Greater Arrow-Pratt (Absolute) Risk Aversion of Higher Orders *

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Higher-order risk attitudes are related to higher-order moments of risk, and are unequivocally characterized by the signs and levels of higher-order derivatives of utility functions. In contrast to the direction of higher-degree risk aversion, the intensity of higher-degree risk aversion beyond the Arrow-Pratt measure of absolute risk aversion is far from conclusive. The purpose of this paper is to develop a unified framework of greater (m, n)th-degree mixed risk aversion in the Arrow-Pratt tradition, which includes many competing notions of greater higher-degree (absolute) risk aversion proposed in the extant literature as special cases. Properties of greater (m, n)th-degree mixed risk aversion are studied, a choice-based characterization is established, and several applications are presented.

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1. Introduction

While economists have long recognized that risk aversion plays a pivotal role in decision making under uncertainty (Bernoulli 1954), measuring the intensity of risk aversion and investigating its determinants are relatively new scientific endeavors.[1] Consider two individuals, u and v, with increasing utility functions, $u(x)$ and $v(x)$, defined over their wealth, $x \in [a, b]$, respectively. The seminal work of Arrow (1971) and Pratt (1964) in the expected utility framework provides the following three equivalent statements of $v(x)$ being more (second-degree) risk averse than $u(x)$:

(i) Individual v is always willing to pay a (weakly) larger risk premium than individual u to avoid an introduction of risk.

(ii) There exists a transformation function, $\phi(y)$, such that $v(x) = \phi\Big(u(x)\Big)$ and $\phi''(y) \leq 0$ for all $y \in [u(a), u(b)]$.

(iii) The Arrow-Pratt measure of absolute risk aversion for $v(x)$ is uniformly larger than that for $u(x)$, i.e., $-v''(x)/v'(x) \geq -u''(x)/u'(x)$ for all $x \in [a, b]$.

The theoretical foundation of comparative second-degree risk aversion laid by Arrow (1971) and Pratt (1964) is known to be necessary and sufficient for a variety of interesting comparative statics results. For example, Arrow (1971) shows that, in a two-asset portfolio choice problem with one safe and one risky assets, individual v invests less in the risky asset and more in the safe asset than individual u.

Recent experimental studies have demonstrated a salient aversion to risk increases of third and even higher degrees.[2] Accompanying the discovery of higher-degree risk aversion, a natural question arises as to how to measure and compare the intensity of risk aversion

[1] See Callen et al. (2014), Ebert and Wiesen (2014), Eckel and Grossman (2002), Grossman and Eckel (2015), and Holt and Laury (2002), among many others.

[2] See Deck and Schlesinger (2010, 2014), Ebert and Wiesen (2011), Maier and Ruger (2011), and Noussair et al. (2014), to name just a few.

beyond the second degree. In this paper, we focus on how to measure and compare the intensity of higher-degree absolute risk aversion, and use “risk aversion” as a synonym for “absolute risk aversion.”

Moving from the second to the third degree, the extant literature has thus far proposed competing notions of greater third-degree (or greater downside) risk aversion. For example, following the spirit of statement (i), Modica and Scarsini (2005) define individual v to be more third-degree risk averse than individual u if the former is always willing to pay a (weakly) larger risk premium than the latter to avoid a third-degree increase in risk in the sense of Menezes et al. (1980). Extending Ross (1981), Modica and Scarsini (2005) characterize their notion of greater third-degree risk aversion by a third-degree Ross condition on the two utility functions.[3] On the other hand, following the spirit of statement (ii), Keenan and Snow (2016) combine $\phi''(y) \leq 0$ and $\phi'''(y) \geq 0$ to define $v(x)$ being more third-degree risk averse than $u(x)$. They characterize their notion of greater third-degree risk aversion by consistent dislike of changes in the wealth distribution that induce third-order stochastic dominance shifts in the utility distribution. Finally, following the spirit of statement (iii), Chiu (2005) uses $-v'''(x)/v''(x) \geq -u'''(x)/u''(x)$, whereas Crainich and Eeckhoudt (2008) use $v'''(x)/v'(x) \geq u'''(x)/u'(x)$, to define $v(x)$ being more third-degree risk averse than $u(x)$.[4] Chiu (2005) characterizes his notion of greater third-degree risk aversion by a single-crossing property that any shifts in the wealth distribution have to satisfy.

The approach that follows the direction represented by statement (i) to generalizing greater second-degree risk aversion to higher degrees seems to be the most successful. Specifically, Denuit and Eeckhoudt (2010b), Jindapon and Neilson (2007), and Li (2009) provide a Ross-type characterizing condition on utility functions under which individual v is always willing to pay a (weakly) larger risk premium than individual u to avoid an nth-degree increase in risk in the sense of Ekern (1980).[5] We refer to this notion of greater nth-

[3]As is well known, while one can increase the downside risk in random wealth, one cannot introduce a downside risk to otherwise non-random wealth.

[4]Kimball (1990) first proposes $-u'''(x)/u''(x)$ as the measure of absolute prudence to quantify the strength of the precautionary saving motive in an inter-temporal setting.

[5]Liu and Meyer (2013) generalize the nth-degree Ross condition to the (n/m)th-degree Ross condition

degree risk aversion as greater nth-degree Ross risk aversion. In contrast, the other two approaches to generalizing greater second-degree risk aversion, along the directions represented by statements (ii) and (iii) above and confined in the Arrow-Pratt tradition, have not been adequately explored beyond the third degree.[6] The purpose of this paper is to develop a unified framework of greater (m, n)th-degree mixed risk aversion, which includes Chiu (2005), Crainich and Eeckhoudt (2008), and Keenan and Snow (2016) as special cases of the third degree.

We need to introduce two concepts in order to describe what can be achieved with the general notion of greater (m, n)th-degree mixed risk aversion. First, a utility function, $u(x)$, is said to exhibit (m, n)th-degree mixed risk aversion if $(-1)^{k+1}u^{(k)}(x) \geq 0$ for all $x \in [a, b]$ and for all $k = m, ..., n$, where $u^{(k)}(x) = \mathrm{d}^k u(x)/\mathrm{d}x^k$ is the kth derivative of $u(x)$ and $n \geq m \geq 1$.[7] Second, generalizing the Arrow-Pratt measure of absolute risk aversion, $-u''(x)/u'(x)$, to higher degrees, the measure of (n/m)th-degree absolute risk aversion is defined as $A_u^{(n/m)}(x) = (-1)^{n-m}u^{(n)}(x)/u^{(m)}(x)$. It is evident that $A_u^{(n/m)}(x)$ is completely general and unifies all measures found in the extant literature.

We show that our notion of greater (m, n)th-degree mixed risk aversion ranks utility functions in a strict partial ordering, i.e., the ranking is irreflexive, anti-symmetric, and transitive, which is necessary for the concept to be useful for comparative statics analysis. Furthermore, it has some nice properties. First, (m, n)th-degree mixed risk aversion embedded in the reference utility function, $u(x)$, is preserved when $u(x)$ is transformed into $v(x)$ that is more (m, n)th-degree mixed risk averse than $u(x)$. Second, all measures of (k/m)th-degree absolute risk aversion for $v(x)$ are uniformly larger than those for $u(x)$, i.e., $A_v^{(k/m)}(x) \geq A_u^{(k/m)}(x)$ for all $x \in [a, b]$ and for all $k = m + 1, ..., n$. We also provide a

and use it to characterize interpersonal comparisons of the willingness to pay (in terms of an mth-degree increase in risk) for avoiding an nth-degree increase in risk.

[6] To the best of our knowledge, the only exceptions are Denuit and Eeckhoudt (2010a), Huang et al. (2017), and Jindapon and Neilson (2007) that analyze a notion of greater nth-degree Arrow-Pratt risk aversion based on the measure $-u^{(n)}(x)/u^{(n-1)}(x)$. Note, however, that $-u^{(n)}(x)/u^{(n-1)}(x)$ is only one of the many potential candidates for the nth-degree extension of $-u''(x)/u'(x)$.

[7] Throughout the paper, we use the notation, $f^{(k)}(x) = \mathrm{d}^k f(x)/\mathrm{d}x^k$, to denote the kth derivative of the function, $f(x)$. For the first, second, and third derivatives of $f(x)$, we use the usual notation, $f'(x)$, $f''(x)$, and $f'''(x)$, respectively.

choice-based characterization for greater (m, n)th-degree mixed risk aversion.

While our approach is completely general, two special cases that have already received some attention in the extant literature are in order. First, when $m = 1$, $v(x)$ is more $(1, n)$th-degree mixed risk averse than $u(x)$ if there exists a transformation function, $\phi(y)$, such that $v(x) = \phi\Big(u(x)\Big)$ and $(-1)^{k+1}\phi^{(k)}(y) \geq 0$ for all $y \in [u(a), u(b)]$ and for all $k = 1, ..., n$. Obviously, the notion of strong increases in downside risk aversion proposed in Keenan and Snow (2016, 2018) is a special case of greater $(1, n)$th-degree mixed risk aversion wherein $n = 3$. Second, when $m = n - 1$, $v(x)$ is more $(n - 1, n)$th-degree mixed risk averse than $u(x)$ if $-v^{(n)}(x)/v^{(n-1)}(x) \geq -u^{(n)}(x)/u^{(n-1)}(x)$ for all $x \in [a, b]$, corresponding to the greater nth-degree Arrow-Pratt risk aversion proposed in Denuit and Eeckhoudt (2010a), Huang et al. (2017), and Jindapon and Neilson (2007), and including the notion of greater third-degree risk aversion proposed in Chiu (2005) as a special case wherein $n = 3$.

Two recent papers by Wong (2018a, 2018b) are closely related to ours. Wong (2018a) defines the nth-degree utility premium as the pain associated with facing the passage from a more favorable risk to a less favorable risk, where the risk increase is specified by a simple increase in nth-degree risk. He further defines the nth-degree prudence utility premium as the increase in pain when the individual suffers a sure loss. He shows that the nth-degree utility premium, normalized by the $(n-1)$th derivative of the utility function evaluated at the initial wealth has the same ranking as that corresponds to the measure of nth-degree Arrow-Pratt absolute risk aversion. On the other hand, the nth-degree prudence utility premium, normalized by the nth derivative of the utility function evaluated at the initial wealth, has the same ranking as that corresponds to the measure of $(n+1)$th-degree Arrow-Pratt absolute risk aversion. Wong (2018b) applies these concepts to examine the effect of increased higher-order risk on the precautionary saving motive. He derives the necessary and sufficient condition under which saving increases in response to an increase in interest rate risk via (m, n)th-order stochastic dominance, which is shown to describe a trade-off between a prudence effect that favors precautionary saving and a risk aversion effect that limits precautionary saving.

Our paper focuses on the measures of Arrow-Pratt absolute risk aversion of higher orders for the sake of doing comparative risk aversion analysis. This is in contrast to Wong (2018b) that focuses on changes in higher-order risk in a specific (saving/consumption) decision problem. We characterize greater (m, n)th-degree mixed risk aversion in the Arrow-Pratt sense by the ranking of risk distributions of a reference individual that would be obeyed by all individuals who are more (m, n)th-degree mixed risk averse than the reference individual but not vice versa. Wong (2018a) offers an alternative characterization using normalized nth-degree utility premiums for the special case that $m = n - 1$, where the normalization using the $(n - 1)$th derivative of the utility function evaluated at the initial wealth is an arbitrarily chosen procedure that has no justification.

The paper is organized as follows. In Section 2, we define (m, n)th-degree mixed risk aversion and (m, n)th-order stochastic dominance, where these two concepts are shown to be closely related. In Section 3, we define our general notion of greater (m, n)th-degree mixed risk aversion, and examine its properties. A choice-based characterization of greater (m, n)th-degree mixed risk aversion is provided. Then, two special cases of greater (m, n)th-degree mixed risk aversion, one for $m = 1$ and the other for $m = n - 1$, are analyzed in more details. Section 4 offers a few applications. Section 5 concludes.

2. Notation and preliminaries

Consider an individual who has random wealth, $\tilde{x}$, that takes on values in $[a, b]$, where $a < b$. The individual possesses a von Neumann-Morgenstern utility function, $u(x)$, defined over his wealth level, $x \in [a, b]$. We state the definition of (m, n)th-degree mixed risk aversion as follows.

Definition 1. For any two integers, n and m, such that $n \geq m \geq 1$ and any utility function, $u(x)$, we say that $u(x)$ exhibits (m, n)th-degree mixed risk aversion if $(-1)^{k+1}u^{(k)}(x) \geq 0$

for all $x \in [a, b]$ and for all $k = m, ..., n$.

