Stochastic differential games between two insurers with generalized mean-variance premium principle ☆

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Abstract

We study a stochastic differential game problem between two insurers who invest in a financial market and adopt reinsurance to manage their claim risks. Supposing that their reinsurance premium rates are calculated according to the generalized mean-variance principle, we consider the competition between the two insurers as a non-zero sum stochastic differential game. Using dynamic programming technique, we derive a system of coupled Hamilton-Jacobi-Bellman equations and show the existence of equilibrium strategies. For an exponential utility maximizing game and a probability maximizing game, we obtain semi-explicit solutions for the equilibrium strategies and the equilibrium value functions, respectively. Finally, we provide some detailed comparative-static analyses on the equilibrium strategies and illustrate some economic insights.

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1. Introduction

Since the classical work of Gerber (1970), the optimal reinsurance/investment problem of insurers has been extensively studied in the fields of insurance and control theory, see Browne (1995), Yang and Zhang (2005), Golubin (2008), Chiu and Wong (2014), and Zeng et al. (2016). These researches reflect the real practice that insurers use reinsurance to diversify their claim risks from policy holders and invest their surpluses in the financial market to make a profit.

Note that the above studies only consider the optimal investment-reinsurance strategy for a single insurer. However, in practice there are several insurers in the market who compete with each others. Recently, interactions among competing insurers have aroused great interest and have been studied extensively. Zeng (2010), Taksar and Zeng (2011) first consider the optimal reinsurance strategies for two competing insurers who share a single payoff function that depends on both insurers' surpluses. They describe the equilibrium of the zero-sum game, prove a verification theorem for a general payoff function, and present explicit solutions for a *probability maximizing game*.[1] Bensoussan et al. (2014) investigate the optimal investment-reinsurance problem for two insurers under the stochastic differential non-zero sum game framework. They assume that each insurer has different utility function and that both insurers' surpluses are modulated by a continuous-time Markov chain and a market-index. In this line, Meng et al. (2015) consider an optimal reinsurance problem when the two insurers' surpluses are subject to quadratic risk controls. Pun and Wong (2016) consider a reinsurance game problem for two ambiguity-averse insurers and obtain equilibrium under a worst-case scenario framework for the exponential utility functions. Pun et al. (2016) also study the reinsurance game problem for

[1] The probability maximizing game problem is first considered by Browne (2000) for two competing investors, where the payoff function is defined as the probability that the difference between two insurers' surpluses reaches an upper bound before it reaches a lower bound.

two insurers under the assumption that the correlations between insurers are ambiguous. Siu et al. (2016) study the non-zero sum investment-reinsurance game for two competitive insurers who are subject to systematic risks and use excess-of-loss reinsurance for risk control.

The above studies assume that the reinsurance premiums are calculated according to the expected value principle or variance principle (see Schmidli (2008)). In fact, reinsurance/investment strategies under different reinsurance premium calculation principle have attracted much attention of scholars, see Hipp and Taksar (2010), Chi (2012), Zeng and Luo (2013). Recently, Zhang et al. (2016) use the *generalized mean-variance principle* to determine the premium of reinsurance and consider the optimal investment-reinsurance strategies for an insurer. Using dynamic programming principle, Zhang et al. (2016) obtain explicit solutions when the insurer's objective is to maximize the exponential utility of her terminal wealth, or to minimize the probability of ruin.

In line with Bensoussan et al. (2014), in this paper we consider two insurers who compete with each other in the insurance market. Both insurers are subject to common impact from the insurance market. They invest in a risky asset and a risk free asset to make a profit and purchase reinsurance for risk management. Moreover, inspired by Zhang et al. (2016), we assume that the reinsurance premium rate is calculated according to a generalized mean-variance principle. We formulate both insurers' optimization problems as a non-zero sum stochastic differential game. By using dynamic programming approach, we write down a system of coupled Hamilton-Jacobi-Bellman (HJB) equations and present some delicate analyses for the existence and uniqueness of the equilibrium investment-reinsurance strategies. Specially, for the cases of *exponential utility maximizing game*, where the utility functions are exponential utility functions, and the probability maximizing game, we derive equilibrium investment-reinsurance strategies and equilibrium value functions for both insurers in semi-explicit forms. Our results indicate that, for each insurer, competition in the insurance market leads to a decrease in the demand for reinsurance protection and an increase in risky-asset investment. Furthermore, we perform some comparative-static analyses numerically on the equilibrium strategies for the two insurers.

It is worth noting that the reinsurance function adopted in this paper, which is induced by the generalized mean-variance principle, is very general and embeds the proportional reinsurance that is considered by Bensoussan et al. (2014) as a special case, and the excess-of-loss reinsurance that is considered by Siu et al. (2016) as a special case. Moreover, we consider competitive insurers' non-zero sum game for general terminal utilities, but not just exponential utility that are adopted by Bensoussan et al. (2014) and Siu et al. (2016). As such, our paper extends Bensoussan et al. (2014) and Siu et al. (2016). Different utility functions imply different economic backgrounds, serve for different aspects of research, and may provide further actuarial insights. For example, in the probability maximizing game, we show that equilibrium reinsurance strategies interact with investment strategies. However, in the case of exponential utility, reinsurance strategies and investment strategies are independent. In this sense, our results help in further investigating equilibrium strategies of insurers in a competitive insurance market.

The rest of this paper is organized as follows. Section 2 formulates the surplus processes for both insurers and defines the non-zero sum stochastic differential game. In Section 3, we derive a system of coupled HJB equations for the game and analyze the equilibrium strategies. Section 4 provides semi-explicit solutions for the exponential utility maximizing game and the probability maximizing game. Section 5 presents several numerical examples and Section 6 concludes.

2. The model

To make the mathematical formulation rigorous, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a probability space satisfying the usual conditions–that is, the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and $\mathbb{P}$-complete, where $\mathcal{F}_t$ represents the information available up to time t. Besides, let $[0, T]$ be a fixed time horizon during which the insurers can adjust their risk exposures and investment strategies continuously. In what follows, all stochastic processes are assumed to be adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$.

Consider an insurance market with two competing insurers, denoted as insurers

1 and 2, whose surpluses are depicted by the classic risk model:[2]

$$X_k(t) = x_k + p_k t - \sum_{i=1}^{N_k(t)+N(t)} Z_i^k, \quad k = 1, 2,$$

where $x_k \geq 0$ is the initial surplus, $p_k > 0$ is the premium rate, $N_k(t) + N(t)$ represents the number of claims up to time t, and $\{Z_i^k\}$ are the positive claims received by insurer k. The surplus process $\{X_k(t)\}$ indicates that insurers 1 and 2 are subject to common impact that is represented by $\{N(t)\}_{t\geq 0}$.[3] $\{N_1(t)\}_{t\geq 0}$, $\{N_2(t)\}_{t\geq 0}$ and $\{N(t)\}_{t\geq 0}$ are three mutually independent Poisson processes with intensity $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda > 0$. $\{Z_i^k\}$ are independent identically distributed (i.i.d.) random variables and are independent of $\{N_1(t)\}, \{N_2(t)\}$ and $\{N(t)\}$. In addition, we assume that $\{Z_i^k\}$ have finite mean $\mu_k := \mathbb{E}[Z_i^k]$ and second moment $\sigma_k^2 := \mathbb{E}[(Z_i^k)^2]$ with distribution density function $f_k(\cdot)$. Moreover, the premium rates of both insurers are assumed to be calculated according to the *expected value principle*, that is, $p_k = (1 + \theta_k)(\lambda_k + \lambda)\mu_k$, where $\theta_k > 0$ is the *relative safety loading* of insurer k representing the additional premium received by insurer k in each unit of time (see Schmidli (2008)).[4]

2.1. Reinsurance strategy

Suppose that both insurers purchase reinsurance for risk control. That is, for each claim Z_i^k, a proportion $\mathcal{H}_k(Z_i^k)$ specified by a *self-reinsurance function* (see Schmidli (2008)) $\mathcal{H}_k(\cdot)$ is paid by insurer k, and the rest $Z_i^k - \mathcal{H}_k(Z_i^k)$ is paid by the reinsurer. In return, insurer k must allocate a fraction of its premium rate $p_k^{\mathcal{H}}$

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[2] Bai et al. (2013) use a similar model to describe an insurer who has two lines of insurance businesses that are subjected to common shock in the industry. Siu et al. (2016) consider a similar model where $N(t)$ represents the common systematic insurance risk. See also Liang and Yuan (2016), Bi et al. (2016), and so on.

