Submitted to the *Journal of Nonparametric Statistics*
Vol. 00, No. 00, Month 20XX, 1–29

# Tail-weighted dependence measures with limit being the tail dependence coefficient

David Lee[a*], Harry Joe[a] and Pavel Krupskii[b]

[a]*Department of Statistics, University of British Columbia, Vancouver, BC, Canada V6T 1Z4*
[b]*CEMSE Division, King Abdullah University of Science and Technology, Thuwal 23955-6900, Kingdom of Saudi Arabia*

(*Received 00 Month 20XX; accepted 00 Month 20XX*)

For bivariate continuous data, measures of monotonic dependence are based on the rank transformations of the two variables. For bivariate extreme value copulas, there is a family of estimators $\hat{\vartheta}_\alpha$, for $\alpha > 0$, of the extremal coefficient, based on a transform of the absolute difference of the $\alpha$ power of the ranks. In the case of general bivariate copulas, we obtain the probability limit $\zeta_\alpha$ of $\hat{\zeta}_\alpha = 2 - \hat{\vartheta}_\alpha$ as the sample size goes to infinity, and show that (i) $\zeta_\alpha$ for $\alpha = 1$ is a measure of central dependence with properties similar to Kendall's tau and Spearman's rank correlation, (ii) $\zeta_\alpha$ is a tail-weighted dependence measure for large $\alpha$, and (iii) the limit as $\alpha \to \infty$ is the upper tail dependence coefficient. We obtain asymptotic properties for the rank-based measure $\hat{\zeta}_\alpha$, and estimate tail dependence coefficients through extrapolation on $\hat{\zeta}_\alpha$. A data example illustrates the use of the new dependence measures for tail inference.

**Keywords:** copula; extremal coefficient; monotone dependence; tail order; tail-weighted dependence

*AMS Subject Classification*: 62G32; 62H20

## 1. Introduction

Multivariate data sets of continuous variables often have dependence structures different from multivariate Gaussian. Lower and upper tail-weighted dependence measures have been used to quantify departures from multivariate Gaussian in bivariate margins; some aspects of departures relevant for applications include tail asymmetry and tail dependence relative to the Gaussian distribution. For data sets with more than two variables, these measures can be computed for each pair for such an assessment.

Empirical versions of bivariate lower (upper) tail-weighted dependence measures put more weight on data in the joint lower (upper) tail. Examples of such measures include semi-correlations (Gabbi (2005); see Section 2.17 of Joe (2014)), conditional Spearman's $\rho$ (Schmid and Schmidt (2007)) and power-weighted measures (Krupskii and Joe (2015)) in the joint lower/upper quadrant. These are invariant to monotone increasing transforms of the variables and have probability-based counterparts defined via copulas. A copula is a multivariate cumulative distribution function with Uniform(0,1) univariate margins. Any probability-based dependence measure invariant to monotone increasing transforms

*Corresponding author. Email: dav001@gmail.com

can be expressed in terms of the copula, and so copulas have been useful for the analysis of dependence properties (see Nelsen (2006) and Joe (2014)). One advantage of tail-weighted dependence measures over the lower and upper tail dependence coefficients is that the latter are defined via limits and have no obvious simple empirical counterparts.

In this paper, we study a family of dependence measures that arise from the extreme value literature for estimating the extremal coefficient. When these measures are considered for general bivariate distributions rather than just bivariate extreme value distributions, they can be considered as a family of dependence measures $\zeta_\alpha$ indexed by $\alpha > 0$, some of which measure central dependence (similar to Kendall's tau and Spearman's rank correlation) and others measure the strength of dependence in the joint upper tail. Furthermore, when $\alpha \to \infty$, $\zeta_\alpha$ converges to the upper tail dependence coefficient. The quantity $\zeta_\alpha$ for any $\alpha$ has an empirical counterpart $\hat{\zeta}_\alpha$ for data that can be defined based on the rank transforms of the two variables. We can then consider extrapolating $\hat{\zeta}_\alpha$ over a sequence of $\alpha$ values to estimate the tail dependence coefficient of a given bivariate data set.

The rest of this paper is organized as follows. Section 2 introduces $\zeta_\alpha$ and its connection with the extreme value theory, as well as its empirical counterpart. Section 3 gives the properties of $\zeta_\alpha$ and the asymptotic distribution of its rank-based estimator. We suggest a method to estimate the upper tail dependence coefficient from a sample in Section 4. A data example is presented in Section 5 and concluding remarks are in Section 6. Appendix A contains supplementary information on the asymptotic variance of the empirical tail-weighted dependence measures.

## 2. The proposed tail-weighted dependence measures and their relationship with the extremal coefficient

We first provide an overview of the extremal coefficient and the F-madogram estimator of the extremal coefficient in Sections 2.1 and 2.2, respectively. The proposed tail-weighted dependence measures and their relationship with the extremal coefficient are given in Section 2.3.

### 2.1. *Overview of the extremal coefficient*

In data analysis, one common approach is to obtain bivariate dependence measures for every pair of variables in a multivariate data set. In this subsection, we define the extremal coefficient for the bivariate case and show how a family of estimators for the extremal coefficient leads to a family of dependence measures that apply more generally.

Let $F_{12}(y_1, y_2)$ be a bivariate continuous distribution with identical univariate margins $F_1, F_2$. Let $(Y_1, Y_2) \sim F_{12}$ and, for large $t$, let $\theta(t)$ be defined via

$$F_{12}(t,t) = \mathbb{P}(\max\{Y_1, Y_2\} \le t) = [F_1(t)]^{\theta(t)}.$$

If $\theta(t) \to \vartheta$ as $t \to \infty$, then $\vartheta$ is the (limiting) extremal coefficient. For independent $(Y_1, Y_2)$, $\vartheta = 2$; for perfectly dependent $(Y_1, Y_2)$, $\vartheta = 1$; and for positively dependent $(Y_1, Y_2)$ with $F_{12}(y_1, y_2) \ge F_1(y_1)F_2(y_2)$ for all $y_1, y_2$, $1 < \vartheta < 2$ if it exists. The quantity $\vartheta$ can be interpreted as the effective number of independent variables (see, e.g., Smith (1990)). If $F_{12}(t,t) = C(F_1(t), F_2(t))$ for a copula $C$, then $F_{12}(t,t) = C(u,u) = u^{\theta(t)}$ with

$u = F_1(t)$, and

$$\vartheta = \lim_{u \to 1^-} \frac{\log C(u,u)}{\log u}.$$

Let $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$ be independently and identically distributed (i.i.d.) bivariate random vectors from $F$, $\boldsymbol{Y}_i = (Y_{i1}, Y_{i2})^\intercal$. An extreme value copula $C_{EV}$ describes the dependence structure of the location-scale limit of the componentwise maxima $\boldsymbol{M}_n = (M_{1n}, M_{2n})^\intercal = (\bigvee_{i=1}^n Y_{i1}, \bigvee_{i=1}^n Y_{i2})^\intercal$ as $n \to \infty$, assuming $(M_{1n} - a_{1n})/b_{1n}$ and $(M_{2n} - a_{2n})/b_{2n}$ converge in distribution for some real sequences $a_{1n}, a_{2n}$ and positive sequences $b_{1n}, b_{2n}$. Extreme value copulas satisfy the max-stability property; in the bivariate case, the condition is $C_{EV}(u_1^t, u_2^t) = C_{EV}^t(u_1, u_2)$ for all $t > 0$, in which case the stable tail dependence function $A(-\log u_1, -\log u_2) = -\log C_{EV}(u_1, u_2)$ is convex and homogeneous of order 1, with $A(w, 0) = A(0, w) = w$ for $w > 0$. For $C_{EV}$,

$$\frac{\log C_{EV}(u,u)}{\log u} = \frac{-A(-\log u, -\log u)}{\log u} = A(1,1)$$

is constant over $u$ and hence $\vartheta = A(1,1)$.

Meanwhile, for a bivariate copula $C$ with well behaved tails, the lower and upper tail dependence coefficients are defined respectively as

$$\begin{aligned} \lambda_L &= \lambda_L(C) = \lim_{u \to 0^+} \frac{C(u,u)}{u}, \\ \lambda_U &= \lambda_U(C) = \lim_{u \to 0^+} \frac{\overline{C}(1-u, 1-u)}{u} = \lim_{u \to 0^+} \frac{\widehat{C}(u,u)}{u} = \lambda_L(\widehat{C}), \end{aligned} \quad (1)$$

where $\overline{C}(u_1, u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$ is the survival function of $C$, and $\widehat{C}(u_1, u_2) = \overline{C}(1 - u_1, 1 - u_2)$ is the reflected or survival copula of $C$. The tail dependence coefficients satisfy $0 \le \lambda_L, \lambda_U \le 1$, with larger values indicating stronger tail dependence. For a bivariate extreme value copula, $\lambda_U = 2 - A(1,1)$, and $\vartheta = A(1,1) = 2 - \lambda_U$.

### 2.2. *The F-madogram estimator of the extremal coefficient*

For extreme value copulas, there exist many empirical estimators of $\vartheta$ in the literature, see, e.g., Pickands (1981); Deheuvels (1991); Capéraà, Fougères, and Genest (1997); Hall and Tajvidi (2000); Cooley, Naveau, and Poncet (2006); Bücher, Dette, and Volgushev (2011)[1]. In the following, we focus on the class of F-madogram estimators (Cooley et al. (2006)) as it motivates our tail-weighted dependence measures. The name of *F-madogram* comes from (stationary) spatial extreme applications where the dependence depends on the distance between sites and the F-madogram quantifies the decrease in dependence as the distance increases.

Let $(Y_1, Y_2)$ have bivariate extreme value distribution $C_{EV}(F_1(y_1), F_2(y_2))$ where $F_1, F_2$ are univariate extreme value distributions. Then $(U_1, U_2) = (F_1(Y_1), F_2(Y_2)) \sim C_{EV}$. Let $M_\alpha = \max\{U_1^\alpha, U_2^\alpha\}$, so that

$$\mathbb{P}(M_\alpha \le x) = C_{EV}(x^{1/\alpha}, x^{1/\alpha}) = \exp\{-A(-\alpha^{-1}\log x, -\alpha^{-1}\log x)\} = \exp\{-\alpha^{-1}(-\log x)\vartheta\}$$

[1]We remark that many of these methods were initially designed to estimate the Pickands dependence function $B(w) = A(w, 1-w)$, $w \in [0,1]$. Because of the homogeneity property of $A$, we have $\vartheta = 2A(1/2, 1/2)$.

and

$$\mathbb{E}(M_\alpha) = \int_0^1 \mathbb{P}(M_\alpha > x)\mathrm{d}x = \int_0^\infty [1 - \exp\{-\alpha^{-1}\vartheta z\}]e^{-z}\mathrm{d}z = 1 - [1 + \alpha^{-1}\vartheta]^{-1}. \quad (2)$$

Since $|a-b| = 2\max\{a,b\} - a - b$ for real numbers $a, b$, we can write $\max\{U_1^\alpha, U_2^\alpha\} = \frac{1}{2}|U_1^\alpha - U_2^\alpha| + \frac{1}{2}(U_1^\alpha + U_2^\alpha)$, giving

$$\mathbb{E}(M_\alpha) = \frac{1}{2}\mathbb{E}[|U_1^\alpha - U_2^\alpha|] + (\alpha+1)^{-1}. \quad (3)$$

Combining (2) and (3), we obtain the relationship

$$\vartheta = \frac{\alpha + \alpha(1+\alpha)\nu_\alpha}{\alpha - (1+\alpha)\nu_\alpha}, \quad \nu_\alpha := \frac{1}{2}\mathbb{E}[|U_1^\alpha - U_2^\alpha|], \quad \alpha > 0. \quad (4)$$

Hence with data, we can define an estimator $\hat{\vartheta}_\alpha$ of $\vartheta$, based on an estimator of $\nu_\alpha$, for all $\alpha > 0$. Suppose the bivariate extreme value data are $(Y_{i1}, Y_{i2})$, $i = 1, \ldots, n$. Let

$$R_{ik} = n^{-1}\sum_{j=1}^n [1(Y_{jk} \le Y_{ik}) - 1/2], \quad i = 1, \ldots, n \quad (5)$$

be scaled ranks in the interval $[0,1]$ for the $k$th variable, $k = 1, 2$. The rank-based estimator of $\vartheta$, depending on $\alpha$, is

$$\hat{\vartheta}_\alpha = \frac{\alpha + \alpha(1+\alpha)\hat{\nu}_\alpha}{\alpha - (1+\alpha)\hat{\nu}_\alpha}, \quad \hat{\nu}_\alpha = \frac{1}{2n}\sum_{i=1}^n |R_{i1}^\alpha - R_{i2}^\alpha|, \quad (6)$$

Note that $\hat{\vartheta}_\alpha \in [1,2]$ and $2 - \hat{\vartheta}_\alpha \in [0,1]$.

The power of exponentiation is $\alpha = 1$ in the original formulation by Cooley et al. (2006). Naveau, Guillou, Cooley, and Diebolt (2009) use the idea of the F-madogram to estimate the Pickands dependence function in the form $B(w) = A(w, 1-w)$ for $w \in [0,1]$; their estimate of $\vartheta$ corresponds to using $\alpha = 1/2$. Fonseca, Pereira, Ferreira, and Martins (2015) consider the case where the powers of $R_{i1}$ and $R_{i2}$ in (6) can be any numbers $\alpha_1, \alpha_2 > 0$. These are all in the context of extreme value distributions.

### 2.3. *The proposed tail-weighted dependence measures*

Our proposed family of tail-weighted dependence measures is given in Definition 1.

*Definition* 1 For a general bivariate copula $C$, the probability version of the tail-weighted dependence measures is defined as $\zeta_\alpha = \zeta_\alpha(C) := 2 - \vartheta_\alpha$, where $\vartheta_\alpha$ with $\alpha > 0$ is as in (4). For a sample of bivariate observations $(Y_{i1}, Y_{i2})$, $i = 1 \ldots n$, with scaled ranks given in (5), the sample version is defined as $\hat{\zeta}_\alpha := 2 - \hat{\vartheta}_\alpha$ with $\hat{\vartheta}_\alpha$ given in (6).