When $m = n$, the notion of (m, n)th-degree mixed risk aversion reduces to the regular nth-degree risk aversion characterized by $(-1)^{n+1}u^{(n)}(x) \geq 0$ for all $x \in [a, b]$. As shown by Eeckhoudt and Schlesinger (2006), mixed risk aversion characterizes the common preferences in that individuals prefer to disaggregate risks across different states of nature. If $u(x)$ satisfies $(1, n)$th-degree mixed risk aversion, letting n go to infinity allows $u(x)$ to have all odd derivatives positive and all even derivatives negative, thereby rendering $u(x)$ to be completely monotone (Brockett and Golden, 1987; Caballé and Pomansky, 1996).[8]

For the ease of exposition, we hereafter refer to the individual with the utility function, $u(x)$, as individual u, who serves as a reference individual, and the one with another utility function, $v(x)$, as individual v. Following Liu and Meyer (2013), we generalize the Arrow-Pratt measure of absolute risk aversion, $-u''(x)/u'(x)$, to higher orders, leading to the measure of (n/m)th-degree absolute risk aversion for the utility function, $u(x)$, that exhibits mth-degree risk aversion with $(-1)^{m+1}u^{(m)}(x) > 0$ for all $x \in [a, b]$:

$$A_u^{(n/m)}(x) = (-1)^{n-m}\frac{u^{(n)}(x)}{u^{(m)}(x)}, \tag{1}$$

for all $x \in [a, b]$, where $n > m \geq 1$.[9] Eq. (1) is completely general and unifies all measures found in the literature. For example, $A_u^{(2/1)}(x) = -u''(x)/u'(x)$ is the Arrow-Pratt measure of absolute risk aversion, $A_u^{(3/2)}(x) = -u'''(x)/u''(x)$ is Kimball's (1990, 1993) measure of absolute prudence (see also Chiu, 2005), and $A_u^{(4/3)}(x) = -u^{(4)}(x)/u'''(x)$ is the measure of absolute temperance proposed by Eeckhoudt et al. (1996) and Gollier and Pratt (1996). Indeed, Caballé and Pomansky (1996), Denuit and Eeckhoudt (2010a), and Jindapon and Neilson (2007) generalize these three measures to a general index of nth-degree absolute risk aversion, $A_u^{(n/n-1)}(x) = -u^{(n)}(x)/u^{(n-1)}(x)$, which is a special case of Eq. (1) when $m = n - 1$. Crainich and Eeckhoudt (2008) and Modica and Scarsini (2005) define $A_u^{(3/1)}(x) =$

[8]Ebert et al. (2018), Eeckhoudt and Schlesinger (2006), and Menegatti (2015) further explore the implications of the utility functions with these properties.

[9]We adopt a strong rather than weak inequality on the mth derivative of $u(x)$ so as to avoid the issue of division by zero when defining the measure of (n/m)th-degree absolute risk aversion for $u(x)$.

$u'''(x)/u'(x)$ as a local coefficient of downside risk aversion, which is generalized by Denuit and Eeckhoudt (2010b) to a local absolute index of nth-degree risk attitude, $A_u^{(n/1)}(x) = (-1)^{n+1}u^{(n)}(x)/u'(x)$, another special case of Eq. (1) when $m = 1$.

Let $F(x)$ and $G(x)$ be two cumulative distribution functions (CDFs) of $\tilde{x}$ over support $[a, b]$, where $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. We follow Liu (2014) to define (m, n)th-order stochastic dominance as follows (see also Ebert et al., 2018).[10]

Definition 2. For any two integers, n and m, such that $n \geq m \geq 1$ and any two CDFs of the random wealth, $F(x)$ and $G(x)$, we say that $F(x)$ is riskier than $G(x)$ via (m, n)th-order stochastic dominance if

$$\int_a^b u(x)\mathrm{d}G(x) \geq \int_a^b u(x)\mathrm{d}F(x), \tag{2}$$

for every utility function, $u(x)$, that exhibits (m, n)th-degree mixed risk aversion.

Starting from $F_1(x) = F(x)$, we define $F_2(x)$, ..., $F_n(x)$ recursively from repeated integrals:

$$F_k(x) = \int_a^x F_{k-1}(y)\mathrm{d}y, \tag{3}$$

for all $x \in [a, b]$ and for all $k = 2, \ldots, n$, where $n \geq 2$. The following lemma characterizes (m, n)th-order stochastic dominance, where a formal proof can be found in Liu (2014).

Lemma 1. For any two integers, n and m, such that $n \geq m \geq 1$ and any two CDFs of the random wealth, $F(x)$ and $G(x)$, the following two statements are equivalent:

(i) $F(x)$ is riskier than $G(x)$ via (m, n)th-order stochastic dominance.

[10] Liu (2014) refers to (m, n)th-order stochastic dominance as nth-degree first $(m-1)$ moments preserving stochastic dominance, whereas Ebert et al. (2018) refer to this as $(m-1)$-moments preserving nth-order stochastic dominance.

(ii) $F(x)$ and $G(x)$ satisfy that $F_k(b) = G_k(b)$ for all $k = 1, ..., m$, $F_k(b) \geq G_k(b)$ for all $k = m+1, ..., n$, and $F_n(x) \geq G_n(x)$ for all $x \in [a, b]$.

Lemma 1 shows that (m, n)th-order stochastic dominance requires the first $m-1$ moments of $\tilde{x}$ to be preserved, which follows from $F_k(b) = G_k(b)$ for all $k = 2, ..., m$. Many well-known definitions of increased risk are included as special cases. For example, $(1, n)$th-order stochastic dominance reduces to the regular nth-order stochastic dominance for all $n \geq 1$, and $(2, n)$th-order stochastic dominance defines mean-preserving nth-order stochastic dominance as in Denuit and Eeckhoudt (2013) for all $n \geq 2$. When $m = n$, Definition 2 reduces to the definition of more nth-degree risk in the sense of Ekern (1980) for all $n \geq 1$. In this case, more first-degree risk is identical to first-order stochastic dominance. More second-degree risk refers to mean-preserving spreads in the sense of Rothschild and Stiglitz (1970). More third-degree risk is equivalent to an increase in downside risk *à la* Menezes et al. (1980), which moves risk from right to left while keeping the mean and variance of $\tilde{x}$ intact. More fourth-degree risk is an increase in outer risk (Menezes and Wang, 2005) that has higher peaks and longer tails (i.e., more kurtotic) while keeping the mean, variance, and third central moment of $\tilde{x}$ constant.

3. A general notion of greater Arrow-Pratt risk aversion of higher orders

3.1 Definition of greater (m, n)th-degree mixed risk aversion

We state our definition of increases in (m, n)th-degree mixed risk aversion as follows.

Definition 3. For any two integers, n and m, such that $n > m \geq 1$ and any two utility functions, $u(x)$ and $v(x)$, such that $u(x)$ exhibits mth-degree risk aversion, we say that $v(x)$ is more (m, n)th-degree mixed risk averse than $u(x)$ if there exists a transformation function, $\phi(y)$, such that

$$(-1)^{m+1}v^{(m-1)}(x) = \phi\Big((-1)^{m+1}u^{(m-1)}(x)\Big), \tag{4}$$

for all $x \in [a,b]$ and $(-1)^{k+1}\phi^{(k)}(y) \geq 0$ for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$ and for all $k = 1, ..., n-m+1$.

We show in the following lemma that the ranking of utility functions by increases in (m,n)th-degree mixed risk aversion according to Definition 3 exhibits a strict partial ordering, i.e., it is irreflexive, anti-symmetric, and transitive, which is necessary for the ranking to be useful for comparative statics analysis.

Lemma 2. For any two integers, n and m, such that $n > m \geq 1$, greater (m,n)th-degree mixed risk aversion ranks utility functions in a strict partial ordering, and implies greater (m,n')th-degree mixed risk aversion for all $n' = m+1, ..., n$.

Proof. See Appendix A. □

Irreflexivity follows because $(-1)^{m+1}u^{(m-1)}(x) = \phi\Big((-1)^{m+1}u^{(m-1)}(x)\Big)$ only if $\phi(y)$ is the identity function, which obviously violates the conditions for being an increase in (m,n)th-degree mixed risk aversion. Transitivity has been shown by Keenan and Snow (2016) for greater $(1,3)$th-degree mixed risk aversion. Lemma 2 shows that transitivity readily extends to greater (m,n)th-degree mixed risk aversion. Finally, anti-symmetry is implied by irreflexivity and transitivity for a strict partial ordering.

Greater (m,n)th-degree mixed risk aversion necessarily implies greater (m,n')th-degree mixed risk aversion for all $n' = m+1, ..., n$, which is an integral part of Definition 3. As pointed out by Keenan and Snow (2009, 2016) and Liu and Meyer (2012), $\phi'''(y) \geq 0$ alone does not guarantee that $v(x) = \phi\Big(u(x)\Big)$ is more downside risk averse than $u(x)$. Without the additional condition that $\phi''(y) \leq 0$, the relation established by the transformation function, $\phi(y)$, is not transitive. The same idea applies to higher orders, thereby rendering Definition 3 and Lemma 2.

3.2 Properties of greater (m, n)th-degree mixed risk aversion

Greater (m, n)th-degree mixed risk aversion as defined in Definition 3 has a nice property that (m, n)th-degree mixed risk aversion embedded in the reference utility function, $u(x)$, is preserved when $u(x)$ is transformed into a more (m, n)th-degree mixed risk averse utility function, $v(x)$, as is shown in our first proposition.

Proposition 1. For any two integers, n and m, such that $n > m \geq 1$ and any two utility functions, $u(x)$ and $v(x)$, such that $u(x)$ exhibits (m, n)th-degree mixed risk aversion and $v(x)$ is more (m, n)th-degree mixed risk averse than $u(x)$, the following two statements hold:

(i) $v(x)$ inherits (m, n)th-degree mixed risk aversion from $u(x)$.

(ii) Given that $(-1)^{m+1}u^{(m)}(x) > 0$ and $(-1)^{m+1}v^{(m)}(x) > 0$ for all $x \in [a, b]$, individual v's measure of (k/m)th-degree absolute risk aversion is uniformly larger than that of individual u, i.e., $A_v^{(k/m)}(x) \geq A_u^{(k/m)}(x)$ for all $x \in [a, b]$ and for all $k = m + 1$, ..., n.

Proof. See Appendix B. □

The preservation of (m, n)th-degree mixed risk aversion when we transform $u(x)$ to $v(x)$ follows from the transitivity of the greater (m, n)th-degree mixed risk aversion ordering as shown in Lemma 2. Proposition 1 shows further that the measures of (k/m)th-degree absolute risk aversion are unambiguously comparable for utility functions that can be ranked by greater (m, n)th-degree mixed risk aversion for all $k = m+1$, ..., n. Indeed, Crainich and Eeckhoudt (2008) show that if $v(x) = \phi\Big(u(x)\Big)$ is more risk averse than $u(x)$ in the Arrow-Pratt sense, i.e., $\phi'(y) \geq 0$ and $\phi''(y) \leq 0$ for all $y \in [u(a), u(b)]$, the additional condition that $\phi'''(y) \geq 0$ for all $y \in [u(a), u(b)]$ is sufficient but not necessary for $v'''(x)/v'(x) \geq u'''(x)/u'(x)$ for all $x \in [a, b]$. Statement (ii) of Proposition 1 as such generalizes their findings beyond the (3/1)th-degree and shows that the measures of $(k/1)$th-degree absolute risk aversion, $A_u^{(k/1)}(x) = (-1)^{k+1}u^{(k)}(x)/u'(x)$ for all $k = 2, \ldots n$, play a special role when we compare $u(x)$ and $v(x)$ that are related by $v(x) = \phi\Big(u(x)\Big)$.

If $F(x)$ is riskier than $G(x)$ via (m, n)th-order stochastic dominance and $u(x)$ is (m, n)th-degree mixed risk averse, it follows from Definition 2 that individual u prefers $G(x)$ to $F(x)$. From statement (i) of Proposition 1, $v(x)$, which is more (m, n)th-degree mixed risk averse than $u(x)$, must inherit (m, n)th-degree mixed risk aversion from $u(x)$. It then follows immediately from Definition 2 that individual v prefers $G(x)$ to $F(x)$ as well. The converse, however, is not true because $u(x)$ does not necessarily inherit (m, n)th-degree mixed risk aversion from $v(x)$.[11] In other words, for any two CDFs of $\tilde{x}$ that can be ranked by the (m, n)th-order stochastic dominance rule, all individuals who are more (m, n)th-degree mixed risk averse than the reference individual agree with the preferences of the latter, but not vice versa. Hence, Definition 3 indeed captures greater (m, n)th-degree mixed risk aversion.

3.3 Choice-based characterizations of greater (m, n)th-degree mixed risk aversion

For any two CDFs of $\tilde{x}$, $F(x)$ and $G(x)$, and any utility function, $u(x)$, we define $T_0(x; m) = F_m(x) - G_m(x)$ for all $x \in [a, b]$ and $T_1(x; m)$, ..., $T_{n-m}(x; m)$ recursively from repeated integrals:

$$T_k(x; m) = \int_a^x (-1)^{m+1} u^{(m)}(y) T_{k-1}(y; m) \mathrm{d}y, \tag{5}$$

for all $x \in [a, b]$ and for all $k = 1, \ldots, n - m$, where $n > m \geq 1$. The following proposition provides a choice-based characterization of greater (m, n)th-degree mixed risk aversion.

