tilde{T} = \max\left[T, L/\overline{s}\right]$ in this paper. Thus, the proposed RUE model can capture the effects of speed limits on link travel times in an uncertain environment with both supply and demand uncertainties.

Acknowledgement
This work was jointly supported by a grant [No. 201611159067] from the University Research Committee of the University of Hong Kong, a grant from the National Natural Science Foundation of China [No. 71271183], and a grant from the Guangzhou Science Technology and Innovation Commission [No. 201707010292]. The authors are grateful to the associate editor, subject editor, and two reviewers for their constructive comments.

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Appendix A- Proof of Proposition 3

According to Proposition 1, the following is true: $\sum_{p\in P}\left(f_p^*-f_p^0\right)b_p^*\le 0$, or $\sum_{p\in P} f_p^* b_p^* \le \sum_{p\in P} f_p^0 b_p^*$. Subtracting $\sum_{p\in P} f_p^0 b_p^0$ from both sides of the inequality and re-reformulating the resultant expression:

$$\sum_{p\in P} f_p^* b_p^* - \sum_{p\in P} f_p^0 b_p^0 \le \sum_{p\in P} f_p^0\left(q_p^*-q_p^0\right)+\sum_{p\in P} f_p^0\left(\lambda\varsigma_p^*-\lambda\varsigma_p^0\right). \tag{A.1}$$

Because the mapping ς is monotone in terms of the path flows, the following inequality is true: $\sum_{p\in P}\left(f_p^*-f_p^0\right)\left(\lambda\varsigma_p^0-\lambda\varsigma_p^*\right)\le 0$, which can be reframed to $\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^0+\sum_{p\in P} f_p^0\left(\lambda\varsigma_p^*-\lambda\varsigma_p^0\right)\le\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*$. Eliminating the first term in the left side of this inequality, it still holds, i.e., $\sum_{p\in P} f_p^0\left(\lambda\varsigma_p^*-\lambda\varsigma_p^0\right)\le\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*$. Thus, the second term in the right side of (A.1) is bounded above. The first term in the right side of (A.1) can be re-written as $\sum_{p\in P} f_p^0\left(q_p^*-q_p^0\right)=\sum_{a\in A} v_a^0\left(t_a^*-t_a^0\right)$, which is also bounded above by $\mu'\sum_{a\in A} t_a^* v_a^*$ according to Appendix C. Thus, the left side of (A.1) is bounded above:

$$\sum_{p\in P} f_p^* b_p^*-\sum_{p\in P} f_p^0 b_p^0\le\mu'\sum_{a\in A} t_a^* v_a^*+\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*. \tag{A.2}$$

Denote $k'=\left(\mu'+\lambda\varepsilon_{\max}\right)/\left(1+\lambda\varepsilon_{\max}\right)$. k' is smaller than one. The right side of (A.2) is expressed as $\mu'\sum_{a\in A} t_a^* v_a^*+\left(1-k'\right)\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*+k'\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*$. Based on Appendix B, $\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*\le\lambda\varepsilon_{\max}\sum_{a\in A} t_a^* v_a^*$. Thus, $\left(1-k'\right)\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*\le\left(1-k'\right)\cdot\left(\lambda\varepsilon_{\max}\right)\sum_{a\in A} t_a^* v_a^*$. Hence, the right side of (A.2) is bounded above:

$$\mu'\sum_{a\in A} t_a^* v_a^*+\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*\le\left(\mu'+\left(1-k'\right)\cdot\left(\lambda\varepsilon_{\max}\right)\right)\sum_{a\in A} t_a^* v_a^*+k'\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*. \tag{A.3}$$

Furthermore, rearranging $k'=\left(\mu'+\lambda\varepsilon_{\max}\right)/\left(1+\lambda\varepsilon_{\max}\right)$ gives $\mu'+\left(1-k'\right)\cdot\left(\lambda\varepsilon_{\max}\right)=k'$. Thus, (A.3) is equivalent to

$$\mu'\sum_{a\in A} t_a^* v_a^*+\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*\le k'\sum_{a\in A} t_a^* v_a^*+k'\sum_{p\in P} f_p^*\cdot\lambda\varsigma_p^*=k'\left(\sum_{p\in P} f_p^*\cdot b_p^*\right). \tag{A.4}$$