Let $\gamma_\alpha = \gamma_\alpha(C) := \int_0^1 C(u^{1/\alpha}, u^{1/\alpha})\,\mathrm{d}u = \alpha\int_0^1 v^{\alpha-1}C(v,v)\,\mathrm{d}v$. Then we have the following relationships:

$$\nu_\alpha = \mathbb{E}(M_\alpha) - (1+\alpha)^{-1} = \int_0^1 [1 - C(u^{1/\alpha}, u^{1/\alpha})]\, \mathrm{d}u - (1+\alpha)^{-1} = \frac{\alpha}{(1+\alpha)} - \gamma_\alpha, \quad (7)$$

$$\vartheta_\alpha = \frac{\alpha + \alpha(1+\alpha)\nu_\alpha}{\alpha - (1+\alpha)\nu_\alpha} = \alpha(\gamma_\alpha^{-1} - 1),$$

$$\zeta_\alpha = 2 - \alpha(\gamma_\alpha^{-1} - 1). \quad (8)$$

For the comonotonicity copula with $C^+(u,v) = \min\{u,v\}$, we have $\vartheta_\alpha = 1$, $\nu_\alpha = 0$ and $\gamma_\alpha = \alpha/(\alpha+1)$; for the independence copula with $C^\perp(u,v) = uv$, we have $\vartheta_\alpha = 2$, $\nu_\alpha = \alpha/[(\alpha+1)(\alpha+2)]$ and $\gamma_\alpha = \alpha/(\alpha+2)$. For copulas with positive quadrant dependence satisfying $C(u,v) \geq uv$ for $0 \leq u, v \leq 1$, $\vartheta_\alpha \in [1,2]$. Therefore, the definition of $\zeta_\alpha$ is such that $\zeta_\alpha \in [0,1]$ for copulas with positive quadrant dependence, with the lower and upper limits reached at the independence and comonotonicity copulas, respectively. We will show in Section 3 that $\zeta_\alpha$ can be negative for $C$ with negative dependence, with a minimum bound of $(1 - \log 4)/(1 - \log 2)$.

It is easy to see that $\zeta_\alpha = \lambda_U$ for all $\alpha > 0$ when $C$ is a bivariate extreme value copula. When $C$ is the comonotonicity copula so that $R_{i1} = R_{i2}$ for all $i$, we have $\hat{\nu}_\alpha = 0$, $\hat{\vartheta}_\alpha = 1$ and $\hat{\zeta}_\alpha = 1$ for all $\alpha > 0$.

## 3. Properties of the tail-weighted dependence measure

In this section, we investigate the properties of $\zeta_\alpha$ in (8). In particular, its interpretation as a tail-weighted dependence measure by varying $\alpha$ and the desirable properties it satisfies are outlined in Section 3.1. The behaviour of $\zeta_\alpha$ as $\alpha$ approaches the two boundaries, 0 and $\infty$, is derived in Section 3.2, while Section 3.3 gives the asymptotic properties of the estimator $\hat{\zeta}_\alpha$ in Definition 1. We illustrate the role of $\zeta_\alpha$ as a tail-weighted dependence measure and its use in distinguishing between copula families with various strengths of tail dependence in Section 3.4.

### 3.1. *Dependence properties*

When $\alpha = 1$, $\gamma_\alpha$ is an integral along the diagonal of the copula at equal increment $\mathrm{d}u$. When $\alpha > 1$, $u^{1/\alpha} > u$ and more emphasis is on the distribution function at the joint upper tail, whereas the opposite is true when $0 < \alpha < 1$. The measure $\zeta_\alpha$ can thus be interpreted as a tail-weighted summary that puts different weights on the strength of dependence of a copula (in terms of the magnitude of $C(u_1, u_2)$ along the diagonal) at different locations.

Scarsini (1984) proposed a list of desirable criteria that a measure of concordance should satisfy; these are summarized in Definition 2.8 of Joe (2014). Most of these properties are satisfied by $\zeta_\alpha$, as illustrated below:

(1) **Domain** (measure defined for all random variables):
- Satisfied (for all continuous random pairs) as $\zeta_\alpha$ is defined for all bivariate pairs with copula $C$.

(2) **Symmetry (permutation)** (measure invariant to a swap of the order of random variables):
- Satisfied as $\gamma_\alpha = \int_0^1 C(u^{1/\alpha}, u^{1/\alpha})\, \mathrm{d}u$ is symmetric in the arguments. See Remark 1 for comments on reflection symmetry.

(3) **Coherence** (measure increasing in the concordance ordering of the copula):
- Satisfied as $C_1(u_1, u_2) \prec_c C_2(u_1, u_2)$ (i.e., $C_2$ is larger than $C_1$ in the concordance ordering or equivalently $C_2 \geq C_1$ pointwise) implies $\gamma_\alpha$ is larger for $C_2$ than $C_1$, and so is $\zeta_\alpha$.

(4) **Range** (measure within the interval $[-1, 1]$ with $-1$ at the countermonotonicity limit and 1 at the comonotonicity limit):
- The measure is constructed such that $\zeta_\alpha = 1$ at comonotonicity. We show in Remark 2 that $\zeta_\alpha$ is not necessarily $-1$ at countermonotonicity, and hence this property is not completely satisfied in general.

(5) **Independence** (measure equals 0 for the independence copula):
- Satisfied as $\zeta_\alpha = 0$ for the independence copula $C^\perp(u, v) = uv$.

(6) **Sign reversal** (negating one variable results in a sign reversal of the measure):
- As the range condition is not generally satisfied, it is impossible for $\zeta_\alpha$ for all $(U_1, U_2)$ to be the negation of that for $(-U_1, U_2)$.

(7) **Continuity** (if a sequence of bivariate random pairs converges in distribution to $C$, then the sequence of the measures for these random pairs converges to the measure of $C$):
- Satisfied as $\zeta_\alpha$ is defined based on the copula.

(8) **Invariance** (measure invariant to strictly increasing functions on each margin):
- Satisfied as monotonic marginal transformations do not affect the copula.

*Remark* 1 Since the reflected copula $\widehat{C}$ of a bivariate copula $C$ satisfies $\widehat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$, we have

$$\gamma_\alpha(\widehat{C}) = \int_0^1 [2u^{1/\alpha} - 1 + C(1 - u^{1/\alpha}, 1 - u^{1/\alpha})] \, \mathrm{d}u = \frac{\alpha - 1}{\alpha + 1} + \int_0^1 C(v^{1/\alpha}, v^{1/\alpha})(v^{-1/\alpha} - 1)^{\alpha - 1} \, \mathrm{d}v.$$

Observe that $\gamma_\alpha(\widehat{C})$ is equal to $\gamma_\alpha(C)$ for any $\alpha > 0$ if $C = \widehat{C}$ (i.e., if $C$ is reflection symmetric), or when $\alpha = 1$ for any $C$. Otherwise, it is not generally true that $\gamma_\alpha(\widehat{C}) = \gamma_\alpha(C)$.

*Remark* 2 For property 4 (range), because $\zeta_\alpha$ is a coherent dependence measure, the lower bound of its range can be obtained by considering the countermonotonicity copula, i.e., the Fréchet-Hoeffding lower bound of a bivariate copula. The countermonotonicity copula is given by $C^-(u_1, u_2) = \max\{0, u_1 + u_2 - 1\}$ and $\gamma_\alpha^- = \int_0^1 \max\{0, 2u^{1/\alpha} - 1\} \, \mathrm{d}u$ $= (2^{-\alpha} + \alpha - 1)/(1 + \alpha)$, where the minus sign at the superscript denotes the value for countermonotonicity copula. This implies $\zeta_\alpha^- = [2^{-\alpha}(\alpha + 2) - 2] \,/\, (2^{-\alpha} + \alpha - 1)$, an increasing function of $\alpha$. When $\alpha \to 0^+$, applying the L'Hôpital's rule yields $\zeta_\alpha^- \to (1 - \log 4)/(1 - \log 2) \approx -1.259$. When $\alpha \to \infty$, $\zeta_\alpha^- \to 0$, and $\zeta_\alpha^- = -1$ when $\alpha = 1$, i.e., the range requirement at countermonotonicity is only satisfied when $\alpha = 1$.

When $\alpha = 1$, $\zeta_\alpha$ is related to Spearman's footrule (Spearman (1904, 1906)). Its sample version is $\tilde{\varphi} = 1 - \sum_{i=1}^n 3|\tilde{R}_{i1} - \tilde{R}_{i2}|/(n - 1) = 1 - 6\tilde{\nu}_1/(n - 1)$, a function of $\tilde{\nu}_1 = \frac{1}{2n}\sum_{i=1}^n |\tilde{R}_{i1} - \tilde{R}_{i2}|$, where $\tilde{R}_{ik} = (n + 1)^{-1}\sum_{j=1}^n 1(Y_{jk} \leq Y_{ik})$ is a slightly different scaling of the marginal ranks to $[0, 1]$ that does not affect asymptotic properties. The probability version of $\tilde{\varphi}$ is $\varphi = 1 - 3\mathbb{E}|U_1 - U_2| = 1 - 6\nu_1$. The distributional properties of $\tilde{\varphi}$ have been previously studied in Genest, Nešlehová, and Ben Ghorbal (2010). Equations (7) and (8) imply $\zeta_1 = (1 - 6\nu_1)/(1 - 2\nu_1) = \varphi/(1 - 2\nu_1)$, i.e., both $\zeta_1$ and $\varphi$ are 0 at independence and 1 at comonotonicity, but $\zeta_1 \geq \varphi$ for all copulas with positive quadrant dependence as $1 - 2\nu_1 \leq 1$. For the countermonotonicity copula $C^-$, $\nu_1 = 1/4$, $\zeta_1 = -1$

and $\varphi = -\frac{1}{2}$.

### 3.2. *Boundary cases*

We investigate the properties of $\zeta_\alpha$ as $\alpha$ approaches the lower or upper limit, i.e., as $\alpha \to 0^+$ or $\alpha \to \infty$. We show that $\zeta_\alpha$ converges to the upper tail dependence coefficient $\lambda_U$ as $\alpha \to \infty$; this property is used in Section 4 for the estimation of $\lambda_U$.

Proposition 3.1 *As $\alpha \to 0^+$, we have*

$$\lim_{\alpha \to 0^+} \zeta_\alpha = 2 - \left[ \int_0^1 v^{-1} C(v,v)\, \mathrm{d}v \right]^{-1}. \tag{9}$$

*Proof.* When $\alpha \to 0^+$, the integrand of $\gamma_\alpha$, $C(u^{1/\alpha}, u^{1/\alpha})$, tends to zero everywhere for $u \in [0,1)$ and is only 1 at $u = 1$. The integrand is also bounded in $[0,1]$, and thus the exchange of limit and integral is valid. This yields $\lim_{\alpha \to 0^+} \gamma_\alpha = 0$ and, using (8),

$$\lim_{\alpha \to 0^+} \zeta_\alpha = 2 - \lim_{\alpha \to 0^+} \frac{\alpha}{\gamma_\alpha} = 2 - \lim_{\alpha \to 0^+} \left[ \int_0^1 v^{\alpha-1} C(v,v)\, \mathrm{d}v \right]^{-1} = 2 - \left[ \int_0^1 v^{-1} C(v,v)\, \mathrm{d}v \right]^{-1}.$$

The limit exists as $C(v,v) \leq v$ for any copula and $v^{-1}C(v,v) \leq 1$. ■

Note that the integral in (9) can be interpreted as the average ratio between $C(u,v)$ and $C^+(u,v)$ (the comonotonicity copula) along the diagonal $u = v$. As the strength of dependence of $C$ increases, this ratio gets closer to 1 and the integral also gets closer to 1. As $C$ approaches the independence copula, this ratio approaches $v$ and the integral tends towards 1/2.

Proposition 3.2 *Assume that the tail of the bivariate copula $C$ is well-behaved in the sense that $\lambda_U(C)$ in (1) exists. Then, as $\alpha \to \infty$, we have $\lim_{\alpha \to \infty} \zeta_\alpha = \lambda_U$.*

*Proof.* When $\alpha \to \infty$, $C(u^{1/\alpha}, u^{1/\alpha})$ tends to 1 everywhere for $u \in (0,1]$ and is undefined at $u = 0$; $\lim_{\alpha \to \infty} \gamma_\alpha = 1$ and thus $\lim_{\alpha \to \infty} \zeta_\alpha = 2 + \lim_{\alpha \to \infty} \alpha(\gamma_\alpha - 1)$. We have

$$\alpha(\gamma_\alpha - 1) = \int_0^1 \alpha \left[ 2u^{1/\alpha} - 2 + \overline{C}(u^{1/\alpha}, u^{1/\alpha}) \right] \mathrm{d}u = \int_0^1 \alpha \overline{C}(u^{1/\alpha}, u^{1/\alpha})\, \mathrm{d}u - \frac{2\alpha}{\alpha+1}. \tag{10}$$

To find the limit of the integral in (10), first note that the integral is bounded as $\overline{C}(u^{1/\alpha}, u^{1/\alpha}) = 1 - 2u^{1/\alpha} + C(u^{1/\alpha}, u^{1/\alpha}) \leq 1 - u^{1/\alpha}$, meaning that $\int_0^1 \alpha \overline{C}(u^{1/\alpha}, u^{1/\alpha})\, \mathrm{d}u \leq \int_0^1 \alpha(1 - u^{1/\alpha})\, \mathrm{d}u = \alpha/(\alpha+1) \leq 1$ for any positive $\alpha$, and tends to 1 as $\alpha \to \infty$. Then, consider the tail expansion of the survival function of $C$ using the definition of the upper tail dependence coefficient $\lambda_U$, i.e., $\overline{C}(1-v, 1-v) \sim v\lambda_U$ as $v \to 0^+$, where $a(v) \sim b(v)$ as $v \to m$ means $\lim_{v \to m} a(v)/b(v) = 1$. Initially, suppose $0 < \lambda_U < 1$. Then for every small $\epsilon > 0$, there exists $\delta > 0$ such that for every $0 \leq v < \delta$ we have

$$v(\lambda_U - \epsilon) \leq \overline{C}(1-v, 1-v) \leq v(\lambda_U + \epsilon). \tag{11}$$

Also, there exists some $\alpha^*$ such that for every $\alpha > \alpha^*$, $1 - u^{1/\alpha} < \delta$ for every $u > \epsilon$. Write

$$\int_0^1 \alpha\overline{C}(u^{1/\alpha}, u^{1/\alpha})\,\mathrm{d}u = \int_0^{\epsilon} \alpha\overline{C}(u^{1/\alpha}, u^{1/\alpha})\,\mathrm{d}u + \int_{\epsilon}^1 \alpha\overline{C}(u^{1/\alpha}, u^{1/\alpha})\,\mathrm{d}u =: h_1(\epsilon,\alpha) + h_2(\epsilon,\alpha).$$

For $h_1(\epsilon,\alpha)$, $0 \le h_1(\epsilon,\alpha) \le \int_0^{\epsilon} \alpha(1-u^{1/\alpha})\,\mathrm{d}u = \alpha\epsilon\left[1 - \alpha\epsilon^{1/\alpha}/(\alpha+1)\right]$. Because this upper bound tends to $\epsilon\,(1-\log\epsilon)$ as $\alpha \to \infty$, there exists some $\alpha^{**}$ and constant $M > 1$ such that $0 \le h_1(\epsilon,\alpha) \le M\epsilon\,(1-\log\epsilon)$ for all $\alpha > \alpha^{**}$.

