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The coupled Hirota system as an example displaying discrete breathers: Rogue waves, modulation instability and varying cross-phase modulations

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Discrete dynamical systems constitute an elegant branch of nonlinear science, where ingenious techniques provide penetrating insight for vibrations and wave motion on lattices. In terms of applications, such systems can model oscillators with hard quartic nonlinearities and switching of optical pulses on discrete arrays. A two-component Hirota system is investigated as an extension of the widely studied Ablowitz-Ladik equation by including discrete third order dispersion. Breathers (periodic pulsating modes) are derived analytically, and are used to establish conservation laws. Rogue waves (unexpectedly large displacements from equilibrium configurations) exhibit unusual features in undergoing oscillations above and below the mean level, and may even reverse polarity. Coupling produces new regimes of modulation instabilities for discrete evolution equations. The robustness of these novel rogue waves, in terms of sensitivity to initial conditions, is elucidated by numerical simulations. Self-phase modulations and cross-phase modulations of the same or opposite signs will generate nonlinear corrections of the frequency, due to the intensity of the wave train itself and the one in the accompanying waveguide respectively. Such effects have a crucial influence on the evolution of discrete and continuous multi-component dynamical systems. https://doi.org/10.1063/1.5045532

I. INTRODUCTION

Discrete evolution systems have been investigated extensively because of the intrinsic importance in nonlinear science as well as their potential applications. One illustrative example is the Ablowitz-Ladik system, which constitutes a discrete version the nonlinear Schrödinger equation (NLSE).[1] The Ablowitz-Ladik system possesses elegant analytical structures, e.g. a linear scattering pair and solitons.[2] Theoretically, single-component as well as coupled or multi-component versions can be constructed and treated by the inverse scattering transform, bilinear operator and Darboux transformations.[1,3–5] In terms of applications, the Ablowitz-Ladik system can be used to model a variety of physical phenomena, e.g. the effect of randomness on the stability of solitons,[6] switching of solitons in discrete optical waveguides,[7] and periodic Fourier transforms in fiber-optic communications.[8]

The Hirota equation, an extension of the continuous NLSE incorporating third order dispersion, also possesses an analogue on an integer lattice and this corresponding dynamical system will be termed the ***discrete Hirota equation*** here.[9–15] Analytically the discrete Hirota equation arises as a generalization of the Ablowitz-Ladik system by taking the input parameters as complex valued. Both the bilinear method[9] and the Darboux transformation[11,13] have been utilized in the theoretical treatments. 'Non-integrable' discrete Hirota equations can also be formulated and applied to ferromagnetic materials.[16] The transition from a discrete lattice back to a continuum state may involve a

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subtle limit as the step size becomes small. Novel discrete models have been introduced to resolve this issue.[17]

Rogue waves, unexpectedly large displacements from an otherwise calm configuration, have received tremendous attention recently.[18,19] Such surprisingly large-amplitude wave motions have also been established for the single-component discrete Hirota equation.[10,13,14] In terms of practical applications, as an illustrative example, consideration of intrinsic localized modes in an array of oscillators with a hard quartic nonlinearity will lead to this discrete Hirota equation.[20]

The goal here is to study analytically coupled systems involving two arrays of oscillators on integer lattices, incorporating both self-phase modulation (SPM) and cross-phase modulation (XPM).[21] For the coupled Ablowitz-Ladik case, 'integrable' versions involving XPM have been formulated and properties of solitons are clarified.[1] Various variants with exotic features but without the full hierarchy of XPM have been considered recently, e.g. branched dispersion,[22] linear coupling,[23] and self-attractive nonlinearity.[4] For coupled discrete Hirota equations, extensions of the single component case have been put forward in the literature.[24,25] Both bright and dark solitons have been derived analytically,[24] and the connection with the classical Toda equations is clarified.[25] Another perspective is to consider a 'non-autonomous' system of Ablowitz-Ladik equations, where the time varying properties of the medium provide insight on controlling and management of solitary, periodic and rogue pulses.[12,14]

Our objective is to conduct further theoretical analysis on such coupled discrete Hirota systems, revealing properties not known or receiving comparatively less attention for similar dynamical systems:

- pulsating rogue waves,
- cross-phase modulations induced instabilities, and
- conservation laws.

In particular, we wish to show the profound influence of SPM and XPM of different signs on the dynamics and evolution of localized pulses, and demonstrate the exotic features generated.

The bilinear scheme for finding localized modes, which is shown to be effective for the Ablowitz-Ladik case, will be utilized again.[3,9] Pulsating modes periodic in space or time will be established theoretically. 'Discrete breathers' have received intensive attention across many disciplines,[26] and the present coupled Hirota system will hopefully be a valuable addition to breather research. Rogue waves will be generated as a long wave limit of breathers.[27–29]

The connection between modulation instability and existence of unexpectedly large amplitude waves has been studied earlier,[30,31] and will be confirmed again here. Conservation laws of dynamical systems are usually associated with appealing physical properties, and will be examined for the simple case of spatially periodic boundary conditions.[10,32,33] Finally, the robustness of rogue waves is studied computationally and close connection with the modulation instability gain spectrum is demonstrated explicitly.[34,35]

The structure of the paper can now be explained. The coupled discrete Hirota system is first formulated (Section II), with the discrete version of the second order dispersion being of opposite signs. Furthermore, we allow for the freedom of self-phase and cross-phase modulations to act either in constructive or destructive interference modes. Breathers and rogue waves are then computed exactly (Section III). Rogue waves can pulsate and oscillate above and below the mean level. They can even reverse polarity after interactions with modes from other waveguides. Modulation instabilities are investigated and correlated with the existence of rogue waves (Section IV). Analogous to the continuous case, coupled discrete waveguides can produce new regimes of instability. One spatially periodic solution is utilized to highlight the existence of low order conservation laws (Section V). Finally, the robustness of rogue waves to perturbed initial conditions is studied by numerical simulations (Section VI). Conclusions are drawn (Section VII).

II. COUPLED HIROTA EQUATIONS

We shall investigate a discrete Hirota system for complex valued wave envelopes ϕ_n, ψ_n on lattices with integer values for n. The system displays nonlinearities arising from both SPM and XPM.[21] Furthermore, we allow for combinations of SPM and XPM of different signs ($\varepsilon = \pm 1$):

$$i(\phi_n)_t + [(1+i\gamma)\phi_{n+1} + (1-i\gamma)\phi_{n-1}]\left[\beta + \sigma\left(|\phi_n|^2 + \varepsilon|\psi_n|^2\right)\right] = 0, \tag{1a}$$

$$i(\psi_n)_t - [(1+i\gamma)\psi_{n+1} + (1-i\gamma)\psi_{n-1}]\left[\beta + \sigma\left(|\phi_n|^2 + \varepsilon|\psi_n|^2\right)\right] = 0. \tag{1b}$$

In the continuum limit, the parameters β, γ will be related to the second and third order dispersion. The parameter σ will measure the strength of the cubic nonlinearity, while ε will indicate whether SPM and XPM interact constructively or destructively. For γ = 0, Eqs. (1a, 1b) reduce to a coupled Ablowitz-Ladik system with dispersions of opposite signs. In principle a staggering transformation $\Psi_n = (-1)^n\ \psi_n$ can be applied to convert the sign of the dispersion in Eq. (1b). Here this notation is chosen to facilitate treatment of waveguides with dispersion of different signs in the future.

