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A Class of Coupled Causal Differential Equations

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Abstract: In this thesis, we consider the existence of extreme solutions to a class of coupled causal differential equations. By utilizing two comparison theorems and a monotone iterative technique, we have obtained sufficient conditions under which the equations have extreme solutions. One practical example is presented.

Keywords: existence results; causal operators; causal differential equations; coupled lower and upper approximative solutions; monotone iterative technique

1. Introduction

We first study the following causal differential equations:

$$\begin{cases} x'(t) = Q_1(x, y)(t), & t \in I, \\ y'(t) = Q_2(y, x)(t), & t \in I, \\ g_1(x(0), x(Y), y(0), y(Y)) = 0, \\ g_2(y(0), y(Y), x(0), x(Y)) = 0, \end{cases} \quad (1)$$

where $I = [0, Y]$, $E = C(I^2, \mathbb{R})$, $Q_1, Q_2 \in C(E, E)$ are causal operators and $g_1, g_2 \in C(\mathbb{R}^4, \mathbb{R})$ and Q_1, Q_2 are bounded. The term causal operator first appeared in Volterra's work on the integral equations. It is a non-anticipative operator, and it was first applied by Volterra. The definition of the operator was given by Tonelli and Tychonoff. This theory can unify many types of equations. Peer works were reported by Drici et al. [1] and Geng [2]. Compared with traditional models, causal differential equations are more similar to real-world problems and more widely used. It has been tapped in depth by many authors in these years. Interested readers are referred to the monograph [3] and article [4] for more details.

At the same time, the research of a class of coupled differential equations is also crucial as such a system often occurs in real-life applications; it is a physical model of phenomena, in terms of the fluid dynamic traffic model, oscillations of earthquakes and economics and has attracted much attention; see, e.g., [5–7]. It should be noted that, in [8], Jankowski discussed coupled causal difference equations. The author developed the monotone iterative technique and gained the existence of extreme solutions to such equations. Affected by the above results, we study a class of coupled causal differential equations and obtain the extreme solutions.

The following is the frame of this paper. In Section 2, two comparison theorems are presented. In Section 3, we investigate the linear differential equations of (1). In Section 4, we establish the existence of the extremal solutions to (1). In the end, a computed example is given to verify the outcome.

2. Comparison Results

In this part, we set $E = C(I, \mathbb{R})$ and we define:

$$\|f - g\| := \max_{t \in I} |f(t) - g(t)|.$$

Let $\Omega := E \cap C^1(I, \mathbb{R})$. A pair (x, y) , $x, y \in \Omega$ is treated as a solution to Problem (1) if it contains (1).

Definition 1. Suppose that $Q \in C(E, E)$ is called a causal operator if for every pair of elements in E such that $x_i(s) = y_i(s)$, $i = 1, 2$ for $t_0 \leq s \leq t \leq Y$, then:

$$Q(x_1, x_2)(s) = Q(y_1, y_2)(s), \quad t_0 \leq s \leq t.$$

Lemma 1. Suppose that $z \in \Omega$ satisfies:

$$\begin{cases} z'(t) \leq -A(t)z(t) - (\mathcal{L}z)(t), & t \in I, \\ z(0) \leq \lambda z(Y), \end{cases} \quad (2)$$

where $A \in C(I, [0, +\infty))$ and $\mathcal{L} \in C(E, E)$ is a positive linear operator satisfying:

$$\int_0^Y [A(t) + \mathcal{L}(t)] dt \leq \epsilon, \quad 0 < \epsilon \leq 1. \quad (3)$$

then $z(t) \leq 0$, $t \in I$.

Proof. Let us say $z(t) \leq 0 \forall t \in I$ is not true; one has two situations:

Case 1. There exists $\bar{t} \in I$ that meets $z(\bar{t}) > 0$, $z(t) \geq 0$, $t \in I$.

Then, from (2), we have $z'(t) \leq 0$ for $t \in I$; hence, $z(t)$ is nonincreasing on I . If $\epsilon = 1$, $z(0) \leq z(Y)$ shows that $z(t) \equiv$ is constant, so $z'(t) \equiv 0$. As $z(\bar{t}) > 0$, one has $0 \equiv z'(\bar{t}) \leq -A(\bar{t})z(\bar{t}) - (\mathcal{L}z)(\bar{t}) < 0$, which is a conflict. If $0 < \epsilon < 1$, then $z(Y) \leq z(0) \leq \epsilon z(Y)$, so it is also a conflict.

Case 2. There are t_m and $t_n \in I$ satisfying $z(t_m) < 0$ and $z(t_n) > 0$.

Put $\inf_{t \in I} z(t) = -r$, $r > 0$. With generality, we may assume that $z(t_m) = -r$.