For $h_2(\epsilon,\alpha)$, since $1 - u^{1/\alpha} < \delta$ for all $u > \epsilon$ and $\alpha > \alpha^*$, we use (11) to establish the bounds

$$(\lambda_U - \epsilon)\int_{\epsilon}^1 \alpha(1-u^{1/\alpha})\,\mathrm{d}u \le h_2(\epsilon,\alpha) \le (\lambda_U + \epsilon)\int_{\epsilon}^1 \alpha(1-u^{1/\alpha})\,\mathrm{d}u$$

$$\Longrightarrow\ (\lambda_U - \epsilon)\left[\frac{\alpha}{\alpha+1} - \alpha\epsilon\left(1 - \frac{\alpha}{\alpha+1}\epsilon^{1/\alpha}\right)\right] \le h_2(\epsilon,\alpha) \le (\lambda_U + \epsilon),$$

where the upper limit uses the relationship $\int_{\epsilon}^1 \alpha(1-u^{1/\alpha})\,\mathrm{d}u \le \int_0^1 \alpha(1-u^{1/\alpha})\,\mathrm{d}u \le 1$. Thus $(\lambda_U - \epsilon)[\alpha/(\alpha+1) - M\epsilon\,(1-\log\epsilon)] \le h_2(\epsilon,\alpha) \le (\lambda_U + \epsilon)$ for all $\alpha > \max\{\alpha^*, \alpha^{**}\}$, and, as $\alpha \to \infty$,

$$(\lambda_U - \epsilon)\,[1 - M\epsilon\,(1-\log\epsilon)] \le h_1(\epsilon,\infty) + h_2(\epsilon,\infty) \le M\epsilon\,(1-\log\epsilon) + (\lambda_U + \epsilon),$$

where $h_j(\epsilon,\infty) = \lim_{\alpha\to\infty} h_j(\epsilon,\alpha)$, $j = 1,2$. Since $\epsilon > 0$ can be arbitrarily small,

$$\lim_{\alpha\to\infty} \int_0^1 \alpha\overline{C}(u^{1/\alpha}, u^{1/\alpha})\,\mathrm{d}u = \lambda_U.$$

The proof applies to $\lambda_U = 1$ or $0$ by taking (11) as $v(\lambda_U - \epsilon) \le \overline{C}(1-v, 1-v) \le v$ or $0 \le \overline{C}(1-v,1-v) \le v(\lambda_U + \epsilon)$, respectively. Putting this result back into (10), we obtain $\lim_{\alpha\to\infty} \zeta_\alpha = \lambda_U$. ■

Proposition 3.2 reinforces the interpretation of $\zeta_\alpha$ that more weight is put on the upper tail as $\alpha$ increases, eventually coinciding with the upper tail dependence coefficient when $\alpha \to \infty$.

### 3.3. *Asymptotic distribution of the sample tail-weighted dependence measure*

For given $\alpha$, the estimator $\hat{\zeta}_\alpha$ defined in Definition 1 is asymptotically normally distributed. This property makes use of the theory for the empirical copula (Fermanian, Radulović, and Wegkamp (2004); Tsukahara (2005); Segers (2012)), defined as

$$C_n(u_1,u_2) = \frac{1}{n}\sum_{i=1}^{n} 1(R_{i1} \le u_1, R_{i2} \le u_2), \tag{12}$$

where the $R$'s are ranks scaled to the interval $[0, 1]$ as in (5). Following Segers (2012), define the first order partial derivatives of $C$ as

$$\begin{aligned}\dot{C}_1(u_1, u_2) &= \lim_{h \to 0} \frac{C(u_1 + h, u_2) - C(u_1, u_2)}{h}, \quad (u_1, u_2) \in V_1; \\ \dot{C}_2(u_1, u_2) &= \lim_{h \to 0} \frac{C(u_1, u_2 + h) - C(u_1, u_2)}{h}, \quad (u_1, u_2) \in V_2,\end{aligned}$$

where $V_1 = (0, 1) \times [0, 1]$ and $V_2 = [0, 1] \times (0, 1)$. At the boundary points, $\dot{C}_1(u_1, u_2)$ and $\dot{C}_2(u_1, u_2)$ are defined as the one-sided limits by convention. For the result to hold, we need the following assumption.

Assumption 1 *For $j = 1, 2$, the partial derivative $\dot{C}_j$ exists on $[0, 1]^2$ and is continuous on the set $V_j$.*

Assumption 1 is used rather than the more restrictive condition of continuous partial derivatives on $[0, 1]^2$ in Fermanian et al. (2004), as the former is satisfied by a much wider class of parametric copula families. In particular, Segers (2012) demonstrates that the $\dot{C}_1$ and $\dot{C}_2$ of a bivariate copula with lower (resp. upper) tail dependence cannot be continuous at the point $(0, 0)$ (resp. $(1, 1)$).

Proposition 3.3 *If the bivariate copula $C$ satisfies Assumption 1, then we have that*

$$\sqrt{n}\left(\hat{\zeta}_\alpha - \zeta_\alpha\right) \xrightarrow{d} N\left(0, \frac{(\alpha + 2 - \zeta_\alpha)^4}{\alpha^2} \mathrm{Var}(X)\right), \tag{13}$$

*where*

$$X = \frac{1}{2}\int_0^1 \mathbb{G}_C\left(u^{1/\alpha}, 1\right) \mathrm{d}u + \frac{1}{2}\int_0^1 \mathbb{G}_C\left(1, u^{1/\alpha}\right) \mathrm{d}u - \int_0^1 \mathbb{G}_C\left(u^{1/\alpha}, u^{1/\alpha}\right) \mathrm{d}u, \tag{14}$$

*in which*

$$\mathbb{G}_C(u_1, u_2) = \mathbb{B}_C(u_1, u_2) - \mathbb{B}_C(u_1, 1)\dot{C}_1(u_1, u_2) - \mathbb{B}_C(1, u_2)\dot{C}_2(u_1, u_2) \tag{15}$$

*is a Gaussian process that involves a Brownian bridge $\mathbb{B}_C$ with covariance function*

$$\mathbb{E}\left[\mathbb{B}_C(u_1, u_2)\mathbb{B}_C(u_3, u_4)\right] = C(u_1 \wedge u_3, u_2 \wedge u_4) - C(u_1, u_2)C(u_3, u_4). \tag{16}$$

*Proof.* Note that $\hat{\nu}_\alpha = (2n)^{-1} \sum_{i=1}^n |R_{i1}^\alpha - R_{i2}^\alpha|$ can be written in terms of $C_n$, the empirical copula defined in (12), as follows:

$$\begin{aligned}\hat{\nu}_\alpha &= \frac{1}{n}\sum_{i=1}^n \left[\max\{R_{i1}^\alpha, R_{i2}^\alpha\} - \frac{1}{2}R_{i1}^\alpha - \frac{1}{2}R_{i2}^\alpha\right] \\ &= \frac{1}{n}\sum_{i=1}^n \int_0^1 \left(1 - 1\{R_{i1} \le u^{1/\alpha}, R_{i2} \le u^{1/\alpha}\}\right) \mathrm{d}u \\ &\quad -\frac{1}{2n}\sum_{j=0}^1 \sum_{i=1}^n \int_0^1 \left(1 - 1\{R_{i1} \le u^{j/\alpha}, R_{i2} \le u^{(1-j)/\alpha}\}\right) \mathrm{d}u\end{aligned}$$

$$= \frac{1}{2}\int_0^1 \left[C_n(u^{1/\alpha},1) + C_n(1,u^{1/\alpha}) - 2C_n(u^{1/\alpha},u^{1/\alpha})\right] \mathrm{d}u.$$

Under the regularity conditions in Assumption 1 (Segers (2012)), we have $\sqrt{n}[C_n(u_1,u_2) - C(u_1,u_2)] \xrightarrow{d} \mathbb{G}_C(u_1,u_2)$, where $\mathbb{G}_C$ is a Gaussian process satisfying (15) and (16). This establishes the asymptotic distribution of $\hat{\nu}_\alpha$, so that $\sqrt{n}(\hat{\nu}_\alpha - \nu_\alpha) \xrightarrow{d} X$, where $X$ is given in (14). The random variable $X$ is normally distributed, as a consequence of Lemma 3.9.8 of van der Vaart and Wellner (1996) which states that a continuous, linear map of a tight Gaussian process (in this case $\mathbb{G}_C$) is Gaussian. As for $\hat{\zeta}_\alpha$, the rank-based estimator of the tail-weighted dependence measure, observe via (7) and (8) that

$$\sqrt{n}\left(\hat{\zeta}_\alpha - \zeta_\alpha\right) = -\alpha\left(\frac{\sqrt{n}\left(\hat{\nu}_\alpha - \nu_\alpha\right)}{[\alpha/(\alpha+1) - \hat{\nu}_\alpha][\alpha/(\alpha+1) - \nu_\alpha]}\right) = r_\alpha\left[\sqrt{n}\left(\hat{\nu}_\alpha - \nu_\alpha\right)\right] + o_p(1),$$

where $r_\alpha = -\alpha[\alpha/(\alpha+1) - \nu_\alpha]^{-2} = -\alpha\gamma_\alpha^{-2}$. As a result,

$$\sqrt{n}\left(\hat{\zeta}_\alpha - \zeta_\alpha\right) \xrightarrow{d} N\left(0, \frac{(\alpha+2-\zeta_\alpha)^4}{\alpha^2}\mathrm{Var}(X)\right).$$

■

The asymptotic variance in (13) is usually a 2-dimensional integral that can be evaluated numerically (see Appendix A). Some examples of square roots of the asymptotic variances are included in Table 1 in the next subsection. For the independence copula, a simple closed-form asymptotic variance can be obtained, as follows.

Proposition 3.4 *If $C$ is the independence copula with $C(u_1,u_2) = C^{\perp}(u_1,u_2) = u_1u_2$, then we have that*

$$\sqrt{n}\left(\hat{\zeta}_\alpha - \zeta_\alpha\right) \xrightarrow{d} N\left(0, \frac{(2+\alpha)^2}{(1+\alpha)(3+2\alpha)}\right). \tag{17}$$

*Proof.* For the independence copula, we have

$$\mathbb{E}\left[\mathbb{B}_C(u_1,u_2)\mathbb{B}_C(u_3,u_4)\right] = (u_1 \wedge u_3)\,(u_2 \wedge u_4) - u_1u_2u_3u_4$$

from (16) and

$$\mathbb{E}\left[\mathbb{G}_C(u_1,u_2)\mathbb{G}_C(u_3,u_4)\right] = (u_1 \wedge u_3 - u_1u_3)(u_2 \wedge u_4 - u_2u_4)$$

from (A2). It can be easily checked that all but the third term of (A1) are zero, and thus

$$\mathrm{Var}(X) = \int_0^1\int_0^1 \mathbb{E}\left[\mathbb{G}_C\left(u^{1/\alpha},u^{1/\alpha}\right)\mathbb{G}_C\left(v^{1/\alpha},v^{1/\alpha}\right)\right]\mathrm{d}u\mathrm{d}v = \frac{\alpha^2}{(2+\alpha)^2(3+5\alpha+2\alpha^2)},$$

so that $\sqrt{n}\left(\hat{\zeta}_\alpha - \zeta_\alpha\right) \xrightarrow{d} N\left(0, \sigma^2\right)$, where

$$\sigma^2 = \frac{(2+\alpha)^4}{\alpha^2} \cdot \frac{\alpha^2}{(2+\alpha)^2(3+5\alpha+2\alpha^2)} = \frac{(2+\alpha)^2}{(1+\alpha)(3+2\alpha)}.$$

■

Note that the asymptotic variance in (17) is a decreasing function in $\alpha$, from 4/3 when $\alpha \to 0^+$ to 1/2 as $\alpha \to \infty$.

### 3.4. *Uses of the tail-weighted dependence measures*

In this subsection, we demonstrate the idea of using the proposed tail-weighted dependence measures $\zeta_\alpha$ to quantify overall and tail dependence of a copula, as well as their use in distinguishing between copulas with various strengths of dependence in the joint tail.

#### *3.4.1. As measures of overall and tail dependence of copulas*

When $\alpha = 1$, $\zeta_\alpha$ can be interpreted as a central dependence measure much like Kendall's $\tau$ and Spearman's $\rho$. When $\alpha > 1$, $\zeta_\alpha$ puts more weight on the upper tail of the copula. We mainly focus on $\alpha \geq 1$ due to the desirable property that $\zeta_\alpha \to \lambda_U$ as $\alpha \to \infty$; emphasis can be put on the lower tail by using $\alpha > 1$ with the reflected copula.

We mentioned in Section 2 that the strength of tail dependence can be measured by the tail dependence coefficients $\lambda_L$ and $\lambda_U$. When there is no tail dependence, it is still possible to quantify the degree of tail heaviness using the notion of tail order (Hua and Joe (2011)) based on an expansion of the corner tail probabilities. The upper and lower tail orders, $\kappa_L$ and $\kappa_U$, are quantities such that

$$C(u,u) = u^{\kappa_L}\ell(u) + o(u^{\kappa_L}\ell(u)), \quad \overline{C}(1-u,1-u) = u^{\kappa_U}\ell^*(u) + o(u^{\kappa_U}\ell^*(u)) \qquad (18)$$

as $u \to 0^+$, for some slowly varying functions $\ell$ and $\ell^*$ at $0^+$. The tail order is the reciprocal of the coefficient of tail dependence in Ledford and Tawn (1996) and Heffernan (2000), and cannot be smaller than 1; $\lambda_L$ ($\lambda_U$) can only be non-zero if $\kappa_L$ ($\kappa_U$) is 1. Copulas with $\kappa = \kappa_L$ or $\kappa_U$ between 1 and 2 are said to have intermediate tail dependence for the respective tail, and those with $\kappa = 2$ are said to have tail quadrant independence. It is possible for $\kappa > 2$ for some copulas with negative quadrant dependence. Similar to the tail dependence coefficient, the tail order is defined as a limit and has no direct empirical counterpart.

In Table 1, we compute the values of $\zeta_\alpha$ for selected values of $\alpha$ and those of other measures of tail dependence, for several bivariate parametric copulas families with parameters such that Kendall's $\tau = 0.3$ or $\tau = 0.7$. These families cover a range of possibilities in the two-term expansion of $\overline{C}(1-u, 1-u)$ as $u \to 0^+$, i.e., these families cover various tail symmetry/asymmetry and dependence characteristics for copulas with positive quadrant dependence. The families are:

- Gaussian: Reflection symmetric with intermediate tail dependence when $0 < \rho < 1$; $\lambda_U = 0$ and $1 < \kappa_U < 2$.
- Frank: Reflection symmetric with tail quadrant independence; $\lambda_U = 0$ and $\kappa_U = 2$.