[11] For example, if $\phi(y) = -\exp(-y)$ and $u(x) = \exp(x)$, we have $v'(x) > 0$ and $v''(x) < 0$ but $u'(x) > 0$ and $u''(x) > 0$ for all $x > 0$.

Proposition 2. For any two integers, n and m, such that $n > m \geq 1$ and any two utility functions, $u(x)$ and $v(x)$, such that individual u exhibits mth-degree risk aversion, the following two statements are equivalent:

(i) Individual v is more (m,n)th-degree mixed risk averse than individual u.

(ii) Individual v prefers $G(x)$ to $F(x)$ for all CDFs of the random wealth, $F(x)$ and $G(x)$, such that $F_k(b) = G_k(b)$ for all $k = 1, ..., m$, $T_k(b;m) \geq 0$ for all $k = 1, ..., n-m$, and $T_{n-m}(x;m) \geq 0$ for all $x \in [a,b]$.

Proof. See Appendix C. □

Applying integration by parts yields

$$\int_a^b u(x)\mathrm{d}[G(x) - F(x)] = \int_a^b (-1)^{m+1}u^{(m)}(x)[F_m(x) - G_m(x)]\mathrm{d}x = T_1(b;m), \tag{6}$$

where the first equality follows from $F_k(b) = G_k(b)$ for all $k = 1, ..., m$, and the second equality follows from Eq. (5). For all $F(x)$ and $G(x)$ that satisfy statement (ii) of Proposition 2, it follows from Eq. (6) that individual u prefers $G(x)$ to $F(x)$ since $T_1(b;m) \geq 0$. Statement (ii) of Proposition 2 as such characterizes the set of CDFs of the random wealth that can be ranked by a reference individual such that the resulting ranking is obeyed by all individuals who are more (m,n)th-degree mixed risk averse than the reference individual, an appealing intuitive comparative statics result.

To see the intuition for Proposition 2, we apply integration by parts to yield

$$\int_a^b v(x)\mathrm{d}[G(x) - F(x)] = F_m(b)\int_a^b \phi\Big((-1)^{m+1}u^{(m-1)}(x)\Big)\mathrm{d}\Big[\frac{G_m(x)}{G_m(b)} - \frac{F_m(x)}{F_m(b)}\Big], \tag{7}$$

where we have used $F_k(b) = G_k(b)$ for all $k = 1, ..., m$ and Eq. (4). Define $\hat{F}(x) = F_m(x)/F_m(b)$ and $\hat{G}(x) = G_m(x)/G_m(b)$. We can as such treat $\hat{F}(x)$ and $\hat{G}(x)$ as two pseudo CDFs of $\tilde{x}$ over support $[a,b]$. The CDFs of $(-1)^{m+1}u^{(m-1)}(x)$ when $\tilde{x}$ has CDFs, $\hat{F}(x)$ and

$\hat{G}(x)$, are given by $\hat{F}^{\hat{u}}(y) = \int_{\hat{u}(a)}^{y} \hat{F}'\Big(\hat{u}^{-1}(z)\Big)\hat{u}^{-1\prime}(z)\mathrm{d}z$ and $\hat{G}^{\hat{u}}(y) = \int_{\hat{u}(a)}^{y} \hat{G}'\Big(\hat{u}^{-1}(z)\Big)\hat{u}^{-1\prime}(z)\mathrm{d}z$ for all $y \in [\hat{u}(a), \hat{u}(b)]$, respectively, where $\hat{u}(x) = (-1)^{m+1}u^{(m-1)}(x)$ for all $x \in [a, b]$.[12] Letting $y = \hat{u}(x)$, we can write Eq. (7) as

$$\int_a^b v(x)\mathrm{d}[G(x) - F(x)] = F_m(b) \int_{\hat{u}(a)}^{\hat{u}(b)} \phi(y)\mathrm{d}[\hat{G}^{\hat{u}}(y) - \hat{F}^{\hat{u}}(y)]. \tag{8}$$

We can also write Eq. (5) as

$$T_k(x; m) = F_m(b)\Big[\hat{F}^{\hat{u}}_{k+1}\Big(\hat{u}(x)\Big) - \hat{G}^{\hat{u}}_{k+1}\Big(\hat{u}(x)\Big)\Big], \tag{9}$$

for all $x \in [a, b]$ and for all $k = 1, ..., n - m$. Hence, it follows from Eq. (9) that $T_k(b; m) \geq 0$ for all $k = 1, ..., n - m$ and $T_{n-m}(x; m) \geq 0$ for all $x \in [a, b]$ if, and only if, $\hat{F}^{\hat{u}}(y)$ is riskier than $\hat{G}^{\hat{u}}(y)$ via $(n - m + 1)$th-order stochastic dominance, which in turn follows from Definition 2 that the right-hand side of Eq. (8) is non-negative if, and only if, $\phi(y)$ exhibits $(1, n-m+1)$th-degree mixed risk aversion, i.e., $v(x)$ is more (m, n)th-degree mixed risk averse than $u(x)$ according to Definition 3.

We conclude this subsection with an example in order to make the sufficient and necessary conditions for $v(x)$ being more (m, n)th-degree mixed risk averse than $u(x)$ more concrete and intuitive. Consider the case wherein $m = 2$ and the reference utility function, $u(x)$, is quadratic such that $u''(x) = -\gamma$, where γ is a positive constant. In this case, $v(x)$ is more $(2, n)$th-degree mixed risk averse than $u(x)$ if, and only if, $v(x)$ exhibits $(2, n)$th-degree mixed risk aversion. For any two CDFs of the random wealth, $F(x)$ and $G(x)$, it follows from $-u''(x) = \gamma$ and Eq. (5) that $T_k(x; 2) = \gamma^k[F_{k+2}(x) - G_{k+2}(x)]$ for all $x \in [a, b]$ and for all $k = 0, ..., n - 2$. From Proposition 2, $v(x)$ is more $(2, n)$th-degree mixed risk averse than $u(x)$ if, and only if, individual v prefers $G(x)$ to $F(x)$ for all $F(x)$ and $G(x)$ such that $F_2(b) = G_2(b)$, $T_k(b; 2) = \gamma^k[F_{k+2}(b) - G_{k+2}(b)] \geq 0$ for all $k = 1, ..., n - 2$, and $T_{n-2}(x; 2) = \gamma^{n-2}[F_n(x) - G_n(x)] \geq 0$ for all $x \in [a, b]$, i.e., $F(x)$ is riskier than $G(x)$ via $(2, n)$th-order stochastic dominance (Definition 1), which is just a restatement of Lemma 1.

[12]Since $u(x)$ is mth-degree risk averse, $\hat{u}(x)$ has a positive first derivative. As such, $\hat{u}^{-1}(y)$ is well-defined for all $y \in [\hat{u}(a), \hat{u}(b)]$.

3.4 Two special cases of greater (m, n)th-degree mixed risk aversion

While the results of Proposition 2 are completely general, two special cases that have been put forth in the literature are in order: (i) increases in $(1, n)$th-degree mixed risk aversion, and (ii) increases in $(n-1, n)$th-degree mixed risk aversion.

We consider first the special case of increases in $(1, n)$th-degree mixed risk aversion. Let $F^u(y) = \int_{u(a)}^{y} F'\Big((u^{-1}(z)\Big)u^{-1\prime}(z)\mathrm{d}z$ and $G^u(y) = \int_{u(a)}^{y} G'\Big((u^{-1}(z)\Big)u^{-1\prime}(z)\mathrm{d}z$ be the CDFs of $u(\tilde{x})$ over support $[u(a), u(b)]$ given that $\tilde{x}$ has CDFs, $F(x)$ and $G(x)$, respectively. The following proposition is a corollary of Proposition 2 when $m = 1$, and thus a formal proof is omitted.

Proposition 3. For any integer, $n \geq 2$, and any two increasing utility functions, $u(x)$ and $v(x)$, the following two statements are equivalent:

(i) Individual v is more $(1, n)$th-degree mixed risk averse than individual u.

(ii) Individual v prefers $G(x)$ to $F(x)$ for all CDFs of the random wealth, $F(x)$ and $G(x)$ such that the corresponding CDFs of individual u's utility function, $F^u(y)$ and $G^u(y)$, satisfy that $F^u(y)$ is riskier than $G^u(y)$ via nth-order stochastic dominance.

Greater $(1, 2)$th-degree mixed risk aversion is simply the usual greater Arrow-Pratt risk aversion wherein $v(x) = \phi\Big(u(x)\Big)$ such that $\phi'(y) \geq 0$ and $\phi''(y) \leq 0$ for all $y \in [u(a), u(b)]$. Diamond and Stiglitz (1974) show that increases in Arrow-Pratt risk aversion are characterized by consistent dislike of changes in the CDF of $\tilde{x}$ that induce second-order stochastic dominance shifts in the CDF of $u(\tilde{x})$. On the other hand, increases in $(1, 3)$th-degree mixed risk aversion are equivalent to strong increases in downside risk aversion proposed by Keenan and Snow (2016) wherein $v(x) = \phi\Big(u(x)\Big)$ such that $\phi'(y) \geq 0$, $\phi''(y) \leq 0$, and $\phi'''(y) \geq 0$ for all $y \in [u(a), u(b)]$ (see also Crainich and Eeckhoudt, 2008). Keenan and Snow (2016) show that strong increases in downside risk aversion are characterized by consistent dislike of changes in the CDF of $\tilde{x}$ that induce third-order stochastic dominance shifts in the CDF of $u(\tilde{x})$. Proposition 3 as such extends these results beyond the third-degree in that increases in $(1, n)$th-degree mixed risk aversion are characterized by consistent dislike of changes in the CDF of $\tilde{x}$ that induce nth-order stochastic dominance shifts in the CDF of $u(\tilde{x})$.

We now consider the other special case of increases in $(n-1, n)$th-degree mixed risk aversion. We show in the following proposition that this case is essentially the one that has been explicitly studied by Chiu (2005), Denuit and Eeckhoudt (2010a), and Jewitt (1989).

Proposition 4. For any integer, $n \geq 2$, and any two utility functions, $u(x)$ and $v(x)$, such that $u(x)$ exhibits $(n-1, n)$th-degree mixed risk aversion, and $(-1)^n u^{(n-1)}(x) > 0$ and $(-1)^n v^{(n-1)}(x) > 0$ for all $x \in [a, b]$, the following three statements are equivalent:

(i) Individual v is more $(n-1, n)$th-degree mixed risk averse than individual u.

(ii) Individual v's measure of $(n/n-1)$th-degree absolute risk aversion is uniformly larger than that of individual u, i.e., $A_v^{(n/n-1)}(x) \geq A_u^{(n/n-1)}(x)$ for all $x \in [a, b]$.

(iii) Individual v prefers $G(x)$ to $F(x)$ for all CDFs of the random wealth, $F(x)$ and $G(x)$, such that $F_k(b) = G_k(b)$ for all $k = 1, ..., n-1$ and $T_1(x; n-1) \geq 0$ for all $x \in [a, b]$.

Proof. See Appendix D. □

For all $F(x)$ and $G(x)$ that satisfy statement (iii) of Proposition 4, it follows from Eq. (6) with $m = n-1$ that individual u prefers $G(x)$ to $F(x)$ since $T_1(b; n-1) \geq 0$. Statement (iii) of Proposition 4 as such characterizes the set of CDFs of the random wealth that can be ranked by individual u such that the resulting ranking is obeyed by all individuals who have measures of $(n/n-1)$th-degree absolute risk aversion that are uniformly larger than that of the reference individual.