For simplicity, we assume a special form of the background plane waves, namely wavenumbers of opposite signs, and $\rho_n, \omega_n (n = 1, 2), k$ being the amplitudes, angular frequencies and wavenumber respectively:

$$\phi_n = i^n \rho_1 \exp[i(kn - \omega_1 t)],\ \psi_n = i^n \rho_2 \exp[i(-kn - \omega_2 t)], \tag{2a}$$

$$\omega_1 = -i\left[(1+i\gamma)\exp(ik) - (1-i\gamma)\exp(-ik)\right]\left[\beta + \sigma(\rho_1^2 + \varepsilon\rho_2^2)\right], \tag{2b}$$

$$\omega_2 = i\left[(1+i\gamma)\exp(-ik) - (1-i\gamma)\exp(ik)\right]\left[\beta + \sigma(\rho_1^2 + \varepsilon\rho_2^2)\right]. \tag{2c}$$

The Hirota bilinear transform which is effective for the Ablowitz-Ladik case will now be generalized (G_n, H_n complex, f_n real)

$$\phi_n = i^n \frac{G_n}{f_n} \exp[i(kn - \omega_1 t)], \quad \psi_n = i^n \frac{H_n}{f_n} \exp[i(-kn - \omega_2 t)], \tag{3a}$$

For subsequent calculations, it will be convenient to impose the constraint:

$$\beta + \sigma(\rho_1^2 + \varepsilon\rho_2^2) = -1. \tag{3b}$$

The angular frequency of the plane wave can be simplified to

$$\omega_1 = 2(\gamma\cos k + \sin k)\left[\beta + \sigma(\rho_1^2 + \varepsilon\rho_2^2)\right] = -2(\gamma\cos k + \sin k), \tag{4a}$$

$$\omega_2 = 2(-\gamma\cos k + \sin k)\left[\beta + \sigma(\rho_1^2 + \varepsilon\rho_2^2)\right] = -2(-\gamma\cos k + \sin k). \tag{4b}$$

The bilinear form is then given by

$$D_t G_n \cdot f_n = (1+i\gamma)\exp(ik)[G_{n+1}f_{n-1} - G_n f_n] + (1-i\gamma)\exp(-ik)[G_n f_n - G_{n-1}f_{n+1}], \tag{5a}$$

$$D_t H_n \cdot f_n = (1+i\gamma)\exp(-ik)[H_n f_n - H_{n+1}f_{n-1}] + (1-i\gamma)\exp(ik)[H_{n-1}f_{n+1} - H_n f_n], \tag{5b}$$

$$f_{n+1}f_{n-1} + \beta f_n^2 + \sigma\left(|G_n|^2 + \varepsilon|H_n|^2\right) = 0. \tag{5c}$$

Proceeding along the widely used procedure in bilinear transform, we adopt the expansion (M real, a_n, b_n, n = 1, 2, complex):

$$\begin{aligned} G_n = \rho_1\{&1 + a_1\exp(pn - \Omega t + \eta_1) + a_2\exp(p^* n - \Omega^* t + \eta_2) \\ &+ M a_1 a_2 \exp[(p+p^*)n - (\Omega + \Omega^*)t + \eta_1 + \eta_2]\}, \end{aligned} \tag{6a}$$

$$\begin{aligned} H_n = \rho_2\{&1 + b_1\exp(pn - \Omega t + \eta_1) + b_2\exp(p^* n - \Omega^* t + \eta_2) \\ &+ M b_1 b_2 \exp[(p+p^*)n - (\Omega + \Omega^*)t + \eta_1 + \eta_2]\}, \end{aligned} \tag{6b}$$

$$\begin{aligned} f_n = &1 + \exp(pn - \Omega t + \eta_1) + \exp(p^* n - \Omega^* t + \eta_2) \\ &+ M\exp[(p+p^*)n - (\Omega + \Omega^*)t + \eta_1 + \eta_2]\}, \end{aligned} \tag{6c}$$

with η_1, η_2 being phase factors. Under these assumptions, the formulations simplify considerably:

$$a_1 = \frac{\Omega + (1+i\gamma)\exp(ik)[1-\exp(-p)] + (1-i\gamma)\exp(-ik)[\exp(p)-1]}{\Omega + (1+i\gamma)\exp(ik)[\exp(p)-1] + (1-i\gamma)\exp(-ik)[1-\exp(-p)]}, \tag{7a}$$

$$a_2 = \frac{1}{a_1^*}, \tag{7b}$$

$$b_1 = \frac{\Omega - (1+i\gamma)\exp(-ik)[1-\exp(-p)] - (1-i\gamma)\exp(ik)[\exp(p)-1]}{\Omega - (1+i\gamma)\exp(-ik)[\exp(p)-1] - (1-i\gamma)\exp(ik)[1-\exp(-p)]}, \tag{7c}$$

$$b_2 = \frac{1}{b_1^*}, \tag{7d}$$

$$M = \frac{M_1}{M_2}, \tag{7e}$$

$$\begin{aligned} M_1 = a_1\{(\Omega-\Omega^*) + (1+i\gamma)\exp(ik)[\exp(p-p^*)-1] + (1-i\gamma)\exp(-ik)[1-\exp(p^*-p)]\} \\ + a_2\{(\Omega^*-\Omega) + (1+i\gamma)\exp(ik)[\exp(p^*-p)-1] + (1-i\gamma)\exp(-ik)[1-\exp(p-p^*)]\}, \end{aligned} \tag{7f}$$

$$\begin{aligned} M_2 = a_1 a_2\{(-\Omega-\Omega^*) - (1+i\gamma)\exp(ik)[\exp(p+p^*)-1] - (1-i\gamma)\exp(-ik)[1-\exp(-p-p^*)]\} \\ + \{(\Omega+\Omega^*) - (1+i\gamma)\exp(ik)[\exp(-p-p^*)-1] - (1-i\gamma)\exp(-ik)[1-\exp(p+p^*)]\}. \end{aligned} \tag{7g}$$

The dispersion relation of the breather is given by

$$\exp(p) + \exp(-p) - 2 + \sigma\rho_1^2\left(a_1 + \frac{1}{a_1} - 2\right) + \varepsilon\sigma\rho_2^2\left(b_1 + \frac{1}{b_1} - 2\right) = 0. \tag{8}$$

Eq. (8) will be a polynomial of the fourth order in terms of Ω.