Subcase (a). If $t_m < t_n$, from (2), we gain:

$$\begin{aligned} z(t_n) - z(t_m) &= \int_{t_m}^{t_n} z'(t) dt \leq - \int_{t_m}^{t_n} [A(t)z(t) + (\mathcal{L}z)(t)] dt \\ &\leq r \left\{ \int_0^Y [A(t) + \mathcal{L}(t)] dt \right\} \end{aligned}$$

hence:

$$0 < z(t_n) \leq -r + r \int_0^Y [A(t) + \mathcal{L}(t)] dt = r \left[\int_0^Y [A(t) + \mathcal{L}(t)] dt - 1 \right],$$

which yields:

$$\int_0^Y [A(t) + \mathcal{L}(t)] dt > 1,$$

which contradicts (3).

Subcase (b). If $t_m > t_n$, from (2), we get:

$$z(Y) = z(t_m) + \int_{t_m}^Y z'(t) dt \leq z(t_m) + r \int_{t_m}^Y [A(t) + \mathcal{L}(t)] dt$$

and:

$$z(t_n) = z(0) + \int_0^{t_n} z'(t) dt \leq z(0) + r \int_0^{t_n} [A(t) + \mathcal{L}(t)] dt.$$

Using the fact $z(0) \leq \epsilon z(Y), 0 < \epsilon \leq 1$, we obtain:

$$\begin{aligned} 0 < z(t_n) &\leq \epsilon \left(z(t_m) + r \int_{t_m}^Y [A(t) + \mathcal{L}(t)] dt \right) + r \int_0^{t_n} [A(t) + \mathcal{L}(t)] dt \\ &\leq -r\epsilon + r \left\{ \int_0^Y [A(t) + \mathcal{L}(t)] dt \right\} \\ &= r \left\{ \int_0^Y [A(t) + \mathcal{L}(t)] dt - \epsilon \right\}, \end{aligned}$$

which is a conflict with a given condition. Therefore, $z(t) \leq 0, t \in J$. The proving process is done. \square

Lemma 2. Let $j, k \in \Omega$ be such that:

$$\begin{cases} j'(t) \leq -A_1(t)j(t) + A_2(t)k(t) - (\mathcal{L}_1j)(t) + (\mathcal{L}_2k)(t), & t \in I, \\ k'(t) \leq -A_1(t)k(t) + A_2(t)j(t) - (\mathcal{L}_1k)(t) + (\mathcal{L}_2j)(t), & t \in I, \\ j(0) \leq \epsilon_1j(Y) + \epsilon_2k(Y), \\ k(0) \leq \epsilon_1k(Y) + \epsilon_2j(Y), \end{cases} \tag{4}$$

where $0 < \epsilon_1 + \epsilon_2 \leq 1, 0 < \epsilon_1 - \epsilon_2 \leq 1, A_1, A_2 \in C(I, [0, +\infty)), A_1(t) \geq A_2(t), t \in I, \mathcal{L}_1 - \mathcal{L}_2 \in C(E, E)$ and $\mathcal{L}_1 + \mathcal{L}_2 \in C(E, E)$ are positive linear operators. Additionally,

$$\begin{aligned} \max \left\{ \frac{\int_0^Y [A_1(t) + A_2(t) + \mathcal{L}_1(t) + \mathcal{L}_2(t)] dt}{\epsilon_1 - \epsilon_2}, \frac{\int_0^Y [A_1(t) - A_2(t) + \mathcal{L}_1(t) - \mathcal{L}_2(t)] dt}{\epsilon_1 + \epsilon_2} \right\} \\ \leq 1. \end{aligned} \tag{5}$$

Then, $j(t) \leq 0$ and $k(t) \leq 0, t \in I$.

Proof. Let $c(t) = j(t) + k(t), t \in I$. Then:

$$\begin{aligned} c'(t) = j'(t) + k'(t) &\leq -A_1(t)c(t) + A_2(t)c(t) - [(\mathcal{L}_1 - \mathcal{L}_2)c](t) \\ &\leq -[A_1(t) - A_2(t)]c(t) - [(\mathcal{L}_1 - \mathcal{L}_2)c](t), \quad t \in I \end{aligned}$$

and:

$$c(0) = j(0) + k(0) \leq \epsilon_1c(Y) + \epsilon_2c(Y) = (\epsilon_1 + \epsilon_2)c(Y).$$

From Lemma (1), $c(t) \leq 0$. Hence, $j(t) \leq -k(t)$ and $k(t) \leq -j(t)$, then we have:

$$\begin{cases} j'(t) \leq -[A_1(t) + A_2(t)]j(t) - [(\mathcal{L}_1 + \mathcal{L}_2)j](t), & t \in I, \\ j(0) \leq (\epsilon_1 - \epsilon_2)j(Y) \end{cases}$$

and:

$$\begin{cases} k'(t) \leq -[A_1(t) + A_2(t)]k(t) - [(\mathcal{L}_1 + \mathcal{L}_2)k](t), & t \in I, \\ k(0) \leq (\epsilon_1 - \epsilon_2)k(Y). \end{cases}$$

From Lemma (1), one obtains $j(t) \leq 0, k(t) \leq 0, t \in I$. This is the end. \square

3. Existence Results

Let us consider the linear problem:

$$\begin{cases} x'(t) = -A(t)x(t) - (\mathcal{L}x)(t) + \sigma(t), & t \in I, \\ N_1x(0) - N_2x(Y) = B, \end{cases} \tag{6}$$

where N_1, N_2 are constants, $\sigma \in E, A \in C(I, [0, +\infty)), B \in \mathbb{R}$.