[3] For example, when both insurers offer auto insurance to policy holders, bad weather leads to an increase in claims and vice versa.

[4] Here, we may also consider p_k to be calculated by other premium calculation principles, e.g., variance principle, modified variance principle, and generalized mean-variance principle as presented below, among others. However, since the calculation of p_k has little impact on our results, we choose to adopt expected value principle for simplicity.

(called *reinsurance premium rate*) to the reinsurer and its retention premium rate becomes $p_k - p_k^{\mathcal{H}}$. Here, $p_k^{\mathcal{H}}$ is assumed to be calculated according to the generalized mean-variance principle (see Zhang et al. (2016)), i.e.,

$$p_k^{\mathcal{H}} := (\lambda_k + \lambda)(1 + \eta_k)\Big[\mathbb{E}[Z_i^k - \mathcal{H}_k(Z_i^k)] + \zeta_k \mathbb{E}[Z_i^k - \mathcal{H}_k(Z_i^k)]^2\Big], \tag{1}$$

where $\zeta_k \geq 0$ and $\eta_k \geq 0$ are the *safety loadings* of the reinsurer. With a larger ζ_k, the reinsurer focuses more on the volatility of her risk exposure.[5] Thus, the surplus of insurer k becomes

$$X_k(t) = x_k + (p_k - p_k^{\mathcal{H}})t - \sum_{i=1}^{N_k(t)+N(t)} \mathcal{H}_k(Z_i^k), \quad k = 1, 2. \tag{2}$$

According to Grandell (1991), Bai et al. (2013) and Siu et al. (2016), Eq. (2) can be approximated by the following diffusion process[6]

$$\begin{aligned} X_k(t) = x_k + \int_0^t (\lambda_k + \lambda)\Big[&(\theta_k - \eta_k)\mu_k + \eta_k \mathbb{E}[\mathcal{H}_k(Z_i^k)] \\ &- (1 + \eta_k)\zeta_k \mathbb{E}[Z_i^k - \mathcal{H}_k(Z_i^k)]^2\Big]\mathrm{d}s \\ &+ \int_0^t \sqrt{(\lambda_k + \lambda)\mathbb{E}[\mathcal{H}_k(Z_i^k)]^2}\mathrm{d}B_k(s), \quad \text{for } k = 1, 2, \end{aligned} \tag{3}$$

where $\{B_1(t)\}_{t\geq 0}$ and $\{B_2(t)\}_{t\geq 0}$ are two standard Brownian motions with correlation

[5]Generalized mean-variance principle is closely related to the so called *variance premium principle*, which is detailedly studied by Bühlmann (1979). With variance premium and generalized mean-variance principles, the reinsurer pays attention to not only the expectation but also the volatility of the claims from the insurer. However, as compared to variance premium principle, the generalized mean-variance principle has the desirable property of additivity for comonotonic risks.

[6]The proof of the approximation is challenging but standard. Bai et al. (2013) present a detailed proof when consider optimal reinsurance problem for an insurer. Siu et al. (2016) consider a non-zero sum game problem for two insurers and show that, under the game setting, the Nash equilibrium under the diffusion-approximated process can be approximated by the Nash equilibrium under the general compound Poisson process.

coefficient (see Bai et al. (2013) and Siu et al. (2016))

$$\rho \triangleq \lambda \frac{\mathbb{E}[\mathcal{H}_1(Z_i^1)]}{\sqrt{(\lambda+\lambda_1)\mathbb{E}[\mathcal{H}_1(Z_i^1)]^2}} \frac{\mathbb{E}[\mathcal{H}_2(Z_i^2)]}{\sqrt{(\lambda+\lambda_2)\mathbb{E}[\mathcal{H}_2(Z_i^2)]^2}}.$$

It is clear that $\rho \geq 0$, i.e., the two insurers are positively related to each other.[7] We further assume that both insurers are allowed to dynamically adjust their risk positions and rewrite $\mathcal{H}_k(\cdot)$ in Eq. (3) as $\mathcal{H}_k(s,\cdot)$ to denote insurer k's reinsurance strategy at time s. Specially, we consider the following self-reinsurance function:

$$\widetilde{\mathcal{H}}_k(a_k(t), Z_i^k) \triangleq a_k(t)[\eta_k + 2(1+\eta_k)\zeta_k Z_i^k] \wedge Z_i^k, \tag{4}$$

with $a_k(t) \in [0, \kappa_k]$ and $\kappa_k \triangleq \frac{1}{2(1+\eta_k)\zeta_k}$. The following results are borrowed from Zhang et al. (2016) and the proof is omitted.

Lemma 1. *Given dynamic reinsurance strategy* $\{\mathcal{H}_k(t,\cdot)\}_{t\geq 0}$*, there exists progressively measurable process* $\{a_k(t)\}_{t\geq 0}$ *such that* $\mathbb{E}[\widetilde{\mathcal{H}}_k(a_k(t), Z_i^k)]^2 = \mathbb{E}[\mathcal{H}_k(t, Z_i^k)]^2$ *and*

$$\begin{aligned}&\eta_k\mathbb{E}[\widetilde{\mathcal{H}}_k(a_k(t), Z_i^k)] - (1+\eta_k)\zeta_k\mathbb{E}[Z_i^k - \widetilde{\mathcal{H}}_k(a_k(t), Z_i^k)]^2 \\ &\geq \eta_k\mathbb{E}[\mathcal{H}_k(t, Z_i^k)] - (1+\eta_k)\zeta_k\mathbb{E}[Z_i^k - \mathcal{H}_k(t, Z_i^k)]^2.\end{aligned}$$

Lemma 1 shows that surplus with self-reinsurance function $\mathcal{H}$ is dominated by one with $\widetilde{\mathcal{H}}$. This result still holds when the insurers are allowed to invest in the financial market. Since the utility functions defined below in Eq. (7) are strictly increasing, optimal reinsurance strategy for each insurer is of the form (3). Thus, in the sequel we will only consider reinsurance strategies given by Eq. (4). Moreover, since $\{\widetilde{\mathcal{H}}_k(a_k(t), Z_i^k)\}_{t\geq 0}$ is uniquely characterized by $\{a_k(t)\}_{t\geq 0}$, we shall call $\{a_k(t)\}_{t\geq 0}$ as insurer k's reinsurance strategy. At any time t, with a larger $a_k(t)$, insurer k reduces expenses on reinsurance and pays a larger proportion of each claim by itself. Specially, when $a_k(t) = \kappa_k$, $\widetilde{\mathcal{H}}_k(a_k(t), Z_i^k) = Z_i^k$, i.e., insurer k pays all of

[7] In practice, there are cases where insurers are negatively related to each other. However, the case of negative relationship is out of the scope of this paper and we will leave it for future research.

the claims by herself; when $a_k(t) = 0$, she transfers all claims to the reinsurer.

Remark 1. *Eq. (4) admits two special cases: when $\zeta_k = 0$, $p_k^{\mathcal{H}}$ is calculated according to the expected value principle and the self-reinsurance function $\widetilde{\mathcal{H}}_k(a_k, Z_i^k) = (a_k\eta_k) \wedge Z_i^k$ becomes an excess-of-loss reinsurance type; when $\eta_k = 0$, $\widetilde{\mathcal{H}}_k(a_k, Z_i^k) = 2\zeta_k a_k Z_i^k$ becomes a proportional reinsurance type (see Schmidli (2008) for more details on premium calculation principles and self-reinsurance functions).*

To simplify our notations, we denote (see Zhang et al. (2016))

$$\begin{aligned}\widehat{\mu}_k(a_k(t)) &\triangleq (\lambda_k + \lambda)\Big[\eta_k \mathbb{E}[\widetilde{\mathcal{H}}_k(a_k(t), Z_i^k)] \\ &\qquad - (1+\eta_k)\zeta_k \mathbb{E}\big[(Z_i^k - \widetilde{\mathcal{H}}_k(a_k(t), Z_i^k))^2\big] + (\theta_k - \eta_k)\mu_k\Big], \\ \widehat{\sigma}_k(a_k(t)) &\triangleq \sqrt{(\lambda_k + \lambda)\mathbb{E}\Big[\widetilde{\mathcal{H}}_k(a_k(t), Z_i^k)^2\Big]}.\end{aligned}$$