On further assuming that p is real and numerically small, a series expansion of the dispersion relation:

$$\Omega = \Omega_1 p + \Omega_2 p^2 + O(p^3), \tag{9}$$

will give the leading order angular frequency Ω_1 of a breather of long wavelength as

$$\begin{aligned} &\Omega_1^4 - 8\gamma(\sin k)\Omega_1^3 + 4\left\{\begin{array}{l}\left[\left(\rho_1^2+\varepsilon\rho_2^2\right)\left(1-\gamma^2\right)\sigma - 6\gamma^2 - 2\right]\cos^2 k \\ -\sigma\left(\rho_1^2-\varepsilon\rho_2^2\right)\gamma\sin(2k) - (\rho_1^2+\varepsilon\rho_2^2)\sigma + 6\gamma^2\end{array}\right\}\Omega_1^2 \\ &+16\left\{\begin{array}{l}\sigma\left(\gamma^2+1\right)\left(-\rho_1^2+\varepsilon\rho_2^2\right)\cos^3 k + \left[2+\left(\rho_1^2+\varepsilon\rho_2^2\right)\sigma\right]\left(\gamma^2+1\right)\gamma\sin k\cos^2 k \\ +\sigma\left(\rho_1^2-\varepsilon\rho_2^2\right)\left(2\gamma^2+1\right)\cos k + \gamma\left[\left(\rho_1^2+\varepsilon\rho_2^2\right)\sigma - 2\gamma^2\right]\sin k\end{array}\right\}\Omega_1 \\ &+16\left\{\begin{array}{l}\left[1+\left(\rho_1^2+\varepsilon\rho_2^2\right)\sigma\right]\left(\gamma^2+1\right)^2\cos^4 k - \left[\left(\rho_1^2+\varepsilon\rho_2^2\right)\left(\gamma^2+1\right)^2\sigma + 2\gamma^4 + 2\gamma^2\right]\cos^2 k \\ -\sigma\left(\rho_1^2-\varepsilon\rho_2^2\right)\gamma\left(\gamma^2+1\right)\sin(2k) - \left(\rho_1^2+\varepsilon\rho_2^2\right)\gamma^2\sigma + \gamma^4\end{array}\right\} = 0. \end{aligned} \tag{10}$$

While Eq. (10) may be analytically intractable, the objective is to study the breather and rogue modes with varying amplitudes (ρ_1, ρ_2), 'discrete third order dispersions' (γ) and relative measures of SPM versus XPM (ε). In addition to the lengthy expressions, the main contrast with the Ablowitz-Ladik case ($\gamma = 0$) is a cubic term in Eq. (10).

III. BREATHERS QUASI-PERIODIC IN TIME AND ROGUE WAVES

Breathers are modes periodic in the spatial variable n or time t. To be consistent with widely adopted terminology, modes periodic in space (n) and time (t) are usually termed the Akhmediev and Kuznetsov-Ma breathers respectively. We defer the treatment of spatially periodic modes to

Section V, where a discussion in conjunction with conservation laws will be presented. Instead of pursuing details for the Kuznetsov-Ma breathers, we shall proceed directly to the long wave limit and obtain the rogue waves.

A. Dispersion relation for simplified set of parameters

It will be instructive to first look at simplified cases of the complicated dispersion relation (Eq. (10)). In anticipation of a long wave limit to rogue waves, theoretically we require complex roots for Ω_1 for nonsingular modes.

1. Effects of varying the ratio of SPM and XMP

By assuming $k = \frac{\pi}{2}, \rho_1 = \rho_2 = \rho$, i.e. a period of 4 for the carrier envelope (Eq. (3a)) and equal background amplitudes, the dispersion relation Eq. (10) simplifies to

$$(\Omega_1 - 2\gamma)^2\left[\Omega_1^2 - 4\gamma\Omega_1 + 4\gamma^2 - 4\sigma\rho^2(\varepsilon + 1)\right] = 0. \tag{11}$$

Complex roots are ensured if $\sigma\rho^2(\varepsilon + 1)$ is negative, i.e. either (i) negative cubic nonlinearity ($\sigma < 0$, $\varepsilon + 1 > 0$) or (ii) XPM coefficient (ε) algebraically less than -1 for $\sigma > 0$.

2. Effects of different background amplitudes ($\rho_2 \neq \rho_1$)

On taking the input parameters as $k = \frac{\pi}{2}, \rho_1 = 1, \rho_2 = \rho$, the dispersion relation is

$$(\Omega_1 - 2\gamma)^2\left(\Omega_1^2 - 4\gamma\Omega_1 - 4\sigma + 4\gamma^2 - 4\rho^2\sigma\varepsilon\right) = 0. \tag{12}$$

Complex roots are achieved if $\sigma(1 + \varepsilon\rho^2) < 0$, i.e. (i) $\sigma < 0, \varepsilon > 0$, or (ii) $\sigma > 0, \varepsilon < 0$ and a suitable choice of ρ.

B. Breathers quasi-periodic in time with rogue waves as special limits

Kuznetsov-Ma breathers can be derived by assuming p real and searching for complex Ω in Eqs. (6)–(8). Omitting the details for such breathers, we proceed directly to the rogue wave modes by taking the long wave limit and the phase factors in Eq. (6) as

$$\exp(\eta_1) = \exp(\eta_2) = -1.$$

The rogue waves are given as rational expressions of the lattice coordinate n and time t for general amplitudes of the background of the waveguides (ρ_1, ρ_2) as

$$\begin{aligned}
&\Omega_1 = a + ib \\
&f_n = (n - at)^2 + b^2t^2 + \frac{4\left(\gamma^2 + 1\right) - a^2 - b^2}{4b^2}, \qquad G_n = \rho_1 g_n,\ H_n = \rho_2 h_n, \\
&g_n = f_n - \frac{4\left\{\begin{array}{l}\left[\left(\gamma^2 - 1\right) + 4i\gamma(n - at)\right]\cos^2 k + \left[\gamma - i\left(\gamma^2 - 1\right)(n - at)\right]\sin 2k \\ +i\left[an + t\left(b^2 - a^2\right)\right](\gamma\cos k + \sin k) + 2i\gamma(at - n) + 1\end{array}\right\}}{\left[2(\cos k - \gamma\sin k) + a\right]^2 + b^2}, \\
&h_n = f_n - \frac{4\left\{\begin{array}{l}\left[\left(\gamma^2 - 1\right) + 4i\gamma(n - at)\right]\cos^2 k + \left[i\left(\gamma^2 - 1\right)(n - at) - \gamma\right]\sin 2k \\ +i\left[an + t\left(b^2 - a^2\right)\right](\sin k - \gamma\cos k) + 2i\gamma(at - n) + 1\end{array}\right\}}{\left[2(\cos k + \gamma\sin k) - a\right]^2 + b^2}.
\end{aligned} \tag{13}$$

This rogue wave will remain bounded if $b \neq 0$ ***and*** $\gamma^2 + 1 > (a^2 + b^2)/4$.