Lemma 3. $x \in \Omega$ is one solution of (6) when $x \in E$ is one solution of the equation:

$$x(t) = \frac{B e^{-\int_0^t A(r) dr}}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} + \int_0^Y D(t,s) [\sigma(s) - (\mathcal{L}x)(s)] ds, \tag{7}$$

where $B = -g(\eta(0), \eta(T)) + N_1\eta(0) - N_2\eta(Y)$, $\eta \in \Omega$, $A \in C(I, [0, +\infty))$, N_1, N_2 are constants with $N_1 \neq N_2 e^{-\int_0^Y A(r) dr}$, and:

$$D(t,s) := \begin{cases} \frac{N_2}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} e^{\int_t^0 A(r) dr} e^{\int_Y^s A(r) dr} + e^{\int_t^s A(r) dr}, & 0 \leq s < t \leq Y, \\ \frac{N_2}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} e^{\int_t^0 A(r) dr} e^{\int_Y^s A(r) dr}, & 0 \leq t \leq s \leq Y. \end{cases}$$

Proof. Suppose that $x \in \Omega$ is one solution of Problem (6), one achieves:

$$x(t) = x(0) e^{-\int_0^t A(r) dr} + \int_0^t e^{\int_t^s A(r) dr} [\sigma(s) - (\mathcal{L}x)(s)] ds.$$

Let $t = Y$; we get:

$$x(Y) = x(0) e^{-\int_0^Y A(r) dr} + \int_0^Y e^{\int_Y^s A(r) dr} [\sigma(s) - (\mathcal{L}x)(s)] ds.$$

Since $N_1 x(0) - N_2 x(T) = B$, we obtain:

$$x(0) = \frac{B}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} + \frac{N_2}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} \int_0^Y e^{\int_Y^s A(r) dr} [\sigma(s) - (\mathcal{L}x)(s)] ds.$$

Then:

$$x(t) = \frac{B e^{-\int_0^t A(r) dr}}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} + \int_0^Y D(t,s) [\sigma(s) - (\mathcal{L}x)(s)] ds,$$

where:

$$D(t,s) = \begin{cases} \frac{N_2}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} e^{\int_t^0 A(r) dr} e^{\int_Y^s A(r) dr} + e^{\int_t^s A(r) dr}, & 0 \leq s < t \leq Y, \\ \frac{N_2}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} e^{\int_t^0 A(r) dr} e^{\int_Y^s A(r) dr}, & 0 \leq t \leq s \leq Y. \end{cases}$$

Apparently, if $x(t)$ is a solution of (6), so $x(t)$ will also be a solution of (7). The proof is complete. \square

Clearly, $\|D(t,s)\| = \max \left\{ \left| \frac{N_1}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} \right|, \left| \frac{N_2}{N_1 - N_2 e^{-\int_0^Y A(r) dr}} \right| \right\}$. In the next part, we define $\|D(t,s)\| = d$.

Lemma 4. $A \in C(I, [0, +\infty))$, $N_1 \neq N_2 e^{-\int_0^Y A(r) dr}$, and:

$$d \|\mathcal{L}\| Y < 1.$$

Then, (6) has only one solution.

Proof. Give a mapping $F : E \rightarrow E$ by:

$$(Fx)(t) := \frac{Be^{-\int_0^t A(r)dr}}{N_1 - N_2e^{-\int_0^Y A(r)dr}} + \int_0^Y D(t,s)[\sigma(s) - (\mathcal{L}x)(s)]ds.$$

For any $x_1, x_2 \in E$, we have:

$$\|Fx_1 - Fx_2\| \leq d \|\mathcal{L}\| Y \|x_2 - x_1\|.$$

From the Banach contraction principle, (6) has only one solution. The proof is then finished. \square

Remark 1. If $A \in C(I, [0, +\infty))$, $N_1 \geq N_2 > 0$, and:

$$\frac{N_1}{N_1 - N_2e^{-\int_0^Y A(r)dr}} \|\mathcal{L}\| Y < 1.$$

Then, (6) has only one solution.

$$\text{Let } B(k, h)(t) := A_1(t)k(t) + A_2(t)h(t) + (\mathcal{L}_1k)(t) + (\mathcal{L}_2h)(t).$$