Then Eq. (3) can be rewritten as

$$\mathrm{d}X_k(t) = \widehat{\mu}_k(a_k(t))\mathrm{d}t + \widehat{\sigma}_k(a_k(t))\mathrm{d}B_k(t), \quad X_k(0) = x_k,$$

where $\widehat{\sigma}_k(a_k(t))$ represents the claim risk of insurer k at time t with upper bound $\widehat{\sigma}_k(\kappa_k) = \sqrt{\lambda_k + \lambda}\sigma_k$. Moreover, let

$$\begin{cases} \iota_k(x) \triangleq \frac{\eta_k x}{1 - 2(1+\eta_k)\zeta_k x}, \\ G_k(x) \triangleq \eta_k + 2(1+\eta_k)\zeta_k x. \end{cases}$$

Direct calculation indicates that

$$\begin{cases} \widehat{\mu}_k(a_k(t)) = (\lambda_k + \lambda)\Big[\eta_k h_k(a_k(t)) - (1+\eta_k)\zeta_k \ell_k(a_k(t)) + (\theta_k - \eta_k)\mu_k\Big], \\ \widehat{\sigma}_k(a_k(t)) = \sqrt{(\lambda_k + \lambda)\int_0^{a_k(t)} 2x \int_{\iota_k(x)}^{\infty} G_k^2(z) f_k(z)\mathrm{d}z\mathrm{d}x}, \\ \rho(t) = \lambda \frac{h_1(a_1(t))h_2(a_2(t))}{\widehat{\sigma}_1(a_1(t))\widehat{\sigma}_2(a_2(t))}, \end{cases} \tag{5}$$

where $h_k(a_k(t)) \triangleq \mathbb{E}[\mathcal{H}_k(a_k(t), Z_i^k)] = \int_0^{a_k(t)} \int_{\iota_k(x)}^{\infty} G_k(z) f_k(z) \mathrm{d}z \mathrm{d}x$ and

$$\begin{aligned}\ell_k(a_k(t)) &\triangleq \mathbb{E}[(Z_i^k - \widetilde{\mathcal{H}}_k(a_k(t), Z_i^k))^2] \\ &= \int_{a_k(t)}^{\kappa_k} \int_{\iota_k(x)}^{\infty} 2[z - xG_k(z)] G_k(z) f_k(z) \mathrm{d}z \mathrm{d}x.\end{aligned}$$

2.2. Financial market

In addition to purchasing reinsurance, both insurers invest in the financial market, including a risk-free asset and a risky asset.[8] The dynamics of the risk-free asset, $\{S_0(t)\}_{t\geq 0}$, are given by

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t,$$

where $r > 0$ denotes the constant, risk-free interest. The dynamics of the risky asset, $\{S(t)\}_{t\geq 0}$, are given by

$$\mathrm{d}S(t) = S(t)[m_S \mathrm{d}t + \sigma_S \mathrm{d}B_S(t)],$$

where $m_S > r$ and $\sigma_S > 0$ denote the return and the volatility of the risky asset S, respectively. We assume that $\{B_1(t)\}_{t\geq 0}$ and $\{B_2(t)\}_{t\geq 0}$ are independent of $\{B_S(t)\}_{t\geq 0}$, indicating that the insurance market is independent of the financial market. This is inline with common practice.

Let $b_k(t)$ be the amount that insurer k invests in the risky asset S at time t and denote $\pi_k(t) \triangleq \{a_k(t), b_k(t)\}_{t\geq 0}$. Then, insurer k's surplus can be modeled by

$$\begin{aligned}\mathrm{d}X_k^{\pi_k}(t) = &\Big[rX_k^{\pi_k}(t) + \widehat{\mu}_k(a_k(t)) + b_k(t)(m_S - r)\Big]\mathrm{d}t \\ &+ \widehat{\sigma}_k(a_k(t))\mathrm{d}B_k(t) + b_k(t)\sigma_S \mathrm{d}B_S(t), \quad \text{for } k = 1, 2. \end{aligned} \tag{6}$$

Here, we use "π_k" as a superscript to indicate that $X_k^{\pi_k}(t)$ is a controlled process.

[8] Our results can be easily extended to the general case with multiple risky assets. Since this extension does not provide more economic insights to our problem, for simplicity we only consider model setup with single risky asset.

Definition 2. *The strategy* $\{\pi_k(t)\}_{t\geq 0}$ *is said to be admissible if:*

(i) $\{a_k(t)\}_{t\in[0,T]}$ *and* $\{b_k(t)\}_{t\in[0,T]}$ *are* $\mathcal{F}$*-progressively measurable processes;*

(ii) $a_k(t) \in [0, \kappa_k]$ *and* $\mathbb{E}[\int_0^T b_k^2(t)\mathrm{d}t] < +\infty$.

The set of all admissible strategies for insurer k *is denoted by* Π_k.

Given an admissible strategy π_k, the stochastic differential equation in Eq. (6) admits a unique strong solution.

Inspired by Epsinosa and Touzi (2013), Bensoussan et al. (2014), Meng et al. (2015) and Pun and Wong (2016), we assume that both insurers' objectives are to maximize their expected utilities of relative performance at the terminal time T. That is, given the strategy π_m of insurer m ($m = 1$ or 2), the other insurer k will choose an admissible investment-reinsurance strategy $\pi_k = (a_k, b_k)$ such that

$$\begin{aligned}&\mathbb{E}\Big[U_k((1-\alpha_k)X_k^{\pi_k}(T) + \alpha_k(X_k^{\pi_k}(T) - X_m^{\pi_m}(T)))\Big]\\&= \mathbb{E}\Big[U_k(X_k^{\pi_k}(T) - \alpha_k X_m^{\pi_m}(T))\Big]\end{aligned} \tag{7}$$

is maximized, where $U_k(\cdot)$ is the strictly increasing utility function of insurer k.[9] The two insurers maximize their utilities simultaneously and thus their optimization problems form a stochastic differential game.

Problem 3. *Find a Nash equilibrium* $(\pi_1^*, \pi_2^*) \in \Pi_1 \times \Pi_2$ *such that*

$$\begin{aligned}\mathbb{E}\left[U_1\big(X_1^{\pi_1}(T) - \alpha_1 X_2^{\pi_2^*}(T)\big)\right] &\leq \mathbb{E}\left[U_1\big(X_1^{\pi_1^*}(T) - \alpha_1 X_2^{\pi_2^*}(T)\big)\right],\\ \mathbb{E}\left[U_2\big(X_2^{\pi_2}(T) - \alpha_2 X_1^{\pi_1^*}(T)\big)\right] &\leq \mathbb{E}\left[U_2\big(X_2^{\pi_2^*}(T) - \alpha_2 X_1^{\pi_1^*}(T)\big)\right].\end{aligned}$$

[9] Espinosa and Touzi (2013), Bensoussan et al. (2014) and Pun et al. (2016) assume that the utility functions U_1 and U_2 are strictly concave and satisfy Inada conditions: $\lim_{y\to-\infty} \frac{\partial U_k(y)}{\partial y} = +\infty, \lim_{+\infty} \frac{\partial U_k(y)}{\partial y} = 0$ for $k = 1, 2$. However, to accommodate for more utility functions, e.g. the probability that insurer k's relative performance reaches a higher level before it reaches a lower level, see Section 4.2, we do not impose these conditions here.

Here, $\alpha_k \in [0,1]$ measures the sensitivity of insurer k to the performance of its competitor (insurer m). With a larger α_k, insurer k pays more attention to her relative performance to insurer m, and the game becomes more competitive. The values of α_1 and α_2 constitute three special cases:

Case (i) when $\alpha_1 = \alpha_2 = 1$, Problem 3 becomes a *zero-sum game* (see Browne (2000), Zeng (2010) and Zeng and Luo (2013));

Case (ii) when $\alpha_1\alpha_2 < 1$ and $\alpha_1 + \alpha_2 > 0$, Problem 3 defines a *non-zero sum game* between the two insurers (see Bensoussan et al. (2014), Meng et al. (2015));

Case (iii) when $\alpha_1 = \alpha_2 = 0$, both insurers are indifferent about each other and Problem 3 retreats to two single-player problems (see Zhang et al. (2016)).

In the sequel, we focus on a non-zero sum game problem and assume that $\alpha_k \in [0,1]$, for $k = 1, 2$, and $\alpha_1 + \alpha_2 > 0, \alpha_1\alpha_2 < 1$.