1. General qualitative features

Typical profiles of the rogue modes are shown for representative values of the parameters (Figure 1). The pulse can in general propagate from left to right or vice versa. The amplitude of the rogue mode can change from a value above the mean position to one below the mean position (left panel

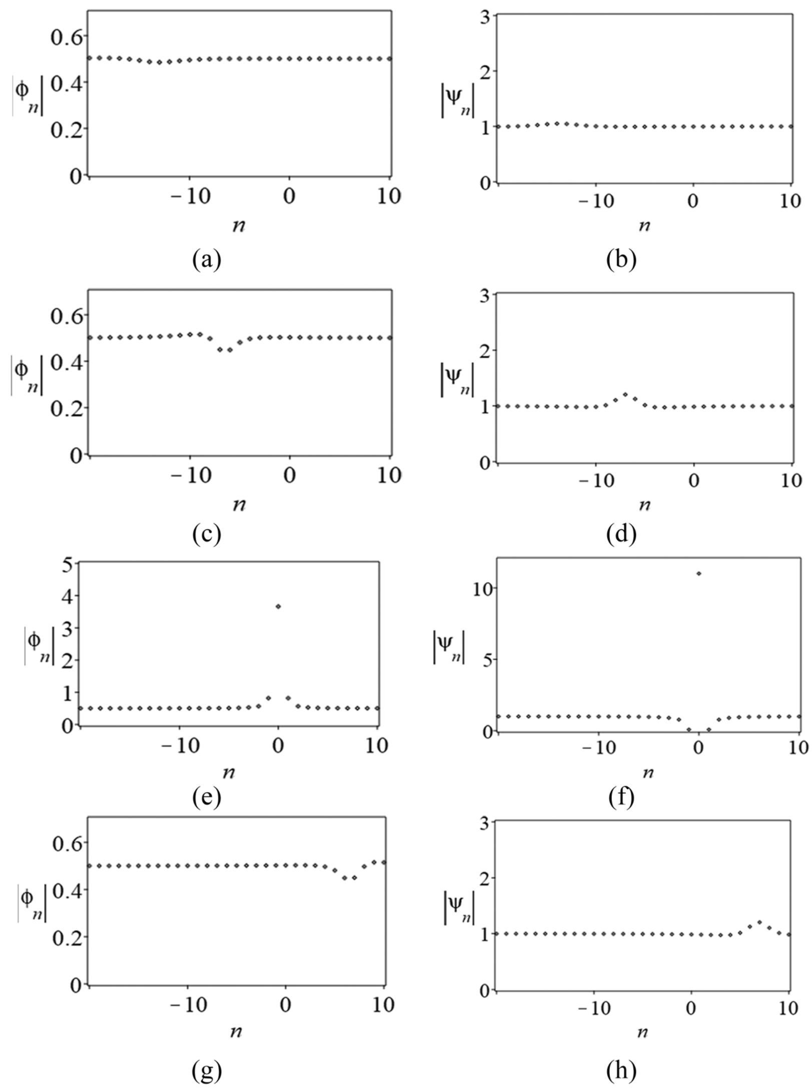

FIG. 1. A rogue wave propagating from left to right with a burst of maximum amplitude around $t = 0$ for parameters $k = 0.5, \sigma = 1, \rho_1 = 0.5, \rho_2 = 1, \gamma = 1, \varepsilon = -1$; From top to bottom are wave profiles at $t = -5$, $t = -2.5$, $t = 0$, $t = 2.5$ with $|\phi_n|$ ($|\psi_n|$) at left (right) panels.

of Figure 1), instead of the more conventional way of 'grow and subside' above (or below) a fixed level. At the time instant of maximum displacement (around $t = 0$), the peaks of ϕ_n and ψ_n can point in the same or opposite directions.

2. Dependence on the parameters γ *and* ε

To illustrate the existence and evolution of 'bright' and 'dark' discrete rogue waves, we utilize three dimensional plots for the profiles of the envelopes ϕ_n and ψ_n (Figure 2). The wave profile can evolve from a displacement below the mean level ('dark rogue wave') to one above the mean level ('bright rogue wave') during the interaction phase, and will regain the dark rogue wave state after the interaction. The opposite can happen for the other waveguide (Figure 2a). On changing the value of γ, such scenarios may persist but just happen for a small period of time around $t = 0$ (Figure 2b).

A varying nature of XPM versus SPM may generate an even more drastic change, e.g. reversal of polarity of discrete rogue waves before and after the interaction. When ε is modified from –1 to +1 (XPM and SPM from different signs to the same sign) for a fixed γ, both $|\phi_n|$ and $|\psi_n|$

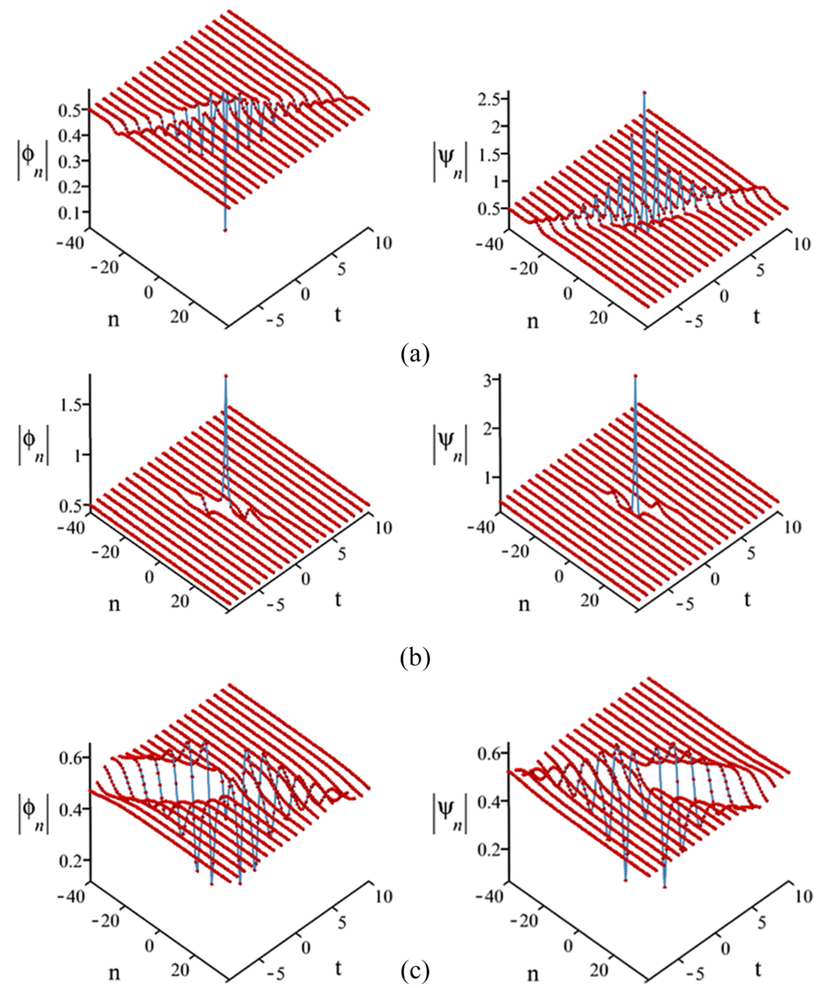

FIG. 2. Various discrete rogue waves with change of polarity; Amplitude versus n and t, with $k = 0.5$, $\sigma = 1$, $\rho_1 = \rho_2 = 0.5$; (a) $\gamma = 1$, $\varepsilon = -1$, $\Omega_1 = 2.72 + 0.43i$; (b) $\gamma = 5$, $\varepsilon = -1$, $\Omega_1 = 7.79 + 3.62i$; (c) $\gamma = 5$, $\varepsilon = +1$, $\Omega_1 = 5.09 + 1.50i$.

reverse polarity (change from dark rogue wave to a bright one after interaction, or vice versa, Figures 2b, 2c).