- Gumbel: Reflection asymmetric with upper tail dependence and intermediate lower tail dependence; $0 < \lambda_U < 1$ and $\kappa_U = 1$; $\lambda_L = 0$ and $1 < \kappa_L < 2$. It is an extreme value copula.
- Student's t: Reflection symmetric with tail dependence; $0 < \lambda_U < 1$ and $\kappa_U = 1$. We consider the t copula with 1 and 5 degrees of freedom; for a given correlation parameter, the tail dependence is stronger with smaller degrees of freedom.
- BB1: Reflection asymmetric with potentially different strengths of dependence in the two tails; $0 < \lambda_U, \lambda_L < 1$; $\kappa_U = \kappa_L = 1$.

Table 1 also has the asymptotic standard error of the rank-based estimator $\hat{\zeta}_\alpha$ for the sample size 500, computed using the results in Section 3.3 and Appendix A. For comparison, we include the corresponding upper semicorrelations of the normal scores for each copula (see Section 2.17 of Joe (2014)), defined as

$$\rho_N^+ = \text{Cor}\left[\Phi^{-1}(U_1), \Phi^{-1}(U_2) | U_1 > 0.5, U_2 > 0.5\right] \tag{19}$$

where $\Phi$ is the standard normal cdf and $(U_1, U_2)$ is a random vector from copula $C$. The normal scores for the $i$th margin are defined as $\Phi^{-1}(U_i)$, $i = 1, 2$. The upper tail-weighted dependence measure of Krupskii and Joe (2015) is also computed; a description of this measure is in Section 3.4.2. These two measures have been developed to quantify the strength of dependence in the joint tails.

In addition to copulas with positive quadrant dependence, we also consider the behaviour of $\zeta_\alpha$ for copula families that admit negative quadrant dependence. Among those listed in Table 1, the Gaussian, Frank and Student's t families allow copula parameters that correspond to a Kendall's $\tau$ of $-0.3$ and $-0.7$. Table 2 lists the analogous results to Table 1 for copulas with negative quadrant dependence. We again observe the convergence of $\zeta_\alpha$ to $\lambda_U$ as $\alpha$ increases, and the purpose of $\zeta_1$ as a measure of central dependence. Student's t copulas have $\lambda_U$ values that are above zero even though there is overall negative dependence; in this case $\zeta_\alpha$ is negative when $\alpha$ is small, and turns positive when $\alpha$ is sufficiently large. This is apparent for the $t_1$ copula but not the $t_5$ copula as the latter has a $\lambda_U$ value that is very close to (but not exactly) zero.

From these results, we observe that $\zeta_1$ is not very different among copulas with the same Kendall $\tau$ value, and can be regarded as a measure of overall dependence strength. As $\alpha$ increases, $\zeta_\alpha$ tends towards the upper tail dependence coefficient $\lambda_U$, although $\zeta_\alpha$ need not be monotone in $\alpha$ as is evident for the $t_1$ copula. By construction, $\zeta_\alpha = \lambda_U$ for all $\alpha$ for an extreme value copula, such as Gumbel. Unlike for the independence copula, the asymptotic standard error generally increases with $\alpha$ for the copula models with positive quadrant dependence; further inspection (not shown) seems to suggest the minimum occurs near $\alpha = 1$. Different copulas exhibit various rates of convergence to $\lambda_U$ as $\alpha$ increases, with slower rates for the Gaussian, reflected Gumbel and $t_\nu$ (large $\nu$) copulas. A more formal investigation of convergence rates will be given in Section 4.

#### *3.4.2. As a tool for distinguishing between copulas with various strengths of dependence in the joint tail*

Because the proposed measures $\zeta_\alpha$ quantify the degree of tail dependence, they can be used to distinguish between copulas with various tail dependence characteristics and assist in the choice of copula families in data modelling.

The semicorrelation (19) and conditional Spearman's $\rho$ (Schmid and Schmidt (2007)) are special cases of a more general class of tail-weighted dependence measures studied in Krupskii and Joe (2015) that are based on conditional correlations; for $(U_1, U_2) \sim C$,

| Kendall's $\tau = 0.3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Measure | Copula | | | | | | | |
| | Gaussian | Frank | Gumbel | rGumbel | $t_1$ | $t_5$ | BB1 | rBB1 |
| $\zeta_1$ | .37 (.03) | .37 (.03) | .38 (.03) | .38 (.03) | .44 (.03) | .38 (.03) | .37 (.03) | .37 (.03) |
| $\zeta_5$ | .29 (.04) | .28 (.04) | .38 (.04) | .24 (.04) | .41 (.04) | .32 (.04) | .33 (.04) | .29 (.04) |
| $\zeta_{20}$ | .20 (.05) | .14 (.04) | .38 (.06) | .14 (.05) | .45 (.06) | .27 (.06) | .31 (.06) | .23 (.06) |
| $\zeta_{100}$ | .12 (.09) | .04 (.05) | .38 (.12) | .06 (.07) | .47 (.12) | .24 (.12) | .30 (.12) | .17 (.10) |
| $\lambda_U$ | .00 | .00 | .38 | .00 | .48 | .18 | .30 | .06 |
| $\kappa_U$ | 1.38 | 2.00 | 1.00 | 1.62 | 1.00 | 1.00 | 1.00 | 1.00 |
| $\rho_N^+$ | .23 | .15 | .46 | .16 | .70 | .37 | .39 | .28 |
| $\varrho_U$ | .22 | .11 | .48 | .14 | .75 | .37 | .40 | .28 |

| Kendall's $\tau = 0.7$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Measure | Copula | | | | | | | |
| | Gaussian | Frank | Gumbel | rGumbel | $t_1$ | $t_5$ | BB1 | rBB1 |
| $\zeta_1$ | .76 (.01) | .77 (.01) | .77 (.01) | .77 (.01) | .79 (.02) | .77 (.01) | .77 (.01) | .77 (.01) |
| $\zeta_5$ | .70 (.02) | .67 (.02) | .77 (.02) | .65 (.02) | .75 (.02) | .72 (.02) | .73 (.02) | .71 (.02) |
| $\zeta_{20}$ | .63 (.04) | .43 (.05) | .77 (.03) | .52 (.05) | .76 (.04) | .67 (.04) | .71 (.04) | .66 (.04) |
| $\zeta_{100}$ | .54 (.10) | .16 (.08) | .77 (.06) | .38 (.11) | .76 (.08) | .63 (.09) | .70 (.08) | .63 (.09) |
| $\lambda_U$ | .00 | .00 | .77 | .00 | .77 | .58 | .70 | .61 |
| $\kappa_U$ | 1.06 | 2.00 | 1.00 | 1.23 | 1.00 | 1.00 | 1.00 | 1.00 |
| $\rho_N^+$ | .75 | .60 | .85 | .64 | .88 | .78 | .80 | .77 |
| $\varrho_U$ | .76 | .51 | .89 | .62 | .90 | .79 | .83 | .79 |

Table 1. Values of the dependence measure $\zeta_\alpha$ for $\alpha = 1, 5, 20, 100$, upper tail dependence coefficient and tail order $\lambda_U$ and $\kappa_U$, upper semicorrelation of the normal scores $\rho_N^+$, and upper tail-weighted dependence measure $\varrho_U$ with weighting function $a(u) = u^6$ (Krupskii and Joe (2015)) for various bivariate parametric copula families with Kendall's $\tau$ equal to 0.3 (above) and 0.7 (below). An "r" in front of the name of the copula family indicates reflection of the copula. For the BB1 copula and its reflection, the parameters are chosen so that the copula has the same upper tail dependence coefficient as Kendall's $\tau$. The numbers in brackets are the asymptotic standard errors of the associated rank-based estimator $\hat{\zeta}_\alpha$ for a sample of size 500.

the lower and upper measures $\varrho_L$ and $\varrho_U$ are defined as

$$\varrho_L = \varrho_L(C) = \mathrm{Cor}\left[a(1 - U_1/p), a(1 - U_2/p) | U_1 < p, U_2 < p\right]; \tag{20}$$

$$\varrho_U = \varrho_U(C) = \mathrm{Cor}\left[a(1 - (1 - U_1)/p), a(1 - (1 - U_2)/p) | U_1 > 1 - p, U_2 > 1 - p\right], \tag{21}$$

respectively, where $a : [0, 1] \to [0, \infty)$ is a continuous increasing weight function with $a(0) = 0$, and $0 < p \leq 0.5$ is the truncation level. The authors considered the class of weight functions $a(u) = u^k$ with $k \geq 1$, for its ease in numerical computation relative to the Gaussian inverse cdf and its property of being a tail-weighted measure.

For the objective of distinguishing between copulas with various strengths of dependence in the joint tail, the authors compared the magnitude of $\varrho_L(C_1; k) - \varrho_L(C_2; k)$ for various copulas $C_1$ with tail dependence and $C_2$ without, against the standard error of the empirical counterpart of this difference (obtained by replacing the $U$'s in (20) and (21) by the scaled ranks $R$'s, and the correlation by sample correlation), for several values of $k$. Based on empirical studies, they found that a value of $k = 6$ generally yields the largest values of $\varrho_L(C_1; k) - \varrho_L(C_2; k)$ relative to the standard errors. We carried out the same sets of simulations for $\zeta_\alpha$, and observe that comparable performance can be obtained when $\alpha$ is between 15 and 20. When $\alpha < 15$, $\zeta_\alpha$ is rather insensitive to the tail behaviour of different copulas relative to the standard error of the difference; when $\alpha > 20$, the standard error increases at a faster rate than the difference $\zeta_\alpha(\widehat{C}_1) - \zeta_\alpha(\widehat{C}_2)$. We also note that both the magnitude of the difference and the standard errors are smaller than those using the tail-weighted dependence measure $\varrho_L$.

| Kendall's $\tau = -0.3$ | | | | |
|---|---|---|---|---|
| Measure | Copula | | | |
| | Gaussian | Frank | $t_1$ | $t_5$ |
| $\zeta_1$ | $-.40$ (.04) | $-.42$ (.04) | $-.26$ (.05) | $-.37$ (.05) |
| $\zeta_5$ | $-.23$ (.02) | $-.24$ (.03) | $-.10$ (.04) | $-.21$ (.03) |
| $\zeta_{20}$ | $-.09$ (.01) | $-.08$ (.02) | .07 (.06) | $-.06$ (.02) |
| $\zeta_{100}$ | $-.02$ ($<$ .01) | $-.02$ (.01) | .13 (.11) | $-.004$ (.04) |
| $\lambda_U$ | .00 | .00 | .15 | .007 |
| $\kappa_U$ | 3.66 | 2.00 | 1.00 | 1.00 |
| $\rho_N^+$ | $-.13$ | $-.07$ | .57 | .06 |
| $\varrho_U$ | $-.09$ | $-.04$ | .63 | .10 |

| Kendall's $\tau = -0.7$ | | | | |
|---|---|---|---|---|
| Measure | Copula | | | |
| | Gaussian | Frank | $t_1$ | $t_5$ |
| $\zeta_1$ | $-.87$ (.02) | $-.90$ (.01) | $-.77$ (.04) | $-.85$ (.02) |
| $\zeta_5$ | $-.41$ (.01) | $-.42$ (.01) | $-.36$ (.02) | $-.40$ (.01) |
| $\zeta_{20}$ | $-.11$ ($<$ .01) | $-.11$ ($<$ .01) | $-.07$ (.03) | $-.10$ ($<$ .01) |
| $\zeta_{100}$ | $-.02$ ($<$ .01) | $-.02$ ($<$ .01) | .01 (.05) | $-.02$ ($<$ .01) |
| $\lambda_U$ | .00 | .00 | .03 | $O(10^{-5})$ |
| $\kappa_U$ | 18.35 | 2.00 | 1.00 | 1.00 |
| $\rho_N^+$ | $-.20$ | $-.08$ | .59 | $O(10^{-4})$ |
| $\varrho_U$ | $-.04$ | $-.01$ | .63 | .08 |

Table 2. Values of the dependence measure $\zeta_\alpha$ for $\alpha = 1, 5, 20, 100$, upper tail dependence coefficient and tail order $\lambda_U$ and $\kappa_U$, upper semicorrelation of the normal scores $\rho_N^+$, and upper tail-weighted dependence measure $\varrho_U$ with weighting function $a(u) = u^6$ (Krupskii and Joe (2015)) for various bivariate parametric copula families with Kendall's $\tau$ equal to $-0.3$ and $-0.7$. The numbers in brackets are the asymptotic standard errors of the associated rank-based estimator $\hat{\zeta}_\alpha$ for a sample of size 500.

## 4. Tail expansion of $\zeta_\alpha$ and estimation of the tail dependence coefficient

The property that $\zeta_\alpha \to \lambda_U$ as $\alpha \to \infty$ makes it relevant to consider the estimation of $\lambda_U$ based on values of $\hat{\zeta}_\alpha$ for several different $\alpha$. Based on the tail expansion of $\zeta_\alpha$ for large $\alpha$, we propose a method to estimate $\lambda_U$ in this section. Unless specified, we focus on the upper tail behaviour in the following because $\lambda_L$ is the same as the upper tail dependence coefficient of the reflected copula.

### 4.1. *Tail expansion of $\overline{C}(1-u, 1-u)$ and $\zeta_\alpha$*

In this subsection, we derive an asymptotic expansion of $\zeta_\alpha - \lambda_U$. This expression will be helpful in devising a method to estimate $\lambda_U$ using our proposed tail-weighted dependence measures.

Proposition 4.1 *Suppose the bivariate copula $C$ is twice continuously differentiable and the upper tail is well-behaved in that $\overline{C}(1-u, 1-u)$ has a tail expansion to the second order that is valid upon differentiation. Then it holds that*

$$\zeta_\alpha - \lambda_U = \frac{(\alpha+1)^{-1}(c_{11} - c_{11}^2) + \alpha^{-(\eta-1)}\xi\Gamma(\eta)\ell^*(\alpha^{-1}) + o(\alpha^{-[(\eta-1)\wedge 1]})}{1 + o(1)} \tag{22}$$

*as $\alpha \to \infty$, where $c_{11} = C_{1|2}(1|1) + C_{2|1}(1|1) = 2 - \lambda_U$ with $C_{i|j} = \partial C(u_i, u_j)/\partial u_j$ being the conditional cdf of $C$, $(i, j) = (1, 2)$ or $(2, 1)$; $\eta$ and $\xi$ are constants with $\eta$ being the upper tail order when $C$ has no upper tail dependence; and $\ell^*$ is a slowly varying function* *at* $0^+$.