Inequality (A.4) indicates that the right side of inequality (A.2) is bounded above, which also means that the left side of (A.2) is also bounded above: $\sum_{p\in P} f_p^* b_p^*-\sum_{p\in P} f_p^0 b_p^0\le k'\left(\sum_{p} f_p^*\cdot b_p^*\right)$, which can be re-arranged to obtain $\sum_{p\in P} f_p^*\cdot b_p^*\Big/\sum_{p\in P} f_p^0\cdot b_p^0\le 1/\left(1-k'\right)=\left(1+\lambda\varepsilon_{\max}\right)/\left(1-\mu'\right)$.■

Appendix B

The travel time standard deviation has an upper bound for the cases with and without speed limits; i.e., $\varsigma_p<\sum_{a\in A}\varepsilon_a t_a\delta_p^a,\ \forall p\in P$.

Proof. Consider two cases for a link: (i) without any speed limit and (ii) with a speed limit.

Case (i)

For a link $b\in\underline{A}$, its mean link travel time is $\underset{\sim}{t}_b(v_b)$, whose expression is given in Eq. (7). Its link travel time standard deviation is $\underset{\sim}{\sigma}_b(v_b)$, whose expression is given in Eq. (8). Dividing its link travel time standard deviation by its mean link travel time gives

$$\underset{\sim}{\sigma}_b(v_b)/\underset{\sim}{t}_b(v_b)=t_b^{\#}\alpha_b v_b^{\beta_b}\cdot\omega_b\Big/\left(t_b^{\#}+t_b^{\#}\alpha_b v_b^{\beta_b}\cdot\theta_b\right),$$

which implies that

$$\underset{\sim}{\sigma}_b(v_b) \big/ \underset{\sim}{t}_b(v_b) < \alpha_b t_b^{\#} {v_b}^{\beta_b} \cdot \omega_b \big/ t_b^{\#} \alpha_b {v_b}^{\beta_b} \cdot \theta_b = \omega_b / \theta_b \,. \tag{B.1}$$

Case (ii)

For a link $a \in \bar{A}$, its mean link travel time is $\tilde{t}_a(v_a)$, whose expression is given in Eqs. (5), (7), and (9). Its link travel time standard deviation is $\tilde{\sigma}_a(v_a)$, whose expression is given in Eqs. (6), (8), and (10). Furthermore, the following hold: 1) $\tilde{\sigma}_a(v_a) \le \underset{\sim}{\sigma}_a(v_a)$; and 2) $\tilde{t}_a(v_a) \ge \underset{\sim}{t}_a(v_a)$. Thus,

$$\tilde{\sigma}_a(v_a) \big/ \tilde{t}_a(v_a) \le \underset{\sim}{\sigma}_a(v_a) \big/ \underset{\sim}{t}_a(v_a) < \omega_a / \theta_a \,. \tag{B.2}$$

Based on conditions (B.1) and (B.2), $\sigma_a < \varepsilon_a t_a$, $a \in A$.

Because the travel time covariance is bounded above ($\sigma_{ac} \le \sigma_a \sigma_c$, $a, c \in A$), the following is true: $\varsigma_p \le \sqrt{\sum_{a\in A} (\sigma_a)^2 \delta_p^a + \sum_{a \in A} \sum_{c \in A, c \ne a} \sigma_a \cdot \sigma_c \cdot \delta_p^a \delta_p^c} = \sqrt{\left(\sum_{a \in A} \sigma_a \delta_p^a \right)^2} = \sum_{a \in A} \sigma_a \delta_p^a$, $\forall p \in P$.

Thus, $\varsigma_p \le \sum_{a \in A} \varepsilon_a t_a \delta_p^a$, $\forall p \in P$.■

Appendix C

Given an RUE link flow pattern $\mathbf{v}^*$ and an RSO link flow pattern $\mathbf{v}^0$, the following must be true: $\sum_{a \in A} \left(t_a^* - t_a^0 \right) v_a^0 \le \mu' \cdot \sum_{a \in A} t_a^* v_a^*$.

