On the compound Poisson risk model with periodic capital injections

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Abstract

The analysis of capital injection strategy in the literature of insurance risk models (e.g. Pafumi (1998), and Dickson and Waters (2004)) typically assumes that whenever the surplus becomes negative, the amount of shortfall is injected so that the company can continue its business forever. Recently, Nie et al. (2011) has proposed an alternative model in which capital is immediately injected to restore the surplus level to a positive level b when the surplus falls between zero and b, and the insurer is still subject to a positive ruin probability. Inspired by the idea of randomized observations in Albrecher et al. (2011b), in this paper we further generalize Nie et al. (2011)'s model by assuming that capital injections are only allowed at a sequence of time points with inter-capital-injection times being Erlang distributed (so that deterministic time intervals can be approximated using the Erlangization technique in Asmussen et al. (2002)). When the claim amount is distributed as a combination of exponentials, explicit formulas for the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)) and the expected total discounted cost of capital injections before ruin are obtained. The derivations rely on a resolvent density associated with an Erlang random variable, which is shown to admit an explicit expression that is of independent interest as well. We shall provide numerical examples, including an application in pricing a perpetual reinsurance contract that makes the capital injections and demonstration of how to minimize the ruin probability via reinsurance. Minimization of the expected discounted capital injections plus a penalty applied at ruin with respect to the frequency of injections and the critical level b will also be illustrated numerically.

Keywords: Compound Poisson risk model; Periodic capital injections; Gerber-Shiu expected discounted penalty function; Resolvent measure; Perpetual reinsurance.

1 Introduction

The classical compound Poisson risk process $U = \{U_t\}_{t\geq 0}$ for an insurance company is defined by

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i, \qquad t \geq 0, \tag{1.1}$$

where $u = U_0 \geq 0$ is the initial surplus, and $c > 0$ is the incoming premium rate per unit time. Furthermore, the number of claims process $N = \{N_t\}_{t\geq 0}$ is assumed to be a Poisson process with

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intensity $\lambda > 0$, whereas the claim amounts $\{X_i\}_{i=1}^{\infty}$, independent of N, are positive continuous random variables that form an independent and identically distributed (i.i.d.) sequence with common density $f_X(\cdot)$ and Laplace transform $\widehat{f}_X(s) = \int_0^{\infty} e^{-sx} f_X(x)dx$. The time of ruin is the first time when the surplus process falls below zero, and the safety loading condition $c > \lambda \mathbb{E}X$ ensures that the (infinite-time) ruin probability is less than one.

In order to keep the business alive forever, Pafumi (1998) and Dickson and Waters (2004, Sections 6.2 and 6.3) considered models in which the necessary amount of capital is injected to restore the surplus process to zero whenever it becomes negative, where the capital injections are made by a reinsurer upon purchase of a reinsurance contract. They utilized the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)) to compute the expected present value of capital injections until ruin. Under the reinsurance setting, it is important to note that the payment of reinsurance premium reduces the company's initial surplus. In this case, the net single premium for the perpetual default reinsurance is related to the expected discounted capital injections until ruin via (5.6) (see e.g. Cheung (2012) and Liu and Cheung (2014) for such a pricing formula in the context of the dual risk model). Interested readers are also referred to e.g. Kulenko and Schmidli (2008) and Eisenberg and Schmidli (2011) for optimal dividend and reinsurance strategies in the compound Poisson model with capital injections. Recently, Nie et al. (2011) has proposed a variant of the model such that whenever the surplus falls between zero and a fixed critical level $b > 0$, capital injection is made to bring the surplus level back to b (instead of zero). In this alternative model, the insurer has a positive ruin probability because capital is not injected when the surplus falls below zero. Nie et al. (2011) derived the ruin probability and the expected discounted capital injections until ruin, and applied these results to determine an optimal reinsurance contract that minimizes the ruin probability numerically. The finite-time ruin probability was then studied by Nie et al. (2015), and this was extended by Dickson and Qazvini (2016) who further incorporated the number of claims until ruin into the analysis. Due to the spatial homogeneity of the compound Poisson process with constant premium rate, by shifting the process downward by b units, Nie et al. (2011)'s model is also equivalent to one that restores the surplus level to zero if it falls between zero and $-b$ but declares ruin if the surplus becomes less than $-b$. In this case, $-b$ can be regarded as a lower bankruptcy barrier such that the business is deemed hopeless once its surplus is below $-b$, so that capital is only injected when the shortfall is less than b with the hope that the business can recover in time.

Apart from the most common understanding that capital is injected by a reinsurer, another possible interpretation is that the injections are made by the shareholders of the insurance company. Following Dickson and Waters (2004, Section 6.1), one can argue that the shareholders are responsible for (1) injecting capital when the insurer's surplus falls below the critical level b but the company still survives; and (2) covering a penalty applied at the ruin time. This leads to the research problem of minimizing the sum of these two contributions, which will be considered in this paper as well.

Traditionally, most contributions to continuous-time insurance risk processes are made on the grounds of continuous monitoring of the surplus over time. For mathematical tractability, cash flows as a result of tax reporting and strategic decisions on dividends and fund raising are usually assumed to occur immediately as long as the surplus level is at its running maximum or falls within a pre-specified region (like all the above works on capital injections). However, in practice it is more reasonable for the tax authority, the board of directors or the reinsurer to inspect the insurer's surplus regularly for taxation, dividend payments and/or capital injections. These led Albrecher et al. (2011b, 2013) to propose the idea of only acting on the process at the discrete time points $\{Z_i\}_{i=1}^{\infty}$ (known as 'observation times') where dividend decisions are made or ruin may be declared. Let $T_i = Z_i - Z_{i-1}$ be the i-th inter-observation time for $i = 1, 2, \ldots$ (with $Z_0 = 0-$). Although the surplus level is often checked at deterministic intervals in practical applications, it is generally very difficult to obtain explicit results for various ruin-related quantities when each T_i is a constant. Therefore, Albrecher et al. (2011b, 2013) assumed that $\{T_i\}_{i=1}^{\infty}$ (independent of other attributes of the surplus process U) form an i.i.d. sequence with common Erlang(m, β) density

$$f_T(t) = \frac{\beta^m t^{m-1} e^{-\beta t}}{(m-1)!}, \qquad t > 0, \tag{1.2}$$

where the shape parameter m is a positive integer, and $\beta > 0$ is a scale parameter. The choice of the Erlang(m, β) distribution is motivated by the Erlangization technique frequently used in finite-time ruin problems (e.g. Asmussen et al. (2002), Stanford et al. (2005, 2011), and Ramaswami et al. (2008)). It is known that if we fix the mean of the generic inter-observation time T to be $\mathbb{E}T = m/\beta = h$ and let $m \to \infty$, then the random variable T converges in distribution to a point mass at h. In other words, one can approximate the situation of e.g. monthly, quarterly or annual observation by increasing m (and β at the same time). Since then, some other variants of Albrecher et al. (2011b)'s model with periodic observations have also been analyzed by others. For example, Avanzi et al. (2013), Zhang (2014) and Zhang and Cheung (2016) worked with a periodic dividend barrier strategy where ruin is monitored continuously, and they respectively looked at the dual risk model, the perturbed compound Poisson risk model and the Markov additive risk process. Moreover, Choi and Cheung (2014) considered a model in which the event of ruin is checked at $\{Z_i\}_{i=1}^{\infty}$ but dividend decisions are only made at $\{Z_{ij}\}_{i=1}^{\infty}$ for some positive integer j, thereby allowing for e.g. monthly checking of ruin and quarterly or semi-annual dividend announcements. We remark that the case of a Poissonian observer (i.e. $\{Z_i\}_{i=1}^{\infty}$ are the arrival epochs of a Poisson process) has been studied by e.g. Boxma et al. (2010), Albrecher and Ivanovs (2013, 2017), Albrecher et al. (2016), and Zhang et al. (2017), and such a model is known to yield simpler formulas and interesting identities. We also refer interested readers to e.g. Albrecher et al. (2011a) and Avanzi et al. (2014) for the study of optimal periodic dividend strategies.

With the idea of randomized observations in mind, this paper aims to propose a periodic capital injection strategy in the classical compound Poisson risk model (1.1), which extends the work by Nie et al. (2011). At the observation times (or capital injection times) $\{Z_i\}_{i=1}^{\infty}$, if the observed surplus level x is such that $x \in [0, b)$ for some pre-specified critical level $b > 0$, then a capital amount of $b - x$ is injected so that the surplus returns to the level b that is deemed safe (see Figure 1). Denoting the modified process as $U^b = \{U_t^b\}_{t \geq 0}$, its dynamics can be jointly described with the auxiliary processes $U^{(i)} = \{U_t^{(i)}\}_{t \geq Z_{i-1}}$ by

$$U_t^{(i)} = \begin{cases} U_t, & i = 1; \quad & t \geq 0, \\ U^b_{Z_{i-1}} + U_t - U_{Z_{i-1}}, & i = 2, 3, \ldots; & t \geq Z_{i-1}, \end{cases}$$

and for $i = 1, 2, \ldots$,

$$U_t^b = \begin{cases} U_t^{(i)}, & Z_{i-1} < t < Z_i. \\ \max(U_{Z_i}^{(i)}, b), & t = Z_i. \end{cases} \tag{1.3}$$

Without loss of generality, we have assumed that $Z_0 = 0-$ (i.e. time zero is not a capital injection time) so that $U_0^b = U_0 = u$ even if $0 \leq u < b$ (see Remark 1). Assuming that the event of ruin is monitored continuously, the ruin time of U^b is defined by $\tau_b = \inf\{t \geq 0 : U_t^b < 0\}$ with the convention $\inf \emptyset = \infty$. It is instructive to note that Nie et al. (2011)'s model can be retrieved by letting $\beta \to \infty$ (with m fixed) so that the generic inter-capital-injection time T converges to a point mass at zero. In this paper, we are interested in the Gerber-Shiu function

$$\phi(u; b) = \mathbb{E}[e^{-\delta \tau_b} w(|U^b_{\tau_b}|) \mathbf{1}_{(\tau_b < \infty)} | U_0^b = u], \qquad u \geq 0, \tag{1.4}$$

where $\delta \geq 0$ is the Laplace transform argument of the ruin time or the force of interest, $w(\cdot)$ is a non-negative penalty function on $(0, \infty)$ that satisfies some mild integrability conditions (see Proposition 4), and $\mathbf{1}_A$ is the indicator function of the event A. Another quantity of interest is expected total discounted cost of capital injections before ruin given by

$$V(u; b) = \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-\delta Z_i} \chi(b - U^b_{Z_i-}) \mathbf{1}_{(Z_i < \tau_b)} \middle| U^b_0 = u\right], \qquad u \geq 0, \tag{1.5}$$

where the function $\chi(\cdot)$ is a non-negative cost function that associates a cost of $\chi(x)$ to an injected capital of size $x \in (0, b]$, with the definition $\chi(x) = 0$ for $x \leq 0$. For example, if $\chi(x) = x$ for $x \in (0, b]$, then $V(u; b)$ represents the expected discounted capital injections until ruin. But if $\delta = 0$ and $\chi(x) = 1$ for $x \in (0, b]$, then $V(u; b)$ becomes the number of times a positive capital is injected.

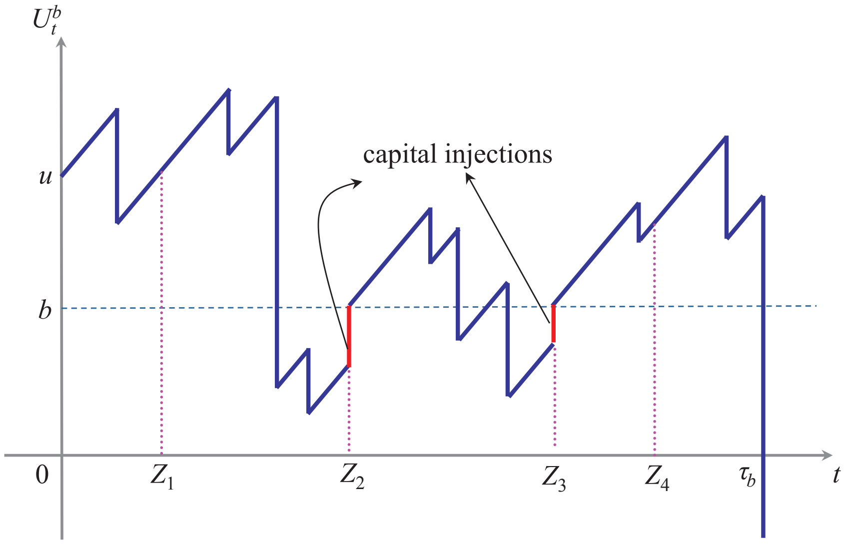


Figure 1: Sample path of $\{U^b_t\}_{t \geq 0}$

Remark 1 Suppose that the initial surplus is such that $0 \leq u < b$. If capital can be injected at time zero, then it is clear that the resulting Gerber-Shiu function and expected total discounted cost of capital injections are simply $\phi(b; b)$ and $\chi(b - u) + V(b; b)$ respectively. $\square$

In order to obtain explicit formulas for $\phi(u; b)$ and $V(u; b)$, we assume for the rest of the paper that the generic claim amount X is distributed as a combination of exponentials with density

$$f_X(x) = \sum_{i=1}^{a} \eta_i \alpha_i e^{-\alpha_i x}, \qquad x > 0, \tag{1.6}$$

where a is a positive integer, α_i's are distinct positive parameters, and η_i's are non-zero constants (which are possibly negative) such that $f_X(\cdot)$ is a proper density function. The above distributional assumption is not restrictive since any positive continuous distribution can be approximated arbitrarily closely by a combination of exponentials (see e.g. Dufresne (2007) for its fitting). See concluding remarks for the discussion of the case where claims have rational Laplace transform.

This paper is organized as follows. In Section 2, preliminary results concerning a resolvent density associated with an Erlang random variable are obtained. Such a resolvent density is shown to admit an explicit expression in Proposition 1, and this plays a crucial role in Sections 3 and 4 to derive full solutions to the Gerber-Shiu function $\phi(u;b)$ and the expected total discounted cost of capital injections before ruin $V(u;b)$ respectively. Although the proofs are tedious, the main results are stated in the form of Propositions for easy reference. The computational procedure of $\phi(u;b)$ and $V(u;b)$ involves recursive calculations of some constant coefficients as well as solving systems of linear equations, which can be readily implemented in software packages like *Mathematica*. Section 5 utilizes the theoretical results to provide some numerical illustrations. In particular, under the reinsurance set-up, a perpetual reinsurance contract that injects the capital is priced and it is applied to determine the optimal reinsurance strategy that minimizes the ruin probability, thereby complementing the results in Nie et al. (2011, Example 5.1.1). On the other hand, under the 'shareholders' interpretation, we aim at minimizing the sum of the expected discounted capital injections and a penalty at ruin, namely $V(u;b)+K\phi(u;b)$ for some constant $K \geq 0$. Because of the periodicity of capital injections introduced in this paper, the minimization of $V(u;b)+K\phi(u;b)$ can also be done with respect to the frequency β of injections (in addition to the critical level b done in the more classical manner). The performance of Erlangization is demonstrated as well. Section 6 ends the paper with some concluding remarks, and the proofs of some intermediate results are collected in the Appendices.