3. General results

This section provides general results on Problem 3. To obtain an equilibrium strategy for this problem, we start by using stochastic dynamic programming approach. To this end, let

$$Y_k^{\pi_k,\pi_m}(t) \triangleq X_k^{\pi_k}(t) - \alpha_k X_m^{\pi_m}(t).$$

Then the difference of the two insurers' surplus processes is governed by the following dynamics

$$\begin{aligned} \mathrm{d}Y_k^{\pi_k,\pi_m}(t) = &\Big[rY_k^{\pi_k,\pi_m}(t) + \widehat{\mu}_k(a_k(t)) - \alpha_k\widehat{\mu}_m(a_m(t)) + (b_k(t) - \alpha_k b_m(t))(m_S - r)\Big]\mathrm{d}t \\ &+ \widehat{\sigma}_k(a_k(t))\mathrm{d}B_k(t) - \alpha_k\widehat{\sigma}_m(a_m(t))\mathrm{d}B_m(t) \\ &+ \big(b_k(t) - \alpha_k b_m(t)\big)\sigma_S\mathrm{d}B_S(t), \end{aligned} \tag{8}$$

with $Y_k^{\pi_k,\pi_m}(0) = y_k \triangleq x_k - \alpha_k x_m$.

Denote $\mathcal{O} \triangleq [0,T)\times\mathbb{R}$ and $\bar{\mathcal{O}} \triangleq [0,T]\times\mathbb{R}$. Then, for $t \in [0,T]$ and $Y_k^{\pi_k,\pi_m}(t) = y_k$, we define the optimal value function $V^k : \mathcal{O} \mapsto \mathbb{R}$ for insurer k by

$$V^k(t, y_k; \pi_m) \triangleq \sup_{\pi_k\in\Pi_k} \mathbb{E}_{t,y_k}\Big[U_k(Y_k^{\pi_k,\pi_m}(T))\Big], \quad k = 1, 2, \tag{9}$$

where $\mathbb{E}_{t,y_k}[\cdot] \triangleq \mathbb{E}_{t,y_k}[\cdot|Y_k^{\pi_k,\pi_m}(t) = y_k]$ is the conditional expectation. Eq. (9) represents the largest utility that insurer k achieves at time t when the difference of surplus is y_k and her competitor adopts strategy π_m. As such, it is a measure of insurer k's welfare.

For $v \in \mathbb{C}^{1,2}(\mathcal{O}) \cap \mathbb{C}^0(\bar{\mathcal{O}})$ and $\pi_k = (a_k, b_k)$, let

$$\begin{aligned}\mathcal{L}_k^{\pi_k,\pi_m} v(t, y_k) \triangleq& \big[ry_k + \widehat{\mu}_k(a_k) - \alpha_k\widehat{\mu}_m(a_m)\big]v_y(t, y_k)\\ &+ \frac{1}{2}\Big[\widehat{\sigma}_k^2(a_k) + \alpha_k^2\widehat{\sigma}_m^2(a_m) - 2\alpha_k\lambda h_k(a_k)h_m(a_m)\Big]v_{yy}(t, y_k)\\ &+ (b_k - \alpha_k b_m)(m_S - r)v_y(t, y_k) + \frac{1}{2}(b_k - \alpha_k b_m)^2\sigma_S^2 v_{yy}(t, y_k),\end{aligned}$$

where $v_y(\cdot,\cdot)$ and $v_{yy}(\cdot,\cdot)$ respectively denote the first- and second-order derivatives with respect to the second derivative. By standard arguments in stochastic dynamic programming (SDP, see Yong and Zhou (1999)), if $V^k \in \mathbb{C}^{1,2}(\mathcal{O}) \cap \mathbb{C}^0(\bar{\mathcal{O}})$, then V^k satisfies the following HJB equation

$$\begin{cases} v_t^k(t, y_k) + \max\limits_{\pi_k\in\Pi_k} \mathcal{L}^{\pi_k,\pi_m} v^k(t, y_k) = 0,\\ v^k(T, y_k) = U_k(y_k).\end{cases} \tag{10}$$

Analogous to Bensoussan et al. (2014), we have the following verification theorem.

Theorem 4 (**Verification Theorem**). *For $k = 1, 2$ and $\pi_m \in \Pi_m$, if Eq. (10) admits solution $v^k \in \mathbb{C}^{1,2}(\mathcal{O}) \cap \mathbb{C}^0(\bar{\mathcal{O}})$ satisfying $\mathbb{E}\int_0^T [v_y^k(t, Y^{\pi_k,\pi_m}(t))]^2 \mathrm{d}t < \infty$ for all $(\pi_k, \pi_m) \in \Pi_k \times \Pi_m$, then $v^k(t, y_k) \geq V^k(t, y_k; \pi_m)$. Moreover, if there exists the pair of admissible strategies (π_1^*, π_2^*), where*

$$\pi_k^* \triangleq \arg\max_{\pi_k\in\Pi_k} \mathcal{L}_k^{\pi_k,\pi_m} v^k(t, y_k), \quad \textit{for } \pi_m \in \Pi_m, k \neq m \in \{1, 2\},$$

such that Eq. (10) holds, then $v^k(t,y_k) = V^k(t,y_k,\pi_m^*)$ and (π_1^*,π_2^*) is the Nash equilibrium for Problem 3.

Proof. See Theorems 2 and 3 of Bensoussan et al. (2014). □

According to Theorem 4, we need to find solutions $v^1, v^2 \in \mathbb{C}^{1,2}(\bar{\mathcal{O}}) \cap \mathbb{C}^0(\mathcal{O})$ (if exist) to the HJB equations (10) and find the corresponding equilibrium strategy pair (π_1^*,π_2^*).

Suppose that $v_y^k(t,y_k) > 0$ and $v_{yy}^k(t,y_k) < 0$ (this assumption will be rigorously verified once we have obtained v^k explicitly). Then, using the first-order condition in Eq. (10), the pair of optimal investment strategies (b_1^*, b_2^*) for insurers 1 and 2 satisfy

$$\begin{cases} b_1^*(t) = \alpha_1 b_2^*(t) - \frac{m_S - r}{\sigma_S^2}\frac{v_y^1(t,y_1)}{v_{yy}^1(t,y_1)}, \\ b_2^*(t) = \alpha_2 b_1^*(t) - \frac{m_S - r}{\sigma_S^2}\frac{v_y^2(t,y_2)}{v_{yy}^2(t,y_2)}. \end{cases} \tag{11}$$

Let (a_1^*, a_2^*) be the pair of optimal reinsurance strategies for insurers 1 and 2. Substituting Eq. (11) into Eq. (10) yields the following system of coupled non-linear partial differential equations (PDEs):

$$\begin{cases} v_t^1(t,y_1) - \frac{(m_S-r)^2}{2\sigma_S^2}\frac{[v_y^1(t,y_1)]^2}{v_{yy}^1(t,y_1)} + \max\limits_{a_1\in[0,\kappa_1]} \mathbb{G}^{a_1,a_2^*} v^1(t,y_1) = 0, \\ v_t^2(t,y_2) - \frac{(m_S-r)^2}{2\sigma_S^2}\frac{[v_y^2(t,y_2)]^2}{v_{yy}^2(t,y_2)} + \max\limits_{a_2\in[0,\kappa_2]} \mathbb{G}^{a_2,a_1^*} v^2(t,y_2) = 0, \end{cases} \tag{12}$$

where the operator $\mathbb{G}$ is defined as

$$\begin{aligned} \mathbb{G}^{a_k,a_m^*} v_k(t,y_k) \triangleq & \Big[r y_k + \widehat{\mu}_k(a_k) - \alpha_k \widehat{\mu}_m(a_m^*) \Big] v_y^k(t,y_k) \\ & + \frac{1}{2}\Big[\widehat{\sigma}_k^2(a_k) + \alpha_k^2 \widehat{\sigma}_m^2(a_m^*) - 2\alpha_k \lambda h_k(a_k) h_m(a_m^*) \Big] v_{yy}^k(t,y_k). \end{aligned}$$

The following lemma shows the existence and uniqueness of (a_1^*, a_2^*) in (12) by assuming that solutions to Eq. (12) exist and satisfy certain conditions.