*Proof.* For $C$ with well-behaved upper tail, the survival function admits the following expansion to the second order:

$$\overline{C}(1-u,1-u) = 2u - 1 + C(1-u,1-u) = \lambda_U u + \xi^* u^\eta \ell(u) + o(u^\eta \ell(u)), \quad u \to 0^+,$$

where $\eta > 1$, $\xi^*$ is a constant and $\ell(u)$ is slowly varying at 0. When $\lambda_U = 0$, this expansion matches that used for the upper tail order in (18).

With well-behaved upper tails, by differentiating the above with respect to $u$, one gets

$$2 - C_{1|2}(1-u|1-u) - C_{2|1}(1-u|1-u) = \lambda_U + \xi u^{\eta-1}\ell(u) + o(u^{\eta-1}\ell(u)), \quad u \to 0^+, \tag{23}$$

where $\xi = \eta\xi^*$. On the other hand, using integration by parts, we have $\gamma_\alpha = 1 - \int_0^1 u^\alpha \left[C_{1|2}(u|u) + C_{2|1}(u|u)\right] \mathrm{d}u =: 1 - I_\alpha$. Note that

$$I_\alpha = \int_0^{1-\epsilon} u^\alpha \left[C_{1|2}(u|u) + C_{2|1}(u|u)\right] \mathrm{d}u + \int_{1-\epsilon}^1 u^\alpha \left[C_{1|2}(u|u) + C_{2|1}(u|u)\right] \mathrm{d}u \tag{24}$$

for any $\epsilon > 0$. The first integral in (24) is bounded by $M(1-\epsilon)^\alpha$ for some positive constant $M$, and can be ignored as $\alpha \to \infty$. For the second integral, applying (23) gives

$$\begin{aligned}
&\int_{1-\epsilon}^1 u^\alpha \left[C_{1|2}(u|u) + C_{2|1}(u|u)\right] \mathrm{d}u \\
=\ &\int_{1-\epsilon}^1 u^\alpha \left[(2-\lambda_U) - \xi(1-u)^{\eta-1}\ell(1-u) + o((1-u)^{\eta-1}\ell(1-u))\right] \mathrm{d}u \\
=\ &\frac{2-\lambda_U}{\alpha+1} - \xi \int_{1-\epsilon}^1 u^\alpha (1-u)^{\eta-1}\ell(1-u)\,\mathrm{d}u + O\left[\frac{(1-\epsilon)^{\alpha+1}}{\alpha+1}\right].
\end{aligned} \tag{25}$$

For any $\delta \in (0,\eta)$, the integral in (25) can be expressed as

$$\begin{aligned}
\int_{1-\epsilon}^1 u^\alpha (1-u)^{\eta-1}\ell(1-u)\,\mathrm{d}u &= \int_0^{-\log(1-\epsilon)} e^{-(\alpha+1)x}(1-e^{-x})^{\eta-1}\ell(1-e^{-x})\,\mathrm{d}x \\
&= \int_0^\infty f_\Gamma(x;\eta-\delta,\alpha+1)\left(\frac{1-e^{-x}}{x}\right)^{\eta-1}\Gamma(\eta-\delta) \\
&\qquad \cdot(\alpha+1)^{-(\eta-\delta)}x^\delta \ell(1-e^{-x})\cdot 1\{x \le -\log(1-\epsilon)\}\,\mathrm{d}x \\
&:= \mathbb{E}\left[g(X_\alpha)\right],
\end{aligned}$$

where $g(x) = \left(\frac{1-e^{-x}}{x}\right)^{\eta-1}\Gamma(\eta-\delta)(\alpha+1)^{-(\eta-\delta)}x^\delta\ell(1-e^{-x})\cdot 1\{x \le -\log(1-\epsilon)\}$, $f_\Gamma(x;\gamma,\beta)$ is the density function of a gamma random variable with shape parameter $\gamma$ and rate parameter $\beta$, and $X_\alpha \sim \text{Gamma}(\eta-\delta,\alpha+1)$ (in the shape-rate parametrization). Note that $X_\alpha - (\eta-\delta)(\alpha+1)^{-1} \xrightarrow{p} 0$ as $\alpha \to \infty$, and thus $g(X_\alpha) - g\left((\eta-\delta)(\alpha+1)^{-1}\right) \xrightarrow{p} 0$ by the continuous mapping theorem. Since $\left(\frac{1-e^{-x}}{x}\right)^{\eta-1} \le 1$ and $x^\delta\ell(1-e^{-x}) \to 0$ as $x \to 0^+$, $g(X_\alpha)$ is integrable with respect to the Gamma$(\eta-\delta,\alpha+1)$ density and thus we have the convergence in mean

$$\mathbb{E}\left[g(X_\alpha)\right] - g((\eta-\delta)(\alpha+1)^{-1}) \to 0.$$

Observe that, for all $\alpha > \alpha^*$ with $(\eta-\delta)(\alpha^*+1)^{-1} = -\log(1-\epsilon)$,

$$\begin{aligned} g((\eta-\delta)(\alpha+1)^{-1}) &= \Gamma(\eta-\delta)(\alpha+1)^{-(\eta-\delta)}\left[1+O\left((\alpha+1)^{-1}\right)\right]\ell\left(1-e^{-(\eta-\delta)/(\alpha+1)}\right) \\ &= \Gamma(\eta-\delta)(\alpha+1)^{-(\eta-\delta)}\ell\left(1-e^{-(\eta-\delta)/(\alpha+1)}\right) + o\left((\alpha+1)^{-(\eta-\delta)}\right). \end{aligned}$$

Since $\epsilon$ and $\delta$ are arbitrarily small positive numbers, we have that $I_\alpha = (\alpha+1)^{-1}(2-\lambda_U) - \alpha^{-\eta}\xi\Gamma(\eta)\ell^*(\alpha^{-1}) + o\left(\alpha^{-(\eta\wedge 1)}\right)$ for $\alpha$ large, where $\ell^*$ is another slowly varying function at 0. With $c_{11} = 2-\lambda_U$, we have

$$\zeta_\alpha - \lambda_U = 2 + \alpha(1-\gamma_\alpha^{-1}) - \lambda_U = \frac{c_{11} - (c_{11}+\alpha)I_\alpha}{1-I_\alpha} = \frac{[c_{11}-(c_{11}+\alpha)I_\alpha](1+I_\alpha)}{1+o(1)},$$

simplifying to (22) after plugging in the asymptotic expansion of $I_\alpha$. ■

The asymptotic expansion (22) provides guidance on how quickly $\zeta_\alpha$ converges to $\lambda_U$ as $\alpha \to \infty$. There are three possibilities:

(1) When $1 < \eta < 2$, the middle term in the numerator of (22) dominates and the rate is $\alpha^{-(\eta-1)}$; this is the case for all bivariate copulas with intermediate tail dependence and some with tail dependence. Whether $\zeta_\alpha$ is increasing or decreasing to the limit as $\alpha$ increases, for $\alpha$ large, depends on the sign of that term.
(2) When $\eta > 2$, the first term dominates and the rate is $\alpha^{-1}$; this is the case for some copulas with tail dependence. This also implies $\zeta_\alpha$ is increasing to the limit as $\alpha$ increases, for large $\alpha$, as $c_{11} \geq 1$ and $c_{11} - c_{11}^2 \leq 0$. It should also be noted that, for copulas with negative quadrant dependence and zero tail dependence (such as the Gaussian copula with negative parameter), typically $\eta > 2$ and $\zeta_\alpha$ increases to the limit of zero as $\alpha$ increases.
(3) When $\eta = 2$, the first two terms have the same order $\alpha^{-1}$; this is the case for copulas with tail quadrant independence and some with tail dependence. The trend of $\zeta_\alpha$ for $\alpha$ large depends on the magnitudes and signs of the two terms.

We illustrate with several examples below for the parametric copula families in Section 3.4.

(1) Gaussian copula (intermediate tail dependent). The (lower) tail is $C(u,u;\rho) \sim u^{2/(1+\rho)}(-\log u)^{-\rho/(1+\rho)}$ (Hua and Joe (2011)), with $\eta = 2/(1+\rho)$, $\ell(u) = (-\log u)^{-\rho/(1+\rho)}$ and $\lambda_L = \lambda_U = 0$. Hence, $\zeta_\alpha$ is decreasing for large $\alpha$ if $\rho > 0$, and increasing if $\rho < 0$.
(2) Frank copula (tail quadrant independent). The (lower) tail expansion is $C(u,u;\theta) \sim \theta(1-e^{-\theta})^{-1}u^2$, with $\eta = 2$, $\lambda_L = \lambda_U = 0$, $\xi = 2\theta(1-e^{-\theta})^{-1}$ and $\ell \equiv 1$. Note that $c_{11} - c_{11}^2 + \xi = 2\left[\theta(1-e^{-\theta})^{-1} - 1\right]$; this is greater than zero if $\theta > 0$ (positive quadrant dependence) and less than zero if $\theta < 0$ (negative quadrant dependence). Hence, $\zeta_\alpha$ is decreasing for large $\alpha$ if $\theta > 0$, and increasing if $\theta < 0$.
(3) Upper tail of Gumbel copula (tail dependent) with parameter $\theta > 1$. The tail expansion is $\overline{C}(1-u, 1-u;\theta) = (2-2^{1/\theta})u + 2^{1/\theta-1}(2^{1/\theta}-1)u^2 + O(u^3)$, so that $\lambda_U = 2-2^{1/\theta}$, $\eta = 2$, $\xi = 2^{1/\theta}(2^{1/\theta}-1)$ and $\ell \equiv 1$. Note that $c_{11} - c_{11}^2 + \xi = 0$

indicating quick convergence — $\zeta_\alpha$ is in fact a constant as this is an extreme value copula.

(4) Lower tail of Gumbel copula (intermediate tail dependent). The copula can be simplified to $C(u,u;\theta) = u^{2^{1/\theta}}$, so that $\lambda_U = 0$, $\eta = \xi = 2^{1/\theta}$ and $\ell \equiv 1$. Since $\xi > 0$, $\zeta_\alpha$ is decreasing for large $\alpha$ (which agrees with intuition since $\zeta_\alpha \geq 0$ for copulas with non-negative dependence).

(5) Student's t copula (tail dependent). It is easier to work with the conditional cdf of the t distribution. Let $F_{12}$ be the bivariate t cdf with correlation parameter $\rho$ and $\nu$ degrees of freedom, and let $T_\nu$ and $t_\nu$ be the univariate t cdf and pdf with $\nu$ degrees of freedom. Let $\gamma = \sqrt{(1-\rho)(\nu+1)/(1+\rho)}$. The tail expansion of the conditional distribution $F_{2|1}$ as $x \to -\infty$ is

$$F_{2|1}(x|x;\rho,\nu) = T_{\nu+1}\left[x\sqrt{\frac{(1-\rho)(\nu+1)}{(1+\rho)(\nu+x^2)}}\right] = T_{\nu+1}\left[-\sqrt{\frac{(1-\rho)(\nu+1)}{1+\rho}}\left(1+\frac{\nu}{x^2}\right)^{-1/2}\right]$$
$$= T_{\nu+1}(-\gamma) + \frac{\nu\gamma}{2x^2}t_{\nu+1}(-\gamma) + O(x^{-4}). \tag{26}$$

Note that $\lambda_U = \lambda_L = 2\lim_{x\to-\infty} F_{2|1}(x|x) = 2T_{\nu+1}(-\gamma)$, so that $2F_{2|1}(x|x;\rho,\nu) \sim \lambda_L + \nu\gamma x^{-2}t_{\nu+1}(-\gamma)$ as $x \to -\infty$. To convert (26) to copula scale, let $x = T_\nu^{-1}(u)$ with $u \to 0^+$ so that

$$x = \nu^{(\nu-1)/2}\frac{\Gamma\left[(\nu+1)/2\right]}{\Gamma(\nu/2)\sqrt{\pi\nu}}u^{-1/\nu} \tag{27}$$

(see Nikoloulopoulos, Joe, and Li (2009)). Therefore $\eta = 1 + 2/\nu$ and $\zeta_\alpha - \lambda = O(\alpha^{-1}) + O(\alpha^{-2/\nu})$ as $\alpha \to \infty$. The convergence rate is $\alpha^{-1}$ when $0 < \nu \leq 2$, and $\alpha^{-2/\nu}$ for $\nu > 2$. For fixed $\rho$, the tail dependence is stronger with smaller $\nu$, and a smaller $\nu$ leads to quicker convergence of $\zeta_\alpha$ to $\lambda_L$ (or $\lambda_U$).

(6) Upper tail of BB1 copula (tail dependent) with $C(u_1,u_2;\theta,\delta) = \left(1+\left[(u_1^{-\theta}-1)^\delta + (u_2^{-\theta}-1)^\delta\right]^{1/\delta}\right)^{-1/\theta}$, for $\theta > 0$ and $\delta > 1$. It has $\lambda_U = 2 - 2^{1/\delta}$, $c_{11} = 2^{1/\delta}$ and the tail expansion is

$$\overline{C}(1-u,1-u;\theta,\delta) = (2-2^{1/\delta})u + \frac{1}{2}(\theta+1)(2^{2/\delta}-2^{1/\delta})u^2 + O(u^3),$$

so that $\eta = 2$, $\xi = (\theta+1)(2^{2/\delta}-2^{1/\delta}) > 0$ and $\ell \equiv 1$. Both the first and second terms in (22) have order $\alpha^{-1}$; the combined coefficient is $\theta(2^{2/\delta}-2^{1/\delta}) > 0$ and thus $\zeta_\alpha$ is decreasing to the limit as $\alpha$ (large) increases.

(7) Lower tail of BB1 copula (tail dependent). The tail expansion at $u \to 0^+$ is $C(u,u;\theta,\delta) = 2^{-1/(\theta\delta)}u + 2^{-1/(\theta\delta)}\theta^{-1}(1-2^{-1/\delta})u^{\theta+1} + O(u^{2\theta+1})$, so that $\lambda_L = 2^{-1/(\theta\delta)}$, $\eta = \theta+1$, $\xi = 2^{-1/(\theta\delta)}(\theta+1)\theta^{-1}(1-2^{-1/\delta})$ and $\ell \equiv 1$. There are three situations:

  (a) If $0 < \theta < 1$ (weaker dependence), the dominating term of $\zeta_\alpha - \lambda_U$ has order $\alpha^{-\theta}$ and the coefficient $2^{-1/(\theta\delta)}(\theta+1)\theta^{-1}(1-2^{-1/\delta})\Gamma(\theta+1) > 0$, and hence $\zeta_\alpha$ is decreasing for large $\alpha$.

  (b) If $\theta = 1$, the dominating term has order $\alpha^{-1}$, with coefficient $c_{11} - c_{11}^2 + \xi = \left(2-2^{-1/\delta}\right) - \left(2-2^{-1/\delta}\right)^2 + 2^{1-1/\delta}(1-2^{-1/\delta}) = (2-3\cdot 2^{-1/\delta})(2^{-1/\delta}-1)$ whose sign depends on $\delta$.