2 A resolvent density at an Erlang time

For $q \geq 0$, the q-scale function associated with the compound Poisson process U, denoted by $W^{(q)}(\cdot)$, is a continuous function on $[0,\infty)$ such that $W^{(q)}(x)=0$ for $x<0$. For $x \geq 0$ it is characterized by the Laplace transform

$$\int_0^\infty e^{-sx}W^{(q)}(x)dx = \frac{1}{\psi(s)-q}, \tag{2.1}$$

where $\psi(s)=cs-\lambda(1-\widehat{f}_X(s))$ is the Laplace exponent of U. See Kyprianou (2013, Chapter 4). Since $\widehat{f}_X(s)=\sum_{i=1}^{a}\eta_i\alpha_i/(\alpha_i+s)$ under the claim assumption (1.6), one can write the right-hand side of (2.1) using partial fractions as

$$\frac{1}{\psi(s)-q} = \frac{\prod_{i=1}^{a}(s+\alpha_i)}{c\prod_{i=1}^{a+1}(s-\rho_{q,i})} = \sum_{i=1}^{a+1}\frac{C_{q,i}}{s-\rho_{q,i}}. \tag{2.2}$$

Here $\{\rho_{q,i}\}_{i=1}^{a+1}$ are the (assumed distinct) roots of the Lundberg's fundamental equation (in ξ)

$$c\xi-(\lambda+q)+\lambda\sum_{i=1}^{a}\frac{\eta_i\alpha_i}{\alpha_i+\xi}=0, \tag{2.3}$$

and

$$C_{q,i} = \frac{\prod_{j=1}^{a}(\rho_{q,i}+\alpha_j)}{c\prod_{j=1,j\neq i}^{a+1}(\rho_{q,i}-\rho_{q,j})}, \qquad i=1,2,\ldots,a+1. \tag{2.4}$$

Therefore, by Laplace transform inversion, we obtain

$$W^{(q)}(x) = \sum_{i=1}^{a+1}C_{q,i}e^{\rho_{q,i}x}, \qquad x,q \geq 0. \tag{2.5}$$

It is well known that (2.3) has a unique non-negative root, denoted by $\rho_{q,a+1}$, which is also called the right inverse of $\psi(\cdot)$ (see Kyprianou (2013, Chapter 2)). All other roots of (2.3), namely $\{\rho_{q,i}\}_{i=1}^{a}$, have negative real parts.

Further define $\tau = \inf\{t \geq 0 : U_t < 0\}$ to be the ruin time of process U without capital injections. Then the q-resolvent measure of U killed on exiting $[0, \infty)$ is defined by

$$\mathcal{R}^{(q)}(u, dx) = \int_0^\infty e^{-qt}\mathbb{P}(U_t \in dx, \tau > t | U_0 = u)dt, \qquad u, x, q \geq 0. \tag{2.6}$$

From Kyprianou (2013, Theorem 5.2), there exists a density $r^{(q)}(u, x)$ such that $\mathcal{R}^{(q)}(u, dx) = r^{(q)}(u, x)dx$, and it is given by

$$r^{(q)}(u, x) = e^{-\rho_{q,a+1}x}W^{(q)}(u) - W^{(q)}(u - x) = \sum_{i=1}^{a+1} C_{q,i}(e^{\rho_{q,i}u-\rho_{q,a+1}x} - e^{\rho_{q,i}(u-x)}\mathbf{1}_{(u\geq x)}), \qquad u, x, q \geq 0, \tag{2.7}$$

where the last equality follows from (2.5). When q is positive, it is noted that the definition (2.6) can be rewritten as

$$\mathcal{R}^{(q)}(u, dx) = \frac{1}{q}\mathbb{P}(U_{\boldsymbol{e}_q} \in dx, \tau > \boldsymbol{e}_q | U_0 = u), \qquad u, x \geq 0;\ q > 0,$$

where $\boldsymbol{e}_q$ is an independent exponential random variable with mean $1/q$. For positive integer n, the above resolvent measure is extended to

$$\mathcal{R}_n^{(q)}(u, dx) = \frac{1}{q^n}\mathbb{P}\left(U_{\sum_{i=1}^n \boldsymbol{e}_{q,i}} \in dx, \tau > \sum_{i=1}^n \boldsymbol{e}_{q,i} \middle| U_0 = u\right), \qquad u, x \geq 0;\ q > 0, \tag{2.8}$$

where $\{\boldsymbol{e}_{q,i}\}_{i=1}^\infty$, independent of U, are i.i.d. with the same distribution as $\boldsymbol{e}_q$. Note that $\mathcal{R}_1^{(q)}(u, dx) = \mathcal{R}^{(q)}(u, dx)$. By Markov property, one has the recursive relationship

$$r_n^{(q)}(u, x) = \int_0^\infty r_{n-1}^{(q)}(u, y)r_1^{(q)}(y, x)dy, \qquad n = 2, 3, \ldots, \tag{2.9}$$

with the starting point $r_1^{(q)}(u, x) = r^{(q)}(u, x)$, where $r_n^{(q)}(u, x) = \mathcal{R}_n^{(q)}(u, dx)/dx$ is the resolvent density of the resolvent measure $\mathcal{R}_n^{(q)}(u, \cdot)$.

In the remainder of this paper, we will only encounter the case $q = \beta + \delta > 0$ (since $\beta > 0$ and $\delta \geq 0$). For convenience, the abbreviations $\rho_{\beta+\delta,i} = \rho_i$ and $C_{\beta+\delta,i} = C_i$ will be used for $i = 1, 2, \ldots, a + 1$. The following Proposition gives explicit results for the resolvent density $r_n^{(\beta+\delta)}(u, x)$, and the proof is given in Appendix A.

Proposition 1 *Suppose that each claim amount in the surplus process* (1.1) *is distributed as a combination of exponentials with density* (1.6)*. Then the resolvent measure $r_n^{(\beta+\delta)}(u, x)$ admits the representations, for $u < x$,*

$$r_n^{(\beta+\delta)}(u, x) = \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j} D_{n,i,j,k}\frac{u^{j-1}}{(j-1)!}\frac{x^{k-1}}{(k-1)!}e^{\rho_i u-\rho_{a+1}x} - \sum_{j=1}^{n} E_{n,a+1,j}\frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_{a+1}(u-x)}, \tag{2.10}$$

and for $u \geq x$,

$$r_n^{(\beta+\delta)}(u,x) = \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j} D_{n,i,j,k}\frac{u^{j-1}}{(j-1)!}\frac{x^{k-1}}{(k-1)!}e^{\rho_i u-\rho_{a+1}x} + \sum_{i=1}^{a}\sum_{j=1}^{n} E_{n,i,j}\frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_i(u-x)}. \quad (2.11)$$

The constants in (2.10) *and* (2.11) *for the starting point $n = 1$ are given by*

$$D_{1,i,1,1} = C_i, \qquad i = 1,2,\ldots,a, \quad (2.12)$$
$$E_{1,i,1} = -\,C_i, \qquad i = 1,2,\ldots,a+1. \quad (2.13)$$

For $n = 1, 2, \ldots$, the constants can be computed recursively via

$$\begin{aligned} D_{n+1,i,j,k} = & \sum_{l=1}^{n+1-j}\sum_{z=1}^{a}\frac{D_{n,i,j,l}C_z}{(\rho_{a+1}-\rho_z)^l}\mathbf{1}_{(k=1)} + D_{n,i,j,k-1}C_{a+1}\mathbf{1}_{(k\neq1)} - \sum_{l=k}^{n+1-j}\sum_{z=1}^{a}\frac{D_{n,i,j,l}C_z}{(\rho_{a+1}-\rho_z)^{l+1-k}} \\ & + E_{n,i,j-1}C_i\mathbf{1}_{(j\neq1,k=1)} + \sum_{l=1}^{n}\sum_{z=1,z\neq i}^{a}\frac{E_{n,z,l}C_i}{(\rho_i-\rho_z)^l}\mathbf{1}_{(j=1,k=1)} - \sum_{z=1,z\neq i}^{a+1}\sum_{l=j}^{n}\frac{E_{n,i,l}C_z}{(\rho_z-\rho_i)^{l+1-j}}\mathbf{1}_{(k=1)} \\ & + \sum_{l=1}^{n}\frac{E_{n,a+1,l}C_i}{(\rho_i-\rho_{a+1})^l}\mathbf{1}_{(j=1,k=1)}, \qquad i=1,2,\ldots,a;\ j=1,2,\ldots,n;\ k=1,2,\ldots,n+1-j, \end{aligned} \quad (2.14)$$

$$D_{n+1,i,j,n+2-j} = D_{n,i,j,n+1-j}C_{a+1}, \qquad i=1,2,\ldots,a;\ j=1,2,\ldots,n, \quad (2.15)$$
$$D_{n+1,i,n+1,1} = E_{n,i,n}C_i, \qquad i=1,2,\ldots,a, \quad (2.16)$$

$$\begin{aligned} E_{n+1,i,j} = & -E_{n,i,j-1}C_i\mathbf{1}_{(j\neq1)} - \sum_{l=1}^{n}\sum_{k=1,k\neq i}^{a}\frac{E_{n,k,l}C_i}{(\rho_i-\rho_k)^l}\mathbf{1}_{(j=1)} + \sum_{k=1,k\neq i}^{a+1}\sum_{l=j}^{n}\frac{E_{n,i,l}C_k}{(\rho_k-\rho_i)^{l+1-j}}\mathbf{1}_{(j\neq n+1)} \\ & - \sum_{k=1}^{n}\frac{E_{n,a+1,k}C_i}{(\rho_i-\rho_{a+1})^k}\mathbf{1}_{(j=1)}, \qquad i=1,2,\ldots,a;\ j=1,2,\ldots,n+1, \end{aligned} \quad (2.17)$$

$$\begin{aligned} E_{n+1,a+1,j} = & -\sum_{l=1}^{n}\sum_{k=1}^{a}\frac{E_{n,k,l}C_{a+1}}{(\rho_{a+1}-\rho_k)^l}\mathbf{1}_{(j=1)} - E_{n,a+1,j-1}C_{a+1}\mathbf{1}_{(j\neq1)} \\ & + \sum_{k=1}^{a}\sum_{l=j}^{n}\frac{E_{n,a+1,l}C_k}{(\rho_k-\rho_{a+1})^{l+1-j}}\mathbf{1}_{(j\neq n+1)}, \qquad j=1,2,\ldots,n+1. \end{aligned} \quad (2.18)$$

3 Analysis of the Gerber-Shiu function

To analyze the Gerber-Shiu function $\phi(u;b)$ defined in (1.4) for the process U^b with capital injections, we condition on whether or not ruin occurs before the first capital injection time, leading to

$$\begin{aligned} \phi(u;b) = & \int_0^\infty e^{-\delta t}\int_0^\infty [\phi(b;b)\mathbf{1}_{(0\leq x\leq b)} + \phi(x;b)\mathbf{1}_{(x>b)}]\mathbb{P}(U_t\in dx, \tau>t|U_0=u)f_T(t)dt \\ & + \int_0^\infty \mathbb{E}[e^{-\delta\tau}w(|U_\tau|);\tau<t|U_0=u]f_T(t)dt, \qquad u\geq0. \end{aligned} \quad (3.1)$$

Under the Erlang density (1.2), it is clear from the definition (2.8) and the fact that $\mathcal{R}_m^{(\beta+\delta)}(u, dx) = r_m^{(\beta+\delta)}(u, x)dx$ one can write

$$\int_0^\infty e^{-\delta t}\mathbb{P}(U_t \in dx, \tau > t | U_0 = u) f_T(t)dt = \left(\frac{\beta}{\beta+\delta}\right)^m \int_0^\infty \mathbb{P}(U_t \in dx, \tau > t | U_0 = u)\frac{(\beta+\delta)^m t^{m-1} e^{-(\beta+\delta)t}}{(m-1)!}dt$$
$$= \beta^m r_m^{(\beta+\delta)}(u, x)dx. \tag{3.2}$$

In addition, we define, for positive integer m,

$$\zeta_m(u) = \mathbb{E}\left[e^{-\delta\tau} w(|U_\tau|); \tau < \sum_{j=1}^m \boldsymbol{e}_{\beta,j} \middle| U_0 = u\right], \qquad u \geq 0, \tag{3.3}$$

which is the Gerber-Shiu function in the classical model U for ruin occurring before an independent Erlang(m, β) time. With (3.2) and (3.3), the integral equation (3.1) becomes

$$\phi(u; b) = \beta^m \phi(b; b) \int_0^b r_m^{(\beta+\delta)}(u, x)dx + \beta^m \int_b^\infty r_m^{(\beta+\delta)}(u, x)\phi(x; b)dx + \zeta_m(u). \tag{3.4}$$

To solve the above integral equation satisfied by $\phi(\cdot; b)$, we turn our focus to the quantity $\zeta_m(u)$ appearing there. Using Markov property and (3.2), we arrive at

$$\zeta_m(u) = \sum_{n=1}^{m-1} \mathbb{E}\left[e^{-\delta\tau} w(|U_\tau|); \sum_{j=1}^n \boldsymbol{e}_{\beta,j} < \tau < \sum_{j=1}^{n+1} \boldsymbol{e}_{\beta,j} \middle| U_0 = u\right] + \mathbb{E}[e^{-\delta\tau} w(|U_\tau|); \tau < \boldsymbol{e}_{\beta,1} | U_0 = u]$$
$$= \sum_{n=1}^{m-1} \beta^n \int_0^\infty r_n^{(\beta+\delta)}(u, x)\zeta_1(x)dx + \zeta_1(u). \tag{3.5}$$

It is instructive to note that

$$\zeta_1(u) = \mathbb{E}[e^{-(\beta+\delta)\tau} w(|U_\tau|); \tau < \infty | U_0 = u] \tag{3.6}$$

is simply the Gerber-Shiu function in U under the force of interest $\beta + \delta$. Therefore, it follows from Kyprianou (2013, Theorem 5.5) along with the claim density (1.6) and the resolvent density (2.7) that

$$\zeta_1(u) = \lambda \int_0^\infty \int_0^\infty w(y) r^{(\beta+\delta)}(u, x) f_X(x+y)dydx = \sum_{j=1}^a \lambda\eta_j\alpha_j \left(\int_0^\infty w(y)e^{-\alpha_j y}dy\right)\left(\int_0^\infty e^{-\alpha_j x} r^{(\beta+\delta)}(u, x)dx\right)$$
$$= \sum_{j=1}^a \lambda\eta_j\alpha_j \left(\int_0^\infty w(y)e^{-\alpha_j y}dy\right)\left(\sum_{l=1}^a C_l \frac{\rho_l - \rho_{a+1}}{(\alpha_j + \rho_l)(\alpha_j + \rho_{a+1})} e^{\rho_l u} + e^{-\alpha_j u} \sum_{l=1}^{a+1} \frac{C_l}{\alpha_j + \rho_l}\right)$$
$$= \sum_{l=1}^a F_{w,l} e^{\rho_l u}, \tag{3.7}$$

where

$$F_{w,l} = C_l \sum_{j=1}^a \frac{\lambda\eta_j\alpha_j(\rho_l - \rho_{a+1})}{(\alpha_j + \rho_l)(\alpha_j + \rho_{a+1})} \int_0^\infty w(y)e^{-\alpha_j y}dy, \qquad l = 1, 2, \ldots, a. \tag{3.8}$$