Jewitt (1989) shows that individual u's preference of $G(x)$ over $F(x)$ is preserved by individual v who is more risk averse than individual u in the Arrow-Pratt sense if, and only if, $F(x)$ and $G(x)$ satisfy the single-crossing property in that there exists a point, $x^\circ \in (a, b]$, such that $F_2(x) \geq G_2(x)$ for all $x \in [a, x^\circ]$ and $F(x) \leq G(x)$ for all $x \in (x^\circ, b]$. Indeed, Denuit and Eeckhoudt (2010a) generalize the single-crossing property of Jewitt (1989) to the case wherein individual v's measure of $(n/n-1)$th-degree absolute risk aversion is uniformly larger than that of individual u (for $n = 3$, see Chiu, 2005). Specifically, the characterization of Denuit and Eeckhoudt (2010a) requires that $F_k(b) = G_k(b)$ for all $k = 1$, ..., $n-1$, $T_1(b; n-1) = 0$, and there exists a point, $x^\circ \in (a, b]$, such that $F_n(x) \geq G_n(x)$ for all $x \in [a, x^\circ]$ and $F_{n-1}(x) \leq G_{n-1}(x)$ for all $x \in (x^\circ, b]$. To show that this characterization implies statement (iii) of Proposition 4, we apply integration by parts to yield

$$T_1(x; n-1) = (-1)^n u^{(n-1)}(x)[F_n(x) - G_n(x)] + \int_a^x (-1)^{n+1} u^{(n)}(y)[F_n(y) - G_n(y)]\mathrm{d}y,$$

which is non-negative for all $x \in [a, x^\circ]$ since $u(x)$ exhibits $(n-1, n)$th-degree mixed risk aversion and $F_n(x) \geq G_n(x)$ for all $x \in [a, x^\circ]$. For all $x \in (x^\circ, b]$, we have

$$\begin{aligned} T_1(x; n-1) &= T_1(x^\circ; n-1) + \int_{x^\circ}^x (-1)^n u^{(n-1)}(y)[F_{n-1}(y) - G_{n-1}(y)]\mathrm{d}y \\ &\geq T_1(x^\circ; n-1) + \int_{x^\circ}^b (-1)^n u^{(n-1)}(y)[F_{n-1}(y) - G_{n-1}(y)]\mathrm{d}y \\ &= T_1(b; n-1) = 0, \end{aligned}$$

where the inequality follows from the fact that $u(x)$ exhibits $(n-1)$th-degree risk aversion and $F_{n-1}(x) \leq G_{n-1}(x)$ for all $x \in (x^\circ, b]$. Hence, we conclude that $T_1(x; n-1) \geq 0$ for all $x \in [a, b]$. Statement (iii) of Proposition 4, however, does not necessarily imply the characterization of Denuit and Eeckhoudt (2010a). To see this, consider the following example for $n = 2$. Individual u has a quadratic utility function, $u(x) = x - x^2/2$ for all $x \in [0, 1]$. $F(x)$ is the uniform distribution over $[0, 1]$ and

$$G(x) = \begin{cases} 0 & \text{for all } x \in [0, 0.125], \\ 2.5x - 0.3125 & \text{for all } x \in [0.125, 0.325], \\ 0.5 & \text{for all } x \in [0.325, 0.675], \\ 2.5x - 1.1875 & \text{for all } x \in [0.675, 0.875], \\ 1 & \text{for all } x \in [0.875, 1]. \end{cases}$$

It is easily verified that $T_1(x; 1) \geq 0$ for all $x \in [0, 1]$ and $T_1(1; 1) = 0$ so that individual u is indifferent between $F(x)$ and $G(x)$ and all individuals who are more risk averse than individual u in the Arrow-Pratt sense prefer $G(x)$ to $F(x)$. However, we have $F_2(x) \geq G_2(x)$ for all $x \in [0, 0.3419]$ but $F(x) > G(x)$ for all $x \in (0.5, 0.7917)$, thereby violating the single-crossing property of Jewitt (1989). In other words, the single-crossing property (Chiu, 2005; Denuit and Eeckhoudt, 2010a; Jewitt, 1989) put forth in the extant literature is sufficient but not necessary for the intuitive comparative statics result with respect to an increase in $(n-1, n)$th-degree mixed risk aversion.

4 Applications

In this section, we illustrate how the concept of greater (m, n)th-degree mixed risk aversion developed in Section 3 can be applied to three well-known comparative statics problems.

4.1 Greater $(1, 3)$th-degree mixed risk aversion in rent-seeking games

Consider the rent-seeking game of Konrad and Schlesinger (1997).[13] There are N con-

[13]Since the rent-seeking game of Konrad and Schlesinger (1997) has similar structure as the single-period model of self-protection (Treich, 2010), our results are readily applicable to the comparative statics problem of self-protection (Eeckhoudt and Gollier, 2005; Menegatti, 2009).

testants, indexed by $i = 1, ..., N$, who compete for a fixed rent, $\beta > 0$, where $N \geq 2$. Each of them is endowed with the same initial wealth level, $x_\circ > 0$, and possesses the same von Neumann-Morgenstern utility function, $u(x)$, defined over the final wealth level, $x \in [0, x_\circ + \beta]$, such that $u'(x) \geq 0$ and $u''(x) \leq 0$ for all $x \in [0, x_\circ + \beta]$. Denote $(e_1, ..., e_N)$ as the N-tuple of rent-seeking efforts of the N contestants, where $e_i \in [0, x_\circ]$ is measured in monetary terms for all $i = 1, ..., N$.

The contest is characterized by a contest success function, $p_i(e_1, ..., e_N)$, for contestant i, which is differentiable and symmetric such that $p_i(e_1, ..., e_N) \in [0, 1]$ for all $i = 1, ..., N$ and $\sum_{i=1}^N p_i(e_1, ..., e_N) = 1$. We assume that $\partial p_i(e_1, ..., e_N)/\partial e_i > 0$ and $\partial p_i(e_1, ..., e_N)/\partial e_j < 0$ for $i \neq j$ so that $p_i(e_1, ..., e_N)$ is positively related to one's own effort and negatively related to the efforts of the others. We further assume that $\partial^2 p_i(e_1, ..., e_N)/\partial e_i^2 < 0$ and $\partial^2 p_i(e_1, ..., e_N)/\partial e_i \partial e_j \leq 0$ for $i \neq j$, reflecting the common assumptions that marginal returns to effort are decreasing. Finally, we assume that $p_i(e, ..., e) = 1/N$ for all $e \in [0, x_\circ]$, i.e., all contestants have the same chance of winning if they exert the same level of effort.

Taking other contestants' rent-seeking efforts as given, contestant i's ex-ante decision problem is given by

$$\max_{e_i \in [0, x_\circ]} \; p_i(e_1, ..., e_N)u(x_\circ + \beta - e_i) + [1 - p_i(e_1, ..., e_N)]u(x_\circ - e_i). \tag{10}$$

The first-order condition for program (10) is given by

$$\frac{\partial p_i(e_1, ..., e_N)}{\partial e_i}[u(x_\circ + \beta - e_i) - u(x_\circ - e_i)]$$

$$-\{p_i(e_1, ..., e_N)u'(x_\circ + \beta - e_i) + [1 - p_i(e_1, ..., e_N)]u'(x_\circ - e_i)\} = 0. \tag{11}$$

We assume that the objective function of program (10) is single peaked for all $i = 1, ..., N$.

A pure-strategy Nash equilibrium in the rent-seeking game is an N-tuple of rent-seeking efforts, $(e_1^*, ..., e_N^*)$, such that given the $(N-1)$-tuple of rent-seeking efforts of the other contestants, $(e_1^*, ..., e_{i-1}^*, e_{i+1}^*, ..., e_N^*)$, e_i^* is the solution to Eq. (11) for all $i = 1, ..., N$. Since all contestants are identical and the objective function of program (10) is single peaked for all $i = 1, ..., N$, the Nash equilibrium is unique and symmetric such that the equilibrium effort level, e_u, solves the following equation:

$$\left.\frac{\partial p_i(e_1, ..., e_N)}{\partial e_i}\right|_{(e_1,...,e_N)=(e_u,...,e_u)} [u(x_\circ + \beta - e_u) - u(x_\circ - e_u)]$$

$$-\Big[\frac{1}{N}u'(x_\circ + \beta - e_u) + \Big(1 - \frac{1}{N}\Big)u'(x_\circ - e_u)\Big] = 0, \tag{12}$$

where we have used the fact that $p_i(e_u, ..., e_u) = 1/N$.

Consider now that all contestants become more risk averse so that their identical utility function is $v(x) = \phi\big(u(x)\big)$, where $\phi(y)$ is a transformation function such that $\phi'(y) \geq 0$ and $\phi''(y) \leq 0$ for all $y \in [u(0), u(x_\circ + \beta)]$. The unique symmetric Nash equilibrium, e_v, is given by

$$\left.\frac{\partial p_i(e_1, ..., e_N)}{\partial e_i}\right|_{(e_1,...,e_N)=(e_v,...,e_v)} \Big[\phi\Big(u(x_\circ + \beta - e_v)\Big) - \phi\Big(u(x_\circ - e_v)\Big)\Big]$$

$$-\Big[\frac{1}{N}\phi'\Big(u(x_\circ + \beta - e_v)\Big)u'(x_\circ + \beta - e_v) + \Big(1 - \frac{1}{N}\Big)\phi'\Big(u(x_\circ - e_v)\Big)u'(x_\circ - e_v)\Big] = 0. \tag{13}$$

The following proposition shows that $e_v \leq e_u$ if $v(x)$ is more $(1, 3)$th-degree mixed risk averse than $u(x)$.

Proposition 5. Given that the symmetric Nash equilibrium in the rent-seeking game is unique, the contestants exert less rent-seeking effort, i.e., $e_v \leq e_u$, if they become more $(1, 3)$th-degree mixed risk averse.

Proof. See Appendix E. □

Proposition 5 extends the comparative statics result of Treich (2010) to the case wherein contestants in two rent-seeking games can be ranked by greater $(1, 3)$th-degree mixed risk aversion. To see this, we take the reference contestants as those who are risk neutral. Proposition 5 then says that all risk-averse and prudent contestants, who are by definition more $(1,3)$th-degree mixed risk averse than the risk-neutral contestants, optimally exert less rent-seeking effort than the risk-neutral equilibrium level, a result that has been shown by Treich (2010).[14]

4.2 Greater $(1,n)$th-degree mixed risk aversion and equilibrium interest rates

Following Crainich and Eeckhoudt (2008) and Keenan and Snow (2016), we consider an exchange economy with consumers who have identical preferences and endowments. The representative consumer, individual u, faces the following ex-ante decision problem:

$$\max_{s}\ u(\overline{x}-s)+\frac{1}{1+\rho^u}\int_a^b u\Big((1+r)s+x\Big)\mathrm{d}F(x;\theta), \tag{14}$$

where ρ^u is the pure rate of individual u's time preferences, s denotes saving, r is the market interest rate, $\overline{x}$ is the current income, and $\theta>0$ is a shift parameter that signifies changes in risk of future income. The optimal saving, s^*, solves the first-order condition for program (14):

$$-u'(\overline{x}-s^*)+\frac{1+r}{1+\rho^u}\int_a^b u'\Big((1+r)s^*+x\Big)\mathrm{d}F(x;\theta)=0. \tag{15}$$

The equilibrium interest rate, r^u, is the one at which $s^*=0$. Substituting $s^*=0$ into Eq. (15) yields

$$r^u=(1+\rho^u)u'(\overline{x})\Big[\int_a^b u'(x)\mathrm{d}F(x;\theta)\Big]^{-1}-1. \tag{16}$$

We focus on the case that an increase in θ indicates a simple increase in nth-degree risk in that $\partial F_k(b,\theta)/\partial\theta=0$ for all $k=1$, ..., n, $\partial F_{n-1}(x,\theta)/\partial\theta\geq 0$ for all $x\in[a,\overline{x}]$,

[14] Jindapon and Whaley (2015) complement the comparative statics result of Treich (2010) by showing that risk-loving and imprudent contestants optimally exert more rent-seeking effort than the risk-neutral equilibrium level.

$\partial F_{n-1}(x, \theta)/\partial\theta \leq 0$ for all $x \in [\overline{x}, b]$, and $\partial F_n(x, \theta)/\partial\theta \geq 0$ for all $x \in [a, b]$ (see Jindapon and Neilson, 2007). Differentiating Eq. (16) with respect to θ yields

$$\frac{\partial r^u}{\partial\theta} = -\frac{(1+r^u)^2}{(1+\rho^u)u'(\overline{x})}\int_a^b (-1)^{n+1}u^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial\theta}\mathrm{d}x$$

$$= -\frac{(1+r^u)^2}{(1+\rho^u)u'(\overline{x})}\int_a^b (-1)^{n+2}u^{(n+1)}(x)\frac{\partial F_n(x;\theta)}{\partial\theta}\mathrm{d}x \leq 0. \tag{17}$$

Hence, a simple increase in nth-degree risk of future income induces a decline in the equilibrium interest rate if $u(x)$ exhibits $(n+1)$th-degree risk aversion. Using Eq. (17), we obtain the elasticity of the equilibrium interest rate with respect to changes in risk of future income as follows:

$$\eta^u = -\left(\frac{\theta}{1+r^u}\right)\frac{\partial r^u}{\partial\theta} = \left(\frac{1+r^u}{1+\rho^u}\right)\frac{\theta}{u'(\overline{x})}\int_a^b (-1)^{n+1}u^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial\theta}\mathrm{d}x. \tag{18}$$

We now consider another exchange economy wherein the representative consumer, individual v, is more $(1, n+1)$th-degree mixed risk averse than individual u in the original economy. We refer to the former as economy v and the latter as economy u. In economy v, the equilibrium interest rate is given by

$$r^v = (1+\rho^v)v'(\overline{x})\left[\int_a^b v'(x)\mathrm{d}F(x;\theta)\right]^{-1} - 1, \tag{19}$$

and the elasticity of the equilibrium interest rate with respect to changes in risk of future income becomes

$$\eta^v = -\left(\frac{\theta}{1+r^v}\right)\frac{\partial r^v}{\partial\theta} = \left(\frac{1+r^v}{1+\rho^v}\right)\frac{\theta}{v'(\overline{x})}\int_a^b (-1)^{n+1}v^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial\theta}\mathrm{d}x. \tag{20}$$

The following proposition shows that the elasticity of the equilibrium interest rate with respect to changes in risk of future income, as characterized by simple increases in nth-degree risk, is greater in economy v than in economy u.