(c) If $\theta > 1$ (stronger dependence), the dominating term also has order $\alpha^{-1}$ but the coefficient $c_{11} - c_{11}^2$ is always negative, meaning that $\zeta_\alpha$ is increasing for large $\alpha$.

The tail properties of the above copula families are summarized in Table 3.

| Copula (tail) | Parameter | $\lambda$ | $\eta$ | $\ell(u)$ as $u \to 0^+$ | Order |
|---|---|---|---|---|---|
| Gaussian (either) | $-1 < \rho < 1$ | $0$ | $\frac{2}{1+\rho}$ | $(-\log u)^{-\frac{\rho}{1+\rho}}$ | $\alpha^{-\left(\frac{1-\rho}{1+\rho} \wedge 1\right)}$ |
| Frank (either) | all $\theta$ | $0$ | $2$ | $1$ | $\alpha^{-1}$ |
| Gumbel (upper) | all $\theta$ | $2 - 2^{1/\theta}$ | $2$ | $1$ | † |
| Gumbel (lower) | all $\theta$ | $0$ | $2^{1/\theta}$ | $1$ | $\alpha^{-(2^{1/\theta}-1)}$ |
| t (either) | $0 < \nu < 2$ | $2T_{\nu+1}(-\gamma)^*$ | $1 + \frac{2}{\nu}$ | ** | $\alpha^{-1}$ |
| | $\nu \geq 2$ | | | | $\alpha^{-2/\nu}$ |
| BB1 (upper) | all $(\theta, \delta)$ | $2 - 2^{1/\delta}$ | $2$ | $1$ | $\alpha^{-1}$ |
| BB1 (lower) | $0 < \theta < 1$ | $2^{-1/(\theta\delta)}$ | $1 + \theta$ | $1$ | $\alpha^{-\theta}$ |
| | $\theta = 1$ | | | | $\alpha^{-1}$ |
| | $\theta > 1$ | | | | $\alpha^{-1}$ |

* Here $\gamma = \sqrt{(1-\rho)(\nu+1)/(1+\rho)}$.

** The slowly varying function of the t copula is a function of $\nu$ and $\rho$ but not $u$; this can be seen from (26) and (27).

† The value of $\zeta_\alpha$ is a constant for the upper tail of the Gumbel copula since it is an extreme value copula.

Table 3. Summary of the tail properties of several commonly used bivariate parametric copula families. The column "Order" gives the asymptotic order of convergence of $\zeta_\alpha$ to the tail dependence coefficient $\lambda$ as $\alpha \to \infty$.

### 4.2. *Estimation of the tail dependence coefficient*

From the results on the rate of convergence of $\zeta_\alpha$ to $\lambda_U$, we propose a method to estimate the tail dependence coefficient based on a regression using the sequence of $\hat{\zeta}_\alpha$ for various large values of $\alpha$.

There is no direct way of estimating the tail dependence coefficient of a general bivariate copula as it is defined as a limit. Dobrić and Schmid (2005) attempt an empirical counterpart of (1) with small values of $u$, a weighted least squares estimate and another one based on a mixture of the independence and comonotonicity copulas. Frahm, Junker, and Schmidt (2005) discuss the challenges of estimating the tail dependence coefficient and have estimates based on various assumptions on the copula models.

We consider the following regression equations in our proposed approach, with $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ being the error terms:

(M1) $\hat{\zeta}_\alpha = b_1 + b_2/\alpha + \epsilon_1$, valid for (22) with $\eta = 2$. The estimated $\lambda_U$ is given by $\hat{b}_1$.

(M2) $\hat{\zeta}_\alpha = b_1 + b_2/\alpha^{b_3} + \epsilon_2$, an extension of M1 for other values of $\eta$. The estimated $\lambda_U$ is also given by $\hat{b}_1$.

(M3) $\hat{\zeta}_\alpha = (2-b) + (b-b^2)/(\alpha+1-b) + \epsilon_3$, which is obtained from (22) for $\eta > 2$, using the asymptotic relationship $I_\alpha \sim c_{11}(\alpha+1)^{-1}$ with $b = c_{11}$, where $I_\alpha$ is given in (24). The estimated $\lambda_U$ is given by $2 - \hat{b}$ in this case.

For a fixed $\alpha$, $\hat{\zeta}_\alpha$ is consistent for $\zeta_\alpha$ as the sample size increases to infinity. If the copula is such that (22) holds with $\eta > 2$, then regression M3 leads to a consistent estimator of $\lambda_U$ when the grid of $\alpha$ values increases appropriately as the sample size increases. When

$\eta \leq 2$, a similar conclusion holds for regressions M1 and M2. However, the asymptotic consistency of the estimator may not be relevant in practice, when the extent of tail dependence must be estimated from finite (and usually small) samples. This is because the variability of $\hat{\zeta}_\alpha$ (and therefore, the variability of the estimated $\lambda_U$) increases as $\alpha$ increases for a given sample size. Therefore, we are more interested in the finite-sample behavior of the estimator.

The grid of $\alpha$ values for which the corresponding $\hat{\zeta}_\alpha$'s are computed can depend on the sample size, with larger values of $\alpha$ when the sample size is larger. A preliminary investigation based on the bivariate parametric copula families in Table 3 suggests that a range of $\alpha$ values between 10 and 20 yields better performance in terms of the root mean square error (RMSE) of the estimate for small to moderate sample sizes (in the hundreds to thousands); larger values of $\alpha$ result in a larger variance while smaller values of $\alpha$ result in a larger bias. With $10 \leq \alpha \leq 20$, the rate of $\alpha$ in (22) might not be accurate but the sign of $\xi$ is generally correct.

Empirically, regression M3 works best when the copula has $\eta > 2$; for copulas with positive dependence, this only happens when the copula has tail dependence. M1 works better for tail dependent copulas ($\lambda_U > 0$) when $\eta = 2$, and also in some cases with $\eta < 2$. Meanwhile, M2 works better for copulas with intermediate tail dependence (which has $1 < \eta < 2$ and $\lambda_U = 0$) or tail quadrant independence (which has $\eta = 2$ and $\lambda_U = 0$); in the latter case, M2 generally yields estimates that are closer to zero than for M1 and the reduction in RMSE is substantial.

A further check on the theoretical asymptotic variance using expressions in Section 3.3 suggests that the rate of increase in the asymptotic variance of $\hat{\zeta}_\alpha$ as a function of $\alpha$ depends on the strength of tail dependence of the copula; Figure 1 shows several examples, each with Kendall's $\tau$ of the copula equal to 0.5. When the copula is tail dependent, as is the case for $t_3$, the asymptotic variance grows at a rate of around $\alpha$; this can be seen by the fitted line that has an intercept of around zero. For copulas with intermediate tail dependence or tail quadrant independence, the asymptotic variance grows at a rate of less than $\alpha$. This observation prompts us to impose a further refinement to the regression procedure for the estimation of $\lambda_U$; we suggest using weighted least squares (WLS) with weight $\alpha$ when there is evidence of tail dependence, and weight $\alpha^{1/2}$ when the dependence in the joint upper tail is weak[2]. With regression M2 on copulas with intermediate tail dependence or tail quadrant independence, we observe a reduction of the RMSE using WLS with weight $\alpha^{1/2}$ rather than $\alpha$. To check if there may be tail dependence, one empirical approach is to compare the sample semicorrelation to the one corresponding to a Gaussian copula with the same overall dependence (correlation of the normal scores). If the sample semicorrelation is much higher, a linear weighting should be used.

To summarize, we estimate the upper tail dependence coefficient using the following diagnostic procedure:

(1) Compute $\hat{\zeta}_\alpha$ for a grid of $\alpha$ values in $[10, 20]$.
(2) If $\hat{\zeta}_\alpha$ is increasing in $\alpha$, in the sense that an ordinary least squares (OLS) regression of $\hat{\zeta}_\alpha$ against $1/\alpha$ has negative slope, use regression M3 (which is suitable when $\zeta_\alpha$ is increasing to the limit) with WLS and weight $\alpha$.
(3) If $\hat{\zeta}_\alpha$ is decreasing in $\alpha$, there are two possibilities. First obtain the results based on regression M2 with WLS and weight $\alpha^{1/2}$. Also obtain the sample upper semicor-

[2]We observe that the rate of growth of the asymptotic variance is typically between $\alpha^0$ and $\alpha^1$ in the range of $10 < \alpha < 20$ for copulas without tail dependence.

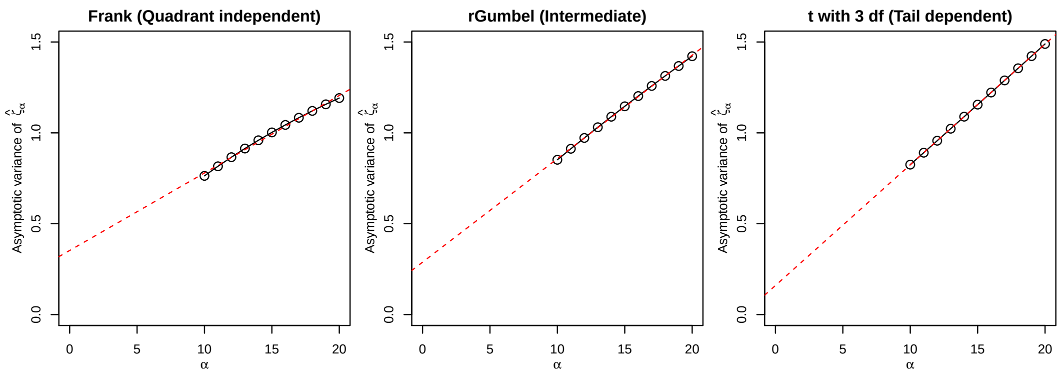


Figure 1. Plots of asymptotic variance of $\hat{\zeta}_\alpha$ against $\alpha$ for the upper tails of three parametric copula families with Kendall's $\tau$ equal to 0.5: Frank, reflected Gumbel and $t_3$. The dashed lines are linear extrapolations based on the computed asymptotic variances (in circles).

relation $\hat{\rho}_N^+$ of the normal scores, and the semicorrelation $\rho_N^+$ of a Gaussian copula (equation (2.59) of Joe (2014)) with dependence parameter being the sample correlation of the normal scores.

(a) If the estimated curvature parameter $b_3$ in regression M2 is larger than $1-\epsilon$ for some small threshold $\epsilon$, or $\hat{\rho}_N^+ - \rho_N^+ > \gamma$ for some cutoff $\gamma$, then use regression M1 with WLS and weight $\alpha$.

(b) Otherwise, use the result already obtained based on regression M2.

For easier understanding, the diagnostic procedure is also given in the form of a flowchart in Figure 2.

The threshold parameter $\epsilon$ is usually small; we note little difference in the results with various choices of $\epsilon$ and will adopt $\epsilon = 0.2$ in the following. The cutoff $\gamma$ reflects the variability of $\hat{\rho}_N^+$ in the Gaussian case and hence depends on both the sample size and the strength of overall dependence. We note that the standard error of $\hat{\rho}_N^+$ for a Gaussian copula with correlation parameter 0.6 is around 0.07 for a sample size of 600 (Joe (2014)). We experiment with various values of $\gamma$ based on copulas with Kendall's $\tau = 0.5$, and choose $\gamma = 0.04$ for a sample size of $n = 500$; those for other sample sizes can be obtained using the square root rule, e.g., quadrupling the sample size reduces $\gamma$ by one half.

A simulation study is conducted to investigate the finite-sample performance of the estimator. For each copula family, we use one with Kendall's $\tau$ equal to 0.5 and sample sizes 500 and 2000. Two BB1 copulas with respective dependence parameter $\theta = 1.5$ and 0.5 are considered, as they have different rates of convergence. The RMSE of the estimator based on the above procedure is computed using 1000 replications for each scenario; we also record the proportion of instances each regression equation is used. The simulation results are shown in Table 4.

For comparison, we consider an estimator based on the following regressions using the empirical survival copula $\overline{C}_n(1-u, 1-u) = n^{-1}\sum_{i=1}^n 1(R_{i1} > 1-u, R_{i2} > 1-u)$:

(DS1) $\overline{C}_n(1-u, 1-u)/u = b_1 + b_2 u$, with estimator for $\lambda_U$ being $b_1$.
(DS2) $\overline{C}_n(1-u, 1-u)/u = b_1 + b_2 u^{b_3}$, with estimator for $\lambda_U$ being $b_1$.

These relationships are modified from the third estimator of Dobrić and Schmid (2005) that assumes a mixture of the independence and comonotonicity copulas:

$$\overline{C}(1-u, 1-u) = \lambda_U u + (1-\lambda_U)u^2,$$

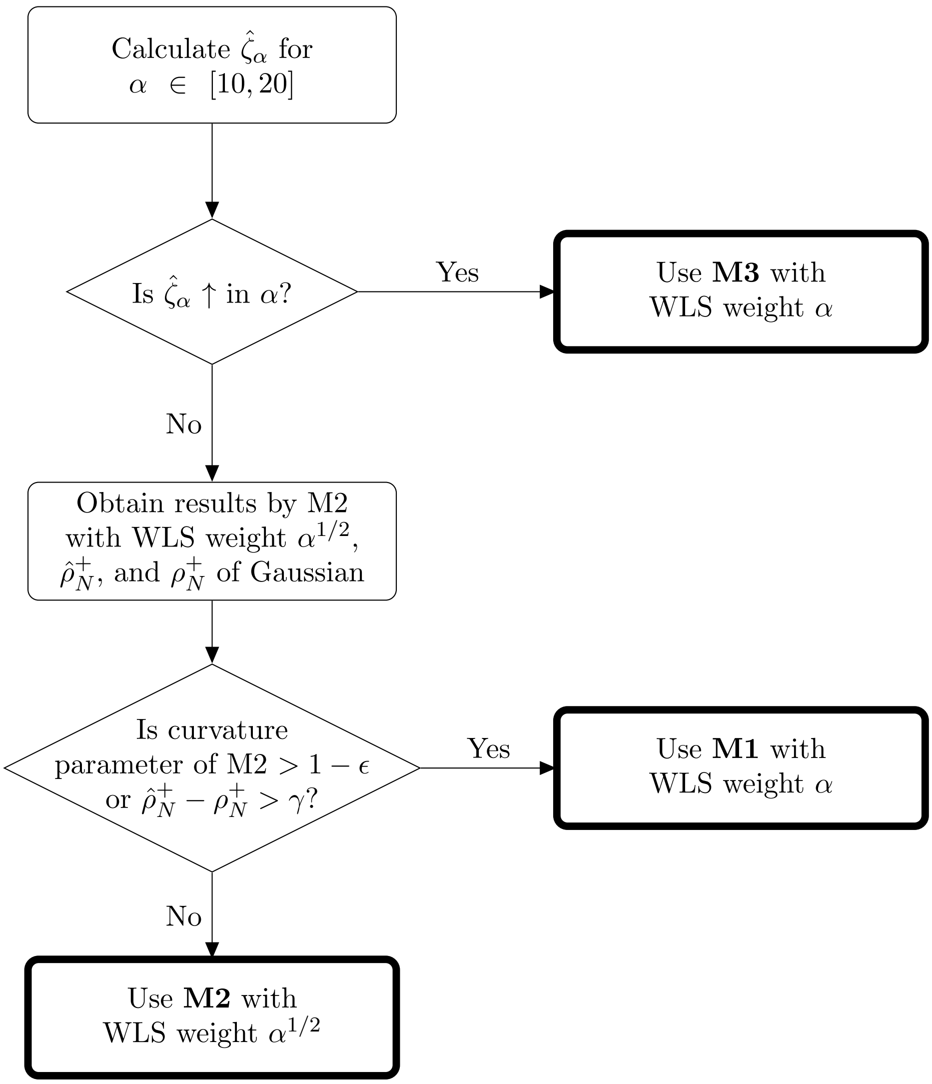


Figure 2. A flowchart of the proposed diagnostic procedure for the estimation of $\lambda_U$ based on a sequence of values of $\zeta_\alpha$.