Theorem 1. Suppose that linear operators $\mathcal{L}_1 - \mathcal{L}_2 \in C(E, E)$ and $\mathcal{L}_1 + \mathcal{L}_2 \in C(E, E)$ are positive. Let $\sigma_1, \sigma_2 \in E$, $C_1, C_2 \in \mathbb{R}$, $A_1, A_2 \in C(I, [0, +\infty))$ and $A_1(t) \geq A_2(t)$ for $t \in I$. Moreover, we assume that (5) holds, N_1, N_2, N_3 are constants with $N_1 \neq (N_2 - N_3)e^{-\int_0^Y [A_1(r)+A_2(r)]dr}$, $N_1 \neq (N_2 + N_3)e^{-\int_0^Y [A_1(r)-A_2(r)]dr}$, and:

$$\max \left\{ \frac{N_1 \|\mathcal{L}_1 + \mathcal{L}_2\| Y}{N_1 - (N_2 - N_3)e^{-\int_0^Y [A_1(r)+A_2(r)]dr}}, \frac{N_1 \|\mathcal{L}_1 - \mathcal{L}_2\| Y}{N_1 - (N_2 + N_3)e^{-\int_0^Y [A_1(r)-A_2(r)]dr}} \right\} < 1. \quad (8)$$

Then, the system:

$$\begin{cases} k'(t) = -B(k, h)(t) + \sigma_1(t), & t \in I, \\ h'(t) = -B(h, k)(t) + \sigma_2(t), & t \in I, \\ N_1 k(0) = N_2 k(Y) - N_3 h(Y) + A_1, \\ N_1 h(0) = N_2 h(Y) - N_3 k(Y) + A_2, \end{cases} \quad (9)$$

has only one solution.

Proof. Set $e = k + h, l = k - h$. Then, we have:

$$\begin{cases} e'(t) = -[A_1(t) + A_2(t)]e(t) - [(\mathcal{L}_1 + \mathcal{L}_2)e](t) + \sigma_1(t) + \sigma_2(t), \\ N_1 e(0) = (N_2 - N_3)e(Y) + C_1 + C_2, \end{cases} \quad (10)$$

and:

$$\begin{cases} l'(t) = -[A_1(t) - A_2(t)]l(t) - [(\mathcal{L}_1 - \mathcal{L}_2)l](t) + \sigma_1(t) - \sigma_2(t), \\ N_1 l(0) = (N_2 + N_3)l(Y) + C_1 - C_2. \end{cases} \quad (11)$$

By Lemma 4, both (10) and (11) have a unique solution.

Note that $k = \frac{e+l}{2}, h = \frac{e-l}{2}$, and the pair (k, h) is the only solution of Problem (9). This ends the proof. \square

4. Main Results

Definition 2. A pair (S, W) is seen as a coupled lower and upper approximative solution to (1) if:

$$\begin{cases} S'(t) \leq Q_1(S, W)(t), & t \in I, \\ W'(t) \geq Q_2(W, S)(t), & t \in I, \\ g_1(S(0), S(Y), W(0), W(Y)) \leq 0, \\ g_2(W(0), W(Y), S(0), S(Y)) \geq 0. \end{cases}$$

Theorem 2. Suppose that (5), (8) hold, $Q_1, Q_2 \in C(E, E)$ and (H_1) the function $(S, W) \in \Omega$ is the coupled lower and upper approximative solution to Problem (1) satisfying $S(t) \leq W(t), t \in I$; (H_2) there exist $A_1, A_2 \in C(I, [0, +\infty))$ with $A_1(t) \geq A_2(t) \geq 0$ for $t \in I$, and the positive linear operators $\mathcal{L}_1 - \mathcal{L}_2, \mathcal{L}_1 + \mathcal{L}_2 \in C(E, E)$ satisfy:

$$\begin{aligned} & Q_i(x, y)(t) - Q_j(v, w)(t) \\ & \leq A_1(t)[v(t) - x(t)] + A_2(t)[w(t) - y(t)] + [\mathcal{L}_1(v - x)](t) + [\mathcal{L}_2(w - y)](t) \end{aligned}$$

for $i \geq j, i, j = 1, 2$ for $t \in I, S \leq x \leq v \leq W, S \leq w \leq y \leq W$; (H_3) the functions $g_1, g_2 \in C(\mathbb{R}^4, \mathbb{R})$ are non-decreasing of the third variable and satisfy:

$$g_i(\bar{x}_1, \bar{x}_2, y, y_1) - g_j(x_1, x_2, y, \bar{y}_1) \leq N_1(\bar{x}_1 - x_1) - N_2(\bar{x}_2 - x_2) + N_3(y_1 - \bar{y}_1)$$

for $i \geq j, i, j = 1, 2, S(0) \leq x_1 \leq \bar{x}_1 \leq W(0), S(0) \leq x_2 \leq \bar{x}_2 \leq W(0), S(Y) \leq y \leq \bar{y} \leq W(Y), S(Y) \leq y_1 \leq \bar{y}_1 \leq W(Y)$, where $N_1 \geq N_2 + N_3, N_2 > 0, N_3 > 0$ and $\epsilon_1 = \frac{N_2}{N_1}, \epsilon_2 = \frac{N_3}{N_1}$.