(The subscript 'w' emphasizes the dependence of $F_{w,l}$ on the choice of penalty function $w(\cdot)$, and similar notation will be used for other constants that subsequently appear.) In the final step of (3.7), we have used (2.4) to see that

$$\sum_{l=1}^{a+1} \frac{C_l}{\alpha_j + \rho_l} = \frac{1}{c} \sum_{l=1}^{a+1} \frac{\prod_{k=1,k\neq j}^{a}(\rho_l + \alpha_k)}{\prod_{k=1,k\neq l}^{a+1}(\rho_l - \rho_k)} = 0, \qquad j = 1, 2, \ldots, a,$$

where the last identity follows from e.g. Cheung (2010, p.444). Then, applying (2.10), (2.11) and (3.7), the integral in (3.5) is evaluated as

$$\begin{aligned}
\int_0^\infty r_n^{(\beta+\delta)}(u,x)\zeta_1(x)dx =& \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j}\sum_{l=1}^{a} D_{n,i,j,k}F_{w,l}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}\int_0^\infty \frac{x^{k-1}}{(k-1)!}e^{-(\rho_{a+1}-\rho_l)x}dx \\
&+ \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{l=1}^{a} E_{n,i,j}F_{w,l}\int_0^u \frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_i(u-x)+\rho_l x}dx \\
&- \sum_{j=1}^{n}\sum_{l=1}^{a} E_{n,a+1,j}F_{w,l}\int_u^\infty \frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_{a+1}(u-x)+\rho_l x}dx \\
=& \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j}\sum_{l=1}^{a} \frac{D_{n,i,j,k}F_{w,l}}{(\rho_{a+1}-\rho_l)^k}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u} + \sum_{i=1}^{a}\sum_{j=2}^{n+1} E_{n,i,j-1}F_{w,i}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u} \\
&+ \sum_{i=1}^{a}\sum_{l=1}^{n}\sum_{k=1,k\neq i}^{a} \frac{E_{n,k,l}F_{w,i}}{(\rho_i-\rho_k)^l}e^{\rho_i u} - \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{l=1,l\neq i}^{a}\sum_{k=j}^{n} \frac{E_{n,i,k}F_{w,l}}{(\rho_l-\rho_i)^{k+1-j}}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u} \\
&+ \sum_{i=1}^{a}\sum_{l=1}^{n} \frac{E_{n,a+1,l}F_{w,i}}{(\rho_i-\rho_{a+1})^l}e^{\rho_i u} \\
=& \sum_{i=1}^{a}\sum_{j=1}^{n+1} G_{w,n,i,j}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}, \qquad (3.9)
\end{aligned}$$

where

$$\begin{aligned}
G_{w,n,i,j} =& \sum_{k=1}^{n+1-j}\sum_{l=1}^{a} \frac{D_{n,i,j,k}F_{w,l}}{(\rho_{a+1}-\rho_l)^k} + E_{n,i,j-1}F_{w,i}\mathbf{1}_{(j\neq 1)} + \sum_{l=1}^{n}\sum_{k=1,k\neq i}^{a} \frac{E_{n,k,l}F_{w,i}}{(\rho_i-\rho_k)^l}\mathbf{1}_{(j=1)} \\
&- \sum_{l=1,l\neq i}^{a}\sum_{k=j}^{n} \frac{E_{n,i,k}F_{w,l}}{(\rho_l-\rho_i)^{k+1-j}} + \sum_{l=1}^{n} \frac{E_{n,a+1,l}F_{w,i}}{(\rho_i-\rho_{a+1})^l}\mathbf{1}_{(j=1)}, \qquad i = 1, 2\ldots, a;\ j = 1, 2, \ldots, n, \\
G_{w,n,i,n+1} =& E_{n,i,n}F_{w,i}, \qquad i = 1, 2, \ldots, a.
\end{aligned}$$

Some details have been omitted in obtaining the second equality of (3.9) since the steps are very similar to the analysis of $I_1(u,x)$ and $I_2(u,x)$ in the proof of Proposition 1. Consolidating (3.5)-(3.9), it is found that $\zeta_m(u)$ can be represented as

$$\zeta_m(u) = \sum_{n=1}^{m-1}\sum_{i=1}^{a}\sum_{j=1}^{n+1} \beta^n G_{w,n,i,j}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u} + \sum_{i=1}^{a} F_{w,i}e^{\rho_i u} = \sum_{i=1}^{a}\sum_{j=1}^{m} H_{w,m,i,j}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}, \qquad (3.10)$$

where

$$H_{w,m,i,j} = \sum_{n=\max(1,j-1)}^{m-1} \beta^n G_{w,n,i,j} + F_{w,i}\mathbf{1}_{(j=1)}, \qquad i = 1, 2, \ldots, a;\ j = 1, 2, \ldots, m. \qquad (3.11)$$

Now we return to the Gerber-Shiu function $\phi(u;b)$. It is important to note that when $0 \le u < b$ equation (3.4) simply expresses $\phi(u;b)$ in terms of $\{\phi(x;b) : x \ge b\}$. Therefore, it suffices to solve the integral equation (3.4) for $\{\phi(u;b) : u \ge b\}$. The following Proposition shows that (3.4) can be transformed into an ordinary differential equation when $u \ge b$.

Proposition 2 *Suppose that each claim amount in the surplus process* (1.3) *is distributed as a combination of exponentials with density* (1.6). *Then the Gerber-Shiu function* $\phi(u;b)$ *satisfies the homogeneous ordinary differential equation*

$$\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\phi(u;b)=\sum_{i=1}^{a+1}\sum_{j=1}^{m}\beta^m E_{m,i,j}\left[\prod_{k=1,k\neq i}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\left(\frac{d}{du}-\rho_i\right)^{m-j}\phi(u;b),\qquad u\ge b, \tag{3.12}$$

where $E_{m,i,j}$*'s are given in Proposition 1.*

Proof. Throughout the proof we consider the domain $u \ge b$. We begin with the analysis of the first term in (3.4). By applying a binomial expansion to the term involving $(u-x)^{j-1}$ in (2.11) followed by a change of order of summations, we arrive at

$$\begin{aligned}\beta^m\int_0^b r_m^{(\beta+\delta)}(u,x)dx=&\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{k=1}^{m+1-j}\beta^m D_{m,i,j,k}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}\int_0^b\frac{x^{k-1}}{(k-1)!}e^{-\rho_{a+1}x}dx\\&+\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{k=1}^{j}\beta^m E_{m,i,j}\frac{u^{k-1}}{(k-1)!}e^{\rho_i u}\int_0^b\frac{(-x)^{j-k}}{(j-k)!}e^{-\rho_i x}dx\\=&\sum_{i=1}^{a}\sum_{j=1}^{m}P_{m,i,j}(b)\frac{u^{j-1}}{(j-1)!}e^{\rho_i u},\end{aligned}\tag{3.13}$$

where

$$P_{m,i,j}(b)=\sum_{k=1}^{m+1-j}\beta^m D_{m,i,j,k}\int_0^b\frac{x^{k-1}}{(k-1)!}e^{-\rho_{a+1}x}dx+\sum_{k=j}^{m}\beta^m E_{m,i,k}\int_0^b\frac{(-x)^{k-j}}{(k-j)!}e^{-\rho_i x}dx \tag{3.14}$$

$$\begin{aligned}=&\sum_{k=1}^{m+1-j}\beta^m D_{m,i,j,k}\left(\frac{1}{\rho_{a+1}^k}-\sum_{l=1}^{k}\frac{1}{\rho_{a+1}^{k+1-l}}\frac{b^{l-1}}{(l-1)!}e^{-\rho_{a+1}b}\right)\\&+\sum_{k=j}^{m}\beta^m E_{m,i,k}(-1)^{k-j}\left(\frac{1}{\rho_i^{k-j+1}}-\sum_{l=1}^{k-j+1}\frac{1}{\rho_i^{k-j+2-l}}\frac{b^{l-1}}{(l-1)!}e^{-\rho_i b}\right),\\&\qquad i=1,2,\ldots,a;\ j=1,2,\ldots,m.\end{aligned}\tag{3.15}$$

Next, using (2.10) and (2.11), the second term in (3.4) can be written as

$$\begin{aligned}\beta^m\int_b^\infty r_m^{(\beta+\delta)}(u,x)\phi(x;b)dx=&\sum_{i=1}^{a}\sum_{j=1}^{m}\left(\sum_{k=1}^{m+1-j}\beta^m D_{m,i,j,k}\int_b^\infty\frac{x^{k-1}}{(k-1)!}e^{-\rho_{a+1}x}\phi(x;b)dx\right)\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}\\&+\sum_{i=1}^{a}\sum_{j=1}^{m}\beta^m E_{m,i,j}\int_b^u\frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_i(u-x)}\phi(x;b)dx\end{aligned}$$

$$-\sum_{j=1}^{m}\beta^m E_{m,a+1,j}\int_u^\infty \frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_{a+1}(u-x)}\phi(x;b)dx. \tag{3.16}$$

We shall proceed by applying the operator $\prod_{k=1}^{a+1}(d/du-\rho_k)^m$ to (3.4). Thanks to the Erlang representations (3.10) and (3.13) along with the fact that $(d/du-\rho_i)^j(u^{j-1}e^{\rho_i u})=0$, one has that

$$\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\zeta_m(u)=0, \tag{3.17}$$

$$\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\beta^m\int_0^b r_m^{(\beta+\delta)}(u,x)dx=0. \tag{3.18}$$

Similarly, as the first summation term in (3.16) also vanishes after operation, we obtain

$$\begin{aligned}
&\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\beta^m\int_b^\infty r_m^{(\beta+\delta)}(u,x)\phi(x;b)dx\\
=&\sum_{i=1}^{a}\sum_{j=1}^{m}\beta^m E_{m,i,j}\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\int_b^u \frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_i(u-x)}\phi(x;b)dx\\
&-\sum_{j=1}^{m}\beta^m E_{m,a+1,j}\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\int_u^\infty \frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_{a+1}(u-x)}\phi(x;b)dx\\
=&\sum_{i=1}^{a+1}\sum_{j=1}^{m}\beta^m E_{m,i,j}\left[\prod_{k=1,k\neq i}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\left(\frac{d}{du}-\rho_i\right)^{m-j}\phi(u;b).
\end{aligned} \tag{3.19}$$

In the last equality, we have used the fact that

$$\left(\frac{d}{du}-\rho_i\right)^j\int_b^u \frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_i(u-x)}\phi(x;b)dx=\phi(u;b),$$

$$\left(\frac{d}{du}-\rho_{a+1}\right)^j\int_u^\infty \frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_{a+1}(u-x)}\phi(x;b)dx=-\phi(u;b),$$

which can be readily verified by successive differentiation. Hence, incorporating (3.17)-(3.19) to (3.4), the result (3.12) follows. ■

The ordinary differential equation (3.12) has characteristic equation (in ξ)

$$\prod_{k=1}^{a+1}(\xi-\rho_k)^m=\sum_{i=1}^{a+1}\sum_{j=1}^{m}\beta^m E_{m,i,j}\left[\prod_{k=1,k\neq i}^{a+1}(\xi-\rho_k)^m\right](\xi-\rho_i)^{m-j}, \tag{3.20}$$

which has a total of $m(a+1)$ roots. The following Proposition provides an equivalent equation to (3.20) such that m of the roots are on the right half of the complex plane. The proof is given in Appendix B.

Proposition 3 *With the claim amount distributed as a combination of exponentials with density* (1.6), *the characteristic equation* (3.20) *is equivalent to*

$$\left(\frac{\beta}{\beta+\delta-\psi(\xi)}\right)^m=1, \tag{3.21}$$

and has exactly m roots with non-negative real parts.

Equation (3.21) can be regarded as the Lundberg's fundamental equation for the present model with periodic capital injections. Although (3.20) and (3.21) are equivalent, it is usually more convenient to solve (3.21) for the Lundberg's roots: (3.21) only involves the Laplace exponent $\psi(\cdot)$ and the model parameters β and δ whereas (3.20) requires the constants $E_{m,i,j}$'s computed recursively from Proposition 1. The roots of (3.20) or (3.21) are denoted by $\{s_{m,n}\}_{n=1}^{m(a+1)}$ and assumed to be distinct. Without loss of generality, we let $\{s_{m,n}\}_{n=ma+1}^{m(a+1)}$ be the m roots with non-negative real parts. The solution to the Gerber-Shiu function $\phi(u;b)$ is given in the next Proposition.

Proposition 4 *Suppose that each claim amount in the surplus process* (1.3) *is distributed as a combination of exponentials with density* (1.6)*, and the regular condition* $\lim_{u\to\infty}\phi(u;b)=0$ *holds. Then the Gerber-Shiu function* $\phi(u;b)$ *admits the explicit representation*

$$\phi(u;b)=\sum_{n=1}^{ma}A_{w,m,n}(b)e^{s_{m,n}u}, \qquad u\geq b, \tag{3.22}$$

where $\{s_{m,n}\}_{n=1}^{ma}$ *are the roots of* (3.21) *with negative real parts, and the constants* $\{A_{w,m,n}(b)\}_{n=1}^{ma}$ *can be solved from the system of* ma *linear equations comprising*

$$\sum_{n=1}^{ma}A_{w,m,n}(b)(P_{m,i,j}(b)e^{s_{m,n}b}+Q_{m,n,i,j}(b))+H_{w,m,i,j}=0, \qquad i=1,2,\ldots,a;\ j=1,2,\ldots,m. \tag{3.23}$$

In (3.23)*, the constants* $Q_{m,n,i,j}(b)$*'s are defined by*

$$Q_{m,n,i,j}(b)=\sum_{k=1}^{m+1-j}\sum_{l=1}^{k}\frac{\beta^m D_{m,i,j,k}}{(\rho_{a+1}-s_{m,n})^{k+1-l}}\frac{b^{l-1}}{(l-1)!}e^{-(\rho_{a+1}-s_{m,n})b}-\sum_{k=j}^{m}\sum_{l=j}^{k}\frac{\beta^m E_{m,i,k}}{(s_{m,n}-\rho_i)^{k+1-l}}\frac{(-b)^{l-j}}{(l-j)!}e^{(s_{m,n}-\rho_i)b},$$
$$i=1,2,\ldots,a;\ j=1,2,\ldots,m, \tag{3.24}$$

whereas $P_{m,i,j}(b)$*'s and* $H_{w,m,i,j}$*'s are given by* (3.15) *and* (3.11) *respectively.*

Proof. The solution form (3.22) is a direct consequence of Propositions 2 and 3 and the assumption $\lim_{u\to\infty}\phi(u;b)=0$. A sufficient condition for the regular condition to hold is that the penalty function $w(\cdot)$ is bounded (say, by a non-negative constant W). Then, it is immediate that $\phi(u;b)\leq W\mathbb{E}[e^{-\delta\tau_b}\mathbf{1}_{(\tau_b<\infty)}|U_0^b=u]\leq W\mathbb{E}[e^{-\delta\tau}\mathbf{1}_{(\tau<\infty)}|U_0=u]$, and the last quantity tends to zero as $u\to\infty$ under $\delta>0$ or the loading condition $c>\lambda\mathbb{E}X$.