where we relax the restriction on the coefficients. Empirical studies (not shown) suggest that this modified empirical copula approach has better finite-sample performance than the estimator by Dobrić and Schmid (2005), and therefore we report the results for the former instead. Similar to the estimator based on $\hat{\zeta}_\alpha$, for a diagnostic procedure, we select the regression DS1 if the estimated curvature parameter $b_3$ in DS2 is larger than $1 - \epsilon$, or if the observed upper semicorrelation is larger than the Gaussian semicorrelation with the same overall dependence (as the correlation of normal scores) by at least $\gamma$. Otherwise, the estimator from regression DS2 is chosen. We conduct OLS with $u = 0.1, 0.15, \ldots, 0.5$ and the same values of $\epsilon$ and $\gamma$ as those for $\hat{\zeta}_\alpha$; a value of $u$ as high as 0.5 is used because it is hard to observe the trend for small values of $u$, due to the large variability of $\overline{C}_n$ near the corner (Figure 3). We emphasize that this modified method only acts as a benchmark for us to observe the typical values of the RMSEs, for currently available nonparametric estimation methods for the tail dependence coefficient.

From Table 4, the RMSE's for the $\hat{\zeta}_\alpha$ approach are comparable to those based on the empirical copula (i.e., using regressions DS1 and DS2); the former generally does better when there is tail dependence (with Gumbel the only exception), while the latter may have better performance when the dependence in the joint upper tail is weak. The approach based on $\hat{\zeta}_\alpha$ appears to be a better diagnostic when $\hat{\zeta}_\alpha$ is increasing in $\alpha$ for $\alpha > 10$ (that is, when regression M3 is chosen), and its RMSE is not much worse when the approach based on the empirical copula has smaller RMSE. Figure 3 has a comparison of plots of $\hat{\zeta}_\alpha$ against $1/\alpha$, versus those of $\overline{C}_n(1-u, 1-u)/u$ against $u$. We observe that the former plots are generally smoother than the latter ones; this contributes to a smaller variance of the estimator in some cases. Regardless of the approach used, we note a higher RMSE for the Gaussian, reflected Gumbel and the $t_5$ copulas; these are the more difficult cases as $\zeta_\alpha$ converges very slowly to $\lambda_U$. For the Gaussian copula, even though the regression M2 with WLS weighting $\alpha^{1/2}$ may be the best choice based on RMSE, most instances based on the diagnostic approach fall into regression M1 as the plot of $\hat{\zeta}_\alpha$ against $1/\alpha$ is not showing sufficient curvature.

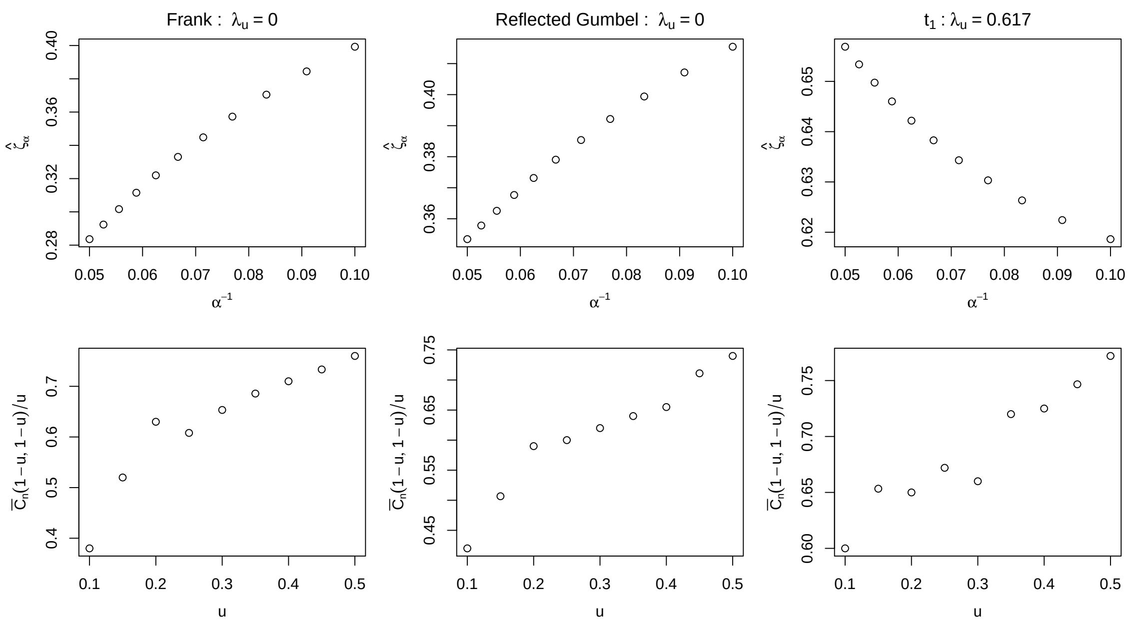


Figure 3. Plots of $\hat{\zeta}_\alpha$ against $1/\alpha$ (top row) and $\overline{C}_n(1-u, 1-u)/u$ against $u$ (bottom row) for simulated data sets of size 500 from the Frank, reflected Gumbel and $t_1$ copulas.

Finally, we would like to point out the following remarks:

- The above procedure serves as a guideline only. For practical applications, it may be relevant to incorporate the nature of the data being studied. For example, if there are reasons to believe there exists tail dependence, then one may choose to use regressions M1 or M3 with linear weighting depending on the trend of $\hat{\zeta}_\alpha$ as a function of $\alpha$. Because of sampling variability, the semicorrelation can be smaller than the Gaussian one even if the data come from a copula with tail dependence.
- Although we focused on copulas with positive quadrant dependence in the above numerical study, we also checked with those with mild negative quadrant dependence and found that the procedures are still applicable. Because our objective is to estimate the joint upper or lower tail dependence coefficient, copulas with strong negative quadrant dependence are less relevant here.

| Copula | Gaussian | Frank | Gumbel | rGumbel | $t_1$ |
|---|---|---|---|---|---|
| Param. | 0.71 | 5.74 | 2 | 2 | 0.71 |
| $\lambda_U$ | 0 | 0 | 0.59 | 0 | 0.62 |
| $\eta$* | 1.17 | 2 | 2 | 1.41 | 3 |
| Rate** | 0.17 | 1 | $1^\dagger$ | 0.41 | 1 |
| $n$ | RMSE based on estimation using $\hat{\zeta}_\alpha$ | | | | |
| 500 | .330 | **.078** | .068 | .199 | **.066** |
| 2000 | .342 | .025 | .042 | .199 | **.033** |
| $n$ | RMSE based on estimation using empirical copula | | | | |
| 500 | **.326** | .098 | **.057** | **.197** | .071 |
| 2000 | **.302** | **.020** | **.028** | **.139** | .048 |
| $n$ | Proportion of method used (using $\hat{\zeta}_\alpha$) (M1/M2/M3 in %) | | | | |
| 500 | 72/26/2 | 5/95/0 | 53/0/47 | 46/54/0 | 37/0/63 |
| 2000 | 91/9/0 | 0/100/0 | 51/0/49 | 57/43/0 | 19/0/81 |

| Copula | $t_5$ | BB1 | rBB1 | BB1 | rBB1 |
|---|---|---|---|---|---|
| Param. | 0.71 | $(1.5, 1.14)$ | $(1.5, 1.14)$ | $(0.5, 1.6)$ | $(0.5, 1.6)$ |
| $\lambda_U$ | 0.35 | 0.17 | 0.67 | 0.46 | 0.42 |
| $\eta$* | 1.4 | 2 | 2.5 | 2 | 1.5 |
| Rate** | 0.4 | 1 | 1 | 1 | 0.5 |
| $n$ | RMSE based on estimation using $\hat{\zeta}_\alpha$ | | | | |
| 500 | **.133** | **.107** | .054 | **.108** | **.111** |
| 2000 | **.093** | **.059** | **.025** | .045 | **.064** |
| $n$ | RMSE based on estimation using empirical copula | | | | |
| 500 | .163 | .139 | **.053** | .156 | .144 |
| 2000 | .118 | .133 | .030 | **.040** | .071 |
| $n$ | Proportion of method used (using $\hat{\zeta}_\alpha$) (M1/M2/M3 in %) | | | | |
| 500 | 84/4/12 | 64/36/1 | 34/0/66 | 71/3/26 | 75/2/22 |
| 2000 | 99/0/1 | 79/21/0 | 16/0/84 | 88/0/12 | 93/0/7 |

* Value of $\eta$ in (22); equal to $\kappa_U$ for copulas with no tail dependence.
** True rate of convergence, i.e., the negative of the dominating power in (22).
† The dominating term for Gumbel copula has power $\alpha^{-1}$, but $\zeta_\alpha \equiv \lambda_U$ for any $\alpha$ as it is an extreme value copula.

Table 4. Results for the estimation of $\lambda_U$ based on regressions using (a) $\hat{\zeta}_\alpha$ and (b) the empirical survival copula, on selected bivariate copulas with Kendall's $\tau$ equal to 0.5 and sample sizes $n = 500$ and 2000. In each case, 1000 replications are conducted. The root mean square errors (RMSE's) are reported and the smaller value in each comparison (between the two approaches) is shown in boldface. The proportion of the regression equations (methods) used (based on $\hat{\zeta}_\alpha$) is also displayed for each copula. These proportions may not sum to 100 due to rounding.

## 5. Data example

In this section, we use a data example to illustrate the estimation of tail dependence coefficients using our tail-weighted dependence measure, and the insight on tail inference using different parametric copula families. The data set consists of 1,500 bivariate observations of insurance loss and the associated allocated loss adjustment expense (ALAE) (Frees and Valdez (1998)). There are 34 censored observations whose claims reach the policy limit, and are dropped in our subsequent illustration. In comparison of parameter estimates with Section 7.4 of Joe (2014), these 34 observations make almost no difference in the best fitting bivariate copula families and their maximum likelihood estimates.

A scatterplot of the normal scores (with correlation $\rho_N = 0.455$) for the remaining 1,466 observations (Figure 4) shows a more peaked joint upper tail, indicating possible upper tail dependence. This is also supported by an upper semicorrelation of 0.415, much higher than the semicorrelation of 0.235 for a Gaussian copula with correlation parameter 0.455.

For each tail, we compute $\hat{\zeta}_\alpha$ for $\alpha = 10, 11, \ldots, 20$ and plot these values against $1/\alpha$ in Figure 5. Both plots suggest that $\hat{\zeta}_\alpha$ decreases with $\alpha$, and hence regression M3 in Section 4.2 (i.e., the regression $\hat{\zeta}_\alpha = (2-b) + (b-b^2)/(\alpha+1-b)$) is not to be used. The estimated curvature parameters for regression M2 are 1.000 (reaching the upper bound) and 0.977 for the upper and lower tails, respectively. We thus use the result from regression M1, which assumes a linear rate of convergence. The estimated tail dependence coefficients are given by $\hat{\lambda}_U = 0.331$ and $\hat{\lambda}_L = 0.081$. To get some idea on the variability of these estimates, we conduct a delete-$k$ jackknife (see, for example, Shao and Wu (1989) for its use on potentially non-smooth estimators). Here we choose $k = 5$, and note that the results are similar for other values of $k > 1$ attempted. Using the jackknife variability estimates, we compute the 95% confidence intervals of the tail dependence coefficient estimates as $(0.247, 0.416)$ and $(0.003, 0.159)$ for the upper and lower tails, respectively. This seems to support the initial diagnostics that there is upper tail dependence. For comparison, we also compute the confidence intervals (based on the same jackknife samples) for Kendall's $\tau$ (length 0.061), $\zeta_1$ (length 0.071) and $\zeta_{20}$ (length 0.129), shown in Table 5. It is clear that the estimation of the tail dependence coefficient is more difficult as it has the longest confidence interval.

For dependence modelling, we consider several parametric copula families with tail asymmetry skewed to the joint upper tail. The three copulas that yield the smallest values of the Akaike information criterion (AIC) are 1-parameter Gumbel, 1-parameter Galambos, and the reflection of the 2-parameter Archimedean copula family based on an integral of the Mittag-Leffler Laplace transform (see Section 4.31.1 of Joe (2014)). The latter is $C(u, v; \theta, \delta) = u + v - 1 + \psi(\psi^{-1}(1-u) + \psi^{-1}(1-v))$, where $\psi(s; \theta, \delta) = 1 - F_B(s^{1/\delta}/(1+s^{1/\delta}); \delta, \theta^{-1})$ and $F_B(\cdot; a, b)$ is the cdf of the Beta$(a, b)$ random variable. The 2-parameter BB1 and BB6 copula families were also fitted but their maximum likelihood estimates were at the boundary corresponding to a Gumbel copula.

With a 2-parameter Pareto marginal distribution for loss and a 3-parameter Burr marginal distribution for ALAE, the maximum likelihood estimates of the copula parameters for the three families are given in Table 5, as well as model-based estimated values of $\tau$, $\zeta_1$, $\zeta_{20}$ and $\lambda_U$.

| Parametric copula | AIC | Cop. param.(s) | $\widetilde{\tau}$ | $\widetilde{\zeta}_1$ | $\widetilde{\zeta}_{20}$ | $\widetilde{\lambda}_U$ |
|---|---|---|---|---|---|---|
| Galambos | 8541.7 | 0.701 | 0.301 | 0.372 | 0.372 | 0.372 |
| Gumbel | 8543.0 | 1.427 | 0.299 | 0.375 | 0.375 | 0.375 |
| imitlefAr | 8541.0 | (0.385,1.386) | 0.303 | 0.377 | 0.351 | 0.273 |
| Non-parametric 95% CI (lower) | — | — | 0.278 | 0.336 | 0.282 | 0.247 |
| Non-parametric 95% CI (upper) | — | — | 0.339 | 0.407 | 0.411 | 0.416 |

Table 5. Non-parametric and model-based estimates of dependence measures. The model-based estimates (with tildes) of dependence measures are calculated for the three best-fitting copula families; imitlefAr refers to the reflected Archimedean copula with integrated Mittag-Leffler Laplace transform. The corresponding 95% non-parametric confidence intervals are based on the same delete-5 jackknife samples.