C. Bifurcation from the Ablowitz-Ladik systems

When the effect of 'discrete third order dispersion' is absent (or $\gamma = 0$), the discrete Hirota equation would reduce to the intensively studied Ablowitz-Ladik system. It is thus instructive to elucidate the 'bifurcation' in the parameter space around $\gamma = 0$. We adopt again as illustrative example the set of input parameters of $k = 0.5$, $\sigma = 1$, $\rho_1 = \rho_2 = 0.5$. For the existence of nonsingular rogue modes, we need to check two constraints. The first is that the imaginary part of Ω_1 should be distinct from zero ($b \neq 0$ in Eq. (13)). The second condition is $4\left(\gamma^2 + 1\right) - a^2 - b^2 > 0$ (Figure 3).

For SPM and XPM of opposite signs, rogue wave can occur for almost arbitrary values of γ (Figure 3b). However, for the more conventional case with SPM and XPM of the same sign, rogue waves bifurcate 'abruptly' from a finite value of γ, but are otherwise prohibited for values of γ numerically close to zero (Figure 3a).

IV. MODULATION INSTABILITIES

Modulation instability refers to the spontaneous growth of small disturbances imposed upon the continuous wave of a system. Such instability in the long wavelength limit has been demonstrated to

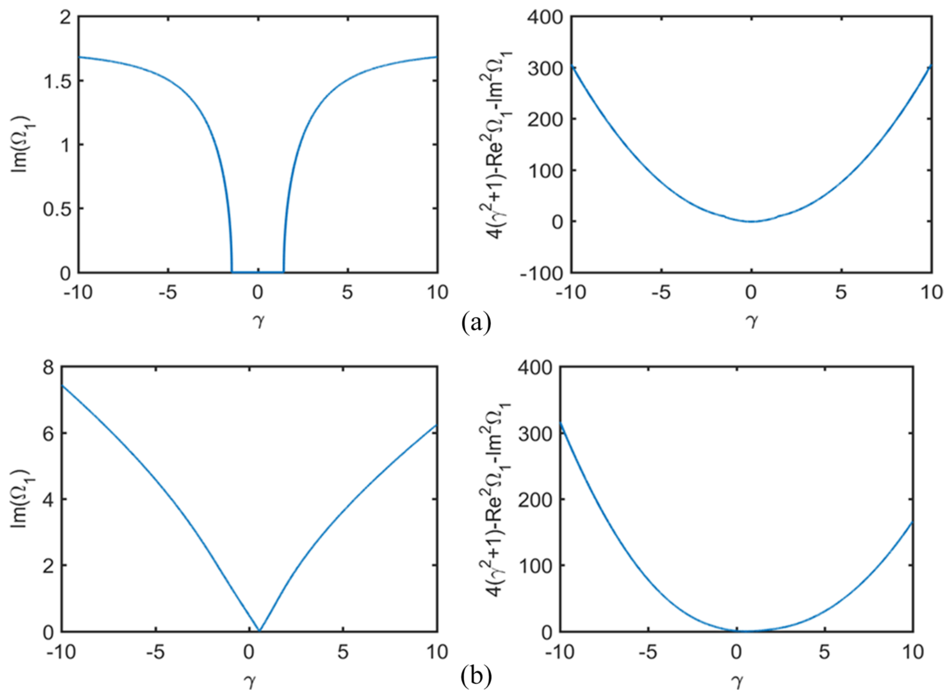

FIG. 3. a: The relationship between the imaginary part of Ω_1 or Im(Ω_1) and γ (left) and the condition for nonsingular rogue wave $4(\gamma^2 + 1) - a^2 - b^2 > 0$ versus γ (right); Input parameters chosen are $k = 0.5$, $\sigma = 1$, $\rho_1 = \rho_2 = 0.5$, and $\varepsilon = +1$ (self-phase modulation and cross-phase modulation of the *same* sign). b: The relationship between the imaginary part of Ω_1 or Im(Ω_1) and γ (left) and the condition for nonsingular rogue wave $4(\gamma^2 + 1) - a^2 - b^2 > 0$ versus γ (right); Input parameters chosen are $k = 0.5$, $\sigma = 1$, $\rho_1 = \rho_2 = 0.5$, and $\varepsilon = -1$ (self-phase modulation and cross-phase modulation of the *opposite* sign).

be intimately connected with the existence condition of rogue waves.[30,31,34,35] Moreover, coupling in the dynamics of wave envelopes usually produces new regimes of instability (and hence new rogue wave).[30,31] A similar scenario is shown to be valid here for discrete evolution systems too.

A. Stability analysis

Disturbances imposed on a train of continuous wave (Eq. (2)) is taken as[21]

$$\phi_n = i^n\rho_1(1 + \phi_n')\exp[i(kn - \omega_1 t)],\ \psi_n = i^n\rho_2(1 + \psi_n')\exp[i(-kn - \omega_2 t)],$$

where the primed quantities are assumed to be small and separated by a modal dependence of the form $\exp[i(rn - st)]$. Standard procedures now lead to a dispersion relation for r, s:

$$[(s - \xi_1)(s - \xi_2) - \xi_3][(s - \lambda_1)(s - \lambda_2) - \lambda_3] = C, \tag{14}$$

$$\xi_1 = 2\left[\left(\beta + \sigma\rho_1{}^2 + \varepsilon\sigma\rho_2{}^2\right)[\sin(r + k) + \gamma\cos(r + k)] - \left(\beta + \varepsilon\sigma\rho_2{}^2\right)(\sin k + \gamma\cos k)\right],$$

$$\xi_2 = 2\left[\left(\beta + \sigma\rho_1{}^2 + \varepsilon\sigma\rho_2{}^2\right)[\sin(r - k) - \gamma\cos(r - k)] + \left(\beta + \varepsilon\sigma\rho_2{}^2\right)(\sin k + \gamma\cos k)\right],$$

$$\xi_3 = -\left[2\sigma\rho_1{}^2(\sin k + \gamma\cos k)\right]^2,$$

$$\lambda_1 = -2\left[\left(\beta + \sigma\rho_1{}^2 + \varepsilon\sigma\rho_2{}^2\right)[\sin(r - k) + \gamma\cos(r - k)] + \left(\beta + \sigma\rho_1{}^2\right)(\sin k - \gamma\cos k)\right],$$

$$\lambda_2 = -2\left[\left(\beta + \sigma\rho_1{}^2 + \varepsilon\sigma\rho_2{}^2\right)[\sin(r + k) - \gamma\cos(r + k)] - \left(\beta + \sigma\rho_1{}^2\right)(\sin k - \gamma\cos k)\right],$$

$$\lambda_3 = -\left[2\varepsilon\sigma\rho_2{}^2(\sin k - \gamma\cos k)\right]^2,$$

$$C = 64(\sigma\rho_1)(\varepsilon\sigma\rho_2)\left\{\left(\beta + \sigma\rho_1{}^2 + \varepsilon\sigma\rho_2{}^2\right)(1 - \cos r)\left[1 - (1 + \gamma^2)\cos^2 k\right]\right\}^2.$$

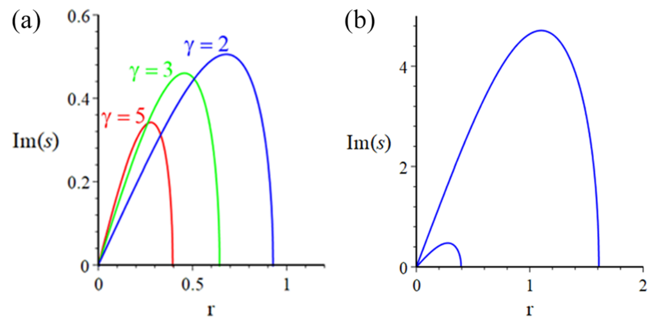

FIG. 4. The modulation instability growth rate (imaginary part of the complex frequency s) versus the disturbance wavenumber r with $k = 0$, $\rho_1 = \rho_2 = 0.5$, $\varepsilon = 1$; (a) varying the third order dispersion parameter γ for $\sigma = 1$; (b) $\gamma = 5$ and $\sigma = -1$.