Now, it remains to determine the unknown constants $\{A_{w,m,n}(b)\}_{n=1}^{ma}$. To this end, we shall use the integral equation (3.4) for $u\geq b$. Since the first and third terms on the right-hand side of (3.4) are already explicitly known from (3.13) and (3.10), we focus on using (3.22) to compute the second term, or equivalently (3.16). By (A.7), the first summation term in (3.16) equals

$$\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{k=1}^{m+1-j}\sum_{l=1}^{k}\sum_{n=1}^{ma}A_{w,m,n}(b)\frac{\beta^m D_{m,i,j,k}}{(\rho_{a+1}-s_{m,n})^{k+1-l}}\frac{b^{l-1}}{(l-1)!}e^{-(\rho_{a+1}-s_{m,n})b}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}.$$

Next, applying (A.6) along with a binomial expansion and a change of order of summations, the second summation term in (3.16) is found to be

$$\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{n=1}^{ma}A_{w,m,n}(b)\beta^m E_{m,i,j}e^{s_{m,n}u}\int_0^{u-b}\frac{x^{j-1}}{(j-1)!}e^{-(s_{m,n}-\rho_i)x}dx$$

$$=\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{n=1}^{ma}A_{w,m,n}(b)\frac{\beta^m E_{m,i,j}}{(s_{m,n}-\rho_i)^j}e^{s_{m,n}u}-\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{n=1}^{ma}\sum_{k=1}^{j}A_{w,m,n}(b)\frac{\beta^m E_{m,i,j}e^{(s_{m,n}-\rho_i)b}}{(s_{m,n}-\rho_i)^{j+1-k}}\frac{(u-b)^{k-1}}{(k-1)!}e^{\rho_i u}$$
$$=\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{n=1}^{ma}A_{w,m,n}(b)\frac{\beta^m E_{m,i,j}}{(s_{m,n}-\rho_i)^j}e^{s_{m,n}u}$$
$$-\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{k=j}^{m}\sum_{n=1}^{ma}\sum_{l=j}^{k}A_{w,m,n}(b)\frac{\beta^m E_{m,i,k}}{(s_{m,n}-\rho_i)^{k+1-l}}\frac{(-b)^{l-j}}{(l-j)!}e^{(s_{m,n}-\rho_i)b}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}.$$

Finally, the third summation term in (3.16) (including the minus sign) is

$$\sum_{j=1}^{m}\sum_{n=1}^{ma}A_{w,m,n}(b)\frac{\beta^m E_{m,a+1,j}}{(s_{m,n}-\rho_{a+1})^j}e^{s_{m,n}u}.$$

Combining the above three expressions, (3.16) becomes

$$\beta^m\int_b^\infty r_m^{(\beta+\delta)}(u,x)\phi(x;b)dx=\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{n=1}^{ma}A_{w,m,n}(b)Q_{m,n,i,j}(b)\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}$$
$$+\sum_{i=1}^{a+1}\sum_{j=1}^{m}\sum_{n=1}^{ma}A_{w,m,n}(b)\frac{\beta^m E_{m,i,j}}{(s_{m,n}-\rho_i)^j}e^{s_{m,n}u},\tag{3.25}$$

where $Q_{m,n,i,j}(b)$'s are given by (3.24).

With (3.10), (3.13), (3.22) and (3.25), the integral equation (3.4) reduces to

$$\sum_{n=1}^{ma}A_{w,m,n}(b)e^{s_{m,n}u}=\sum_{i=1}^{a+1}\sum_{j=1}^{m}\sum_{n=1}^{ma}A_{w,m,n}(b)\frac{\beta^m E_{m,i,j}}{(s_{m,n}-\rho_i)^j}e^{s_{m,n}u}+\sum_{i=1}^{a}\sum_{j=1}^{m}H_{w,m,i,j}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}$$
$$+\sum_{i=1}^{a}\sum_{j=1}^{m}\sum_{n=1}^{ma}A_{w,m,n}(b)(P_{m,i,j}(b)e^{s_{m,n}b}+Q_{m,n,i,j}(b))\frac{u^{j-1}}{(j-1)!}e^{\rho_i u},$$

which is true for $u\geq b$. Comparing the coefficients of $e^{s_{m,n}u}$ leads to no information because each $s_{m,n}$ is a root of the characteristic equation (3.20). On the other hand, by matching the coefficients of $\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}$, one asserts that $\{A_{w,m,n}(b)\}_{n=1}^{ma}$ satisfy (3.23). ■

4 Analysis of expected discounted cost of capital injections

In this section, we study the expected total discounted cost incurred by capital injections $V(u;b)$ defined in (1.5). Since much of the derivation is very similar to that in Section 3, some details will be omitted. But we require the following Lemma which is concerned with the limiting behaviour of $V(u;b)$ as $u\to\infty$. The proof is provided in Appendix C.

Lemma 1 *Suppose that the cost function $\chi(\cdot)$ is bounded on $(0,b]$. If $\delta>0$ or the loading condition $c>\lambda\mathbb{E}X$ holds, then*

$$\lim_{u\to\infty}V(u;b)=0.\tag{4.1}$$

Next, using the logic leading to (3.4), it is clear that $V(\cdot;b)$ satisfies the integral equation

$$V(u;b)=\beta^m\int_0^b r_m^{(\beta+\delta)}(u,x)[\chi(b-x)+V(b;b)]dx+\beta^m\int_b^\infty r_m^{(\beta+\delta)}(u,x)V(x;b)dx,\qquad u\geq 0. \tag{4.2}$$

Although the above equation is valid for $u \geq 0$, following the comments before Proposition 2, we only need to solve it for $\{V(u;b) : u \geq b\}$. The next two Propositions are the analogues of Propositions 2 and 4.

Proposition 5 *Suppose that each claim amount in the surplus process* (1.3) *is distributed as a combination of exponentials with density* (1.6)*. Then the expected discounted cost of capital injections* $V(u;b)$ *satisfies the homogeneous ordinary differential equation*

$$\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]V(u;b)=\sum_{i=1}^{a+1}\sum_{j=1}^{m}\beta^m E_{m,i,j}\left[\prod_{k=1,k\neq i}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\left(\frac{d}{du}-\rho_i\right)^{m-j}V(u;b),\qquad u\geq b, \tag{4.3}$$

where $E_{m,i,j}$*'s are given in Proposition 1.*

Proof. To prove (4.3), the operator $\prod_{k=1}^{a+1}(d/du-\rho_k)^m$ is applied to (4.2). We start with the integral term in (4.2) involving $\chi(\cdot)$. By slightly modifying (3.13) and (3.14), we obtain

$$\beta^m\int_0^b r_m^{(\beta+\delta)}(u,x)\chi(b-x)dx=\sum_{i=1}^{a}\sum_{j=1}^{m}H_{\chi,m,i,j}(b)\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}, \tag{4.4}$$

where

$$H_{\chi,m,i,j}(b)=\sum_{k=1}^{m+1-j}\beta^m D_{m,i,j,k}\int_0^b\chi(b-x)\frac{x^{k-1}}{(k-1)!}e^{-\rho_{a+1}x}dx+\sum_{k=j}^{m}\beta^m E_{m,i,k}\int_0^b\chi(b-x)\frac{(-x)^{k-j}}{(k-j)!}e^{-\rho_i x}dx,$$
$$i=1,2,\ldots,a;\ j=1,2,\ldots,m, \tag{4.5}$$

depend on the choice of cost function $\chi(\cdot)$. The Erlang form of (4.4) implies that this term vanishes after applying the operator $\prod_{k=1}^{a+1}(d/du-\rho_k)^m$. In addition, following the proof of Proposition 2, it is immediate to see that (3.19) is still valid (with $\phi(\cdot;b)$ replaced by $V(\cdot;b)$), and so is (3.18). Combining these observations yields (4.3). ■

Proposition 6 *Suppose that each claim amount in the surplus process* (1.3) *is distributed as a combination of exponentials with density* (1.6)*, and the regular condition* $\lim_{u\to\infty}V(u;b)=0$ *in Lemma 1 holds. Then the expected total discounted cost of capital injections before ruin* $V(u;b)$ *admits the explicit representation*

$$V(u;b)=\sum_{n=1}^{ma}B_{\chi,m,n}(b)e^{s_{m,n}u},\qquad u\geq b, \tag{4.6}$$

where $\{s_{m,n}\}_{n=1}^{ma}$ *are the roots of* (3.21) *with negative real parts, and the constants* $\{B_{\chi,m,n}(b)\}_{n=1}^{ma}$ *can be solved from the system of ma linear equations comprising*

$$\sum_{n=1}^{ma}B_{\chi,m,n}(b)(P_{m,i,j}(b)e^{s_{m,n}b}+Q_{m,n,i,j}(b))+H_{\chi,m,i,j}(b)=0,\qquad i=1,2,\ldots,a;\ j=1,2,\ldots,m. \tag{4.7}$$

In (4.7)*, the constants* $P_{m,i,j}(b)$*'s,* $Q_{m,n,i,j}(b)$*'s and* $H_{\chi,m,i,j}(b)$*'s are given by* (3.15)*,* (3.24) *and* (4.5) *respectively.*

Proof. As the characteristic equation of the homogeneous differential equation (4.3) is identical to (3.20) (and also (3.21) according to Proposition 3), the solution form (4.6) follows from condition (4.1). To determine the unknown constants $\{B_{\chi,m,n}(b)\}_{n=1}^{ma}$, (4.6) is substituted back into (4.2). Note that (3.25) still holds true but with $\phi(\cdot;b)$ and $A_{w,m,n}(b)$ replaced by $V(\cdot;b)$ and $B_{\chi,m,n}(b)$ respectively. Further utilizing (3.13), equating the coefficients of $\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}$ in the substituted (4.2) results in (4.7). ■

5 Numerical examples

In this section, we shall apply the theoretical results derived in Sections 2-4 to provide some numerical examples. For $u \geq b$, the Gerber-Shiu function $\phi(u;b)$ and the expected discounted cost of capital injections before ruin $V(u;b)$ are calculated by Propositions 4 and 6. Then, for $0 \leq u < b$, the values of $\phi(u;b)$ and $V(u;b)$ are computed via (3.4) and (4.2). (Although the integrals in (3.4) and (4.2) can be evaluated explicitly since the components involved are known functions, the tedious but straightforward details are omitted for brevity.)

5.1 Poissonian observer and exponential claims

We start with the simplest case where both the inter-capital-injection times and the claim amounts are exponentially distributed, i.e. their densities are $f_T(t) = \beta e^{-\beta t}$ and $f_X(x) = \alpha_1 e^{-\alpha_1 x}$ respectively by letting $m = 1$ and $a = 1$ in (1.2) and (1.6). Due to the memoryless property of exponential claims (and the fact that the event of ruin is always monitored continuously), given that ruin occurs the deficit at ruin $|U^b_{\tau_b}|$ follows the same exponential distribution as the individual claim. Therefore, as far as the Gerber-Shiu function $\phi(u;b)$ is concerned, it is sufficient to consider the special case $w(\cdot) \equiv 1$ so that $\phi(u;b)$ corresponds to the Laplace transform of the ruin time.

5.1.1 Exact results for continuous observation

Note that if $\beta \to \infty$ then the surplus process U^b is observed continuously for capital injections, i.e. whenever the surplus falls below the critical level b, capital is injected immediately to restore it to b, which is exactly the model considered by Nie et al. (2011). At the limit, the functions $\phi(u;b)$ and $V(u;b)$ will be respectively denoted by $\phi_\infty(u;b)$ and $V_\infty(u;b)$, which are provided below for the sake of completeness. Under the cost function $\chi(x) = x$ for $x \in (0,b]$, the quantity $V_\infty(u;b)$ becomes the expected present value of capital injected before ruin, which was derived by Nie et al. (2011, Section 5.1) as

$$V_\infty(u;b) = \phi(u-b)\left\{\frac{1}{\alpha_1}[1-e^{-\alpha_1 b}(1+\alpha_1 b)] + V_\infty(b;b)(1-e^{-\alpha_1 b})\right\}, \qquad u \geq b, \tag{5.1}$$

where

$$V_\infty(b;b) = \frac{\phi(0)\frac{1}{\alpha_1}[1-e^{-\alpha_1 b}(1+\alpha_1 b)]}{1-\phi(0)(1-e^{-\alpha_1 b})}.$$

Here

$$\phi(u) = \frac{\alpha_1 + s_{1,1}}{\alpha_1}e^{s_{1,1}u}, \qquad u \geq 0, \tag{5.2}$$

is the Laplace transform of the ruin time pertaining to the classical model U without capital injections given by Gerber and Shiu (1998, Equation (5.38)), where

$$s_{1,1} = -\frac{(c\alpha_1 - \lambda - \delta) + \sqrt{(c\alpha_1-\lambda-\delta)^2 + 4c\alpha_1\delta}}{2c}$$

is the negative root of (3.21) (which is now a quadratic equation). For Laplace transform of the ruin time $\phi_\infty(u;b)$, we need to slightly modify the results in Nie et al. (2011) who considered the ruin probability. Omitting the details, it is found that $\phi_\infty(u;b)$ admits the explicit expression

$$\phi_\infty(u;b) = \phi(u-b)\big\{e^{-\alpha_1 b} + \phi_\infty(b;b)(1-e^{-\alpha_1 b})\big\}, \qquad u \geq b, \tag{5.3}$$

where

$$\phi_\infty(b;b) = \frac{\phi(0)e^{-\alpha_1 b}}{1-\phi(0)(1-e^{-\alpha_1 b})}.$$

For $0 \leq u < b$, capital is immediately injected at time zero if the surplus starts below b under continuous checking for capital injections. Hence, one has the definitions

$$V_\infty(u;b) = b - u + V_\infty(b;b), \qquad 0 \leq u < b, \tag{5.4}$$

and

$$\phi_\infty(u;b) = \phi_\infty(b;b), \qquad 0 \leq u < b. \tag{5.5}$$

5.1.2 Impact of injection frequency β and optimization from shareholders' viewpoint

We begin by looking at the impact of the parameter β on $\phi(u;b)$ and $V(u;b)$ in Example 1, where β can be interpreted as the frequency of checking the process for capital injections.

Example 1 *(Impact of β on $\phi(u;b)$ and $V(u;b)$)* In this example, the Poisson claim arrival rate is assumed to be $\lambda = 1$ and each claim size is exponential with parameter $\alpha_1 = 1$. Let $c = 1.2$ be the premium rate (so that the loading condition holds). The critical level for capital injections is chosen to be $b = 5$, and both the Laplace transform argument in $\phi(u;b)$ and the force of interest used to discount the injections are $\delta = 0.1$. In Figure 2, $\phi(u;b)$ (under the penalty $w(\cdot) \equiv 1$) and $V(u;b)$ (under the cost function $\chi(x) = x$ for $x \in (0,b]$) are plotted against $u \geq 0$ for exponential inter-capital-injection times with parameters $\beta = 1, 5, 10, 50, 100$, and the curves for $\phi_\infty(u;b)$ and $V_\infty(u;b)$ computed by (5.1) and (5.3)-(5.5) are also provided and labelled as $\beta = \infty$.