Proposition 6. For any integer, $n > 1$, and any two utility functions, $u(x)$ and $v(x)$, such that $u(x)$ exhibits $(1, n+1)$th-degree mixed risk aversion and $v(x) = \phi\big(u(x)\big)$ is more

$(1, n+1)$th-degree mixed risk averse than $u(x)$, if the current income, $\overline{x}$, solves the following equation:

$$\int_a^b \Big[\phi'\Big(u(x)\Big) - \phi'\Big(u(\overline{x})\Big)\Big] u'(x)\mathrm{d}F(x;\theta) = 0, \tag{21}$$

the elasticity of the equilibrium interest rate with respect to a simple increase in nth-degree risk of future income is greater in economy v than in economy u, i.e., $\eta^v \geq \eta^u \geq 0$.

Proof. See Appendix F. □

It follows from Eqs. (16) and (19) and the fact that $v(x) = \phi\Big(u(x)\Big)$ that Eq. (21) is equivalent to $(1 + r^u)/(1 + \rho^u) = (1 + r^v)/(1 + \rho^v)$. Hence, from Eqs. (18) and (20), we have $\eta^v \geq \eta^u$ if, and only if,

$$\frac{1}{v'(\overline{x})}\int_a^b (-1)^{n+1} v^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial\theta}\mathrm{d}x \geq \frac{1}{u'(\overline{x})}\int_a^b (-1)^{n+1} u^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial\theta}\mathrm{d}x,$$

which holds given that $u(x)$ exhibits $(1, n+1)$th-degree mixed risk aversion and $v(x)$ is more $(1, n+1)$th-degree mixed risk averse than $u(x)$. Proposition 6 as such extends the comparative statics results of Crainich and Eeckhoudt (2008) and Keenan and Snow (2016) to a simple increase in nth-degree risk of future income in the case wherein the two economies can be ranked by greater $(1, n+1)$th-degree mixed risk aversion.

4.3 Greater $(n-1, n)$th-degree mixed risk aversion and effort choices

We examine the effort choice problem of Jindapon and Neilson (2007) wherein individual u can exert effort, $e \in [0, 1]$, by incurring a monetary cost, $c(e)$, to shift the initial CDF of $\tilde{x}$ from $F(x)$ to a mixture, $H(x|e)$, where

$$H(x|e) = eG(x) + (1 - e)F(x), \tag{22}$$

for all $x \in [a, b]$. We assume that $c(e)$ satisfies that $c(0) = 0$, $c'(0) = 0$, and $c'(e) \geq 0$ and $c''(e) \geq 0$ for all $e \in [0, 1]$. Individual u prefers $G(x)$ to $F(x)$, and is non-satiated and risk averse so that $u'(x) \geq 0$ and $u''(x) \leq 0$ for all $x \in [a - c(1), b]$. By the betweenness property of expected utility, individual u prefers $H(x|e_2)$ to $H(x|e_1)$ for all e_1 and $e_2 \in [0, 1]$ such that $e_2 > e_1$.

Individual u's ex-ante decision problem is given by

$$\max_{e \in [0,1]} \int_a^b u\Big(x - c(e)\Big) \mathrm{d}H(x|e). \tag{23}$$

The first-order condition for program (23) is given by

$$\int_a^b u\Big(x - c(e_u)\Big) \mathrm{d}[G(x) - F(x)] - \int_a^b u'\Big(x - c(e_u)\Big) c'(e_u) \mathrm{d}H(x|e_u) = 0, \tag{24}$$

where e_u is individual u's optimal effort. The second-order condition for program (23) is given by

$$\begin{aligned} &-2\int_a^b u'\Big(x - c(e_u)\Big) c'(e_u) \mathrm{d}[G(x) - F(x)] \\ &- \int_a^b \Big[u'\Big(x - c(e_u)\Big) c''(e_u) - u''\Big(x - c(e_u)\Big) c'(e_u)^2\Big] \mathrm{d}H(x|e_u) \leq 0, \end{aligned} \tag{25}$$

which is satisfied given that $u'(x) \geq 0$ and $u''(x) \leq 0$ for all $x \in [a - c(1), b]$, and $c'(e) \geq 0$ and $c''(e) \geq 0$ for all $e \in [0, 1]$. Since individual u prefers $G(x)$ to $F(x)$, the first term on the left-hand side of Eq. (24) is positive. Hence, we have $e_u \geq 0$.

We write Eq. (24) as

$$\frac{\int_a^b u\Big(x - c(e_u)\Big) \mathrm{d}[G(x) - F(x)]}{\int_a^b u'\Big(x - c(e_u)\Big) \mathrm{d}H(x|e_u)} = c'(e_u), \tag{26}$$

so that the marginal benefit of effort on the left-hand side of Eq. (26) is normalized in a way that it is now expressed in monetary terms. Once this normalization has been made, it is legitimate for us to conduct interpersonal comparisons.

We now consider individual v who faces the same ex-ante decision problem as individual u. Individual v's optimal effort level, e_v, as such solves the following first-order condition:

$$\frac{\int_a^b v\Big(x - c(e_v)\Big)\mathrm{d}[G(x) - F(x)]}{\int_a^b v'\Big(x - c(e_v)\Big)\mathrm{d}H(x|e_v)} = c'(e_v). \tag{27}$$

We state and prove the following proposition.

Proposition 7. For any integer, $n > 1$, and any two utility functions, $u(x)$ and $v(x)$, such that $(-1)^n u^{(n-1)}(x) > 0$, $(-1)^{n+1}u^{(n)}(x) \geq 0$, and $(-1)^n v^{(n-1)}(x) > 0$ for all $x \in [a, b]$, the following two statements are equivalent:

(i) Individual v is more $(n-1, n)$th-degree mixed risk averse than individual u.

(ii) Individual v optimally exerts more effort than individual u, i.e., $e_v \geq e_u$, for all $F(x)$ and $G(x)$ such that $F_k(b) = G_k(b)$ for all $k = 1, ..., n-1$, and there exists a point, $x^\circ \in (a, b)$, at which $F_n(x^\circ) = G_n(x^\circ)$ such that $F_n(x^\circ) \geq G_n(x^\circ)$ for all $x \in [a, x^\circ]$, $F_{n-1}(x) \leq G_{n-1}(x)$ for all $x \in [x^\circ, b]$, and

$$\frac{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)} \geq \frac{\int_a^b v'\Big(x - c(e_u)\Big)\mathrm{d}H(x|e_u)}{\int_a^b u'\Big(x - c(e_u)\Big)\mathrm{d}H(x|e_u)}. \tag{28}$$

Proof. See Appendix G. □

As shown in Proposition 4, statement (ii) of Proposition 7 implies statement (iii) of Proposition 4, but not vice versa, so that individuals u and v both prefer $G(x)$ to $F(x)$. To ensure individual v to exert more effort than individual u, more stringent restrictions have to be imposed to $F(x)$ and $G(x)$. To illustrate the results of Proposition 7, we use the example of Jindapon and Neilson (2007), where $u(x) = -\exp(-7x)$, $v(x) = -\exp(-9x)$, $c(e) = e^2$, and $F(x) = x$ for all $x \in [0, 1]$. We modify $G(x)$ by a shift parameter, ε, which is non-negative:

$$G(x) = \begin{cases} 0 & \text{for all } x \in [0, 0.4 - \varepsilon], \\ 5x - 2 + 5\varepsilon & \text{for all } x \in [0.4 - \varepsilon, 0.6 - \varepsilon], \\ 1 & \text{for all } x \in [0.6 - \varepsilon, 1]. \end{cases}$$

Since individual u prefers $G(x)$ to $F(x)$, we must have $\varepsilon \in [0, 0.2104]$. When $\varepsilon = 0$, we have $e_u = 0.0576$ and $e_v = 0.0526$ so that $e_u > e_v$, which are the findings of Jindapon and Neilson (2007). When $\varepsilon \in (0, 0.2104]$, $G_2(x)$ crosses $F_2(x)$ from below at $x^\circ = 1 - \sqrt{2\varepsilon}$ for all $\varepsilon \in (0, 0.1528]$ and $x^\circ = (0.1 - 0.25\varepsilon)(5 + \sqrt{5})$ for all $\varepsilon \in (0.1528, 0.2104]$. It is easily shown that $e_u > e_v$ for all $\varepsilon \in (0, 0.0787)$, i.e., $x^\circ \in (0.6033, 1)$, and $e_u < e_v$ for all $\varepsilon \in (0.0787, 0.2104]$, i.e., $x^\circ \in [0.3430, 0.6033)$. When $\varepsilon = 0.0787$, we have $x^\circ = 0.6033$ and $e_u = e_v = 0.0442$. The restriction given by Eq. (28) ensures that $x^\circ \leq 0.6033$ so that $e_v \geq e_u$.

5. Conclusion

In this paper, we develop a unified framework of greater (m, n)th-degree mixed risk aversion, where $n > m \geq 1$, in the tradition of Arrow (1971) and Pratt (1964). We also define the (n/m)th-degree measure of absolute risk aversion, $A_u^{(n/m)}(x) = (-1)^{n-m}u^{(n)}(x)/u^{(m)}(x)$, for a utility function, $u(x)$. Our notion of greater (m, n)th-degree mixed risk aversion defines a strict partial ordering over utility functions, which includes many existing notions of greater higher-degree risk aversion as special cases. In addition, it has some nice properties. First, (m, n)th-degree mixed risk aversion embedded in the reference utility function, $u(x)$, is preserved when $u(x)$ is transformed into a more (m, n)th-degree mixed risk averse utility function, $v(x)$. Second, all measures of (k/m)th-degree absolute risk aversion for $v(x)$ are uniformly larger than those for $u(x)$, i.e., $A_v^{(k/m)}(x) \geq A_u^{(k/m)}(x)$ for all $k = m + 1$, ..., n. We also provide a choice-based characterization for greater (m, n)th-degree mixed risk aversion.

While our approach is completely general, two special cases that have already received some attention in the extant literature are in order. When $m = 1$, $v(x)$ is more $(1, n)$th-degree mixed risk averse than $u(x)$ if there exists a transformation function, $\phi(y)$, such that $v(x) = \phi\Big(u(x)\Big)$ and the successive derivatives of $\phi(y)$ are alternate in sign, which includes the notion of strong increases in downside risk aversion proposed in Keenan and Snow (2016) as a special case wherein $n = 3$. When $m = n - 1$, $v(x)$ is more $(n - 1, n)$th-degree mixed risk averse than $u(x)$ if $A_v^{(n/n-1)}(x) \geq A_u^{(n/n-1)}(x)$, corresponding to the greater nth-degree Arrow-Pratt risk aversion proposed in Denuit and Eeckhoudt (2010a), Huang et al. (2017), and Jindapon and Neilson (2007), and including the special case of $n = 3$ considered in Chiu (2005).

We apply the notion of greater (m, n)th-degree mixed risk aversion to three well-known comparative statics problems. First, we consider a standard rent-seeking game and show that contestants optimally exert less rent-seeking effort should they become more $(1, 3)$th-degree mixed risk averse. Second, we consider a general equilibrium model that determines the equilibrium interest rate. We show that the elasticity of the equilibrium interest rate with respect to a simple increase in nth-degree risk of future income is greater in an economy that is more $(1, n)$th-degree mixed risk averse than the reference economy. Finally, we consider the effort choice problem of Jindapon and Neilson (2007) wherein individuals can shift from an initial payoff distribution to a preferred payoff distribution at a monetary cost. We derive necessary and sufficient conditions on the distributions under which individuals who are more $(n - 1, n)$th-degree mixed risk averse than the reference individual always exert more effort. These applications illustrate that the notion of greater (m, n)th-degree mixed risk aversion is useful in yielding intuitive comparative statics results, thereby making it appealing to many other decision problems under uncertainty.