Instability will be present in general (Figure 4). In particular, a long wave ($r \to 0$) expansion of Eq. (14), with $s = rs_1 + O(r^2)$, will yield the governing equation for s_1 as

$$s_1^4 - 8\gamma(\sin k)s_1^3 + 4\left\{\begin{array}{l}\left[\left(\rho_1^2+\varepsilon\rho_2^2\right)\left(1-\gamma^2\right)\sigma - 6\gamma^2 - 2\right]\cos^2 k \\ -\sigma\left(\rho_1^2-\varepsilon\rho_2^2\right)\gamma\sin(2k) - (\rho_1^2+\varepsilon\rho_2^2)\sigma + 6\gamma^2\end{array}\right\}s_1^2$$
$$+16\left\{\begin{array}{l}\sigma\left(\gamma^2+1\right)\left(-\rho_1^2+\varepsilon\rho_2^2\right)\cos^3 k + \left[2+\left(\rho_1^2+\varepsilon\rho_2^2\right)\sigma\right]\left(\gamma^2+1\right)\gamma\sin k\cos^2 k \\ +\sigma\left(\rho_1^2-\varepsilon\rho_2^2\right)\left(2\gamma^2+1\right)\cos k + \gamma\left[\left(\rho_1^2+\varepsilon\rho_2^2\right)\sigma - 2\gamma^2\right]\sin k\end{array}\right\}s_1$$
$$+16\left\{\begin{array}{l}\left[1+\left(\rho_1^2+\varepsilon\rho_2^2\right)\sigma\right]\left(\gamma^2+1\right)^2\cos^4 k - \left[\left(\rho_1^2+\varepsilon\rho_2^2\right)\left(\gamma^2+1\right)^2\sigma + 2\gamma^4 + 2\gamma^2\right]\cos^2 k \\ -\left(\rho_1^2-\varepsilon\rho_2^2\right)\left(\gamma^2+1\right)\sigma\gamma\sin(2k) - \left(\rho_1^2+\varepsilon\rho_2^2\right)\gamma^2\sigma + \gamma^4\end{array}\right\} = 0.$$

which is the same as Eq. (10), confirming again the intimate connection between the onset of modulation instability and the existence condition of rogue waves.

B. Comparison with single component case

The dynamical system Eq. (1) will degenerate to the single component case if the individual waveguide has no effect on the other. Theoretically the system of modulation instability equations is decoupled if $C = 0$ in Eq. (14). This can be achieved by choosing either $\rho_1 = 0$ or $\rho_2 = 0$. Other possible cases will be studied in the future.

For $\rho_2 = 0$, Eq. (14) gives the dispersion relation for modulation of plane waves for a single component discrete Hirota system as

$$i(\phi_n)_t + [(1+i\gamma)\phi_{n+1} + (1-i\gamma)\phi_{n-1}]\left(\beta + \sigma|\phi_n|^2\right) = 0. \tag{15a}$$

Similarly, for $\rho_1 = 0$, Eq. (14) yields the condition for modulation instability of plane waves for another one component case:

$$i(\psi_n)_t - [(1+i\gamma)\psi_{n+1} + (1-i\gamma)\psi_{n-1}]\left(\beta + \sigma\varepsilon|\psi_n|^2\right) = 0. \tag{15b}$$

Similar to the situation for continuous dynamical system, XPM effects may induce new modulation instability in the coupled case, even though the uncoupled systems themselves are stable. As an illustrative example, for $\sigma = 1$, $\rho_1 = 0.5$, $\varepsilon = 1$, $\gamma = 5$, $k = 0$, there is no modulation instability for the one component system Eq. (15a). The same remark applies to the case of $\sigma = 1$, $\rho_2 = 0.5$, $\varepsilon = 1$, $\gamma = 5$, $k = 0$ for the single component case Eq. (15b). However, for the fully coupled discrete case, Eq. (14), with $\sigma = 1$, $\rho_1 = \rho_2 = 0.5$, $\varepsilon = 1$, $\gamma = 5$, $k = 0$, instability is present (Figure 4).

V. BREATHERS PERIODIC IN THE LATTICE COORDINATE AND CONSERVATION LAWS

A. Spatially periodic breathers

Breathers periodic in the lattice coordinate n can be obtained by choosing the wavenumber p in the formulation of Eq. (6) to be purely imaginary. To demonstrate the concrete existence of such pulsating modes, we choose $p = i\pi/4$ and Eq. (6) will yield a periodic expansion. However, to maintain an overall spatially periodic structure in view of Eq. (3), we need to specify k to be $\pi/4$ too. The breather will then attain a period of 8.