Then, Problem (1) has solution $(x, y) \in [S, W] \times [S, W]$, where $[S, W] = \{\xi \in \Omega : S(t) \leq \xi(t) \leq W(t), t \in I\}$.

Proof. First, using the following equations, one constructs two sequences $\{S_n(t)\}$ and $\{W_n(t)\}$:

$$\begin{cases} S'_n(t) = -B(S_n, W_n)(t) + F_1(S_{n-1}, W_{n-1})(t), & t \in I, \\ W'_n(t) = -B(W_n, S_n)(t) + F_2(W_{n-1}, S_{n-1})(t), & t \in I, \\ N_1 S_n(0) = N_2 S_n(Y) - N_3 W_n(Y) + \varphi_1(S_{n-1}, W_{n-1}), \\ N_1 W_n(0) = N_2 W_n(Y) - N_3 S_n(Y) + \varphi_2(W_{n-1}, S_{n-1}), \end{cases} \quad (12)$$

with $n = 1, 2, \dots$, where $S_0 = S, W_0 = W, F_i(x, y)(t) = Q_i(x, y)(t) + A_1(t)x(t) + A_2(t)y(t) + (\mathcal{L}_1 x)(t) + (\mathcal{L}_1 y)(t)$ and $\varphi_i(u, v) = N_1 u(0) - N_2 u(Y) + N_3 v(Y) - g_i(u(0), u(Y), v(0), v(Y)), i = 1, 2$.

Apparently, from Theorem 1, Problem (12) has only one solution. Now, we are going to divide this proof into five parts.

Step 1. One shows that $S \leq S_1$ and $W_1 \leq W$.

Set $r(t) = S(t) - S_1(t)$, $s(t) = W_1(t) - W(t)$. From (H_1) , we get:

$$\begin{aligned} r'(t) &= S'(t) - S_1'(t) \\ &\leq Q_1(S, W)(t) + B(S_1, W_1)(t) - F_1(S, W)(t) \\ &\leq -A_1(t)r(t) + A_2(t)s(t) - (\mathcal{L}_1 r)(t) + (\mathcal{L}_2 s)(t), \quad t \in I, \\ s'(t) &= W_1'(t) - W'(t) \\ &\leq B(W_1, S_1)(t) + F_2(W, S)(t) - Q_2(W, S)(t) \\ &\leq -A_1(t)s(t) + A_2(t)r(t) - (\mathcal{L}_1 s)(t) + (\mathcal{L}_2 r)(t), \quad t \in I, \\ r(0) &= S(0) - S_1(0) \\ &\leq \frac{1}{N_1}g_1(S(0), S(Y), W(0), W(Y)) + \frac{N_2}{N_1}r(Y) + \frac{N_3}{N_1}s(Y) \\ &\leq \frac{N_2}{N_1}r(Y) + \frac{N_3}{N_1}s(Y), \end{aligned}$$

and:

$$\begin{aligned} s(0) &= W_1(0) - W(0) \\ &\leq -\frac{1}{N_1}g_2(W(0), W(Y), S(0), W(Y)) + \frac{N_2}{N_1}s(Y) + \frac{N_3}{N_1}r(Y) \\ &\leq \frac{N_2}{N_1}s(Y) + \frac{N_3}{N_1}r(Y). \end{aligned}$$

From Lemma 2 and $N_1 \geq N_2 > 0$, $N_1 \geq N_3 > 0$, we have $r(t) \leq 0$ and $s(t) \leq 0$, so $S \leq S_1$ and $W_1 \leq W$.

Step 2. We show that $W_n \leq W_{n-1}$ and $S_{n-1} \leq S_n$ if $W_{n-1} \leq W_{n-2}$, $S_{n-2} \leq S_{n-1}$, $n = 2, 3, \dots$