Proof. The proof is divided into two main parts: (i) and (ii).

Part (i): Without any speed limit

For a link $b \in \underline{A}$ without any speed limit, the expected link travel time $t_b(v_b)$ is expressed by $\underset{\sim}{t}_b(v_b)$ according to Eq. (7). That is, $\underset{\sim}{t}_b(v_b) = t_b^{\#} + t_b^{\#} \alpha_b \left(v_b \right)^{\beta_b} \theta_b$. Based on this, consider the following maximization problem:

$$\max_{x_b \ge 0} \; F_b(x_b) = \underset{\sim}{t}_b(v_b^*) x_b - \underset{\sim}{t}_b(x_b) x_b \,.$$

The first-order derivative of $F_b(x_b)$ with respect to x_b is

$$dF_b(x_b) \big/ dx_b = t_b^{\#} \alpha_b \theta_b \left[\left(v_b^* \right)^{\beta_b} - \left(\beta_b + 1 \right) \left(x_b \right)^{\beta_b} \right]. \tag{C.1}$$

If v_b^* is larger than zero, the square bracket term in the right side of Eq. (C.1) is zero or larger on $0 \le x_b \le (1+\beta_b)^{-1/\beta_b} v_b^*$ and smaller than zero on $x_b > (1+\beta_b)^{-1/\beta_b} v_b^*$. So is $dF_b(x_b)/dx_b$. The objective function $F_b(x_b)$ is strictly increasing on $0 \le x_b \le (1+\beta_b)^{-1/\beta_b} v_b^*$ and decreasing on $x_b > (1+\beta_b)^{-1/\beta_b} v_b^*$. If v_b^* equals zero, $dF_b(x_b)/dx_b$ equals zero at $x_b = 0$ and is smaller than zero for $x_b > 0$. The function $F_b(x_b)$ is decreasing on $x_b \ge 0$. In either case, the global maximum $\hat{x}_b$ of the objective function exists and is also unique and satisfies the following condition: $dF_b(\hat{x}_b) / d\hat{x}_b = 0$, i.e., $\hat{x}_b = (1+\beta_b)^{-1/\beta_b} v_b^*$. Substituting the global maximum into the objective function $F_b(x_b)$, the maximum value of the objective function is

$$F_b(\hat{x}_b) = \beta_b\left(1+\beta_b\right)^{-1/\beta_b-1}\cdot\left[t_b^{\#}\alpha_b\theta_b\left(v_b^*\right)^{\beta_b}v_b^*\right]. \quad \text{(C.2)}$$

The square bracket term in the right side of Eq. (C.2) is smaller than $\underset{\sim}{t}_b(v_b^*)v_b^*$. Thus,

$$F_b(\hat{x}_b) \le \beta_b\left(1+\beta_b\right)^{-1/\beta_b-1}\cdot\underset{\sim}{t}_b(v_b^*)v_b^*. \quad \text{(C.3)}$$

Because $F_b(\hat{x}_b)$ is the global maximum objective value,

$$F_b(v_b^0) \le F_b(\hat{x}_b) \quad \text{(C.4)}$$

Denote $\mu' = \beta_{\max}\left(1+\beta_{\max}\right)^{-1/\beta_{\max}-1}$. Then,

$$\beta_b\left(1+\beta_b\right)^{-1/\beta_b-1} \le \mu'. \quad \text{(C.5)}$$

Based on conditions (C.3) – (C.5), $F_b(v_b^0) \le \mu'\cdot\underset{\sim}{t}_b(v_b^*)v_b^*$, or equivalently,

$$\underset{\sim}{t}_b(v_b^*)v_b^0 - \underset{\sim}{t}_b(v_b^0)v_b^0 \le \mu'\cdot\underset{\sim}{t}_b(v_b^*)v_b^*. \quad \text{(C.6)}$$

The above proves that for link $b\in\underline{A}$, $\left(t_b^* - t_b^0\right)v_b^0 \le \mu'\cdot t_b^*v_b^*$.