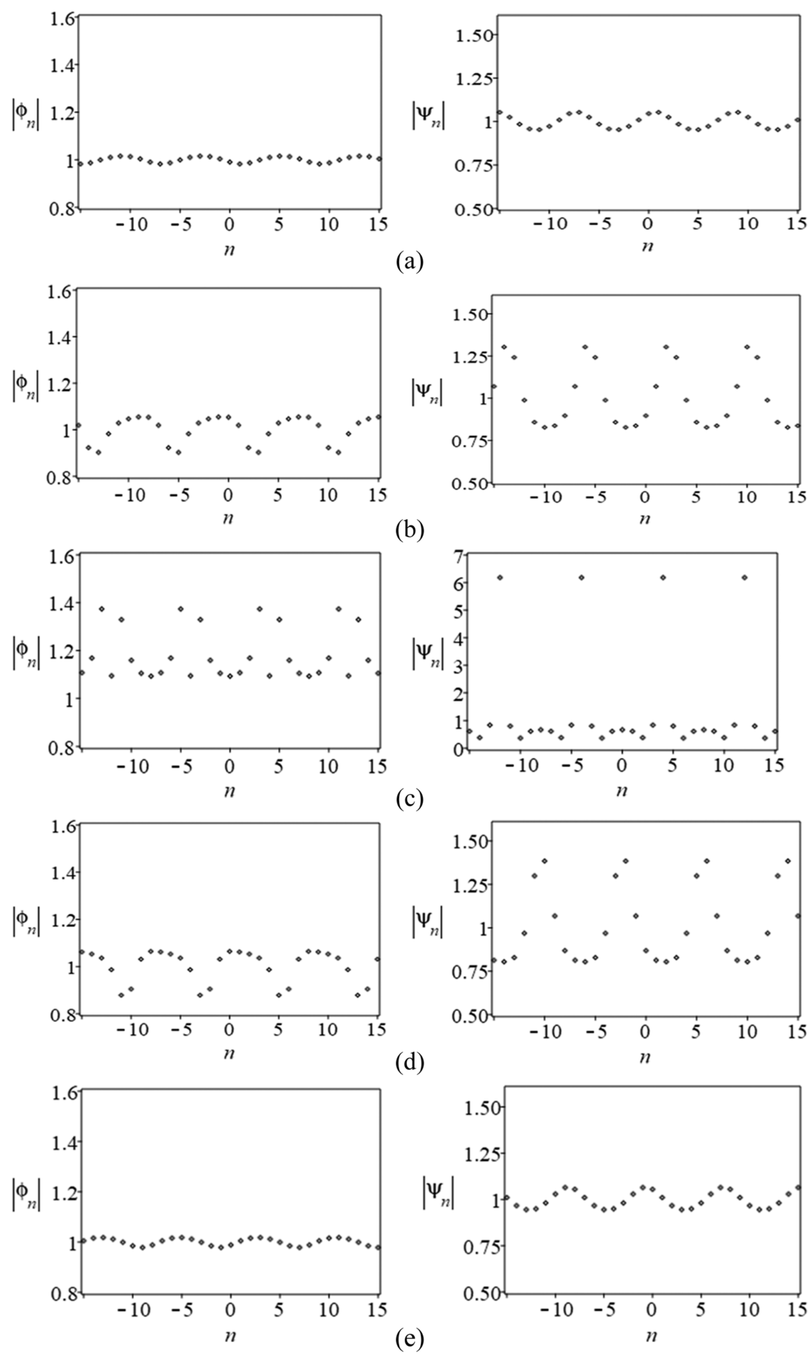

FIG. 5. Typical wave profiles of the breather modes with spatial period 8 for parameters $\sigma = 0.2$, $\rho_1 = \rho_2 = 1$, $k = \pi/4$, $p = i\pi/4$, $\gamma = 0.5$, $\varepsilon = -2$, $\Omega = 0.30 + 1.51i$, (a) $t = -10$, (b) $t = -5$, (c) $t = 0$, (d) $t = 5$, (e) $t = 10$.

Such pulsating modes will usually be localized in time, and may display unexpected features. As illustrative example, the wave profile may possess two distinct peaks in the time periods during maximum displacements (Figure 5).

B. Conservation laws

The existence of spatially periodic breathers provides a beneficial platform in elucidating conservation laws of this system of discrete Hirota equations. For the widely studied Ablowitz-Ladik equations, conservation laws like

$$\sum\left(\phi_n\phi_{n+1}^* + \phi_n^*\phi_{n+1}\right),\ i\sum\left(\phi_n\phi_{n+1}^* - \phi_n^*\phi_{n+1}\right)$$

associated with 'energy' and 'momentum' respectively, do *not* hold for the present Hirota system.[33]

Theoretically, conservation laws are derived by writing the evolution equations for two consecutive grid points and combining quadratic factors in appropriate ways.[33,34] In the present case, energy and momentum conservation principles are combined analytically into one single expression, namely,

$$Y_1 = \sum\left[\left(\phi_n\phi_{n+1}^* + \phi_n^*\phi_{n+1}\right) - i\gamma\left(\phi_n\phi_{n+1}^* - \phi_n^*\phi_{n+1}\right)\right]. \tag{16}$$

The other class of conservation laws can be obtained by combining the evolution equations for both waveguides:

$$Y_2 = \sum\left[\left(\phi_n\phi_{n+1}^* + \phi_n^*\phi_{n+1}\right) - \varepsilon\left(\psi_n\psi_{n+1}^* + \psi_n^*\psi_{n+1}\right)\right]. \tag{17}$$

In both cases, the summation is taken over one spatial period. For spatially localized modes, the summation might be performed over the entire domain and vanishing boundary condition would be needed, but such details would not be pursued here. To verify that the quantities Y_1, Y_2 are indeed invariant in time, numerical computations over one spatial period confirm the accuracy of our claim (Figure 6). Higher order conservation laws probably exist, but algebraic verification will be involved. The existence of these low order conservation laws enhances the confidence on the special character of the present coupled discrete Hirota equations.

These considerations also carry additional theoretical significance from another perspective. If we perform a transformation (ξ = a real constant)

$$\phi_n = \exp(in\xi)u_n, \quad \psi_n = \exp(in\xi)v_n, \tag{18}$$

then we would obtain a new coupled Ablowitz-Ladik system

$$i(u_n)_t + (u_{n+1} + u_{n-1})(1+\gamma^2)^{1/2}[\beta + \sigma(|u_n|^2 + \varepsilon|v_n|^2)] = 0, \tag{19a}$$

$$i(v_n)_t + (v_{n+1} + v_{n-1})(1+\gamma^2)^{1/2}[\beta + \sigma(|u_n|^2 + \varepsilon|v_n|^2)] = 0, \tag{19b}$$

provided that ξ is chosen as

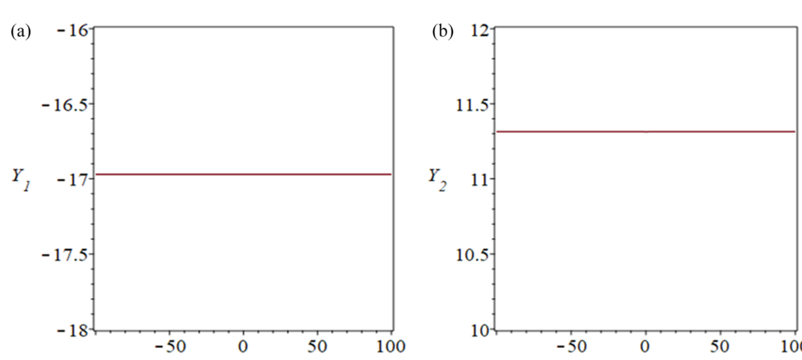

FIG. 6. The time invariance of the conserved quantities Y_1 and Y_2 defined by Eq. (16) and Eq. (17). The input parameters are the same as those in Figure 5.

$$\exp(i\xi) = (1 - i\gamma)/(1 + \gamma^2)^{1/2}.$$

In many practical applications, we need to impose a periodic boundary condition $\phi_n = \phi_{n+2N}$, $\psi_n = \psi_{n+2N}$, for a range of lattice points of length $2N$ (N = an integer). In view of Eq. (18), a periodic solution of the Ablowitz-Ladik system will ***not*** solve the discrete Hirota equation (or vice versa), except for very special values of γ.