Lemma 5. *Suppose that solutions to Eq. (10), $v^1, v^2 \in \mathbb{C}^{1,2}(\mathcal{O}) \cap \mathbb{C}^0(\bar{\mathcal{O}})$, exist and are strictly increasing and concave. Then, at any time $t \in [0,T]$, the insurers'* *equilibrium reinsurance strategies $a_1^*(t) \in [0, \kappa_1]$ and $a_2^*(t) \in [0, \kappa_2]$ exist and are uniquely characterized by the following system of non-linear equations:*

$$\begin{cases} a_1(\frac{v^1_{yy}(t,y_1)}{v^1_y(t,y_1)} - \frac{1}{\kappa_1}) + 1 = \alpha_1 \frac{\lambda}{\lambda+\lambda_1}\psi_1(a_1)h_2(a_2)\frac{v^1_{yy}(t,y_1)}{v^1_y(t,y_1)}, \\ a_2(\frac{v^2_{yy}(t,y_2)}{v^2_y(t,y_2)} - \frac{1}{\kappa_2}) + 1 = \alpha_2 \frac{\lambda}{\lambda+\lambda_2}\psi_2(a_2)h_1(a_1)\frac{v^2_{yy}(t,y_2)}{v^2_y(t,y_2)}, \end{cases} \tag{13}$$

where $\psi_k(a_k) \triangleq \frac{\int_{\iota_k(a_k)}^{\infty} G_k(z) f_k(z) \mathrm{d}z}{\int_{\iota_k(a_k)}^{\infty} G_k^2(z) f_k(z) \mathrm{d}z}$.

Proof. See Appendix A.1. □

Since $\{Z_i^k\}$ has finite mean and second moment, ψ_k in Lemma 5 is well defined. Also, the proof of Lemmas 5 provides a hint for constructing the equilibrium reinsurance strategies (a_1^*, a_2^*) numerically.

4. Examples

In this section, we consider the special cases of exponential utility maximizing game and probability maximizing game, and present semi-explicit solutions to illustrate our results.

4.1. Exponential utility maximizing game

Firstly, we consider the case where both insurers adopt exponential utility functions

$$U_k(x) = -\frac{e^{-\gamma_k x}}{\gamma_k}, \quad k = 1, 2, \tag{14}$$

where the two positive constants γ_1 and γ_2 are coefficients of absolute risk aversion for both insurers. Since Eq. (14) is the only utility function under which the principle of "zero utility" gives a fair premium that is independent of the level of reserves of an insurer, Eq. (14) plays an important role in insurance mathematics and actuarial practice and has been widely adopted by Yang and Zhang (2005), Bensoussan et al. (2010), and so on.

Inspired by Zhang et al. (2016), we make the following *Ansatz*:

$$v^k(t, y_k) = -\frac{A_k(t)}{\gamma_k} \exp\big\{ -\gamma_k y_k e^{r(T-t)} \big\}, \tag{15}$$

where $A_k(t) > 0$ is to be determined and satisfies $A_k(T) = 1$. This leads to

$$\begin{cases} v_t^k(t, y_k) = -\left[\frac{A_k'(t)}{\gamma_k} + A_k(t) r y_k e^{r(T-t)}\right] e^{-\gamma_k y e^{r(T-t)}}, \\ v_y^k(t, y_k) = A_k(t) e^{r(T-t)} e^{-\gamma_k y_k e^{r(T-t)}}, \\ v_{yy}^k(t, y_k) = -\gamma_k A_k(t) e^{2r(T-t)} e^{-\gamma_k y_k e^{r(T-t)}}. \end{cases} \tag{16}$$

Thus, by substituting these expressions into Eqs. (11) and (13), we can obtain the equilibrium reinsurance strategies $\{a_1^*(t)\}$ and $\{a_2^*(t)\}$ by solving the system of equations

$$\begin{cases} -a_1(\gamma_1 e^{r(T-t)} + \frac{1}{\kappa_1}) + 1 = -\alpha_1 \frac{\lambda}{\lambda+\lambda_1} \psi_1(a_1) h_2(a_2) \gamma_1 e^{r(T-t)}, \\ -a_2(\gamma_2 e^{r(T-t)} + \frac{1}{\kappa_2}) + 1 = -\alpha_2 \frac{\lambda}{\lambda+\lambda_2} \psi_2(a_2) h_1(a_1) \gamma_2 e^{r(T-t)} \end{cases} \tag{17}$$

numerically, and obtain the equilibrium investment strategies $\{b_1^*(t)\}$ and $\{b_2^*(t)\}$ explicitly as following:

$$\begin{cases} b_1^*(t) = (\frac{1}{\gamma_1} + \alpha_1 \frac{1}{\gamma_2}) \frac{1}{(1-\alpha_1\alpha_2)} \frac{(m_S - r)}{\sigma_S^2} e^{-r(T-t)}, \\ b_2^*(t) = (\frac{1}{\gamma_2} + \alpha_2 \frac{1}{\gamma_1}) \frac{1}{(1-\alpha_1\alpha_2)} \frac{(m_S - r)}{\sigma_S^2} e^{-r(T-t)}. \end{cases} \tag{18}$$

Finally, by substituting the values of $(a_k^*(t), b_k^*(t))$ and Eq. (16) into Eq. (12), the PDE for v_k becomes an ordinary differential equation (ODE) for A_k:

$$\frac{A_k'(t)}{\gamma_k} = A_k(t) Q_k(t), \quad \text{for } k = 1, 2,$$

where

$$Q_k(t) \triangleq \frac{(m_S - r)^2}{2\sigma_S^2} + \widehat{\mu}_k(a_k^*) - \alpha_k \widehat{\mu}_m(a_m^*)]e^{r(T-t)}$$
$$- \frac{\gamma_k}{2} \left[\widehat{\sigma}_k^2(a_k^*) + \alpha_k^2 \widehat{\sigma}_m^2(a_m^*) - 2\alpha_k \lambda h_k(a_k^*) h_m(a_m^*)\right] e^{2r(T-t)}.$$

By using $A_k(T) = 1$, we have

$$A_k(t) = e^{-\int_t^T \gamma_k Q_k(s) \mathrm{d}s}. \tag{19}$$

It is clear that $v^k(t, y_k)$ is strictly increasing and concave in y_k and that $v_y^k(t, y_k)$ is bounded above. Thus, all conditions in Theorem 4 are satisfied and the following results are straightforward.

Proposition 6. *Suppose that insurers 1 and 2 have exponential utilities* (14)*, in equilibrium the value functions are given by Eq.* (15) *with* $A_k(t)$ *given by Eq.* (19)*, the corresponding investment-reinsurance strategy,* $\pi_k^* = \{a_k^*(t), b_k^*(t)\}_{t\in[0,T]}, k = 1, 2,$ *is given by Eqs.* (17) *and* (18)*.*

When $\zeta_1 = \zeta_2 = 0$, the equilibrium strategies (π_1^*, π_2^*) retreat to that in Bensoussan et al. (2014) and that in Pun et al. (2016) without ambiguous correlations and risky asset investment.

Eq. (18) indicates that each insurer's risk attitude and sensitivity parameter have great impact on both insurers' investment strategies. When insurer $k(k = 1, 2)$ becomes more risk-aversive (represented by a larger γ_k), insure k invests less money in the risky asset and allocates more surplus in the risk-free asset. Conversely, when insurer k becomes more competitive and is more sensitive to her relative performance at time T (represented by a larger α_k), insurer k invests more money in the risky asset. Accordingly, the other insurer (insurer $m(m \neq k)$) also increases her risky asset investment. Thus, insurers in a more competitive insurance market should invest more money in the risky asset, as opposed to insurers in a less competitive insurance market.