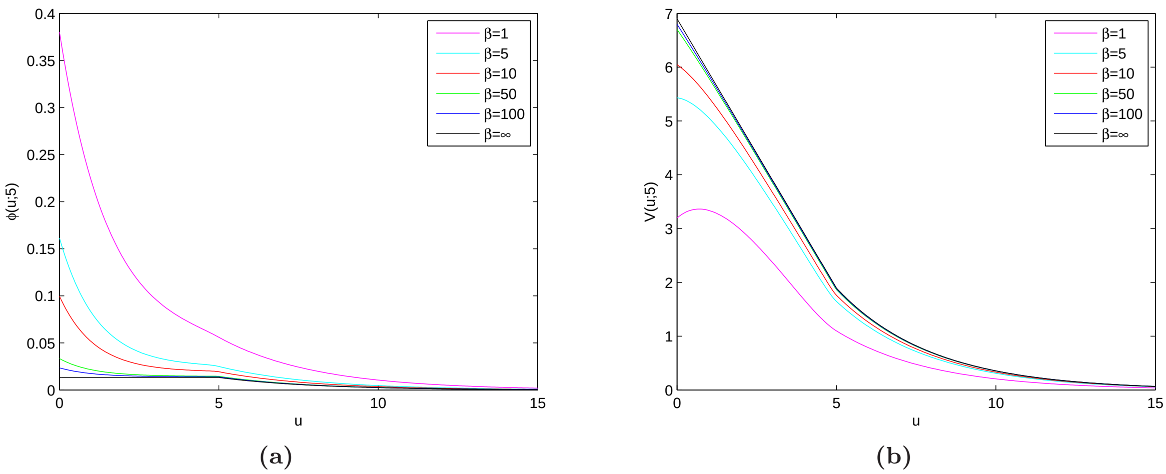


Figure 2: The impact of the parameter β: (a) the Laplace transform of the ruin time; (b) the expected discounted capital injections until ruin

For fixed initial surplus u, we first observe from Figure 2(a) that $\phi(u;b)$ is decreasing in β while $V(u;b)$ in Figure 2(b) is increasing in β. The reason is clear: a larger β means more frequent (check for) capital injections and hence the event of ruin is delayed. As $\beta \to \infty$, one retrieves the case of continuous checking, and it is noted that $\phi(u;b) \to \phi_\infty(u;b)$ and $V(u;b) \to V_\infty(u;b)$ for $u \geq 0$.

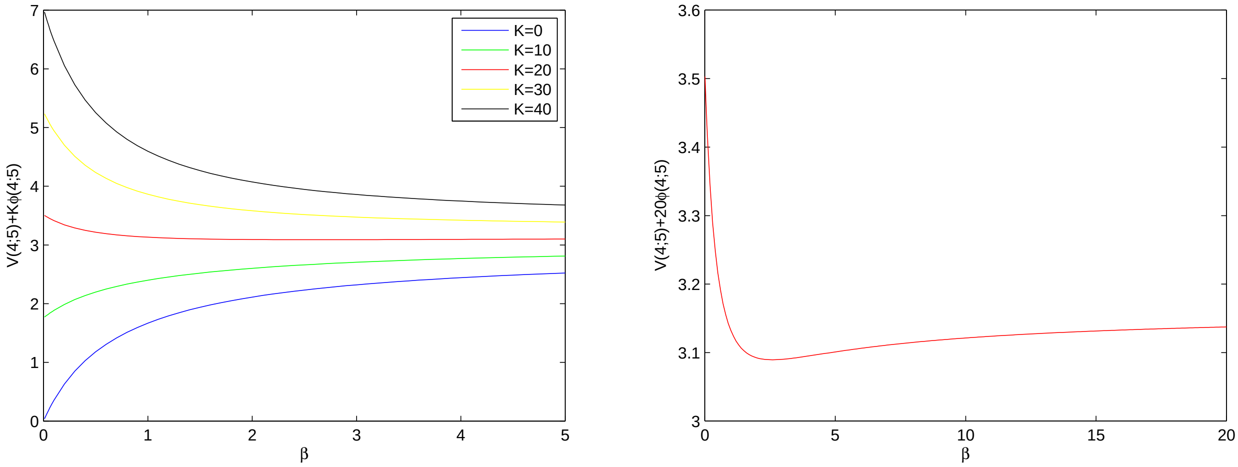
From Figure 2(a), it is seen that for each fixed β the Laplace transform of the ruin time $\phi(u;b)$ is decreasing in u. This must be the case because (for the same realization of the aggregate claims process $\{\sum_{i=1}^{N_t} X_i\}_{t\geq 0}$ and capital injection times $\{Z_i\}_{i=1}^\infty$) an insurer possessing a larger amount of initial surplus must ruin no earlier than one with smaller initial surplus. However, Figure 2(b) shows that $V(u;b)$ is decreasing in u only for $\beta = 5, 10, 50, 100, \infty$, but it first increases and then decreases in u when $\beta = 1$. Intuitively, when the initial surplus u increases, there are two opposing effects to the amount of injected capital. On one hand, with a larger initial surplus the process is more likely to be above the critical level b at the capital injection times, thereby reducing the need for capital injections. On the other hand, a larger u also means that the surplus process survives longer and hence there could be potentially more capital injections at later times. Under these competing factors, $V(u;b)$ is not necessarily monotone in u. Figure 2(b) suggests that the former effect always dominates when $\beta = 5, 10, 50, 100, \infty$. However, when $\beta = 1$ the latter effect dominates for small initial surplus. This is because when u is quite small and checking for capital injections occurs infrequently, the process may just ruin before the surplus is ever checked for capital injections, but a slight increase in u could increase the chance of ever having a capital injection and the amount of the first injection in such case could be large as the surplus starts far below the critical level b. Although each curve in both Figures 2(a) and 2(b) seemingly has a kink at $u = 5$, we have zoomed in the figures (which are not reproduced here) and found that the functions are indeed smoothly pasted at $u = 5$ when $\beta = 5, 10, 50, 100$ but not when $\beta = \infty$ (see Remark 2 below for more explanations). □

Remark 2 The smooth pasting property can indeed be analyzed for general claim density $f_X(\cdot)$ and Erlang(n) inter-capital-injection times using the same steps as in Avanzi et al. (2013, Sections 2.1 and 3.1). While $\phi'(b-;b) = \phi'(b+;b)$ is found to be valid for any penalty function $w(\cdot)$, the smooth pasting $V'(b-;b) = V'(b+;b)$ holds true only when the cost function is such that $\chi(0+) = 0$, which is satisfied by $\chi(x) = x$ in the case of expected discounted capital injections. (It is understood that the above derivatives are taken with respect to the first argument.) The details are omitted here for brevity. □

As discussed in the introduction, it is of the shareholders' interest to minimize $V(u;b) + K\phi(u;b)$ for some constant $K \geq 0$ as the owners of the insurance company are responsible for making capital injections and paying a penalty applied at ruin that possibly depends on the deficit. See Dickson and Waters (2004) for related discussions. The constant K can be regarded as a weight assigned to $\phi(u;b)$: if K is small then one puts more emphasis on minimizing the expected discounted cost of capital injections; but if K is set to be large then the insurer is more concerned about the penalty in the event of ruin. The optimization with respect to the injection frequency is performed in the next example.

Example 2 *(Minimizing capital injections plus penalty at ruin with respect to β)* We follow the same model parameters as in Example 1 in that $\lambda = 1$, $\alpha_1 = 1$, $c = 1.2$, $b = 5$, $\delta = 0.1$, $w(\cdot) \equiv 1$ and $\chi(x) = x$ for $x \in (0, b]$. The initial surplus is fixed to be $u = 4$ throughout. (Note that in this setting $\phi(u;b)$ also represents the expected discounted deficit since the deficit follows an independent exponential distribution with mean 1 given that ruin occurs.)

Figure 3(a) plots $V(u;b) + K\phi(u;b)$ as a function of β for fixed $K = 0, 10, 20, 30, 40$. Recall from Example 1 that $\phi(u;b)$ is decreasing while $V(u;b)$ is increasing in β. When $K = 0, 10$, the increasing



(a) **(b)**

Figure 3: Plot of $V(4;5) + K\phi(4;5)$ against β: (a) $K = 0, 10, 20, 30, 40$; (b) optimization at a positive finite β when $K = 20$

pattern of $V(4;5)$ dominates and hence $V(4;5) + K\phi(4;5)$ is increasing in β so that $\beta^* = 0$ (i.e. the process is never checked for capital injections) will minimize $V(4;5) + K\phi(4;5)$. On the other hand, when K is large at $K = 30, 40$, the decreasing pattern of $\phi(4;5)$ dominates such that $V(4;5) + K\phi(4;5)$ is decreasing in β and therefore $\beta^* = \infty$ (i.e. continuous check for capital injections) will minimize $V(4;5) + K\phi(4;5)$. Among the five values of K chosen, $V(4;5) + K\phi(4;5)$ first decreases and then increases in β only when $K = 20$. Since this is not obvious from Figure 3(a), such a case is plotted again in Figure 3(b) with a magnified y-axis in order to depict that there is a positive finite value β^* that minimizes $V(4;5) + 20\phi(4;5)$. It is found that $\beta^* = 2.5874$ and the minimized value of $V(4;5) + 20\phi(4;5)$ equals 3.0895. □

5.1.3 Optimal reinsurance to minimize ruin probability

We shall first demonstrate how $V(u;b)$ can be applied to price a perpetual reinsurance contract. The idea of perpetual reinsurance was proposed by Pafumi (1998) and Dickson and Waters (2004), where a reinsurer immediately makes the necessary payments to bring the insurer's surplus back to zero whenever it drops below zero due to a claim. In this paper, the above perpetual reinsurance is readily modified as follows. At each capital injection time Z_i, if the surplus process drops below the critical level $b > 0$, then the reinsurer will make the payments to restore the surplus to level b provided that ruin has not occurred in the interim. Assume that the insurance company possesses an initial surplus of $U^* \geq 0$, and it pays part of U^* as a net single premium to a reinsurer at time zero in return for the reinsurance payments. Taking into account the fact that paying the reinsurance premium reduces the insurer's initial surplus and denoting the price of the perpetual reinsurance by RP, then RP$\in (0, U^*]$ (if it exists) satisfies the equation

$$\mathrm{RP} = V(U^* - \mathrm{RP}; b), \tag{5.6}$$

where $V(u;b)$ is calculated under a cost function $\chi(\cdot)$ agreed by the insurer and the reinsurer. Note that the insurer effectively starts at the surplus level $U^* - \mathrm{RP}$ after purchasing reinsurance. In general, the above equation has to be solved numerically although explicit expressions for $V(u;b)$ is available.

Example 3 *(Pricing reinsurance contract)* We follow similar parameter settings as in Example 1, and set $\lambda = 1$ and $c = 1.2$ and assume exponential claims with $\alpha_1 = 1$. The inter-capital-injection times are also exponential but with parameter $\beta = 2$. We are interested in calculating the reinsurance premium RP using (5.6). Assuming that the reinsurer has a 50% loading factor, $V(u; b)$ appearing in (5.6) is computed under the cost function $\chi(x) = 1.5x$ for $x \in (0, b]$, and a force of interest $\delta = 0.1$ is used to discount the injections.

Table 1: Exact values of reinsurance premium

$b \setminus U^*$	2	4	6	8	10	12	14	16	18	20
1	0.10816	0.05455	0.02778	0.01419	0.00727	0.00373	0.00191	0.00098	0.00050	0.00026
2	NA	0.38060	0.18295	0.09110	0.04607	0.02348	0.01201	0.00615	0.00316	0.00162
3	NA	3.43539	0.55877	0.25966	0.12757	0.06413	0.03258	0.01664	0.00852	0.00437
4	NA	NA	1.46495	0.55543	0.25827	0.12692	0.06381	0.03242	0.01656	0.00848
5	NA	NA	NA	1.10256	0.45638	0.21629	0.10708	0.05401	0.02749	0.01405
6	NA	NA	NA	7.74752	0.77389	0.34432	0.16661	0.08320	0.04213	0.02148
7	NA	NA	NA	NA	**1.38925**	0.53685	0.25054	0.12329	0.06202	0.03152
8	NA	NA	NA	NA	9.93625	0.85310	0.37325	0.17966	0.08951	0.04528
9	NA	NA	NA	NA	NA	**1.49204**	0.56181	0.26091	0.12816	0.06442
10	NA	NA	NA	NA	NA	NA	**0.87654**	0.38158	0.18339	0.09131
11	NA	NA	NA	NA	NA	NA	**1.52225**	**0.56875**	0.26378	0.12950
12	NA	NA	NA	NA	NA	NA	NA	**0.88292**	0.38383	0.18439

Table 2: Combinations of (U^*, b) with multiple reinsurance premiums

(U^*, b)	RP_1	RP_2	RP_3
(10,7)	1.38925	5.25865	9.11847
(12,9)	1.49204	4.87474	11.51230
(14,10)	0.87654	8.27653	12.97092
(14,11)	1.52225	4.78336	13.68106
(16,11)	0.56875	12.01898	14.03225
(16,12)	0.88292	8.13849	15.30299

The exact values of the reinsurance premium RP for various pairs of (U^*, b) are presented in Table 1. The word 'NA' indicates cases where RP does not exist, i.e. even the entire initial surplus U^* is insufficient to purchase reinsurance. These entries happen mostly when the critical level b is no less than U^*. Although $V(u; b)$ is not always decreasing in u (see the blue line in Figure 4 as well as the discussions in Example 1), within Table 1 one observes that RP is decreasing in U^* across each row, suggesting that an insurer with higher initial surplus pays less reinsurance premium. Furthermore, from each column of Table 1, we see that RP is increasing in b, meaning that it is more expensive for the insurer to purchase a reinsurance contract that injects capital to bring the surplus to a higher level b at a capital injection time. Interestingly, it is found that the RP value exists but is not unique for six pairs of (U^*, b), and the corresponding cells are in **bold** in Table 1 (where we have chosen the smallest value of RP). In each case, there are three values of RP satisfying (5.6) (denoted by RP_i for $i = 1, 2, 3$ in increasing order), and the results are summarized in Table 2. In particular, to see how multiple solutions occur when $(U^*, b) = (10, 7)$, we plot x and $V(U^* - x; b)$ as function of x in Figure 4, and the intersections of the two curves satisfy (5.6). (The plots for the other five cases are very similar and thus omitted.) For each of the six cases, after paying RP_1 as reinsurance premium, the process still starts above the critical level b

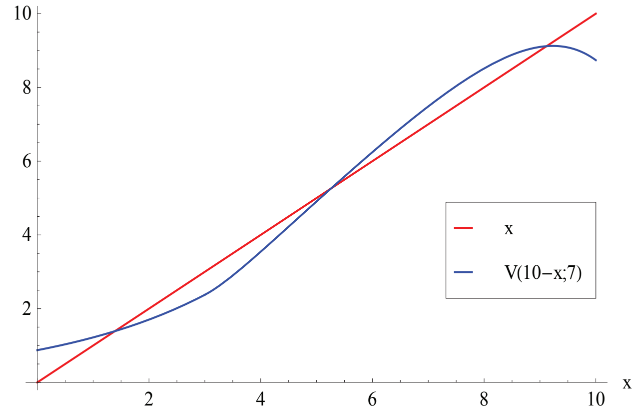


Figure 4: Illustration of multiple reinsurance premiums when $(U^*, b) = (10, 7)$

Table 3: Ruin probabilities after reinsurance

$b \setminus U^*$	2	4	6	8	10	12	14	16	18	20
Classical	0.59711	0.42785	0.30657	0.21966	0.15740	0.11278	0.08081	0.05790	0.04149	0.02973
1	0.59243	0.42072	0.30012	0.21456	0.15356	0.10996	0.07877	0.05643	0.04043	0.02897
2	NA	0.38955	0.27008	0.19058	0.13554	0.09675	0.06919	0.04953	0.03547	0.02541
3	NA	**0.44419**	0.22057	0.15036	0.10539	0.07472	0.05326	0.03806	0.02724	0.01950
4	NA	NA	0.17446	0.10742	0.07325	0.05135	0.03641	0.02595	0.01855	0.01327
5	NA	NA	NA	0.07348	0.04727	0.03254	0.02290	0.01626	0.01160	0.00829
6	NA	NA	NA	**0.25285**	0.02957	0.01972	0.01372	0.00970	0.00690	0.00493
7	NA	NA	NA	NA	0.01891	0.01176	0.00803	0.00563	0.00400	0.00285
8	NA	NA	NA	NA	**0.25407**	0.00706	0.00467	0.00324	0.00229	0.00163
9	NA	NA	NA	NA	NA	0.00444	0.00272	0.00186	0.00130	0.00092
10	NA	NA	NA	NA	NA	NA	0.00162	0.00107	0.00074	0.00052
11	NA	NA	NA	NA	NA	NA	0.00101	0.00062	0.00042	0.00030
12	NA	NA	NA	NA	NA	NA	NA	0.00037	0.00024	0.00017

as $U^* - \mathrm{RP}_1 > b$, and this leads to the lowest ruin probability (given in Table 3) compared to paying RP_2 or RP_3. But if the insurer instead chooses to pay RP_2 or RP_3, then its surplus will go below the critical level b at the beginning. (Recall that time zero is not assumed to be a capital injection time.) This not only results in higher ruin probability but also brings more extremes to the amount of injection: there could potentially be a big injection especially at the first capital injection time if ruin has not occurred yet; but the process may have ruined before the first capital injection time due to insufficient surplus and then no capital will ever be injected.