Set $e(t) = S_{n-1}(t) - S_n(t)$, $l(t) = W_n(t) - W_{n-1}(t)$. From (H_1) , (H_2) , we get:

$$\begin{aligned} e'(t) &= S'_{n-1}(t) - S'_n(t) \\ &= -B(S_{n-1}, W_{n-1})(t) + F_1(S_{n-2}, W_{n-2})(t) + B(S_n, W_n)(t) - F_1(S_{n-1}, W_{n-1})(t) \\ &\leq -A_1(t)e(t) + A_2(t)l(t) - (\mathcal{L}_1 e)(t) + (\mathcal{L}_2 l)(t) + Q_1(S_{n-2}, W_{n-2}) \\ &\quad - Q_1(S_{n-1}, W_{n-1}) + A_1(t)[S_{n-2}(t) - S_{n-1}(t)] + A_2(t)[W_{n-2}(t) - W_{n-1}(t)] \\ &\quad + [\mathcal{L}_1(S_{n-2} - S_{n-1})](t) + [\mathcal{L}_2(W_{n-2} - W_{n-1})](t) \\ &\leq -A_1(t)e(t) + A_2(t)l(t) - (\mathcal{L}_1 e)(t) + (\mathcal{L}_2 l)(t), \quad t \in I, \end{aligned}$$

and:

$$\begin{aligned} l'(t) &= W'_n(t) - W'_{n-1}(t) \\ &= -B(W_n, S_n)(t) + F_2(W_{n-1}, S_{n-1})(t) + B(W_{n-1}, S_{n-1})(t) - F_2(W_{n-2}, S_{n-2})(t) \\ &\leq -A_1(t)l(t) + A_2(t)e(t) - (\mathcal{L}_1 l)(t) + (\mathcal{L}_2 e)(t) + Q_2(W_{n-2}, S_{n-2}) \\ &\quad - Q_2(W_{n-1}, S_{n-1}) + A_1(t)[W_{n-1}(t) - W_{n-2}(t)] + A_2(t)[S_{n-1}(t) - S_{n-2}(t)] \\ &\quad + [\mathcal{L}_1(W_{n-1} - W_{n-2})](t) + [\mathcal{L}_2(S_{n-1} - S_{n-2})](t) \\ &\leq -A_1(t)l(t) + A_2(t)e(t) - (\mathcal{L}_1 l)(t) + (\mathcal{L}_2 e)(t), \quad t \in I. \end{aligned}$$

By (H_1) and (H_3) , we have:

$$\begin{aligned} N_1 e(0) &= N_1[S_{n-1}(0) - S_n(0)] \\ &= N_1 S_{n-1}(0) - N_2 S_n(Y) + N_3 W_n(Y) - \varphi_1(S_{n-1}, W_{n-1}) \\ &= N_2 e(Y) + N_3 l(Y) + g_1(S_{n-1}(0), S_{n-1}(Y), W_{n-1}(0), W_{n-1}(Y)) \\ &\quad - g_1(S_{n-2}(0), S_{n-2}(Y), W_{n-2}(0), W_{n-2}(Y)) + N_1[S_{n-2}(0) - S_{n-1}(0)] \\ &\quad + N_2[S_{n-1}(Y) - S_{n-2}(Y)] + N_3[W_{n-2}(Y) - W_{n-1}(Y)] \\ &\leq N_2 e(Y) + N_3 l(Y), \end{aligned}$$

and:

$$\begin{aligned} N_1 l(0) &= N_1[W_n(0) - W_{n-1}(0)] \\ &= N_2 W_n(Y) - N_3 S_n(Y) + \varphi_2(W_{n-1}, S_{n-1}) - N_1 W_{n-1}(0) \\ &= N_2 l(Y) + N_3 e(Y) - g_2(W_{n-1}(0), W_{n-1}(Y), S_{n-1}(0), S_{n-1}(Y)) \\ &\quad + g_2(W_{n-2}(0), W_{n-2}(Y), S_{n-2}(0), S_{n-2}(Y)) + N_1[W_{n-1}(0) - W_{n-2}(0)] \\ &\quad - N_2[W_{n-1}(Y) - W_{n-2}(Y)] + N_3[S_{n-1}(Y) - S_{n-2}(Y)] \\ &\leq N_2 l(Y) + N_3 e(Y). \end{aligned}$$

Using Lemma 2 and $N_1 \geq N_2 > 0, N_1 \geq N_3 > 0$, we have $e \leq 0, l \leq 0$, so $W_n \leq W_{n-1}, S_{n-1} \leq S_n$. From mathematical induction, one gets that $\{S_n\}$ is a nondecreasing sequence, and $\{W_n\}$ is a nonincreasing sequence.

Step 3. We claim that $S_1 \leq W_1$ if $S \leq W$.

Let $u(t) = S_1(t) - W_1(t)$. Using (H_2) and (H_3) , one receives:

$$\begin{aligned} u'(t) &= S_1'(t) - W_1'(t) \\ &= -B(S_1, W_1)(t) + F_1(S, W)(t) + B(W_1, S_1)(t) - F_2(W, S)(t) \\ &= -[A_1(t) - A_2(t)]u(t) - [(\mathcal{L}_1 - \mathcal{L}_2)u](t) + Q_1(S, W)(t) + A_1(t)S(t) \\ &\quad + A_2(t)W(t) + (\mathcal{L}_1 S)(t) + (\mathcal{L}_2 W)(t) - Q_2(W, S)(t) \\ &\quad - A_1(t)W(t) - A_2(t)S(t) - (\mathcal{L}_1 W)(t) - (\mathcal{L}_2 S)(t) \\ &\leq -[A_1(t) - A_2(t)]u(t) - [(\mathcal{L}_1 - \mathcal{L}_2)u](t), \quad t \in I, \end{aligned}$$