Two interesting research issues arise from our work here, which may prove worthwhile for future research. First, it is noted in the introduction that Arrow (1971) and Pratt (1964) establish statements (i), (ii), and (iii) as equivalent ways of ordering utility functions according to (second-degree) risk aversion, and that our paper provides a unified generalization of statements (ii) and (iii) to the notion of more (m,n)th-degree mixed risk averse (Definition 3). Then a natural question is: what connection can be made between the notion of more (m,n)th-degree mixed risk averse and statement (i)? More specifically, is a more (m,n)th-degree mixed risk averse individual always willing to pay a larger risk premium to avoid an (m,n)th-degree risk increase (Definition 2), provided that some restrictions are placed on the (m,n)th-degree risk increase? Second, we have focused in this paper on providing a comprehensive generalization of the Arrow-Pratt measure of second-degree absolute risk aversion to higher orders. A parallel comprehensive generalization of the Arrow-Pratt measure of second-degree relative risk aversion to higher orders seems to be in order. Such a comprehensive generalization would shed new light on some recent studies on higher-order relative risk aversion in specific contexts (Chiu et al., 2012; Denuit and Rey, 2014; and Huang et al., 2017).

Appendix

A. Proof of Lemma 2

Irreflexivity follows because $(-1)^{m+1}u^{(m-1)}(x) = \phi\Big((-1)^{m+1}u^{(m-1)}(x)\Big)$ only if $\phi(y)$ is the identity function, which obviously violates the conditions for being an increase in (m,n)th-degree mixed risk aversion. For a strict partial ordering, it is well-known that irreflexivity and transitivity imply anti-symmetry. To complete the proof, we remain to show transitivity.

Suppose that $v_1(x)$ is more (m,n)th-degree mixed risk averse than $u(x)$ with respect to a transformation function, $\phi_1(y)$, for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$, and that $v_2(x)$ is more (m,n)th-degree mixed risk averse than $v_1(x)$ with respect to a transformation function, $\phi_2(y)$, for all $y \in [(-1)^{m+1}v_1^{(m-1)}(a), (-1)^{m+1}v_1^{(m-1)}(b)]$. We need to show that $v_2(x)$ is more (m,n)th-degree mixed risk averse than $u(x)$ with respect to the transformation function, $\phi_3(y) = \phi_2\Big(\phi_1(y)\Big)$, for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$. Consider a vector of non-negative integers, $(j_1, ..., j_k)$. For any integer, $j \in [1, k]$, we refer to $(j_1, ..., j_k)$

as a (k,j)-multiindex if $\sum_{i=1}^{k} j_i = j$ and $\sum_{i=1}^{k} ij_i = k$. Let $\Omega(k,j)$ be the set of all (k,j)-multiindexes. For example, $\Omega(1,1) = \{(1)\}$, $\Omega(2,1) = \{(0,1)\}$, $\Omega(2,2) = \{(2,0)\}$, and $\Omega(4,2) = \{(1,0,1,0),(0,2,0,0)\}$. According to the formula of Faà di Bruno (see, e.g., Johnson, 2002; Slevinsky and Safouhi, 2009), we have

$$\phi_3^{(k)}(y) = \sum_{j=1}^{k} \phi_2^{(j)}\Big(\phi_1(y)\Big) \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! \prod_{i=1}^{k} \frac{1}{j_i!}\Big[\frac{\phi_1^{(i)}(y)}{i!}\Big]^{j_i}. \tag{A.1}$$

Using Eq. (A.1), we have

$$\begin{aligned}
(-1)^{k+1}\phi_3^{(k)}(y) &= (-1)^{k+1}\sum_{j=1}^{k} \phi_2^{(j)}\Big(\phi_1(y)\Big) \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! \prod_{i=1}^{k} \frac{1}{j_i!}\Big[\frac{\phi_1^{(i)}(y)}{i!}\Big]^{j_i} \\
&= (-1)^{k+1}\sum_{j=1}^{k} \phi_2^{(j)}\Big(\phi_1(y)\Big) \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! \prod_{i=1}^{k} (-1)^{j_i - ij_i}\frac{1}{j_i!}\Big[\frac{(-1)^{i-1}\phi_1^{(i)}(y)}{i!}\Big]^{j_i} \\
&= (-1)^{k+1}\sum_{j=1}^{k} \phi_2^{(j)}\Big(\phi_1(y)\Big) \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! (-1)^{\sum_{i=1}^{k} j_i - ij_i} \prod_{i=1}^{k} \frac{1}{j_i!}\Big[\frac{(-1)^{i-1}\phi_1^{(i)}(y)}{i!}\Big]^{j_i} \\
&= \sum_{j=1}^{k} (-1)^{j+1}\phi_2^{(j)}\Big(\phi_1(y)\Big) \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! \prod_{i=1}^{k} \frac{1}{j_i!}\Big[\frac{(-1)^{i-1}\phi_1^{(i)}(y)}{i!}\Big]^{j_i},
\end{aligned} \tag{A.2}$$

where the last equality follows from the fact that $\sum_{i=1}^{k} j_i = j$ and $\sum_{i=1}^{k} ij_i = k$. Since $(-1)^{k+1}\phi_1^{(k)}(y) \geq 0$ for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$ and $(-1)^{k+1}\phi_2^{(k)}(y) \geq 0$ for all $y \in [(-1)^{m+1}v_1^{(m-1)}(a), (-1)^{m+1}v_1^{(m-1)}(b)]$ and for all $k = 1, \ldots, n-m+1$, it follows from Eq. (A.2) that $(-1)^{k+1}\phi_3^{(k)}(y) \geq 0$ for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$ and for all $k = 1, \ldots, n-m+1$. We as such conclude that $v_2(x)$ is more (m,n)th-degree mixed risk averse than $u(x)$.

B. Proof of Proposition 1

According to the formula of Faà di Bruno (see, e.g., Johnson, 2002; Slevinsky and Safouhi, 2009), we have

$$(-1)^{m+1}v^{(m+k-1)}(x)$$

$$= \sum_{j=1}^{k} \phi^{(j)}\Big((-1)^{m+1}u^{(m-1)}(x)\Big) \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! \prod_{i=1}^{k} \frac{1}{j_i!}\left[\frac{(-1)^{m+1}u^{(m+i-1)}(x)}{i!}\right]^{j_i}. \quad \text{(A.3)}$$

Using Eq. (A.3), we have

$$\begin{aligned}
&(-1)^{m+k}v^{(m+k-1)}(x)\\
&= (-1)^{k-1}\sum_{j=1}^{k} \phi^{(j)}\Big((-1)^{m+1}u^{(m-1)}(x)\Big)\\
&\qquad \times \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! \prod_{i=1}^{k} \frac{1}{j_i!}\left[\frac{(-1)^{m+1}u^{(m+i-1)}(x)}{i!}\right]^{j_i}\\
&= (-1)^{k-1}\sum_{j=1}^{k} \phi^{(j)}\Big((-1)^{m+1}u^{(m-1)}(x)\Big)\\
&\qquad \times \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! \prod_{i=1}^{k} (-1)^{j_i - ij_i}\frac{1}{j_i!}\left[\frac{(-1)^{m+i}u^{(m+i-1)}(x)}{i!}\right]^{j_i}\\
&= (-1)^{k-1}\sum_{j=1}^{k} \phi^{(j)}\Big((-1)^{m+1}u^{(m-1)}(x)\Big)\\
&\qquad \times \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! (-1)^{\sum_{i=1}^{k} j_i - ij_i} \prod_{i=1}^{k} \frac{1}{j_i!}\left[\frac{(-1)^{m+i}u^{(m+i-1)}(x)}{i!}\right]^{j_i}\\
&= \sum_{j=1}^{k} (-1)^{j-1}\phi^{(j)}\Big((-1)^{m+1}u^{(m-1)}(x)\Big)\\
&\qquad \times \sum_{(j_1,\ldots,j_k)\in\Omega(k,j)} k! \prod_{i=1}^{k} \frac{1}{j_i!}\left[\frac{(-1)^{m+i}u^{(m+i-1)}(x)}{i!}\right]^{j_i}, \qquad \text{(A.4)}
\end{aligned}$$

where the last equality follows from the fact that $\sum_{i=1}^{k} j_i = j$ and $\sum_{i=1}^{k} ij_i = k$. Since $(-1)^{k+1}\phi^{(k)}(y) \geq 0$ for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$ for all $k = 1$, ..., $n-m+1$, and $(-1)^{k+1}u^{(k)}(x) \geq 0$ for all $x \in [a,b]$ and for all $k = m$, ..., n, it follows from Eq. (A.4) that $(-1)^{k+1}v^{(k)}(x) \geq 0$ for all $x \in [a,b]$ and for all $k = m$, ..., n.

Since $\Omega(k,1)=\{(0,...,0,1)\}$, it follows from Eq. (A.4) that

$$(-1)^{m+k}v^{(m+k-1)}(x)=\phi'\Big((-1)^{m+1}u^{(m-1)}(x)\Big)(-1)^{m+k}u^{(m+k-1)}(x)$$

$$+\sum_{j=2}^{k}(-1)^{j-1}\phi^{(j)}\Big((-1)^{m+1}u^{(m-1)}(x)\Big)$$

$$\times\sum_{(j_1,\ldots,j_k)\in\Omega(k,j)}k!\prod_{i=1}^{k}\frac{1}{j_i!}\left[\frac{(-1)^{m+i}u^{(m+i-1)}(x)}{i!}\right]^{j_i}, \tag{A.5}$$

for $k\geq 2$. Given that $(-1)^{m+1}u^{(m)}(x)>0$ and $(-1)^{m+1}v^{(m)}(x)>0$ for all $x\in[a,b]$, we divide Eq. (A.5) by $(-1)^{m+1}v^{(m)}(x)=\phi'\Big((-1)^{m+1}u^{(m-1)}(x)\Big)(-1)^{m+1}u^{(m)}(x)$ to yield

$$(-1)^{k-1}\frac{v^{(m+k-1)}(x)}{v^{(m)}(x)}=(-1)^{k-1}\frac{u^{(m+k-1)}(x)}{u^{(m)}(x)}$$

$$+\sum_{j=2}^{k}\frac{(-1)^{j-1}\phi^{(j)}\Big((-1)^{m+1}u^{(m-1)}(x)\Big)}{\phi'\Big((-1)^{m+1}u^{(m-1)}(x)\Big)(-1)^{m+1}u^{(m)}(x)}$$

$$\times\sum_{(j_1,\ldots,j_k)\in\Omega(k,j)}k!\prod_{i=1}^{k}\frac{1}{j_i!}\left[\frac{(-1)^{m+i}u^{(m+i-1)}(x)}{i!}\right]^{j_i}. \tag{A.6}$$

Eq. (A.6) implies that $A_v^{(k/m)}(x)\geq A_u^{(k/m)}(x)$ for all $x\in[a,b]$ and for all $k=m+1,\ ...,\ n$.

C. Proof of Proposition 2

(ii) $\Rightarrow$ (i). Applying integration by parts yields

$$\int_a^b v(x)\mathrm{d}[G(x)-F(x)]$$

$$=\int_a^b(-1)^m v^{(m-1)}(x)[F_{m-1}(x)-G_{m-1}(x)]\mathrm{d}x$$

$$=\int_a^b-\phi\Big((-1)^{m+1}u^{(m-1)}(x)\Big)[F_{m-1}(x)-G_{m-1}(x)]\mathrm{d}x$$

$$= \sum_{k=1}^{n-m} (-1)^{k+1} \phi^{(k)}\Big((-1)^{m+1} u^{(m-1)}(b)\Big) T_k(b;m)$$

$$+ \int_a^b (-1)^{n-m} \phi^{(n-m+1)}\Big((-1)^{m+1} u^{(m-1)}(x)\Big)(-1)^{m+1} u^{(m)}(x) T_{n-m}(x;m) \mathrm{d}x, \quad \text{(A.7)}$$

where the first equality follows from $F_k(b) = G_k(b)$ for all $k = 1, ..., m-1$, the second equality follows from Eq. (4), and the last equality follows from integration by parts, $F_m(b) = G_m(b)$, and Eq. (5). Given that $(-1)^{k+1}\phi^{(k)}(y) \geq 0$ for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$ and for all $k = 1, ..., n-m+1$, $T_k(b;m) \geq 0$ for all $k = 1, ..., n-m$, and $T_{n-m}(x;m) \geq 0$ for all $x \in [a,b]$, the right-hand side of Eq. (A.7) is non-negative and thus individual v prefers $G(x)$ to $F(x)$.

(i) $\Rightarrow$ (ii). It is trivial that $F(b) = G(b)$. To show that $F_2(b) = G_2(b)$ when $m \geq 2$, we suppose the contrary that $F_2(b) < (>)\ G_2(b)$. Consider $\hat{v}(x)$ with $\hat{v}'(x) = v'(x) + c$ for all $x \in [a,b]$, where $v(x)$ is more (m,n)th-degree mixed risk averse than $u(x)$, and c is a positive (negative) constant. It follows that $\hat{v}(x)$ is also more (m,n)th-degree mixed risk averse than $u(x)$ and

$$\int_a^b \hat{v}(x)\mathrm{d}[G(x) - F(x)] = \int_a^b v(x)\mathrm{d}[G(x) - F(x)] + c[F_2(b) - G_2(b)]. \quad \text{(A.8)}$$

Since $c[F_2(b) - G_2(b)] < 0$, we can choose an appropriate value for c such that the right-hand side of Eq. (A.8) is negative, a contradiction. Hence, it must be true that $F_2(b) = G_2(b)$. Repeating this argument, we verify that $F_k(b) = G_k(b)$ for all $k = 1, ..., m$.