Part (ii): With a speed limit

Consider two mutually exclusive cases for a link $a\in\overline{A}$ with the speed limit $\overline{s}_a$ and the mean link travel time $t_a(v_a)$ expressed by $\tilde{t}_a(v_a)$ according to Eqs. (5), (7), and (9):

Case i: $v_a^* \le v_a^0$

For the case of $v_a^* \le v_a^0$, due to the monotonicity of the function $\tilde{t}_a(v_a)$, $\tilde{t}_a(v_a^*) \le \tilde{t}_a(v_a^0)$. Then $\left(\tilde{t}_a(v_a^*) - \tilde{t}_a(v_a^0)\right)v_a^0 \le 0$, which implies that $\left(\tilde{t}_a(v_a^*) - \tilde{t}_a(v_a^0)\right)v_a^0 \le \mu'\cdot\tilde{t}_a(v_a^*)v_a^*$, because $\mu'\cdot\tilde{t}_a(v_a^*)v_a^*$ and v_a^0 are larger than or equal to zero.

Case ii: $v_a^* > v_a^0$

For the second case (i.e., $v_a^* > v_a^0$), based on inequality (C.6), replacing $\underset{\sim}{t}_b(v_b^*)$ with $\underset{\sim}{t}_a(v_a^*)$, $\underset{\sim}{t}_b(v_b^0)$ with $\underset{\sim}{t}_a(v_a^0)$, v_b^* with v_a^*, and v_b^0 with v_a^0, the following inequality is obtained:

$$\underset{\sim}{t}_a(v_a^*)v_a^0 - \underset{\sim}{t}_a(v_a^0)v_a^0 \le \mu'\cdot\underset{\sim}{t}_a(v_a^*)v_a^*. \quad \text{(C.7)}$$

Note that $\underset{\sim}{t}_a(\cdot)$ represents the original mean link travel time function of link a without any speed limit.

Furthermore, the following holds:

$$\tilde{t}_a(v_a^*)\big/\underset{\sim}{t}_a(v_a^*) \le \tilde{t}_a(v_a^0)\big/\underset{\sim}{t}_a(v_a^0) \ . \quad \text{(C.8)}$$

Multiplying each term in inequality (C.7) by $\tilde{t}_a(v_a^*)\big/\underset{\sim}{t}_a(v_a^*)$, which is the left side of inequality (C.8), the following is obtained:

$$\tilde{t}_a(v_a^*)v_a^0 - \left(\tilde{t}_a(v_a^*)\big/\underset{\sim}{t}_a(v_a^*)\right)\underset{\sim}{t}_a(v_a^0)v_a^0 \le \mu'\cdot\tilde{t}_a(v_a^*)v_a^*. \quad \text{(C.9)}$$

Multiplying $-v_a^0$ to each side of inequality (C.8) and rearranging the resultant expression,

$$-\left(\tilde{t}_a(v_a^*)\big/\underset{\sim}{t}_a(v_a^*)\right)\underset{\sim}{t}_a(v_a^0)\cdot v_a^0 \ge -\tilde{t}_a(v_a^0)\cdot v_a^0. \quad \text{(C.10)}$$

Condition (C.10) indicates that the second term in the left side of inequality (C.9) is larger than or equal to $-\tilde{t}_a(v_a^0)v_a^0$. Replacing the second term in the left side of inequality (C.9) with $-\tilde{t}_a(v_a^0)v_a^0$, inequality (C.9) still holds, i.e.,

$$\tilde{t}_a(v_a^*)v_a^0 - \tilde{t}_a(v_a^0)v_a^0 \le \mu' \cdot \tilde{t}_a(v_a^*)v_a^*.$$

The above proves that for link $a \in \bar{A}$ with the speed limit $\bar{s}_a$, the following is true: $\left(t_a^* - t_a^0\right)v_a^0 \le \mu' \cdot t_a^* v_a^*$.

Conclusion

In summary, for any link $a \in A$ in the network at RUE, the following holds:

$$\left(t_a^* - t_a^0\right)v_a^0 \le \mu' \cdot t_a^* v_a^*,\ \forall a \in A.$$

Summing up the inequalities of all links in the network, condition $\sum_{a \in A}\left(t_a^* - t_a^0\right)v_a^0 \le \mu' \cdot \sum_{a \in A} t_a^* v_a^*$ is obtained. ■