Finally, the resulting ruin probabilities after applying the reinsurance premium in Table 1 are given in Table 3, and these are calculated as $\phi(U^* - \mathrm{RP}; b)$ under the penalty $w(\cdot) \equiv 1$ and Laplace transform argument $\delta = 0$. For comparison, we have additionally provided the ruin probabilities $\phi(U^*)$ for the classical case without reinsurance computed via (5.2) under $\delta = 0$. (One may regard each 'NA' entry in Table 3 to follow the classical ruin probability in the same column, since the non-existence of RP means that no reinsurance is purchased.) When RP exists, we note that reinsurance (compared to the absence of reinsurance) always reduces the ruin probability except for three pairs of (U^*, b), namely $(4, 3)$, $(8, 6)$ or $(10, 8)$, which are in **bold**. Further examination of Table 1 reveals that these exceptions are the only cases where purchasing reinsurance brings the initial surplus below the critical level b (i.e. $U^* - \mathrm{RP} < b$). In these three cases, RP is unique but it is so big such that the value of $U^* - \mathrm{RP}$ is quite small, which gives rise to a higher ruin probability than without reinsurance. □

Understanding from the above Example 3 that ruin probability can be reduced via reinsurance, it is of the insurer's interest to find the reinsurance contract that minimizes the ruin probability. This can be formulated as follows. At time zero, the insurer holds an amount of capital U^* and decides to allocate u as the initial surplus. The remaining $U^* - u = V(u;b)$ is used to buy a reinsurance with critical level b (that is described before Example 3). Therefore, we would like to search for the optimal pair of (u, b), namely (u^*, b^*), that minimizes the ruin probability $\phi(u;b)$ (under $w(\cdot) \equiv 1$ and $\delta = 0$) subject to the constraint $U^* = u + V(u;b)$. See also Nie et al. (2011, Section 4) for the descriptions of such an optimal reinsurance strategy under continuous check for injections. Intuitively, a higher (resp. lower) reinsurance premium $V(u;b)$ can be used to purchase better (worse) reinsurance contract which leads to higher (resp. lower) safety, but at the same time this results in a lower (resp. higher) retained capital u which makes the process more (resp. less) dangerous. Such a trade-off makes optimization possible. More idea is illustrated in the next example, which complements Nie et al. (2011, Example 5.1.1).

Example 4 *(Optimal reinsurance strategy)* We use the same parameters as in Example 3 and set $\lambda = 1$, $c = 1.2$, $\alpha_1 = 1$ and $\beta = 2$. The price of reinsurance $V(u;b)$ is calculated under $\chi(x) = 1.5x$ (for $x \in (0, b])$ and $\delta = 0.1$ whereas the ruin probability is computed from the Gerber-Shiu function $\phi(u;b)$ with $w(\cdot) \equiv 1$ and $\delta = 0$.

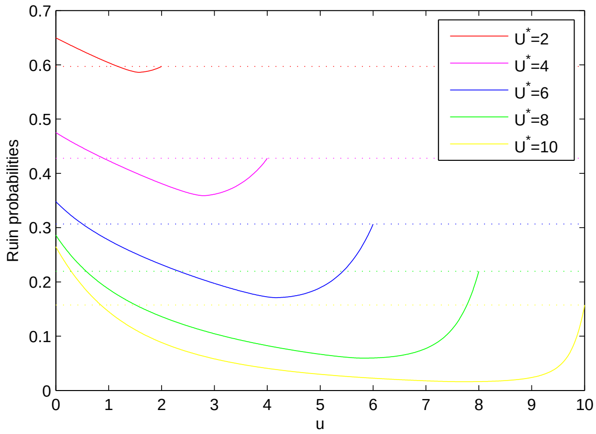


Figure 5: Ruin probabilities subject to the constraint $U^* = u + V(u;b)$

Table 4: Optimal reinsurance strategy (u^*, b^*)

U^*	u^*	b^*	$\phi(u^*;b^*)$	$\phi(U^*)$
2	1.5734	1.6026	0.58618	0.59711
4	2.8013	2.8532	0.35888	0.42785
6	4.1651	4.1845	0.17101	0.30657
8	5.8319	5.7482	0.05957	0.21966
10	7.7200	7.5495	0.01612	0.15740

For $U^* = 2, 4, 6, 8, 10$, the ruin probabilities under the constraint $U^* = u + V(u;b)$ are plotted in

Figure 5, where the dotted lines represent the values of $\phi(U^*)$ (i.e. ruin probabilities without reinsurance). Specifically, to obtain each curve (fixed U^*) in Figure 5, we first fix $u \in [0, U^*]$ and find the value of b (say b_u) such that $U^* - u = V(u; b)$. Since $V(u; b)$ is increasing in b (see also Example 6), we always find a unique value of b_u, and then the ruin probability is given by $\phi(u; b_u)$. (Note that if $u = U^*$ then no money is allocated for reinsurance and therefore $b_{U^*} = 0$.) It is found that the curves in Figure 5 are first decreasing and then increasing in u (and they are all convex in u). Hence, for fixed U^* the ruin probability $\phi(u; b_u)$ is minimized at a unique point $u = u^*$ and the resulting critical level is b_{u^*} (i.e. the optimal pair of (u, b) is $(u^*, b^*) = (u^*, b_{u^*})$). The values of (u^*, b^*) and the resulting ruin probabilities $\phi(u^*, b^*)$ are presented in Table 4, where the ruin probabilities $\phi(U^*)$ without reinsurance are also provided for easy reference. It is clear that the ruin probabilities can be significantly reduced after pursuing optimal reinsurance (except when $U^* = 2$). □

Remark 3 Indeed, the optimal reinsurance strategy can in principle be determined by following the steps in Example 3 as well. More precisely, for fixed U^*, one can first compute RP via (5.6) as a function of b and then calculate the resulting ruin probability as $\phi(U^* - \mathrm{RP}; b)$. The optimal value b^* is chosen to minimize $\phi(U^* - \mathrm{RP}; b)$ with respect to b. (For example, when $U^* = 8, 10$, by examining Table 3 one sees that the optimal values b^* should not be far from 5 and 7 respectively.) Finally, u^* is calculated as $U^* - \mathrm{RP}$ where RP is the reinsurance premium under the optimal critical level b^*. Although this will lead to the same optimal reinsurance strategy as in Example 4, it is more straightforward to implement the procedure in Example 4 because solving (5.6) (which is the starting point of Example 3) may result in no solution or multiple solutions. □

5.2 Erlang inter-capital-injection times

In this subsection, we assume Erlang inter-capital-injection times. The Erlangization technique discussed in Section 1 will be first demonstrated with the following numerical example.

5.2.1 Performance of Erlangization

Example 5 *(Erlangization)* In this example, three claim size distributions will be considered, namely

(1) a sum of two exponentials with means 1/3 and 2/3 (i.e. $f_X(x) = 3e^{-1.5x} - 3e^{-3x}$);

(2) an exponential distribution with mean 1 (i.e. $f_X(x) = e^{-x}$); and

(3) a mixture of two exponentials: with mixing probability 1/3 it is exponential with mean 2 and with mixing probability 2/3 it is exponential with mean 1/2 (i.e. $f_X(x) = (1/6)e^{-(1/2)x} + (4/3)e^{-2x}$).

Note that the above distributions all belong to the class of combinations of exponentials, and therefore our results in earlier sections are applicable. The common mean of these distributions is 1 but their variances are 0.56, 1 and 2 respectively. Furthermore, we set $c = 1.5$, $\lambda = 1$, and $b = 8$. As mentioned in Section 1, the Erlangization procedure can be used to approximate deterministic inter-capital-injection times by fixing the mean $\mathbb{E}T = m/\beta$ and increasing m (and β at the same time). For illustration, we fix $m/\beta = 1$ and set $m = 1, 2, \ldots, 9$.

For the initial surplus levels $u = 0, 5, 10, 15, 20$, the exact values of ruin probabilities (i.e. $\phi(u; b)$ with $w(\cdot) \equiv 1$ and $\delta = 0$) are presented in Tables 5-7 whereas those for the expected discounted capital injections until ruin (calculated as $V(u; b)$ under $\chi(x) = x$ for $x \in (0, b]$ and $\delta = 0.1$) are given in Tables

Table 5: Ruin probabilities when $f_X(x) = 3e^{-1.5x} - 3e^{-3x}$

$m \setminus u$	0	5	10	15	20
1	0.36085080	0.0072682848	0.00072917712	0.000079851108	0.0000087444241
2	0.40163573	0.0064666971	0.00049601606	0.000053758142	0.0000058851280
3	0.41786679	0.0061486253	0.00041714467	0.000045065247	0.0000049337279
4	0.42651947	0.0059774340	0.00037783915	0.000040760897	0.0000044627259
5	0.43187283	0.0058703580	0.00035436570	0.000038199470	0.0000041824512
6	0.43550242	0.0057970267	0.00033878157	0.000036502784	0.0000039967961
7	0.43812127	0.0057436479	0.00032768847	0.000035296946	0.0000038648484
8	0.44009800	0.0057030493	0.00031939238	0.000034396186	0.0000037662818
9	0.44164195	0.0056711299	0.00031295518	0.000033697875	0.0000036898670

Table 6: Ruin probabilities when $f_X(x) = e^{-x}$

$m \setminus u$	0	5	10	15	20
1	0.33865446	0.019699285	0.0040982776	0.00077406465	0.000146201930
2	0.37380968	0.018176075	0.0032574347	0.00061009063	0.000115153970
3	0.38773077	0.017582094	0.0029547935	0.00055177613	0.000104138550
4	0.39515265	0.017264898	0.0027992449	0.00052196332	0.000098511248
5	0.39975293	0.017067357	0.0027045919	0.00050387716	0.000095098617
6	0.40287930	0.016932437	0.0026409505	0.00049174085	0.000092809107
7	0.40514054	0.016834412	0.0025952335	0.00048303490	0.000091166944
8	0.40685130	0.016759961	0.0025608065	0.00047648576	0.000089931719
9	0.40819039	0.016701493	0.0025339487	0.00047138064	0.000088968909

Table 7: Ruin probabilities when $f_X(x) = (1/6)e^{-(1/2)x} + (4/3)e^{-2x}$

$m \setminus u$	0	5	10	15	20
1	0.33833838	0.074895097	0.033979610	0.012217314	0.0043908496
2	0.36523397	0.072193550	0.031298192	0.011220800	0.0040295346
3	0.37564407	0.071129991	0.030283847	0.010846282	0.0038941495
4	0.38110808	0.070556952	0.029750311	0.010649856	0.0038232294
5	0.38445773	0.070197345	0.029421214	0.010528895	0.0037795852
6	0.38671583	0.069950206	0.029197946	0.010446919	0.0037500200
7	0.38833903	0.069769768	0.029036533	0.010387699	0.0037286680
8	0.38956117	0.069632206	0.028914398	0.010342914	0.0037125242
9	0.39051412	0.069523949	0.028818794	0.010307863	0.0036998905

8-10. In these tables, the values are converging as one moves down along each column, showing the effect of Erlangization. Looking across Tables 5-7, it is noted that the ruin probability appears to be higher when the variance of the claim amount is larger (except when $u = 0$). This is natural since a larger claim variance represents a higher risk to the insurer. For $u = 0$, although the ruin probabilities are in reverse order of the claim variances, they are still quite close to each other. This is due to the possibility that when $u = 0$ there is considerable chance the process ruins before the first capital injection time, and the ruin probability with zero initial surplus in the classical model without capital injections equals $\mathbb{P}(\tau < \infty | U_0 = 0) = \lambda \mathbb{E}X/c$ (i.e. independent of the individual claim distribution). Next, when comparing Tables 8-10, the expected present value of injected capital is mostly increasing with the claim variance. Intuitively, the surplus process is likely to fall below critical level b more frequently under a higher claim variance, thereby increasing the chance for capital injections to occur. However, we also note the exceptional case of $u = 5$ when moving from Table 8 to Table 9 where $V(u; b)$ decreases as

Table 8: $V(u;8)$ under $\chi(x) = x$ when $f_X(x) = 3e^{-1.5x} - 3e^{-3x}$

$m \setminus u$	0	5	10	15	20
1	4.6320339	2.9181685	0.23338436	0.015408883	0.0010173399
2	4.2157705	2.9122645	0.24809564	0.016470958	0.0010877590
3	4.0502436	2.9111756	0.25365974	0.016866332	0.0011138429
4	3.9620409	2.9109585	0.25656756	0.017071356	0.0011273467
5	3.9074828	2.9109844	0.25834997	0.017196438	0.0011355787
6	3.8704962	2.9110851	0.25955283	0.017280579	0.0011411138
7	3.8438104	2.9112044	0.26041869	0.017341005	0.0011450878
8	3.8236677	2.9113217	0.26107151	0.017386483	0.0011480781
9	3.8079345	2.9114298	0.26158120	0.017421940	0.0011504092