and:

$$\begin{aligned} N_1 u(0) &= N_1[S_1(0) - W_1(0)] \\ &= (N_2 + N_3)u(Y) + \varphi_1(S, W) - \varphi_2(W, S) \\ &= (N_2 + N_3)u(Y) + N_1[S(0) - W(0)] + (N_2 + N_3)[W(Y) - S(Y)] \\ &\quad - g_1(S(0), S(Y), W(0), W(Y)) + g_2(W(0), W(Y), S(0), S(Y)) \\ &\leq (N_2 + N_3)u(Y). \end{aligned}$$

From Lemma 1, one arrives at $u \leq 0$, and $S_1 \leq W_1$. Utilizing the mathematical induction, we gain $W_n \leq S_n, n = 1, 2, \dots$

Step 4. By the above three steps, we obtain:

$$S_0 \leq S_1 \leq \dots \leq S_n \leq \dots \leq W_n \leq \dots \leq W_1 \leq W_0,$$

and every S_n, W_n satisfies (12) for all n . We can get that the sequences $\{S_n(t)\}, \{W_n(t)\}$ are monotonically and bounded. Then, there are ρ, r satisfying that $\lim_{n \rightarrow \infty} S_n(t) = \rho(t), \lim_{n \rightarrow \infty} W_n(t) = r(t)$ and satisfying the equations:

$$\begin{cases} \rho'(t) = Q_1(\rho, r)(t), & t \in I, \\ r'(t) = Q_2(r, \rho)(t), & t \in I, \\ g(\rho(0), \rho(Y), r(0), r(Y)) = 0, \\ g(r(0), r(Y), \rho(0), \rho(Y)) = 0. \end{cases}$$

It proves that a pair (ρ, r) is the solution of Problem (1).

Step 5. One verifies that (ρ, r) is an extremal solution of (1) in $[S, W]$.

Let (x, y) be any pair of solutions for (1) satisfying $S(t) \leq x(t), y(t) \leq W(t)$. We assume that there is a positive integer n satisfying $S_{n-1}(t) \leq x(t), y(t) \leq W_{n-1}(t)$. Set $u(t) = S_n(t) - x(t)$ and $v(t) = y(t) - W_n(t)$. By using (H_2) and (H_3) , one gets:

$$\begin{aligned} u'(t) &= S'_n(t) - x'(t) \\ &= -B(S_n, W_n)(t) + F_1(S_{n-1}, W_{n-1}) - Q_1(x, y)(t) \\ &= -A_1(t)S_n(t) - A_2(t)W_n(t) - (\mathcal{L}_1 S_n)(t) - (\mathcal{L}_2 W_n)(t) + Q_1(S_{n-1}, W_{n-1})(t) \\ &\quad + A_1(t)S_{n-1}(t) + A_2(t)W_{n-1}(t) + (\mathcal{L}_1 S_{n-1})(t) + (\mathcal{L}_2 W_{n-1})(t) - Q_1(x, y)(t) \\ &\leq -A_1(t)u(t) + A_2(t)v(t) - (\mathcal{L}_1 u)(t) + (\mathcal{L}_2 v)(t), \quad t \in I, \\ v'(t) &= y'(t) - W'_n(t) \\ &= Q_2(x, y)(t) + B(W_n, S_n)(t) - F_2(W_{n-1}, S_{n-1}) \\ &= Q_2(y, x)(t) + A_1(t)W_n(t) + A_2(t)S_n(t) + (\mathcal{L}_1 W_n)(t) + (\mathcal{L}_2 S_n)(t) \\ &\quad - Q_2(W_{n-1}, S_{n-1})(t) - A_1(t)W_{n-1}(t) - A_2(t)S_{n-1}(t) \\ &\quad - (\mathcal{L}_1 W_{n-1})(t) - (\mathcal{L}_2 S_{n-1})(t) \\ &\leq -A_1(t)v(t) + A_2(t)u(t) - (\mathcal{L}_1 v)(t) + (\mathcal{L}_2 u)(t), \quad t \in I, \\ N_1 u(0) &= N_1[S_n(0) - x(0)] \\ &= N_2 S_n(Y) - N_3 W_n(Y) + \varphi_1(S_{n-1}, W_{n-1}) - N_1 x(0) \\ &= N_2 S_n(Y) - N_3 W_n(Y) + N_1 S_{n-1}(0) - N_2 S_{n-1}(Y) + N_3 W_{n-1}(Y) - N_1 x(0) \\ &\quad - g_1(S_{n-1}(0), S_{n-1}(Y), W_{n-1}(0), W_{n-1}(Y)) + g_1(x(0), x(Y), y(0), y(Y)) \\ &\leq N_2 S_n(Y) - N_2 x(Y) - N_3 W_n(Y) + N_3 y(Y) \\ &\leq N_2 u(Y) + N_3 v(Y), \end{aligned}$$