As an example, we assume that claims for both insurers are exponentially dis-

Table 1: Default model parameters

σ	r	m_S	T	λ
0.4	0.07	0.1	10	0.8

	θ_k	η_k	λ_k	γ_k	α_k	ζ_k	δ_k
Insurer 1	0.5	0.7	1.0	0.3	0.2	0.1	0.5
Insurer 2	0.5	0.7	0.8	0.5	0.5	0.1	0.5

tributed, i.e., $f_k(z) = \delta_k e^{-\delta_k z}(\delta_k > 0)$, for $k = 1, 2$. Moreover, we set the default parameter values as given in Table 1. Fig. 1 displays the equilibrium investment-reinsurance strategies of both insurers. The numerical results show that both a_k^*

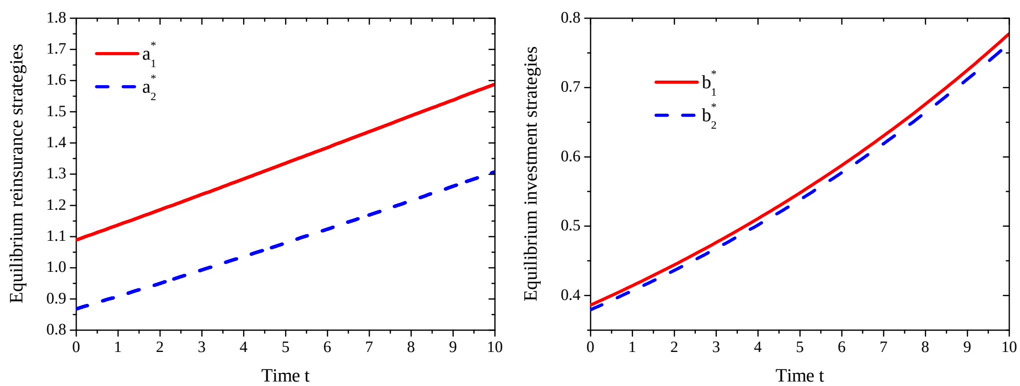

Fig. 1: Equilibrium strategies for exponential utility functions.

and b_k^* increase with t. That is, as the deadline gets closer both insurers decrease their demand for reinsurance and increase their amounts of risky-asset investment. A possible reason is that, when time t is close to the deadline T, both insurers face less uncertainty and thus act more boldly.

4.2. The probability maximizing game

Secondly, we consider a special utility function, where for given constants $0 \leq l_k < u_k$ and her competitor's strategy π_m, insurer k aims to maximize the probability of $Y_k^{\pi_k, \pi_m}(t)$ reaching the upper bound u_k before reaching the lower bound l_k. Problems with related objectives are first considered by Browne (2000), and later on by Zeng (2010) and Taksar and Zeng (2013).

Define stopping times $\tau_l^k \triangleq \inf\{t \geq 0 : Y^{\pi_k,\pi_m}(t) \leq l_k\}$ and $\tau_u^k \triangleq \inf\{t \geq 0 : Y^{\pi_k,\pi_m}(t) \geq u_k\}$. Then, given her competitor's investment-reinsurance strategy π_m, insurer k's optimal value function is defined as

$$V_k(y_k;\pi_m) \triangleq \sup_{\pi_k\in\Pi_k} \mathbb{P}(\tau_u^k < \tau_l^k).$$

In this case, V_k and the associated investment-reinsurance strategies are independent of time t. Besides, V_k satisfies the following boundary conditions:

$$V_k(u_k;\pi_m) = 1, \quad V_k(l_k;\pi_m) = 0. \tag{20}$$

We also assume that $r = 0$ so as to obtain explicit solution for the non-zero sum game problem. For this case we observe that, once the pair of equilibrium reinsurance strategies a_1^* and a_2^* are determined according to Eq. (13) or Eq. (24) below, the system of ODEs (12) becomes

$$\begin{cases} -\frac{1}{2}\frac{m_S^2}{\sigma_S^2}\frac{(v_y^1(y_1))^2}{(v_{yy}^1(y_1))^2} + (\widehat{\mu}_1 - \alpha_1\widehat{\mu}_2)\frac{v_y^1(y_1)}{v_{yy}^1(y_1)} + \frac{1}{2}(\widehat{\sigma}_1^2 + \alpha_1^2\widehat{\sigma}_2^2 - 2\alpha_1\lambda h_1 h_2) = 0, \\ -\frac{1}{2}\frac{m_S^2}{\sigma_S^2}\frac{(v_y^2(y_2))^2}{(v_{yy}^2(y_2))^2} + (\widehat{\mu}_2 - \alpha_2\widehat{\mu}_1)\frac{v_y^2(y_2)}{v_{yy}^2(y_2)} + \frac{1}{2}(\widehat{\sigma}_2^2 + \alpha_2^2\widehat{\sigma}_1^2 - 2\alpha_2\lambda h_1 h_2) = 0, \end{cases} \tag{21}$$

where we drop the dependence of $\widehat{\mu}_k, \widehat{\sigma}_k$ and h_k on a_k to simplify our notations. By solving Eq. (21) with boundary condition (20), we have

$$v^k(y_k) = \frac{1 - e^{-\Lambda_k(y_k - l_k)}}{1 - e^{-\Lambda_k(u_k - l_k)}}, \quad \text{for } k = 1, 2, \tag{22}$$

where

$$\Lambda_k := \frac{\frac{m_S^2}{\sigma_S^2}}{\sqrt{(\widehat{\mu}_k - \alpha_k\widehat{\mu}_m)^2 + \frac{m_S^2}{\sigma_S^2}(\widehat{\sigma}_k^2 + \alpha_k^2\widehat{\sigma}_m^2 - 2\alpha_k\lambda h_k h_m)} - (\widehat{\mu}_k - \alpha_k\widehat{\mu}_m)} > 0. \tag{23}$$

It is straight forward to verify that v^k is strictly increasing and concave. Thus, according to Theorem 4, v^1 and v^2 coincide with the equilibrium value functions V^1 and V^2. By substituting v^1 and v^2 into Eqs. (11) and (13) and then simplifying both systems of equations, we obtain the equilibrium reinsurance strategies a_1^* and a_2^* by solving the system of equations

$$\begin{cases} -a_1(\Lambda_1 + \frac{1}{\kappa_1}) + 1 = -\alpha_1 \frac{\lambda}{\lambda+\lambda_1} \psi_1(a_1) h_2(a_2) \Lambda_1, \\ -a_2(\Lambda_2 + \frac{1}{\kappa_2}) + 1 = -\alpha_2 \frac{\lambda}{\lambda+\lambda_2} \psi_2(a_2) h_1(a_1) \Lambda_2 \end{cases} \tag{24}$$

numerically, and the equilibrium investment strategies b_1^* and b_2^* as following:

$$\begin{cases} b_1^* = \frac{1}{1-\alpha_1\alpha_2} \big[\alpha_1 \frac{m_S}{\sigma_S^2} \Lambda_2 + \Lambda_1\big], \\ b_2^* = \frac{1}{1-\alpha_1\alpha_2} \big[\alpha_2 \frac{m_S}{\sigma_S^2} \Lambda_1 + \Lambda_2\big]. \end{cases} \tag{25}$$

To sum up, we have the following results.

Proposition 7. *For the probability maximizing game, in equilibrium the value functions are given by Eq.* (22)*, and the corresponding investment-reinsurance strategies,* $(a_k^*, b_k^*), k = 1, 2$*, are given by Eqs.* (24) *and* (25)*.*

It is interesting to see from Eqs. (24) and (25) that, while we have set the upper- and lower- boundaries u_k and l_k for insurer k, both boundaries don't play any roles in characterizing the equilibrium investment-reinsurance strategies for both insurers. Also, the investment strategies (b_1^*, b_2^*) in (25) are different from those in (18) in that they are connected to the reinsurance strategies (a_1^*, a_2^*) and are dependent on the model parameters: μ_k, θ_k, η_k and $\zeta_k,\ k = 1, 2$. With parameters given in Table 1, we obtain the equilibrium reinsurance strategies $(a_1^*, a_2^*) = (2.3646, 2.7523)$ and the equilibrium investment strategies $(b_1^*, b_2^*) = (0.0969, 0.0556)$.

5. Numerical example

In this section, we present some numerical examples to better understand the effects of the model parameters on the equilibrium investment-reinsurance strategy $\{a_k^*, b_k^*\}, k = 1, 2$. We assume that both insurers adopt exponential utility functions and set the default model parameters as given in Table 1. Due to the symmetric nature of the game, in the following we only vary insurer 1's parameters to analyze their impact on both insurers' equilibrium strategies. Besides, since at any time $t \in [0, T]$ the relative value of equilibrium strategies exhibit in a similar pattern as when $t = 0$ (see Fig. 1), we only consider the impact of model parameters on $(a_k^*(0), b_k^*(0)), k = 1, 2$.