Table 9: $V(u;8)$ under $\chi(x) = x$ when $f_X(x) = e^{-x}$

$m \setminus u$	0	5	10	15	20
1	4.8919184	3.0119179	0.36270548	0.043552111	0.0052295498
2	4.5114376	3.0106634	0.38626730	0.046649690	0.0056054608
3	4.3605839	3.0117011	0.39529091	0.047822708	0.0057468776
4	4.2800769	3.0127717	0.40004638	0.048437707	0.0058208240
5	4.2301410	3.0136779	0.40297872	0.048815802	0.0058662198
6	4.1961864	3.0144228	0.40496632	0.049071583	0.0058969021
7	4.1716178	3.0150347	0.40640185	0.049256064	0.0059190177
8	4.1530242	3.0155408	0.40748702	0.049395379	0.0059357111
9	4.1384665	3.0159633	0.40833602	0.049504288	0.0059487566

Table 10: $V(u;8)$ under $\chi(x) = x$ when $f_X(x) = (1/6)e^{-(1/2)x} + (4/3)e^{-2x}$

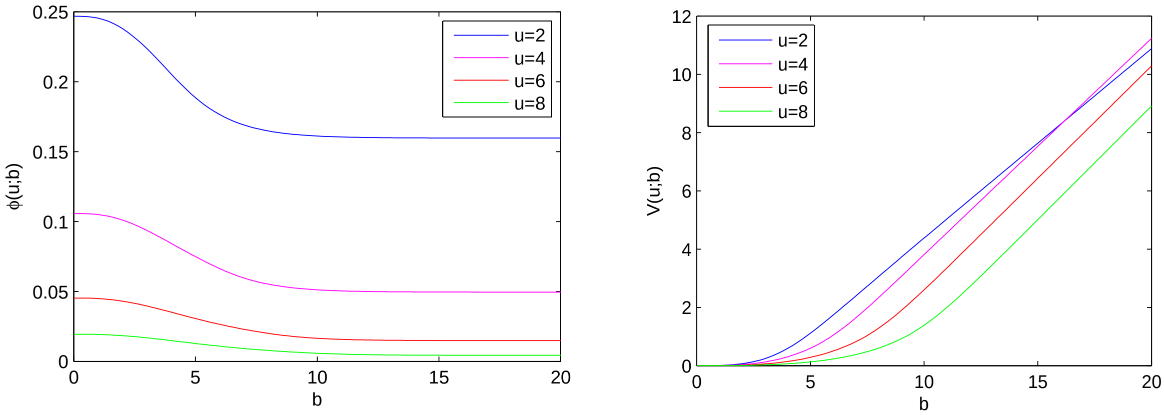
$m \setminus u$	0	5	10	15	20
1	5.1598654	3.0004264	0.55194624	0.13973267	0.035424000
2	4.8293697	2.9951112	0.59128302	0.15038064	0.038188492
3	4.7012623	2.9949608	0.60664039	0.15450608	0.039253211
4	4.6338259	2.9956075	0.61482248	0.15669623	0.039816974
5	4.5923805	2.9963593	0.61990458	0.15805378	0.040165893
6	4.5643836	2.9970581	0.62336746	0.15897754	0.040403089
7	4.5442250	2.9976703	0.62587847	0.15964674	0.040574802
8	4.5290263	2.9981957	0.62778226	0.16015381	0.040704852
9	4.5171577	2.9986381	0.62927238	0.16055102	0.040806734

claim variance increases. This is attributed to the fact that a higher ruin probability resulting from increased claim variance could lead to less capital injections in the long run since injections stop after ruin occurrence. See Example 1 for similar comments on Figure 2(b) where there are two competing factors. □

5.2.2 Impact of critical level b and optimization from shareholders' viewpoint

In the final two examples, we study the impact of the parameter b on $\phi(u;b)$ and $V(u;b)$ as well as the minimization of $V(u;b) + K\phi(u;b)$. The interpretation of the minimization has been given before Example 2 (except that optimization in Example 2 is done with respect to the injection frequency).

Example 6 *(Impact of b on $\phi(u;b)$ and $V(u;b)$)* In this example, it is assumed that each claim is exponentially distributed with density $f_X(x) = e^{-x}$ and each inter-capital-injection time is Erlang with parameters $m = 5$ and $\beta = 2$. Let $c = 1.5$ and $\lambda = 1$. The Laplace transform of the ruin time (i.e. $\phi(u; b)$ with $w(\cdot) \equiv 1$) and the expected discounted capital injections until ruin (i.e. $V(u; b)$ with $\chi(x) = x$ for $x \in (0, b]$), both calculated under $\delta = 0.1$, are depicted in Figure 6 for $u = 2, 4, 6, 8$ and $b \in [0, 20]$.



(a) **(b)**

Figure 6: The impact of the parameter b: (a) the Laplace transform of the ruin time; (b) the expected discounted capital injections until ruin

For each fixed u, Figure 6(a) shows that $\phi(u; b)$ decreases as b increases. This is because a higher critical level b means that the surplus process is brought to a higher and hence safer level whenever capital is injected, and therefore ruin is less likely to occur (or occurs later). On the other hand, $V(u; b)$ in Figure 6(b) is increasing in b when u is fixed. Clearly, a larger b implies that (1) a larger amount of capital needs to be injected to restore the surplus to b when capital injection occurs; and (2) there could be more injections in the long run as the surplus process survives for a longer period. As b increases further, $\phi(u; b)$ tends to a constant independent of b (which is denoted by $\phi(u; \infty)$) whereas each curve in Figure 6(b) approaches a straight line with positive slope. Indeed, when b is very large, a huge amount of capital (at the order of b) will be injected at the first Erlang capital injection time as long as ruin has not occurred before that. Then, the surplus process (at the high level b) is very unlikely to ruin after the first injection. As a result, if ruin occurs it has to happen before the first injection time, so that $\phi(u; \infty)$ is the Laplace transform of the ruin time for ruin occurring before an independent Erlang(m, β) time, namely $\mathbb{E}[e^{-\delta\tau}\mathbf{1}_{(\tau<T)}|U_0 = u]$, which equals $\zeta_m(u)$ (when $w(\cdot) \equiv 1$) given in (3.3) or (3.10). For similar reason, we can argue that $V(u; b) \sim \mathbb{E}[e^{-\delta T}\mathbf{1}_{(\tau>T)}|U_0 = u]b$ as $b \to \infty$ since the first injection (which is of order b) will dominate subsequent ones. This explains the linearity of $V(u; b)$ for large b in Figure 6(b). For fixed b, we also note from Figure 6 that $\phi(u; b)$ is decreasing in u but $V(u; b)$ is not always monotone in u. The reasons have been discussed in Example 1 and are not repeated here. □

Example 7 *(Minimizing capital injections plus penalty at ruin with respect to b)* We aim at searching for an optimal level b_{opt} that minimizes $V(u; b) + K\phi(u; b)$ with respect to b under identical parameters as in Example 6. In Figure 7, the function $V(u; b) + K\phi(u; b)$ is plotted against b for $K = 1$ and $K = 30$. It is found that b_{opt} depends on K but is independent of the initial surplus u for each fixed K. Specifically, our numerical results suggest that b_{opt} equals 0 and 4.455 for $K = 1$ and $K = 30$ respectively. When $K = 1$, the shape of the function $V(u; b)$ always dominates that of $\phi(u; b)$ such that $V(u; b) + \phi(u; b)$ is increasing in b and therefore $b_{\text{opt}} = 0$. When $K = 30$, the decreasing pattern of $\phi(u; b)$ first dominates

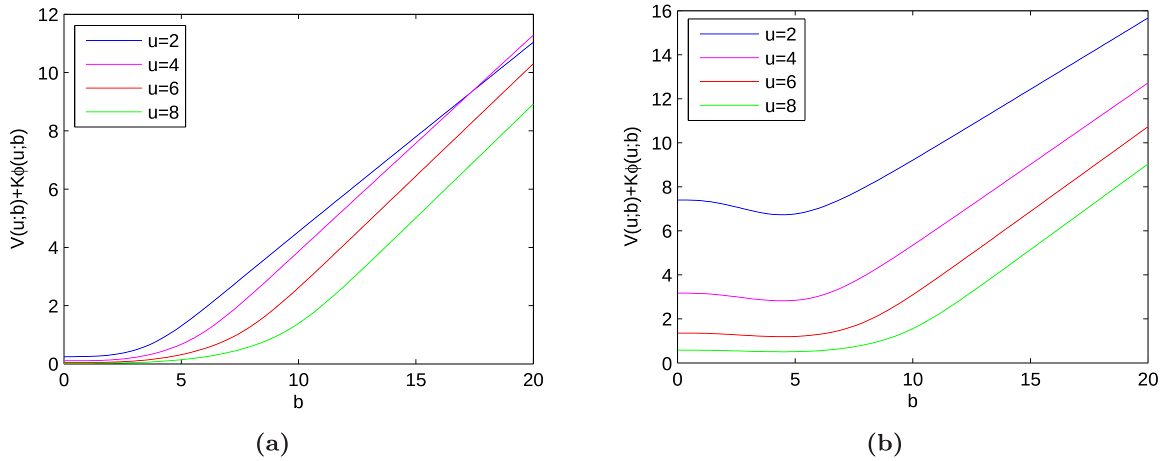


Figure 7: Plot of $V(u;b) + K\phi(u;b)$ against b: (a) $K = 1$; (b) $K = 30$

$V(u;b)$ for small values of b and then the increasing behaviour of $V(u;b)$ dominates $\phi(u;b)$ for large b, and consequently $V(u;b) + 30\phi(u;b)$ attains minimum at a positive value of b, namely $b_{\text{opt}} = 4.455$. $\square$

6 Concluding remarks

In the context of the classical compound Poisson risk model, we have extended Nie et al. (2011)'s model so that capital injections are only possible at the arrival epochs $\{Z_i\}_{i=1}^{\infty}$ of an Erlang renewal process. Even we have assumed that each claim amount follows a combination of exponentials with probability density (1.6), our results are indeed applicable to the case where the claims have rational Laplace transform (of order a). In such a case, the Lundberg's equation $\psi(\xi) = q$ also has $a + 1$ (assumed distinct) roots, and the scale function $W^{(q)}(x)$ is still in the form of (2.5) but the constants $C_{q,i}$'s will not be as explicit as (2.4). Consequently, the resolvent density $r_n^{(\beta+\delta)}(u, x)$ given in Proposition 1 is still valid. Another modification that needs to be made is that although $\zeta_1(u)$ in (3.6) still takes on the form (3.7) according to Landriault and Willmot (2008, Corollary 4), the coefficients $F_{w,l}$'s will be solved from a system of linear equations (see Landriault and Willmot (2008, Theorem 6)) instead of being given explicitly in (3.8). With the afore-mentioned adjustments, our main results (namely Propositions 4 and 6) concerning the Gerber-Shiu function and the expected discounted cost of capital injections still hold true.

In this paper, we have assumed that solvency is continuously checked, i.e. ruin is declared immediately when the insurer's surplus level becomes negative. On the other hand, for the case where the event of ruin is only monitored at the time points $\{Z_i\}_{i=1}^{\infty}$, it is clear that the discounted increment of the process observed at Erlang intervals (e.g. Albrecher et al. (2011, 2013)) can be utilized to solve the problem. More generally, one may consider the case where capital injection may only be made every j times ruin is checked (e.g. monthly balancing of books but annual capital injection if necessary). See Choi and Cheung (2014) for a similar dividend-ruin problem. The derivations, however, will be much more tedious.

Throughout this paper, it is assumed that the critical level b is pre-specified. It will be interesting to analyze optimal control problems involving periodic capital injection. This includes e.g. finding the optimal dividend and capital injection strategies that maximize the expected discounted dividends minus the expected discounted capital injections if both types of payments are only made periodically. We leave these as open problems for future research.

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A Appendix: Proof of Proposition 1

Proposition 1 can be proved by mathematical induction. To begin, the starting point (2.7) can be reexpressed as

$$r_1^{(\beta+\delta)}(u,x) = \begin{cases} \sum_{i=1}^{a+1} C_i e^{\rho_i u - \rho_{a+1}x}, & u < x. \\ \sum_{i=1}^{a} C_i e^{\rho_i u - \rho_{a+1}x} - \sum_{i=1}^{a} C_i e^{\rho_i(u-x)}, & u \geq x. \end{cases} \tag{A.1}$$

Hence, (2.10) and (2.11) hold true for $n = 1$ with the constants given by (2.12) and (2.13). Assuming that $r_n^{(\beta+\delta)}(u,x)$ is given by (2.10) and (2.11) for some positive integer n, we shall look at $r_{n+1}^{(\beta+\delta)}(u,x)$. By the first equality of (2.9) along with the induction assumption, one can write

$$r_{n+1}^{(\beta+\delta)}(u,x) = I_0(u,x) + I_1(u,x) - I_2(u,x), \tag{A.2}$$

where

$$I_0(u,x) = \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j} D_{n,i,j,k}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u}\int_0^\infty \frac{y^{k-1}}{(k-1)!}e^{-\rho_{a+1}y}r_1^{(\beta+\delta)}(y,x)dy, \tag{A.3}$$

$$I_1(u,x) = \sum_{i=1}^{a}\sum_{j=1}^{n} E_{n,i,j}\int_0^u \frac{(u-y)^{j-1}}{(j-1)!}e^{\rho_i(u-y)}r_1^{(\beta+\delta)}(y,x)dy, \tag{A.4}$$

$$I_2(u,x) = \sum_{j=1}^{n} E_{n,a+1,j}\int_u^\infty \frac{(u-y)^{j-1}}{(j-1)!}e^{\rho_{a+1}(u-y)}r_1^{(\beta+\delta)}(y,x)dy. \tag{A.5}$$