and:

$$\begin{aligned} N_1 v(0) &= N_1[y(0) - W_n(0)] \\ &= N_1 y(0) - N_2 W_n(Y) + N_3 S_n(Y) + \varphi_1(W_{n-1}, S_{n-1}) \\ &= N_1 y(0) - N_2 W_n(Y) + N_3 S_n(Y) - N_1 W_{n-1}(0) + N_2 W_{n-1}(Y) - N_3 S_{n-1}(Y) \\ &\quad + g_1(W_{n-1}(0), W_{n-1}(Y), S_{n-1}(0), S_{n-1}(Y)) - g_1(y(0), y(Y), x(0), x(Y)) \\ &\leq -N_2 W_n(Y) + N_2 y(Y) + N_3 S_n(Y) - N_3 x(Y) \\ &\leq N_2 v(Y) + N_3 u(Y). \end{aligned}$$

From Lemma 2, one see that $S_n(t) \leq x(t), y(t) \leq W_n(t)$ on I . Since $S(t) \leq x(t), y(t) \leq W(t)$, by induction, one gets $S_n(t) \leq x(t), y(t) \leq W_n(t)$ for every $n \in \mathbb{N}$. Now, take the limit as $n \rightarrow \infty$, then $\rho(t) \leq x(t), y(t) \leq r(t)$. The proof is over. \square

5. Example

Example 1. Assume that:

$$\begin{cases} z'(t) = Q_1(z, w)(t), & t \in [0, Y], \\ w'(t) = Q_2(w, z)(t), & t \in [0, Y], \\ g_1(z(0), z(Y), w(0), w(Y)) = 0, \\ g_2(w(0), w(Y), z(0), z(Y)) = 0, \end{cases} \quad (13)$$

with:

$$\begin{aligned} Q_1(z, w)(t) &= -tz(t) - \frac{Y}{5t} \int_0^t z(s)ds + \frac{3Y}{10t^3} \int_0^t s^2 w(s)ds, \\ Q_2(w, z)(t) &= -tw(t) - \frac{Y}{5t} \int_0^t w(s)ds + \frac{3Y}{10t^3} \int_0^t s^2 z(s)ds, \\ g(z(0), z(Y), w(0), w(Y)) &= 4z(0) - 2z(Y) + w(Y) - 2, \\ g(w(0), w(Y), z(0), z(Y)) &= 4w(0) - 2w(Y) + z(Y) - 2. \end{aligned}$$

Set $S = 0$, $W = 2$. It is easily verified that W is an upper solution, S is a lower solution and $S \leq W$. Take $(\mathcal{L}_1 z)(t) = \frac{Y}{5t} \int_0^t z(s)ds$, $(\mathcal{L}_2 w)(t) = \frac{3Y}{10t^3} \int_0^t s^2 w(s)ds$, $A_1(t) = t$, $A_2(t) = 0$, $N_1 = 4$, $N_2 = 2$, $N_3 = 1$ and $Y = \frac{1}{2}$. It is easy to see that (H_1) , (H_2) and (H_3) are satisfied. By simple computation, we obtain $\mathcal{L}_1(t) = 0.2Y$, $\mathcal{L}_2(t) = 0.1Y$, and:

$$\begin{aligned} \int_0^Y [A_1(t) + A_2(t) + \mathcal{L}_1(t) + \mathcal{L}_2(t)] dt &= 0.2 < \frac{N_2 - N_3}{N_1} = 0.25, \\ \int_0^Y [A_1(t) - A_2(t) + \mathcal{L}_1(t) - \mathcal{L}_2(t)] dt &= 0.15 < \frac{N_2 + N_3}{N_1} = 0.75, \\ \frac{N_1}{N_1 - (N_2 - N_3)e^{-\int_0^Y [A_1(r) + A_2(r)]dr}} \|\mathcal{L}_1 + \mathcal{L}_2\| Y &< 1, \\ \frac{N_1}{N_1 - (N_2 + N_3)e^{-\int_0^Y [A_1(r) + A_2(r)]dr}} \|\mathcal{L}_1 - \mathcal{L}_2\| Y &< 1. \end{aligned}$$

Hence, utilizing Theorem 2, (13) has an extreme solution in $[S, W] \times [S, W]$.

This equation can be used as a simple model of the physical field, and this example shows that the conclusion in this paper applies to Problem (13).

6. Conclusions

In this paper, we obtain the existence results of coupled causal differential equations by means of the monotone iterative technique. In practice, after modeling in our field, if such coupled differential equations are obtained, we do not have to calculate the solution; we only need to use the sufficient conditions in this paper to judge the existence of the solution.

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