$T_1(b;m) \geq 0$ follows from Eq. (6) by setting $u(x) = v(x)$. To show that $T_k(b;m) \geq 0$ for all $k = 2, ..., n-m$, we suppose the contrary that $T_k(b;m) < 0$ for some $k \in [2, n-m]$. Consider $v(x)$ that satisfies Eq. (4) with the following transformation function:

$$\phi(y) = \frac{1}{k!}(-1)^{k+1}[y - (-1)^{m+1}u^{(m-1)}(b)]^k, \quad \text{(A.9)}$$

for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$. It is easily verified that for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$ we have

$$(-1)^{j+1}\phi^{(j)}(y) = \frac{1}{(k-j)!}[(-1)^{m+1}u^{(m-1)}(b) - y]^{k-j} \geq 0, \quad \text{(A.10)}$$

for all $j = 1, ..., k$ and $(-1)^{j+1}\phi^{(j)}(y) = 0$ for all $j = k+1, ..., n-m+1$. Hence, $v(x)$ is more (m, n)th-degree mixed risk averse than $u(x)$. Since $(-1)^{j+1}\phi^{(j)}\Big((-1)^{m+1}u^{(m-1)}(b)\Big) = 0$ for all $j = 1, ..., k-1$ and $(-1)^{k+1}\phi^{(k)}(y) = 1$ for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$, it follows from Eq. (A.7) that

$$\int_a^b v(x)\mathrm{d}[G(x) - F(x)] = \int_a^b (-1)^{m+1}u^{(m)}(x)T_{k-1}(x; m)\mathrm{d}x = T_k(b; m) < 0,$$

a contradiction. Hence, it must be true that $T_k(b; m) \geq 0$ for all $k = 2, ..., n - m$.

Finally, to show that $T_{n-m}(x; m) \geq 0$ for all $x \in [a, b]$, we suppose the contrary that there exists a point, $\hat{x} \in [a, b]$, at which $T_{n-m}(\hat{x}; m) < 0$. By continuity, we have $T_{n-m}(x; m) < 0$ for all $x \in [\hat{x} - \varepsilon_1, \hat{x} + \varepsilon_2]$, where ε_1 and ε_2 are two small non-negative numbers. Consider $v(x)$ that satisfies Eq. (4) with the following transformation function:

$$\phi(y) = \int_{(-1)^{m+1}u^{(m-1)}(a)}^{y} \int_{t_1}^{(-1)^{m+1}u^{(m-1)}(b)} \cdots \int_{t_{n-m}}^{(-1)^{m+1}u^{(m-1)}(b)}$$

$$\times 1_{[(-1)^{m+1}u^{(m-1)}(\hat{x}-\varepsilon_1),(-1)^{m+1}u^{(m-1)}(\hat{x}+\varepsilon_2)]}(t_{n-m+1})\mathrm{d}t_{n-m+1}\cdots\mathrm{d}t_2\mathrm{d}t_1, \qquad \text{(A.11)}$$

for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$, where $1_{[y_1,y_2]}(y)$ is the indicator function such that $1_{[y_1,y_2]}(y) = 1$ for all $y \in [y_1, y_2]$ and $1_{[y_1,y_2]}(y) = 0$ for all $y \notin [y_1, y_2]$. It follows from Eq. (A.11) that

$$\phi^{(k)}(y) = (-1)^{k-1} \int_{y}^{(-1)^{m+1}u^{(m-1)}(b)} \int_{t_{k+1}}^{(-1)^{m+1}u^{(m-1)}(b)} \cdots \int_{t_{n-m}}^{(-1)^{m+1}u^{(m-1)}(b)}$$

$$\times 1_{[(-1)^{m+1}u^{(m-1)}(\hat{x}-\varepsilon_1),(-1)^{m+1}u^{(m-1)}(\hat{x}+\varepsilon_2)]}(t_{n-m+1})\mathrm{d}t_{n-m+1}\cdots\mathrm{d}t_{k+2}\mathrm{d}t_{k+1}, \qquad \text{(A.12)}$$

for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$ and for all $k = 1, ..., n - m$, and

$$\phi^{(n-m+1)}(y) = (-1)^{n-m}1_{[(-1)^{m+1}u^{(m-1)}(\hat{x}-\varepsilon_1),(-1)^{m+1}u^{(m-1)}(\hat{x}+\varepsilon_2)]}(y), \qquad \text{(A.13)}$$

for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$. It is evident from Eqs. (A.12) and (A.13) that $(-1)^{k+1}\phi^{(k)}(y) \geq 0$ for all $y \in [(-1)^{m+1}u^{(m-1)}(a), (-1)^{m+1}u^{(m-1)}(b)]$ and for all $k = 1$, ..., $n-m+1$. Hence, $v(x)$ is more (m, n)th-degree mixed risk averse than $u(x)$. Eq. (A.11) implies that $\phi^{(k)}\Big((-1)^{m+1}u^{(m-1)}(b)\Big) = 0$ for all $k = 1, ..., n - m$, and Eq. (A.13) implies that $\phi^{(n-m+1)}(y) = (-1)^{n-m}$ for all $y \in [(-1)^{m+1}u^{(m-1)}(\hat{x}-\varepsilon_1), (-1)^{m+1}u^{(m-1)}(\hat{x}+\varepsilon_2)]$ and $\phi^{(n-m+1)}(y) = 0$ for all $y \notin [(-1)^{m+1}u^{(m-1)}(\hat{x}-\varepsilon_1), (-1)^{m+1}u^{(m-1)}(\hat{x}+\varepsilon_2)]$. It then follows from Eq. (A.7) that

$$\int_a^b v(x)\mathrm{d}[G(x)-F(x)] = \int_{\hat{x}-\varepsilon_1}^{\hat{x}+\varepsilon_2} (-1)^{m+1}u^{(m)}(x)T_{n-m}(x;m)\mathrm{d}x < 0,$$

a contradiction. Hence, it must be true that $T_{n-m}(x;m) \geq 0$ for all $x \in [a,b]$.

D. Proof of Proposition 4

Statements (i) and (iii) of Proposition 4 are equivalent, which follows from Proposition 2 by setting $m = n-1$. To show that statements (i) and (ii) are equivalent, we consider a transformation function, $\phi(y)$, such that $(-1)^n v^{(n-2)}(x) = \phi\Big((-1)^n u^{(n-2)}(x)\Big)$. Then, we have

$$(-1)^n v^{(n-1)}(x) = \phi'\Big((-1)^n u^{(n-2)}(x)\Big)(-1)^n u^{(n-1)}(x). \tag{A.14}$$

Since $(-1)^n u^{(n-1)}(x) > 0$ and $(-1)^n v^{(n-1)}(x) > 0$ for all $x \in [a,b]$, it follows from Eq. (A.14) that $\phi'(y) > 0$ for all $y \in [(-1)^n u^{(n-2)}(a), (-1)^n u^{(n-2)}(b)]$. Furthermore, we have

$$\begin{aligned}(-1)^n v^{(n)}(x) = \phi''\Big((-1)^n u^{(n-2)}(x)\Big)(-1)^{2n} u^{(n-1)}(x)^2 \\ +\phi'\Big((-1)^n u^{(n-2)}(x)\Big)(-1)^n u^{(n)}(x).\end{aligned} \tag{A.15}$$

Using Eqs. (A.14) and (A.15), we have

$$\begin{aligned}&\phi''\Big((-1)^n u^{(n-2)}(x)\Big)(-1)^n u^{(n-1)}(x) \\ &= -\phi'\Big((-1)^n u^{(n-2)}(x)\Big)[A_v^{(n/n-1)}(x) - A_u^{(n/n-1)}(x)].\end{aligned} \tag{A.16}$$

It then follows from Eq. (A.16) that $\phi''(y) \leq 0$ for all $y \in [(-1)^n u^{(n-2)}(a), (-1)^n u^{(n-2)}(b)]$ if, and only if, $A_v^{(n/n-1)}(x) \geq A_u^{(n/n-1)}(x)$ for all $x \in [a,b]$. Hence, statements (i) and (ii) of Proposition 4 are equivalent.

E. Proof of Proposition 5

Consider the following function:

$$\Phi(e) = \frac{\partial p_i(e_1, ..., e_N)}{\partial e_i}\Big|_{(e_1,...,e_N)=(e,...,e)} \Big[\phi\Big(u(x_\circ + \beta - e)\Big) - \phi\Big(u(x_\circ - e)\Big)\Big]$$

$$-\Big[\frac{1}{N}\phi'\Big(u(x_\circ + \beta - e)\Big)u'(x_\circ + \beta - e) + \Big(1 - \frac{1}{N}\Big)\phi'\Big(u(x_\circ - e)\Big)u'(x_\circ - e)\Big]. \quad \text{(A.17)}$$

Given that the symmetric Nash equilibrium is unique, $\Phi(e)$ must be a strictly decreasing function (Treich, 2010). Since Eq. (13) is equivalent to $\Phi(x_v) = 0$, we have $e_v \leq e_u$ if, and only if, $\Phi(e_u) \leq 0$.

Consider the following function:

$$\Psi(p) = \frac{p\phi'\Big(u(x_\circ + \beta - e_u)\Big)u'(x_\circ + \beta - e_u) + (1-p)\phi'\Big(u(x_\circ - e_u)\Big)u'(x_\circ - e_u)}{pu'(x_\circ + \beta - e_u) + (1-p)u'(x_\circ - e_u)}. \quad \text{(A.18)}$$

From Eq. (12), we have

$$\frac{\partial p_i(e_1, ..., e_N)}{\partial e_i}\Big|_{(e_1,...,e_N)=(e_u,...,e_u)} = \frac{\frac{1}{N}u'(x_\circ + \beta - e_u) + \Big(1 - \frac{1}{N}\Big)u'(x_\circ - e_u)}{u(x_\circ + \beta - e_u) - u(x_\circ - e_u)}. \quad \text{(A.19)}$$

Using Eqs. (A.17), (A.18), and (A.19), we have

$$\Phi(e_u) = \frac{\partial p_i(e_1, ..., e_N)}{\partial e_i}\Big|_{(e_1,...,e_N)=(e_u,...,e_u)} \int_{u(x_\circ - e_u)}^{u(x_\circ + \beta - e_u)} [\phi'(y) - \Psi(1/N)]\mathrm{d}y. \quad \text{(A.20)}$$

Differentiating Eq. (A.18) with respect to p yields

$$\Psi'(p) = \frac{u'(x_\circ + \beta - e_u)u'(x_\circ - e_u)\Big[\phi'\Big(u(x_\circ + \beta - e_u)\Big) - \phi'\Big(u(x_\circ - e_u)\Big)\Big]}{[pu'(x_\circ + \beta - e_u) + (1-p)u'(x_\circ - e_u)]^2} \leq 0, \quad \text{(A.21)}$$

since $\phi''(y) \leq 0$. It then follows from Eq. (A.21) and $1/N \leq 1/2$ that $\Psi(1/N) \geq \Psi(1/2)$. Note that

$$\Psi(1/2) - \frac{1}{2}\Big[\phi'\Big(u(x_\circ + \beta - e_u)\Big) + \phi'\Big(u(x_\circ - e_u)\Big)\Big]$$

$$= \frac{[u'(x_\circ + \beta - e_u) - u'(x_\circ - e_u)]\Big[\phi'\Big(u(x_\circ + \beta - e_u)\Big) - \phi'\Big(u(x_\circ - e_u)\Big)\Big]}{2[u'(x_\circ + \beta - e_u) + u'(x_\circ - e_u)]} \geq 0, \quad \text{(A.22)}$$

since $u''(x) \leq 0$ and $\phi''(y) \leq 0$. It then follows from $\phi'''(y) \geq 0$ and Lemma 1 of Eeckhoudt and Gollier (2005) that

$$\int_{u(x_\circ - e_u)}^{u(x_\circ+\beta-e_u)} \Big\{\frac{1}{2}\Big[\phi'\Big(u(x_\circ + \beta - e_u)\Big) + \phi'\Big(u(x_\circ - e_u)\Big)\Big] - \phi'(y)\Big\}\mathrm{d}y \geq 0. \quad \text{(A.23)}$$

Hence, Eqs. (A.20), (A.22), and (A.23) imply that $\Phi(e_u) \leq 0$ so that $e_v \leq e_u$.