To evaluate these three terms, the identities (which are valid for positive integer k)

$$\int_0^u \frac{y^{k-1}}{(k-1)!}e^{-\gamma y}dy = \frac{1}{\gamma^k} - \sum_{l=1}^{k}\frac{1}{\gamma^{k+1-l}}\frac{u^{l-1}}{(l-1)!}e^{-\gamma u}, \qquad \gamma \neq 0, \tag{A.6}$$

$$\int_u^\infty \frac{y^{k-1}}{(k-1)!}e^{-\gamma y}dy = \sum_{l=1}^{k}\frac{1}{\gamma^{k+1-l}}\frac{u^{l-1}}{(l-1)!}e^{-\gamma u}, \qquad \Re(\gamma) > 0, \tag{A.7}$$

will be useful. First, substitution of (A.1) into (A.3) yields

$$\begin{aligned} I_0(u,x) = &\sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j}\sum_{l=1}^{a} D_{n,i,j,k}C_l\frac{u^{j-1}}{(j-1)!}e^{\rho_i u - \rho_{a+1}x}\int_0^\infty \frac{y^{k-1}}{(k-1)!}e^{\rho_l y - \rho_{a+1}y}dy \\ &+ \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j} D_{n,i,j,k}C_{a+1}\frac{u^{j-1}}{(j-1)!}\frac{x^k}{k!}e^{\rho_i u - \rho_{a+1}x} \\ &- \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j}\sum_{l=1}^{a} D_{n,i,j,k}C_l\frac{u^{j-1}}{(j-1)!}e^{\rho_i u - \rho_l x}\int_x^\infty \frac{y^{k-1}}{(k-1)!}e^{-\rho_{a+1}y + \rho_l y}dy \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j}\sum_{l=1}^{a} \frac{D_{n,i,j,k}C_l}{(\rho_{a+1}-\rho_l)^k}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u-\rho_{a+1}x}\\
&+ \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=2}^{n+2-j} D_{n,i,j,k-1}C_{a+1}\frac{u^{j-1}}{(j-1)!}\frac{x^{k-1}}{(k-1)!}e^{\rho_i u-\rho_{a+1}x}\\
&- \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{n+1-j}\sum_{l=k}^{n+1-j}\sum_{z=1}^{a} \frac{D_{n,i,j,l}C_z}{(\rho_{a+1}-\rho_z)^{l+1-k}}\frac{u^{j-1}}{(j-1)!}\frac{x^{k-1}}{(k-1)!}e^{\rho_i u-\rho_{a+1}x},
\end{aligned}
\tag{A.8}
$$

where we have changed the order of summations in the third term. Second, due to (2.7) the quantity $I_1(u,x)$ defined in (A.4) is given by

$$
I_1(u,x) = \begin{cases} I_{1,1}(u,x), & u < x,\\ I_{1,1}(u,x) - I_{1,2}(u,x), & u \ge x, \end{cases}
\tag{A.9}
$$

where

$$
\begin{aligned}
I_{1,1}(u,x) &= \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{a+1} E_{n,i,j}C_k \int_0^u \frac{(u-y)^{j-1}}{(j-1)!}e^{\rho_i(u-y)+\rho_k y-\rho_{a+1}x}dy\\
&= \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{a+1} E_{n,i,j}C_k e^{\rho_k u-\rho_{a+1}x}\int_0^u \frac{y^{j-1}}{(j-1)!}e^{(\rho_i-\rho_k)y}dy\\
&= \sum_{i=1}^{a}\sum_{j=1}^{n} E_{n,i,j}C_i\frac{u^j}{j!}e^{\rho_i u-\rho_{a+1}x} + \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1,k\neq i}^{a+1}\frac{E_{n,i,j}C_k}{(\rho_k-\rho_i)^j}e^{\rho_k u-\rho_{a+1}x}\\
&\quad - \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1,k\neq i}^{a+1}\sum_{l=1}^{j}\frac{E_{n,i,j}C_k}{(\rho_k-\rho_i)^{j+1-l}}\frac{u^{l-1}}{(l-1)!}e^{\rho_i u-\rho_{a+1}x}\\
&= \sum_{i=1}^{a}\sum_{j=2}^{n+1} E_{n,i,j-1}C_i\frac{u^{j-1}}{(j-1)!}e^{\rho_i u-\rho_{a+1}x} + \sum_{i=1}^{a+1}\sum_{j=1}^{n}\sum_{k=1,k\neq i}^{a}\frac{E_{n,k,j}C_i}{(\rho_i-\rho_k)^j}e^{\rho_i u-\rho_{a+1}x}\\
&\quad - \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1,k\neq i}^{a+1}\sum_{l=j}^{n}\frac{E_{n,i,l}C_k}{(\rho_k-\rho_i)^{l+1-j}}\frac{u^{j-1}}{(j-1)!}e^{\rho_i u-\rho_{a+1}x},
\end{aligned}
\tag{A.10}
$$

and similarly

$$
\begin{aligned}
I_{1,2}(u,x) &= \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1}^{a+1} E_{n,i,j}C_k \int_x^u \frac{(u-y)^{j-1}}{(j-1)!}e^{\rho_i(u-y)+\rho_k(y-x)}dy\\
&= \sum_{i=1}^{a}\sum_{j=2}^{n+1} E_{n,i,j-1}C_i\frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_i(u-x)} + \sum_{i=1}^{a+1}\sum_{j=1}^{n}\sum_{k=1,k\neq i}^{a}\frac{E_{n,k,j}C_i}{(\rho_i-\rho_k)^j}e^{\rho_i(u-x)}\\
&\quad - \sum_{i=1}^{a}\sum_{j=1}^{n}\sum_{k=1,k\neq i}^{a+1}\sum_{l=j}^{n}\frac{E_{n,i,l}C_k}{(\rho_k-\rho_i)^{l+1-j}}\frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_i(u-x)}.
\end{aligned}
\tag{A.11}
$$

Note that (A.6) has been applied along with a change of order of summations in obtaining the above two expressions. Third, we consider $I_2(u,x)$ in (A.5) and utilize (A.1) and (A.7). Omitting some details, it is found that, for $u < x$,

$$I_2(u,x) = \sum_{i=1}^{a}\sum_{j=1}^{n} E_{n,a+1,j}C_i \int_u^\infty \frac{(u-y)^{j-1}}{(j-1)!} e^{\rho_{a+1}(u-y)+\rho_i y - \rho_{a+1}x} dy$$
$$+ \sum_{j=1}^{n} E_{n,a+1,j}C_{a+1} \int_u^x \frac{(u-y)^{j-1}}{(j-1)!} e^{\rho_{a+1}(u-y)+\rho_{a+1}(y-x)} dy$$
$$- \sum_{i=1}^{a}\sum_{j=1}^{n} E_{n,a+1,j}C_i \int_x^\infty \frac{(u-y)^{j-1}}{(j-1)!} e^{\rho_{a+1}(u-y)+\rho_i(y-x)} dy$$
$$= -\sum_{i=1}^{a}\sum_{j=1}^{n} \frac{E_{n,a+1,j}C_i}{(\rho_i-\rho_{a+1})^j} e^{\rho_i u - \rho_{a+1}x} - \sum_{j=2}^{n+1} E_{n,a+1,j-1}C_{a+1}\frac{(u-x)^{j-1}}{(j-1)!} e^{\rho_{a+1}(u-x)}$$
$$+ \sum_{j=1}^{n}\sum_{k=1}^{a}\sum_{l=j}^{n} \frac{E_{n,a+1,l}C_k}{(\rho_k-\rho_{a+1})^{l+1-j}} \frac{(u-x)^{j-1}}{(j-1)!} e^{\rho_{a+1}(u-x)}, \tag{A.12}$$

and for $u \geq x$,

$$I_2(u,x) = \sum_{i=1}^{a}\sum_{j=1}^{n} E_{n,a+1,j}C_i (e^{\rho_i u - \rho_{a+1}x} - e^{\rho_i(u-x)}) \int_u^\infty \frac{(u-y)^{j-1}}{(j-1)!} e^{-(\rho_{a+1}-\rho_i)(y-u)} dy$$
$$= \sum_{i=1}^{a}\sum_{j=1}^{n} \frac{E_{n,a+1,j}C_i}{(\rho_i-\rho_{a+1})^j} e^{\rho_i(u-x)} - \sum_{i=1}^{a}\sum_{j=1}^{n} \frac{E_{n,a+1,j}C_i}{(\rho_i-\rho_{a+1})^j} e^{\rho_i u - \rho_{a+1}x}. \tag{A.13}$$

Finally, combining (A.2) and (A.8)-(A.13) and collecting the coefficients of the terms $\frac{u^{j-1}}{(j-1)!}\frac{x^{k-1}}{(k-1)!}e^{\rho_i u - \rho_{a+1}x}$, $\frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_i(u-x)}$ and $\frac{(u-x)^{j-1}}{(j-1)!}e^{\rho_{a+1}(u-x)}$, one asserts that the representations (2.10) and (2.11) hold true for the case $n+1$ with the constants given by (2.14)-(2.18). This completes the proof.

B Appendix: Proof of Proposition 3

First, we derive a differential equation that is equivalent to (3.12). For $u \geq 0$ and $j = 1, 2, \ldots, m$, let $\phi(u; b, j)$ be the Gerber-Shiu function for a modified process that is identical to U^b except that the first capital injection time is Erlang(j, β) distributed (so that $\phi(u; b) = \phi(u; b, m)$). By treating an Erlang$(m, \beta)$ distribution as the sum of m i.i.d. exponential random variables each with mean $1/\beta$, analogous to (3.1) or (3.4) we obtain

$$\phi(u; b, j) = \beta \int_0^\infty r^{(\beta+\delta)}(u,x)\phi(x; b, j-1)dx + \zeta_1(u), \qquad j = 2, 3, \ldots, m,$$
$$\phi(u; b, 1) = \beta \int_0^\infty r^{(\beta+\delta)}(u,x)[\phi(b; b)\mathbf{1}_{(0\leq x \leq b)} + \phi(x; b)\mathbf{1}_{(x>b)}]dx + \zeta_1(u).$$

Using (2.7) and (3.7), the above two equations can be rewritten as

$$\phi(u; b, j) = \sum_{i=1}^{a+1} \beta C_i e^{\rho_i u} \int_0^\infty e^{-\rho_{a+1}x}\phi(x; b, j-1)dx - \sum_{i=1}^{a+1} \beta C_i \int_0^u e^{\rho_i(u-x)}\phi(x; b, j-1)dx$$
$$+ \sum_{i=1}^{a} F_{w,i}e^{\rho_i u}, \qquad j = 2, 3, \ldots, m,$$

$$\begin{aligned}\phi(u;b,1) =& \sum_{i=1}^{a+1}\beta C_i e^{\rho_i u}\int_0^\infty e^{-\rho_{a+1}x}[\phi(b;b)\mathbf{1}_{(0\le x\le b)}+\phi(x;b)\mathbf{1}_{(x>b)}]dx\\ &-\sum_{i=1}^{a+1}\beta C_i\int_0^u e^{\rho_i(u-x)}[\phi(b;b)\mathbf{1}_{(0\le x\le b)}+\phi(x;b)\mathbf{1}_{(x>b)}]dx+\sum_{i=1}^{a}F_{w,i}e^{\rho_i u}.\end{aligned}$$

Application of the operator $\prod_{k=1}^{a+1}(d/du-\rho_k)$ to them yields

$$\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)\right]\phi(u;b,j) = -\sum_{i=1}^{a+1}\beta C_i\left[\prod_{k=1,k\ne i}^{a+1}\left(\frac{d}{du}-\rho_k\right)\right]\phi(u;b,j-1), \qquad j=2,3,\ldots,m,$$
$$\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)\right]\phi(u;b,1) = -\sum_{i=1}^{a+1}\beta C_i\left[\prod_{k=1,k\ne i}^{a+1}\left(\frac{d}{du}-\rho_k\right)\right]\phi(u;b).$$

Hence, recursively these lead to the ordinary differential equation

$$\left[\prod_{k=1}^{a+1}\left(\frac{d}{du}-\rho_k\right)^m\right]\phi(u;b) = \left[-\sum_{i=1}^{a+1}\beta C_i\prod_{k=1,k\ne i}^{a+1}\left(\frac{d}{du}-\rho_k\right)\right]^m\phi(u;b), \tag{B.1}$$

which has characteristic equation (in ξ)

$$\prod_{k=1}^{a+1}(\xi-\rho_k)^m = \left[-\sum_{i=1}^{a+1}\beta C_i\prod_{k=1,k\ne i}^{a+1}(\xi-\rho_k)\right]^m. \tag{B.2}$$

Since the differential equations (3.12) and (B.1) must be equivalent for $u\ge b$, their characteristic equations (3.20) and (B.2) are also identical. Dividing both sides of (B.2) by $\prod_{k=1}^{a+1}(\xi-\rho_k)^m$ results in

$$\left(\sum_{i=1}^{a+1}\frac{\beta C_i}{\rho_i-\xi}\right)^m = 1.$$

Further application of (2.2) leads the above equation to (3.21). It follows from Albrecher et al. (2013, Appendix A) that (3.21) has exactly m roots with non-negative real parts, and the proof is completed.

C Appendix: Proof of Lemma 1

Without loss of generality, we assume $u\ge b$ in the entire proof (as we will let $u\to\infty$ at the end). Let $L_b=\inf\{Z_i : U^b_{Z_i-}<b\}=\inf\{Z_i : U_{Z_i}<b\}$ be the first time when U drops below b at some observation time point (which is also the first time when a positive capital is injected into the surplus process U^b if ruin has not yet occurred). Conditioning on the time L_b yields

$$\begin{aligned}V(u;b) &= \mathbb{E}\Big[e^{-\delta L_b}\chi(b-U_{L_b});\inf_{0\le t\le L_b}U_t\ge 0, L_b<\infty\Big|U_0=u\Big]+\mathbb{E}\Big[e^{-\delta L_b};\inf_{0\le t\le L_b}U_t\ge 0, L_b<\infty\Big|U_0=u\Big]V(b;b)\\ &\le \mathbb{E}\Big[e^{-\delta L_b};\inf_{0\le t\le L_b}U_t\ge 0, L_b<\infty\Big|U_0=u\Big](\chi_{\max}+V(b;b)),\end{aligned} \tag{C.1}$$

where $\chi_{\max}$ is the upper bound of $\chi(\cdot)$, i.e. $\chi(x) \leq \chi_{\max}$ for $x \in (0, b]$. Note that

$$\mathbb{E}\Big[e^{-\delta L_b}; \inf_{0\leq t\leq L_b} U_t \geq 0, L_b < \infty \big| U_0 = u\Big] \leq \mathbb{E}[e^{-\delta L_b}; L_b < \infty | U_0 = u] \leq \mathbb{E}[e^{-\delta\tau}\mathbf{1}_{(\tau<\infty)} | U_0 = u - b], \quad \text{(C.2)}$$

where the last inequality follows from the fact that L_b must be no less than the first time when the process U falls below b, which is in turn equivalent to the ruin time of U but with initial surplus $u - b$ thanks to the spatial homogeneity of U. Since the process U^b returns to level b after each capital injection, $V(b; b)$ appearing in (C.1) can be represented as

$$\begin{aligned} V(b;b) &= \sum_{k=1}^{\infty} \bigg(\mathbb{E}\Big[e^{-\delta L_b}; \inf_{0\leq t\leq L_b} U_t \geq 0, L_b < \infty \big| U_0 = b\Big] \bigg)^{k-1} \mathbb{E}\Big[e^{-\delta L_b}\chi(b - U_{L_b}); \inf_{0\leq t\leq L_b} U_t \geq 0, L_b < \infty \big| U_0 = b\Big] \\ &\leq \chi_{\max} \sum_{k=1}^{\infty} \bigg(\mathbb{E}\Big[e^{-\delta L_b}; \inf_{0\leq t\leq L_b} U_t \geq 0, L_b < \infty \big| U_0 = b\Big] \bigg)^{k}. \qquad \text{(C.3)} \end{aligned}$$

Under $\delta > 0$ or $c > \lambda\mathbb{E}X$, one must have $\mathbb{E}[e^{-\delta\tau}\mathbf{1}_{(\tau<\infty)} | U_0 = 0] < 1$. Combining this with (C.2) (at $u = b$) confirms that the summation in (C.3) is finite as

$$V(b;b) \leq \frac{\chi_{\max}\mathbb{E}[e^{-\delta L_b}; \inf_{0\leq t\leq L_b} U_t \geq 0, L_b < \infty | U_0 = b]}{1 - \mathbb{E}[e^{-\delta L_b}; \inf_{0\leq t\leq L_b} U_t \geq 0, L_b < \infty | U_0 = b]} < \infty,$$

i.e. $V(b; b)$ is bounded. Finally, incorporating (C.2) into (C.1) and using the boundedness of $V(b; b)$ as well as the fact that $\lim_{u\to\infty} \mathbb{E}[e^{-\delta\tau}\mathbf{1}_{(\tau<\infty)} | U_0 = u - b] = 0$ under $\delta > 0$ or $c > \lambda\mathbb{E}X$, the result (4.1) follows by taking the limit $u \to \infty$.