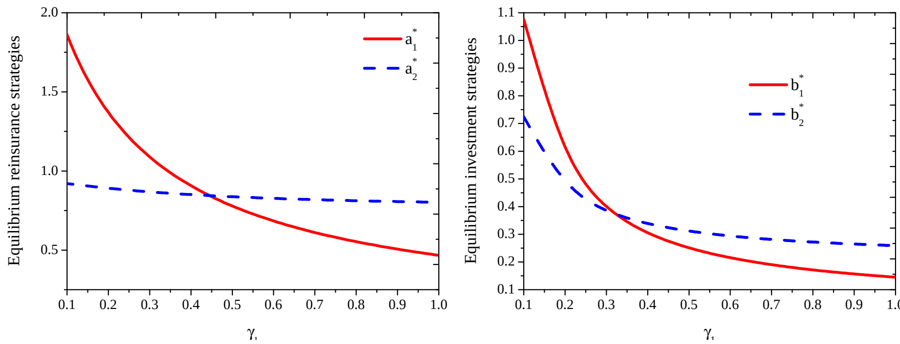

Fig. 2: Effect of γ_1 on the equilibrium strategies.

Fig. 2 shows the effects of risk-aversion parameter γ_1 on the equilibrium strategies. With a larger γ_1, insurer 1 becomes more risk-aversive and thus chooses a smaller retention level a_1^* to transfer more risks to the reinsurer and invests less money on the risky asset. This leads to a decrease in insurer 1's profits and alleviates the pressure of insurer 2. Consequently, the increase of insurer 1's risk attitude also leads to a decrease in insurer 2's retention level and risky-asset investment, but with a smaller extent.

Fig. 3 shows the impact of α_1 on the equilibrium strategies. One can see that, in a competitive insurance market, each insurer becomes more risk-seeking by purchasing less reinsurance and investing more money on the financial market. Indeed, when insurer 1 becomes more competitive (represented by a larger α_1), she chooses to cut down on reinsurance expenditures and invest more money in the risky asset so as to increase her profits and to have a better relative performance with respect to her competitor. As a response, insurer 2 also increases her profits by decreasing her expenditures on the reinsurance and increasing her risky-asset investment. Similar

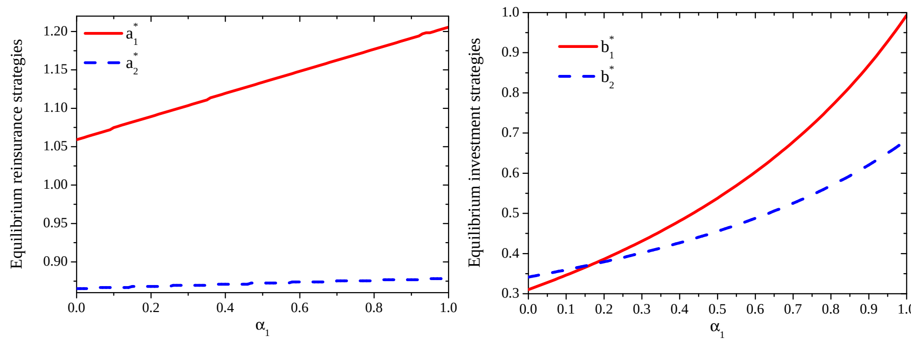

Fig. 3: Effect of α_1 on the equilibrium strategies.

result has been reported by Bensoussan et al. (2014), Siu et al. (2016), and so on. It indicates that the competition between insurers carries considerable impact on the insurers' reinsurance and investment decision-making and should not be neglected. In a competitive insurance market all insurers face a large risk exposure and should be subject to a stricter risk monitoring imposed by the regulatory authority.

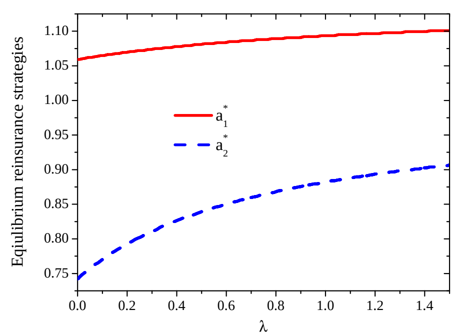

Fig. 4: Effect of λ on the equilibrium strategies.

Fig. 4 shows the effect of λ on the insurers' equilibrium reinsurance strategies (note that λ does not impact both insurers' investment strategies). With a larger λ, both insurers are subject to a larger degree of common impact and become more closely related to each other. In this case, high level of reinsurance protection would not increase their relative terminal surplus, thus both insurers choose to increase their retention levels and pay more claims by themselves.

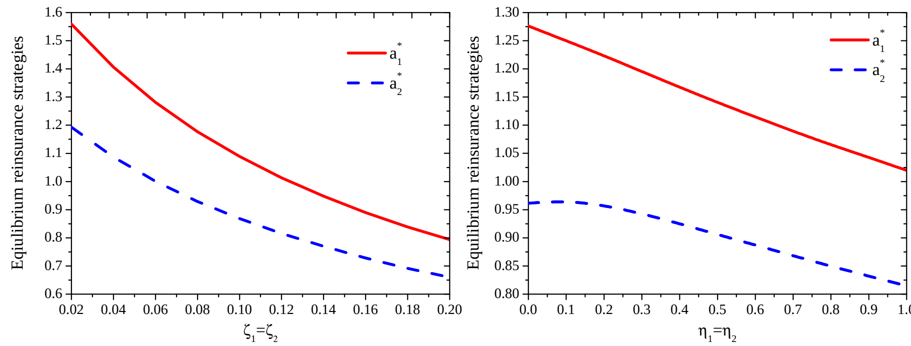

Fig. 5: Effect of ζ_k and η_k on the equilibrium reinsurance strategies.

Assuming that both insurers are subject to the same reinsurance safety loadings, Fig. 5 shows the effect of ζ_k and η_k on the insurers' equilibrium reinsurance strategies. As ζ_k or η_k increase, the reinsurer pays more concern on her risk exposures and charges more for them. Consequently, both insurers decrease their demand for reinsurance and pays more claims by themselves.

6. Conclusion

In this paper, we study a non-zero sum stochastic differential game between two insurers who invest in the financial market and use reinsurance for risk management. We assume that the reinsurance premium rate is calculated according to the generalized mean-variance principle. When the insurers have general utility functions, we derive a system of coupled HJB equations and provide sufficient conditions to guarantee the existence of equilibrium investment-reinsurance strategies. For the cases of exponential maximizing game and probability maximizing game, we obtain semi-explicit solutions, including the value functions and the equilibrium strategies, for both insurers. Our results show that, for each insurer, the competition in insurance market leads to an increase in the demand for the risky-asset investment and a decrease in the demand for reinsurance.

Our results can be extended in two directions. First, instead of considering game problem between two competitive insurers, to make the model more realistic, we may adopt the ideas of Espinosa and Touzi (2013) by considering the game problem for multiple competitive insurers. Second, one may consider the game problem between insurer and reinsurer.

Appendix A. Proof

Appendix A.1. Proof of Lemma 5

Proof. Direct calculation shows

$$
\begin{aligned}
&\frac{\mathrm{d}\widehat{\mu}_k(a_k)}{\mathrm{d}a_k} = (\lambda_k+\lambda)\left[\eta_k\frac{\mathrm{d}h_k(a_k)}{\mathrm{d}a_k} - (1+\eta_k)\zeta_k\frac{\mathrm{d}\ell_k(a_k)}{\mathrm{d}a_k}\right] \\
&= (\lambda_k+\lambda)\Big[\int_{\iota_k(a_k)}^{\infty} G_k(z)f_k(z)(\eta_k+2(1+\eta_k)\zeta_k z)\mathrm{d}z - 2(1+\eta_k)\zeta_k a_k\int_{\iota_k(a_k)}^{\infty} G_k^2(z)f_k(z)\mathrm{d}z\Big] \\
&= (\lambda_k+\lambda)\left[\int_{\iota_k(a_k)}^{\infty} G_k^2(z)f_k(z)\mathrm{d}z(1-2(1+\eta_k)\zeta_k a_k)\right] \\
&= -\frac{1}{2}\frac{\mathrm{d}\widehat{\sigma}_k^2(a_k)}{\mathrm{d}a_k}\Big[2(1+\eta_k)\zeta_k - \frac{1}{a_k}\Big].
\end{aligned}
$$