This analysis agrees with Section 7.1.2 of Joe (2014) that for different fitted parametric copula families with similar AIC values, model-based estimates of central dependence measures are very close, model-based estimates of tail-weighted measures of dependence are not as close, and model-based estimates of tail dependence coefficients can be much farther apart. This is also shown in the 95% confidence intervals of the non-parametric estimates of the $\lambda_U$, $\zeta_{20}$ and $\tau$ in Table 5. It is not surprising that more observations are needed to estimate the tail-based quantities well.

Unless the sample size is very large, it seems the tail-weighted dependence measures, such as the ones in Krupskii and Joe (2015) and in this paper, are more informative. If there is tail dependence, then parametric copula families with tail dependence may yield smaller AIC values, but this does not mean that one can get good model-based estimates of the tail dependence coefficients.

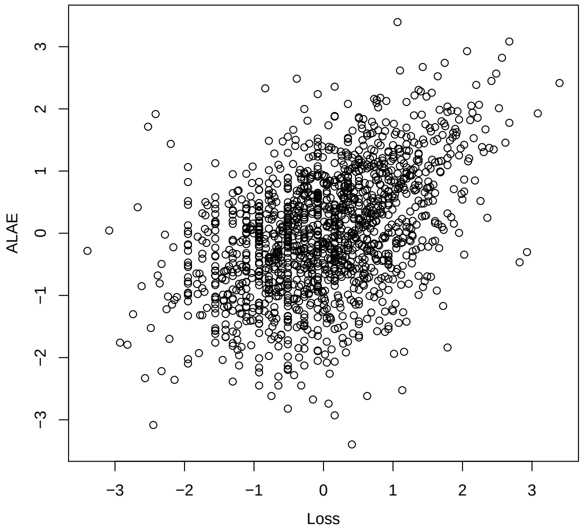


Figure 4. Pairwise scatterplots of the normal scores of the insurance data set

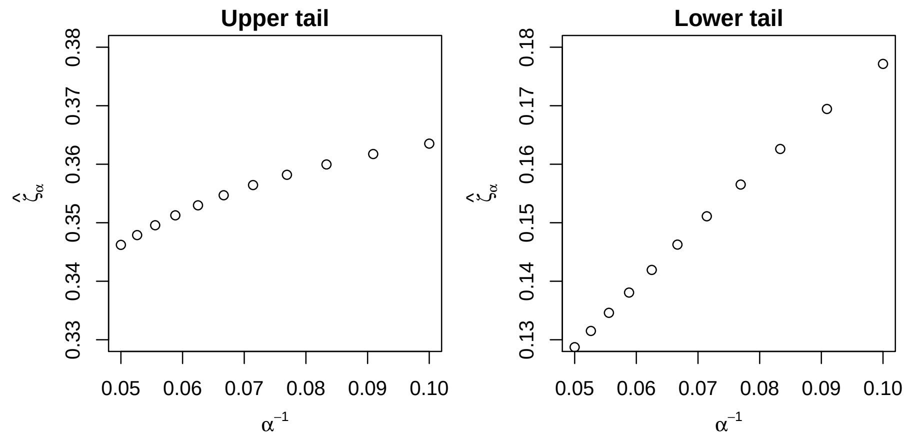


Figure 5. Plots of $\hat{\zeta}_\alpha$ against $1/\alpha$ for the two tails of the insurance data set

## 6. Conclusion

In this paper, we propose tail-weighted dependence measures that are motivated from the extreme value theory where a direct empirical counterpart of the tail dependence coefficient exists. The proposed measures are functions of $\gamma_\alpha$, an integral along the diagonal of the copula. One can control the weighting at different portions of the copula by adjusting the value of $\alpha$, obtaining a central dependence measure when $\alpha = 1$ and a tail-weighted dependence measure when $\alpha$ is far away from 1. In particular, the desirable property that $\zeta_\alpha$ converges to the upper tail dependence coefficient $\lambda_U$ as $\alpha \to \infty$ allows us to devise a method to estimate $\lambda_U$ based on the observed trajectory of $\zeta_\alpha$ at various values of $\alpha$.

The proposed measures can be used to distinguish between copulas with various strengths of dependence in the joint tail, and are useful as diagnostic measures for modelling where inference of the tail is of interest. There are two advantages over the tail-weighted dependence measure $\varrho$ in Krupskii and Joe (2015):

- The probabilistic version of $\zeta_\alpha$ involves a one-dimensional integral and is simpler than that of $\varrho$, which involves a conditional correlation and a two-dimensional integral. It is thus easier to analyze the distributional properties of the empirical estimator of $\zeta_\alpha$.
- Because the upper tail dependence coefficient $\lambda_U$ can be obtained as a limit of $\zeta_\alpha$ as $\alpha \to \infty$, we can extrapolate estimates of $\zeta_\alpha$ for several $\alpha$ to get an estimate of $\lambda_U$. There is no such relationship for the measure $\varrho$.

Through a simulation study, we observe that the estimation of $\lambda_U$ using $\hat{\zeta}_\alpha$ may have better performance than the one based on the empirical copula when there is tail dependence. The data example illustrates the potential uses of the proposed tail-weighted dependence measure; they are especially relevant when tail inference is of interest, such as estimating the value-at-risk or joint exceedance probabilities.

## Acknowledgements

This research has been supported by UBCs Four Year Doctoral Fellowship, NSERC Discovery Grant 8698, a Collaborative Research Team grant for the project: *Copula Dependence Modelling: Theory and Applications* of the Canadian Statistical Sciences Institute, and the King Abdullah University of Science and Technology. We would like to thank the anonymous referees for their useful comments and suggestions.

## References

Bücher, A., Dette, H., and Volgushev, S. (2011), 'New estimators of the Pickands dependence function and a test for extreme-value dependence', *Annals of Statistics*, 39, 1963–2006.

Capéraà, P., Fougères, A.L., and Genest, C. (1997), 'A nonparametric estimation procedure for bivariate extreme value copulas', *Biometrika*, 84, 567–577.

Cooley, D., Naveau, P., and Poncet, P. (2006), 'Variograms for spatial max-stable random fields', in *Dependence in Probability and Statistics*, Springer, New York, pp. 373–390.

Deheuvels, P. (1991), 'On the limiting behavior of the Pickands estimator for bivariate extreme-value distributions', *Statistics & Probability Letters*, 12, 429–439.

Dobrić, J., and Schmid, F. (2005), 'Nonparametric estimation of the lower tail dependence $\lambda_L$ in bivariate copulas', *Journal of Applied Statistics*, 32, 387–407.

Fermanian, J.D., Radulović, D., and Wegkamp, M. (2004), 'Weak convergence of empirical copula processes', *Bernoulli*, 10, 847–860.
Fonseca, C., Pereira, L., Ferreira, H., and Martins, A.P. (2015), 'Generalized madogram and pairwise dependence of maxima over two regions of a random field', *Kybernetika*, 51, 193–211.
Frahm, G., Junker, M., and Schmidt, R. (2005), 'Estimating the tail-dependence coefficient: Properties and pitfalls', *Insurance: Mathematics and Economics*, 37, 80–100.
Frees, E.W., and Valdez, E.A. (1998), 'Understanding relationships using copulas', *North American Actuarial Journal*, 2, 1–25.
Gabbi, G. (2005), 'Semi-correlations as a tool of geographical and sector asset allocation', *European Journal of Finance*, 11, 271–281.
Genest, C., Nešlehová, J., and Ben Ghorbal, N. (2010), 'Spearman's footrule and Gini's gamma: a review with complements', *Journal of Nonparametric Statistics*, 22, 937–954.
Hall, P., and Tajvidi, N. (2000), 'Distribution and dependence-function estimation for bivariate extreme-value distributions', *Bernoulli*, 6, 835–844.
Heffernan, J.E. (2000), 'A directory of coefficients of tail dependence', *Extremes*, 3, 279–290.
Hua, L., and Joe, H. (2011), 'Tail order and intermediate tail dependence of multivariate copulas', *Journal of Multivariate Analysis*, 102, 1454–1471.
Joe, H. (2014), *Dependence Modeling with Copulas*, Boca Raton: Chapman & Hall/CRC.
Krupskii, P., and Joe, H. (2015), 'Tail-weighted measures of dependence', *Journal of Applied Statistics*, 42, 614–629.
Ledford, A.W., and Tawn, J.A. (1996), 'Statistics for near independence in multivariate extreme values', *Biometrika*, 83, 169–187.
Naveau, P., Guillou, A., Cooley, D., and Diebolt, J. (2009), 'Modelling pairwise dependence of maxima in space', *Biometrika*, 96, 1–17.
Nelsen, R.B. (2006), *An Introduction to Copulas*, 2nd ed., New York: Springer.
Nikoloulopoulos, A.K., Joe, H., and Li, H. (2009), 'Extreme value properties of multivariate t copulas', *Extremes*, 12, 129–148.
Pickands, J. (1981), 'Multivariate extreme value distributions', in *Proceedings 43rd Session International Statistical Institute*, Vol. 2, pp. 859–878.
Scarsini, M. (1984), 'On measures of concordance', *Stochastica*, 8, 201–218.
Schmid, F., and Schmidt, R. (2007), 'Multivariate conditional versions of Spearman's rho and related measures of tail dependence', *Journal of Multivariate Analysis*, 98, 1123–1140.
Segers, J. (2012), 'Asymptotics of empirical copula processes under non-restrictive smoothness assumptions', *Bernoulli*, 18, 764–782.
Shao, J., and Wu, C.F.J. (1989), 'A general theory for jackknife variance estimation', *Annals of Statistics*, 17, 1176–1197.
Smith, R.L. (1990), 'Max-stable processes and spatial extremes', Unpublished manuscript.
Spearman, C. (1904), 'The proof and measurement of association between two things', *The American Journal of Psychology*, 15, 72–101.
Spearman, C. (1906), ''Footrule' for measuring correlation', *British Journal of Psychology*, 2, 89–108.
Tsukahara, H. (2005), 'Semiparametric estimation in copula models', *Canadian Journal of Statistics*, 33, 357–375.
van der Vaart, A., and Wellner, J.A. (1996), *Weak Convergence and Empirical Processes: With Applications to Statistics*, New York: Springer.

## Appendix A. Asymptotic variance of $\hat{\zeta}_\alpha$

The asymptotic variance in (13) is usually a 2-dimensional integral that can be evaluated numerically. Note that

$\mathrm{Var}(X) = \mathbb{E}(X^2)$

$$
\begin{aligned}
&= \frac{1}{4}\int_0^1\int_0^1 \mathbb{E}\left[\mathbb{G}_C\left(u^{1/\alpha},1\right)\mathbb{G}_C\left(v^{1/\alpha},1\right)\right]\mathrm{d}u\mathrm{d}v + \frac{1}{4}\int_0^1\int_0^1 \mathbb{E}\left[\mathbb{G}_C\left(1,u^{1/\alpha}\right)\mathbb{G}_C\left(1,v^{1/\alpha}\right)\right]\mathrm{d}u\mathrm{d}v \\
&\quad + \int_0^1\int_0^1 \mathbb{E}\left[\mathbb{G}_C\left(u^{1/\alpha},u^{1/\alpha}\right)\mathbb{G}_C\left(v^{1/\alpha},v^{1/\alpha}\right)\right]\mathrm{d}u\mathrm{d}v + \frac{1}{2}\int_0^1\int_0^1 \mathbb{E}\left[\mathbb{G}_C\left(u^{1/\alpha},1\right)\mathbb{G}_C\left(1,v^{1/\alpha}\right)\right]\mathrm{d}u\mathrm{d}v \\
&\quad - \int_0^1\int_0^1 \mathbb{E}\left[\mathbb{G}_C\left(u^{1/\alpha},1\right)\mathbb{G}_C\left(v^{1/\alpha},v^{1/\alpha}\right)\right]\mathrm{d}u\mathrm{d}v - \int_0^1\int_0^1 \mathbb{E}\left[\mathbb{G}_C\left(1,u^{1/\alpha}\right)\mathbb{G}_C\left(v^{1/\alpha},v^{1/\alpha}\right)\right]\mathrm{d}u\mathrm{d}v.
\end{aligned}
\tag{A1}
$$

Using (15), we have

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{G}_C\left(u_1,u_2\right)\mathbb{G}_C\left(u_3,u_4\right)\right] \\
=&\mathbb{E}\left[\mathbb{B}_C(u_1,u_2)\mathbb{B}_C(u_3,u_4)\right] - C_{2|1}(u_2|u_1)\mathbb{E}\left[\mathbb{B}_C(u_1,1)\mathbb{B}_C(u_3,u_4)\right] - C_{1|2}(u_1|u_2)\mathbb{E}\left[\mathbb{B}_C(1,u_2)\mathbb{B}_C(u_3,u_4)\right] \\
&- C_{2|1}(u_4|u_3)\mathbb{E}\left[\mathbb{B}_C(u_1,u_2)\mathbb{B}_C(u_3,1)\right] + C_{2|1}(u_2|u_1)C_{2|1}(u_4|u_3)\mathbb{E}\left[\mathbb{B}_C(u_1,1)\mathbb{B}_C(u_3,1)\right] \\
&+ C_{1|2}(u_1|u_2)C_{2|1}(u_4|u_3)\mathbb{E}\left[\mathbb{B}_C(1,u_2)\mathbb{B}_C(u_3,1)\right] - C_{1|2}(u_3|u_4)\mathbb{E}\left[\mathbb{B}_C(u_1,u_2)\mathbb{B}_C(1,u_4)\right] \\
&+ C_{2|1}(u_2|u_1)C_{1|2}(u_3|u_4)\mathbb{E}\left[\mathbb{B}_C(u_1,1)\mathbb{B}_C(1,u_4)\right] + C_{1|2}(u_1|u_2)C_{1|2}(u_3|u_4)\mathbb{E}\left[\mathbb{B}_C(1,u_2)\mathbb{B}_C(1,u_4)\right].
\end{aligned}
\tag{A2}
$$

Together with (16), the integrals can be evaluated once the expressions of $C$, $C_{1|2}$ and $C_{2|1}$ are given.