F. Proof of Proposition 6

Let $v(x) = \phi\Big(u(x)\Big)$. Then, we have

$$\begin{aligned}
\frac{\partial r^v}{\partial \theta} &= -\frac{(1+r^v)^2}{(1+\rho^v)v'(\overline{x})}\int_a^b (-1)^{n+1} v^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial \theta}\mathrm{d}x \\
&= -\frac{(1+r^v)^2}{(1+\rho^v)\phi'\Big(u(\overline{x})\Big)u'(\overline{x})}\int_a^b \Big\{ - n\phi''\Big(u(x)\Big)u'(x)(-1)^{n+1}u^{(n)}(x) \\
&\quad -\phi''\Big(u(x)\Big) \sum_{(j_1,\ldots,j_{n+1})\in\Omega(n+1,2)\setminus(1,0,\ldots,0,1,0)} (n+1)! \prod_{i=1}^{n+1} \frac{1}{j_i!}\Big[\frac{(-1)^{i+1}u^{(i)}(x)}{i!}\Big]^{j_i} \\
&\quad + \sum_{j=3}^{n+1}(-1)^{j-1}\phi^{(j)}\Big(u(x)\Big) \sum_{(j_1,\ldots,j_{n+1})\in\Omega(n+1,j)} (n+1)! \prod_{i=1}^{n+1} \frac{1}{j_i!}\Big[\frac{(-1)^{i+1}u^{(i)}(x)}{i!}\Big]^{j_i}\Big\}\frac{\partial F_n(x;\theta)}{\partial \theta}\mathrm{d}x \\
&\quad -\frac{(1+r^v)^2}{(1+\rho^v)\phi'\Big(u(\overline{x})\Big)u'(\overline{x})}\int_a^b \phi'\Big(u(x)\Big)(-1)^{n+1}u^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial \theta}\mathrm{d}x. \quad \text{(A.24)}
\end{aligned}$$

The first term on the right-hand side of Eq. (A.24) is negative given that $u(x)$ exhibits $(1, n+1)$th-degree mixed risk aversion, $v(x)$ is more $(1, n+1)$th-degree mixed risk averse than $u(x)$, and $\partial F_n(x, \theta)/\partial\theta \geq 0$ for all $x \in [a, b]$. Using Eq. (20), we have

$$\eta^v \geq \Big(\frac{1+r^v}{1+\rho^v}\Big)\frac{\theta}{u'(\overline{x})}\int_a^b \frac{\phi'\Big(u(x)\Big)}{\phi'\Big(u(\overline{x})\Big)}(-1)^{n+1}u^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial \theta}\mathrm{d}x. \quad \text{(A.25)}$$

Since $\phi''(y) \leq 0$, we have

$$\frac{\phi'\Big(u(x)\Big)}{\phi'\Big(u(\overline{x})\Big)}(-1)^{n+1}u^{(n)}(x) \geq (\leq)\ (-1)^{n+1}u^{(n)}(x) \tag{A.26}$$

for all $x \leq (\geq)\ \overline{x}$. Since $\partial F_{n-1}(x,\theta)/\partial\theta \geq 0$ for all $x \in [a,\overline{x}]$ and $\partial F_{n-1}(x,\theta)/\partial\theta \leq 0$ for all $x \in [\overline{x}, b]$, it follows from Eq. (A.26) that

$$\int_a^b \frac{\phi'\Big(u(x)\Big)}{\phi'\Big(u(\overline{x})\Big)}(-1)^{n+1}u^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial\theta}\mathrm{d}x$$

$$\geq \int_a^b (-1)^{n+1}u^{(n)}(x)\frac{\partial F_{n-1}(x;\theta)}{\partial\theta}\mathrm{d}x. \tag{A.27}$$

Hence, we conclude from Eqs. (18), (A.25), and (A.27) that $\eta^v \geq \eta^u \geq 0$.

G. Proof of Proposition 7

(i) $\Rightarrow$ (ii). Applying integration by parts, we have

$$\int_a^b \left[\frac{v\Big(x - c(e_u)\Big)}{(-1)^n v^{(n-1)}\Big(x^\circ - c(e_u)\Big)} - \frac{u\Big(x - c(e_u)\Big)}{(-1)^n u^{(n-1)}\Big(x^\circ - c(e_u)\Big)}\right]\mathrm{d}[G(x) - F(x)]$$

$$= \int_a^{x^\circ} \left[\frac{u^{(n)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)} - \frac{v^{(n)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)}\right][F_n(x) - G_n(x)]\mathrm{d}x$$

$$+ \int_{x^\circ}^b \left[\frac{v^{(n-1)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)} - \frac{u^{(n-1)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)}\right][F_{n-1}(x) - G_{n-1}(x)]\mathrm{d}x, \tag{A.28}$$

where we have used $F_k(b) = G_k(b)$ for all $k = 1,\ ...,\ n-1$ and $F_n(x^\circ) = G_n(x^\circ)$. Since $A_v^{(n/n-1)}\Big(x - c(e_u)\Big) \geq A_u^{(n/n-1)}\Big(x - c(e_u)\Big)$ for all $x \in [a,b]$ and $x^\circ \in (a,b)$, Jindapon and Neilson (2007) show that

$$\frac{v^{(n-1)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)} \geq (\leq)\ \frac{u^{(n-1)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)}, \tag{A.29}$$

for all $x \leq (\geq)\ x^\circ$. Since $F_{n-1}(x) \leq G_{n-1}(x)$ for all $x \in [x^\circ, b]$, it follows from Eq. (A.29) that the second term on the right-hand side of Eq. (A.28) is non-negative. For all $x \in [a, x^\circ]$, we have

$$-\frac{v^{(n)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)} = A_v^{(n/n-1)}\Big(x - c(e_u)\Big) \times \frac{v^{(n-1)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)}$$

$$\geq A_u^{(n/n-1)}\Big(x - c(e_u)\Big) \times \frac{u^{(n-1)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)} = -\frac{u^{(n)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)}, \tag{A.30}$$

where the inequality follows from $A_v^{(n/n-1)}\Big(x-c(e_u)\Big) \geq A_u^{(n/n-1)}\Big(x-c(e_u)\Big)$ for all $x \in [a, b]$ and Eq. (A.29). Since $F_n(x) \geq G_n(x)$ for all $x \in [a, x^\circ]$, it follows from Eq. (A.30) that the first term on the right-hand side of Eq. (A.28) is non-negative. It then follows from Eqs. (28) and (A.28) that

$$\frac{\int_a^b v\Big(x - c(e_u)\Big)\mathrm{d}[G(x) - F(x)]}{\int_a^b v'\Big(x - c(e_u)\Big)\mathrm{d}H(x|e_u)} \geq \frac{\int_a^b u\Big(x - c(e_u)\Big)\mathrm{d}[G(x) - F(x)]}{\int_a^b u'\Big(x - c(e_u)\Big)\mathrm{d}H(x|e_u)} = c'(e_u), \tag{A.31}$$

where the equality follows from Eq. (26). It then follows from Eqs. (25), (27), and (A.31) that $e_v \geq e_u$.

(ii) $\Rightarrow$ (i). Suppose the contrary that there exists a point, $\hat{x} \in [a, b]$, at which $A_v^{(n/n-1)}\Big(\hat{x} - c(e_u)\Big) < A_u^{(n/n-1)}\Big(\hat{x} - c(e_u)\Big)$. By continuity, we have $A_v^{(n/n-1)}\Big(x - c(e_u)\Big) < A_u^{(n/n-1)}\Big(x - c(e_u)\Big)$ for all $x \in [\hat{x} - \varepsilon_1, \hat{x} + \varepsilon_2]$, where ε_1 and ε_2 are two small non-negative numbers. We construct two CDFs of $\tilde{x}$, $F(x)$ and $G(x)$, over support $[\hat{x} - \varepsilon_1, \hat{x} + \varepsilon_2]$. Set $F_k(\hat{x} + \varepsilon_2) = G_k(\hat{x} + \varepsilon_2)$ for all $k = 1, ..., n-1$. There is a point, $x^\circ \in (\hat{x} - \varepsilon_1, \hat{x} + \varepsilon_2)$, that solves

$$\frac{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)} = \frac{\int_{\hat{x}-\varepsilon_1}^{\hat{x}+\varepsilon_2} v'\Big(x - c(e_u)\Big)\mathrm{d}H(x|e_u)}{\int_{\hat{x}-\varepsilon_1}^{\hat{x}+\varepsilon_2} u'\Big(x - c(e_u)\Big)\mathrm{d}H(x|e_u)}, \tag{A.32}$$

at which $F_n(x^\circ) = G_n(x^\circ)$ such that $F_n(x) \geq G_n(x)$ for all $x \in [\hat{x} - \varepsilon_1, x^\circ]$ and $F_{n-1}(x) \leq G_{n-1}(x)$ for all $x \in [x^\circ, \hat{x} + \varepsilon_2]$. Applying integration by parts yields

$$\int_{\hat{x}-\varepsilon_1}^{\hat{x}+\varepsilon_2} \left[\frac{v\Big(x - c(e_u)\Big)}{(-1)^n v^{(n-1)}\Big(x^\circ - c(e_u)\Big)} - \frac{u\Big(x - c(e_u)\Big)}{(-1)^n u^{(n-1)}\Big(x^\circ - c(e_u)\Big)}\right]\mathrm{d}[G(x) - F(x)]$$

$$= \int_{\hat{x}-\varepsilon_1}^{x^\circ} \left[\frac{u^{(n)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)} - \frac{v^{(n)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)} \right] [F_n(x) - G_n(x)]\mathrm{d}x$$

$$+ \int_{x^\circ}^{\hat{x}+\varepsilon_2} \left[\frac{v^{(n-1)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)} - \frac{u^{(n-1)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)} \right] [F_{n-1}(x) - G_{n-1}(x)]\mathrm{d}x, \tag{A.33}$$

where we have used $F_k(\hat{x} + \varepsilon_2) = G_k(\hat{x} + \varepsilon_2)$ for all $k = 1, ..., n-1$ and $F_n(x^\circ) = G_n(x^\circ)$. Since $A_v^{(n/n-1)}\Big(x - c(e_u)\Big) < A_u^{(n/n-1)}\Big(x - c(e_u)\Big)$ for all $x \in [\hat{x} - \varepsilon_1, \hat{x} + \varepsilon_2]$, it follows from Jindapon and Neilson (2007) that

$$\frac{v^{(n-1)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)} \leq (\geq) \ \frac{u^{(n-1)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)} \tag{A.34}$$

for all $x \leq (\geq)\ x^\circ$. Since $F_{n-1}(x) \leq G_{n-1}(x)$ for all $x \in [x^\circ, \hat{x} + \varepsilon_2]$, it follows from Eq. (A.34) that the second term on the right-hand side of Eq. (A.33) is negative. For all $x \in [\hat{x} - \varepsilon_1, x^\circ]$, we have

$$-\frac{v^{(n)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)} = A_v^{(n/n-1)}\Big(x - c(e_u)\Big) \times \frac{v^{(n-1)}\Big(x - c(e_u)\Big)}{v^{(n-1)}\Big(x^\circ - c(e_u)\Big)}$$

$$< A_u^{(n/n-1)}\Big(x - c(e_u)\Big) \times \frac{u^{(n-1)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)} = -\frac{u^{(n)}\Big(x - c(e_u)\Big)}{u^{(n-1)}\Big(x^\circ - c(e_u)\Big)}, \tag{A.35}$$

where the inequality follows from $A_v^{(n/n-1)}\Big(x - c(e_u)\Big) < A_u^{(n/n-1)}\Big(x - c(e_u)\Big)$ for all $x \in [\hat{x} - \varepsilon_1, \hat{x} + \varepsilon_2]$ and Eq. (A.34). Since $F_n(x) \geq G_n(x)$ for all $x \in [\hat{x} - \varepsilon_1, x^\circ]$, it follows from Eq. (A.35) that the first term on the right-hand side of Eq. (A.33) is negative. It then follows from Eqs. (A.32) and (A.33) that

$$\frac{\int_{\hat{x}-\varepsilon_1}^{\hat{x}+\varepsilon_2} v\Big(x - c(e_u)\Big)\mathrm{d}[G(x) - F(x)]}{\int_{\hat{x}-\varepsilon_1}^{\hat{x}+\varepsilon_2} v'\Big(x - c(e_u)\Big)\mathrm{d}H(x|e_u)}$$

$$< \frac{\int_{\hat{x}-\varepsilon_1}^{\hat{x}+\varepsilon_2} u\Big(x - c(e_u)\Big)\mathrm{d}[G(x) - F(x)]}{\int_{\hat{x}-\varepsilon_1}^{\hat{x}+\varepsilon_2} u'\Big(x - c(e_u)\Big)\mathrm{d}H(x|e_u)} = c'(e_u), \tag{A.36}$$

where the equality follows from Eq. (26). It then follows from Eqs. (25), (27), and (A.36) that $e_v < e_u$, a contradiction.

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