Thus, by differentiating $\mathbb{G}^{a_k,a_m}v_k(t,y_k)$ with respect to a_k, we obtain that the extreme maximum point satisfies

$$
\begin{aligned}
0 &= v_y^k(t,y_k)\frac{\mathrm{d}\widehat{\mu}_k(a_k)}{\mathrm{d}a_k} + \frac{1}{2}v_{yy}^k(t,y_k)[\frac{\mathrm{d}\widehat{\sigma}_k^2(a_k)}{\mathrm{d}a_k} - 2\alpha_k\lambda h_m(a_m^*)\frac{\mathrm{d}h_k(a_k)}{\mathrm{d}a_k}] \\
&= -\frac{1}{2}v_y^k(t,y_k)\frac{\mathrm{d}\widehat{\sigma}_k^2(a_k)}{\mathrm{d}a_k}[2(1+\eta_k)\zeta_k - \frac{1}{a_k}] + \frac{1}{2}v_{yy}^k(t,y_k)\frac{\mathrm{d}\widehat{\sigma}_k^2(a_k)}{\mathrm{d}a_k} \\
&\quad - \alpha_k\lambda h_m(a_m)v_{yy}^k(t,y_k)\frac{\mathrm{d}h_k(a_k)}{\mathrm{d}a_k} \\
&= \frac{1}{2}\frac{\mathrm{d}\widehat{\sigma}_k^2(a_k)}{\mathrm{d}a_k}\left[v_{yy}^k(t,y_k) - v_y^k(t,y_k)[2(1+\eta_k)\zeta_k - \frac{1}{a_k}]\right] - \alpha_k\lambda h_m(a_m)v_{yy}^k(t,y_k)\frac{\mathrm{d}h_k(a_k)}{\mathrm{d}a_k} \\
&= (\lambda+\lambda_k)\int_{\iota_k(a_k)}^{\infty} G_k^2(z)f_k(z)\mathrm{d}z\left[a_k v_{yy}^k(t,y_k) - v_y^k(t,y_k)(\frac{a_k}{\kappa_k}-1)\right] \\
&\quad - \alpha_k\lambda h_m(a_m)v_{yy}^k(t,y_k)\int_{\iota_k(a_k)}^{\infty} G_k(z)f_k(z)\mathrm{d}z.
\end{aligned}
$$

That is,

$$a_k(\frac{v_{yy}^k(t,y_k)}{v_y^k(t,y_k)} - \frac{1}{\kappa_k}) + 1 = \alpha_k \frac{\lambda}{\lambda+\lambda_k} h_m(a_m) \frac{v_{yy}^k(t,y_k)}{v_y^k(t,y_k)} \psi_k(a_k), \quad k \neq m \in \{1,2\}.$$

For $k = 1, 2$, define

$$F_k(a_k, a_m) \triangleq \alpha_k \frac{\lambda}{\lambda+\lambda_k} h_m(a_m) \frac{v_{yy}^k(t,y_k)}{v_y^k(t,y_k)} \psi_k(a_k) - a_k(\frac{v_{yy}^k(t,y_k)}{v_y^k(t,y_k)} - \frac{1}{\kappa_k}) - 1.$$

For any fixed $a_2 \in [0, \kappa_2]$, since $v_y^1(t,y_1) > 0$ and$v_{yy}^1(t,y_1) < 0$, we have

$$F_1(0, a_2) = \alpha_1 \frac{\lambda}{\lambda+\lambda_1} h_2(a_2) \frac{v_{yy}^1(t,y_1)}{v_y^1(t,y_1)} \psi_1(0) - 1 < 0$$

and

$$\begin{aligned}
\lim_{a_1 \to \kappa_1} F_1(a_1, a_2) &= \alpha_1 \frac{\lambda}{\lambda+\lambda_1} h_2(a_2) \frac{v_{yy}^1(t,y_1)}{v_y^1(t,y_1)} \lim_{a_1 \to \kappa_1} \psi_1(a_1) - \kappa_1 \frac{v_{yy}^1(t,y_1)}{v_y^1(t,y_1)} \\
&= \alpha_1 \frac{\lambda}{\lambda+\lambda_1} h_2(a_2) \frac{v_{yy}^1(t,y_1)}{v_y^1(t,y_1)} \frac{1}{G_1(\iota_1(\kappa_1))} - \kappa_1 \frac{v_{yy}^1(t,y_1)}{v_y^1(t,y_1)} \\
&= -\kappa_1 \frac{v_{yy}^1(t,y_1)}{v_y^1(t,y_1)} > 0.
\end{aligned}$$

Therefore, for fixed $a_2 \in [0, \kappa_2]$, equation $F_1(a_1, a_2) = 0$ admits solutions $a_1 = \mathbf{a}_1(a_2)$ which is dependent on the value of a_2. On the other hand, since $v_{yy}^1(t,y_1) < 0 < v_y^1(t,y_1)$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a_1} F_1(a_1, a_2) &= -(\frac{v_{yy}^1(t,y_1)}{v_y^1(t,y_1)} - \frac{1}{\kappa_k}) + \alpha_1 \frac{\lambda}{\lambda+\lambda_1} h_2(a_2) \frac{v_{yy}^1(t,y_1)}{v_y^1(t,y_1)} \frac{\mathrm{d}\psi_1(a_1)}{\mathrm{d}a_1} \\
&> \alpha_1 \frac{\lambda}{\lambda+\lambda_1} h_2(a_2) \frac{v_{yy}^1(t,y_1)}{v_y^1(t,y_1)} \frac{\mathrm{d}\psi_1(a_1)}{\mathrm{d}a_1},
\end{aligned}$$

where

$$\frac{\mathrm{d}\psi_1(a_1)}{\mathrm{d}a_1} = \frac{1}{(\int_{\iota_1(a_1)}^{\infty} G_1^2(z)f_1(z)\mathrm{d}z)^2}\Big[- G_1(\iota_1(a_1))f_1(\iota_1(a_1))\frac{\mathrm{d}\iota_1(a_1)}{\mathrm{d}a_1}\int_{\iota_1(a_1)}^{\infty} G_1^2(z)f_1(z)\mathrm{d}z$$
$$+ G_1^2(\iota_1(a_1))f_1(\iota_1(a_1))\frac{\mathrm{d}\iota_1(a_1)}{\mathrm{d}a_1}\int_{\iota_1(a_1)}^{\infty} G_1(z)f_1(z)\mathrm{d}z\Big]$$
$$= G_1(\iota_1(a_1))f_1(\iota_1(a_1))\frac{\mathrm{d}\iota_1(a_1)}{\mathrm{d}a_1}\frac{-\int_{\iota_1(a_1)}^{\infty} G_1^2(z)f_1(z)\mathrm{d}z + G_1(\iota_1(a_1))\int_{\iota_1(a_1)}^{\infty} G_1(z)f_1(z)\mathrm{d}z}{(\int_{\iota_1(a_1)}^{\infty} G_1^2(z)f_1(z)\mathrm{d}z)^2}.$$

Since G_1 and ι_1 are strictly increasing, we have $\frac{\mathrm{d}\iota_1(a_1)}{\mathrm{d}a_1} > 0$ and

$$G_1(\iota_1(a_1))\int_{\iota_1(a)}^{\infty} G_1(z)f_1(z)\mathrm{d}z < \int_{\iota_1(a_1)}^{\infty} G_1^2(z)f_1(z)\mathrm{d}z.$$

Thus $\frac{\mathrm{d}}{\mathrm{d}a_1}F_1(a_1, a_2) > 0$, i.e., $F_1(a_1, a_2)$ is strictly increasing in a_1. Based on the above observations, we see that for any fixed a_2, $F_1(a_1, a_2) = 0$ has a unique solution $a_1 = \mathbf{a}_1(a_2)$, which is dependent on a_2 and is a function of a_2 for some function $\mathbf{a}_1$. Moreover, since $F_1(a_1, a_2)$ is strictly decreasing in a_2, $\mathbf{a}_1(a_2)$ is strictly increasing in a_2.

Substituting $a_1 = \mathbf{a}_1(a_2)$ into $F_2(a_1, a_2) = 0$. By a similar procedure, we are able to show that $F_2(\mathbf{a}_1(a_2), a_2)$ is strictly increasing in a_2 and satisfies $F_2(a_1(0), 0) < 0$ and $\lim_{a_2\to\kappa_2} F_2(a_1(a_2), a_2) = -\kappa_2\frac{v_{yy}^2}{v_y^2} > 0$. Therefore, $F_2(\mathbf{a}_1(a_2), a_2) = 0$ has a unique solution a_2^*, with which $a_1^* = \mathbf{a}_1(a_2^*)$ is also uniquely determined. This completes our proof. □

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