VI. STRUCTURAL STABILITY AND TEST FOR ROBUSTNESS

The dynamical evolution of rogue waves in continuous systems under perturbed initial conditions, as well as the possible emergence of rogue modes from a general random state, have been demonstrated to be related to the modulation instability of plane waves.[27,29,34] Usually such instability is necessary

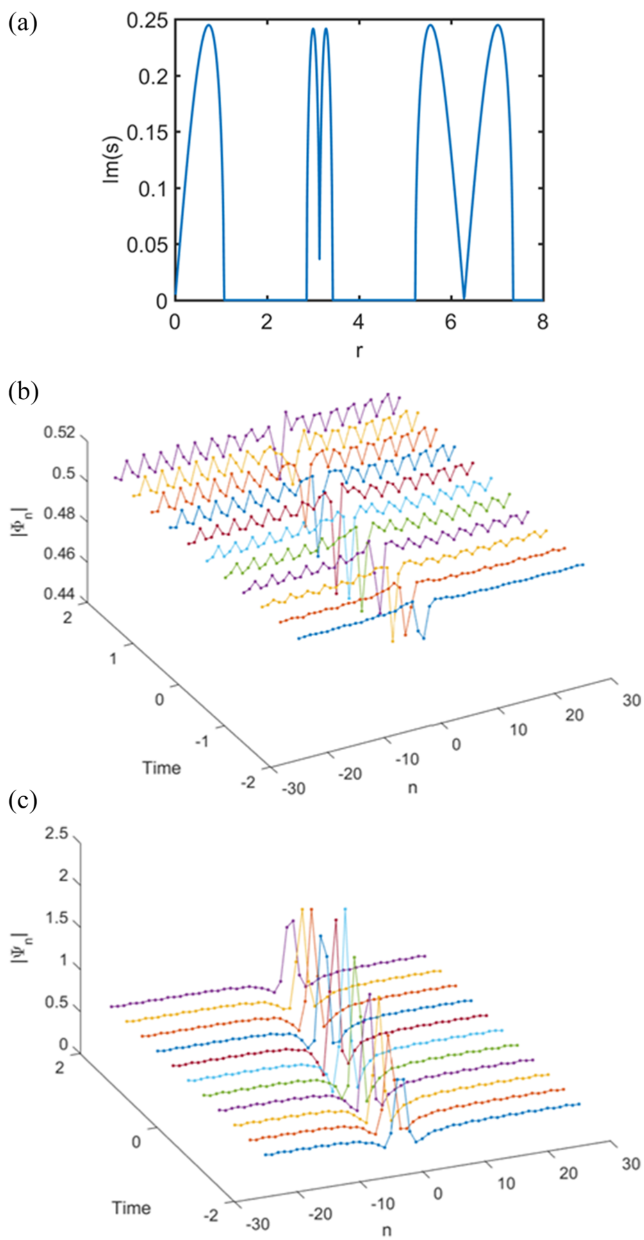

FIG. 7. Persistence and robustness of rogue waves in the presence of 'weak' modulation instability, (a) gain spectrum of growth rate (Im(s)) versus wavenumber r; (b) and (c): evolution of ϕ_n and ψ_n for $k = 0.5$, $\sigma = 1$, $\varepsilon = -1$, $\gamma = 0$, $\rho_1 = \rho_2 = 0.5$.

for the existence of rogue waves. However, sufficiently strong instability might overwhelm the natural development of rogue waves as background noises also grow rapidly, a feature also tested for discrete evolution equations too.[5]

It is thus instructive to examine this idea for the present coupled discrete Hirota equations too, especially for the new regimes. The computational techniques adopted will be a conceptually straightforward fourth order Runge-Kutta scheme.

When the modulation instability is relatively 'weak' (or roughly 0.25 for the illustrative example shown in Figure 7a), the main evolution of the rogue waves in the two waveguides can still be maintained (Figures 7b and 7c), despite the growth of the 1% noise imposed initially.

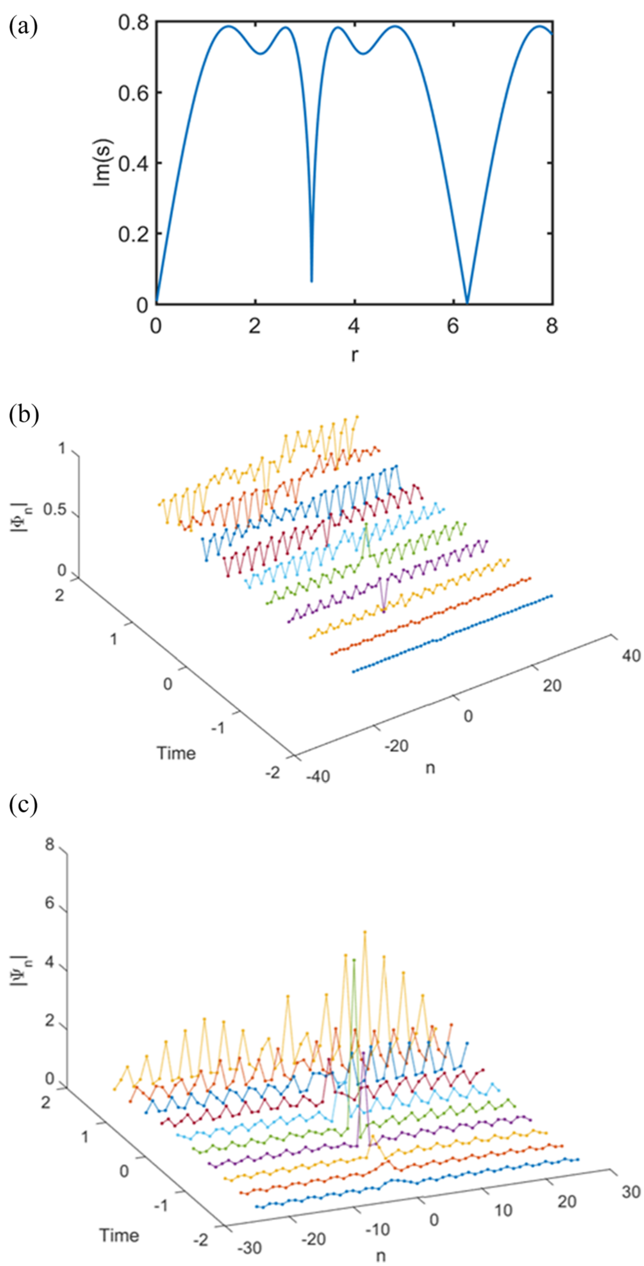

FIG. 8. Significant growth of the background noise and overwhelming of the rogue wave in the presence of 'strong' modulation instability, (a) instability gain spectrum of growth rate (Im(s)) versus wavenumber r; (b) and (c): evolution of of ϕ_n and ψ_n for $k = 0.5, \sigma = 3, \varepsilon = -1, \gamma = 0, \rho_1 = 0.5, \rho_2 = 0.5$.

However, when the modulation instability gain is much stronger (or roughly being tripled to 0.8 by increasing the cubic nonlinearity parameter σ, Figure 8a), the evolution of the rogue wave is being overwhelmed by the much stronger growth of the background disturbance (Figures 8b and 8c).

VII. DISCUSSIONS AND CONCLUSIONS

The two-component discrete Hirota system (a set of coupled evolution equations) is investigated by the bilinear transform. The most striking feature here is that the self-phase modulations and cross-phase modulations are allowed to have opposite signs. Such scenario has been proposed in connection with the electromagnetic wave propagation in left-handed materials (metamaterials), if values of the electric permittivity and magnetic permeability attain different signs.[36] The theoretical formulations for the continuous counterparts under these conditions, namely, coupled nonlinear Schrödinger equations, have revealed that shape changes and energy exchange among localized modes are feasible.[37–40] We have demonstrated here that, on changing the signs of these phase modulations for discrete systems, intriguing dynamics can occur too. These issues will likely generate promising lines of research in nonlinear science.[41]

On the more technical aspects about modes and evolution for discrete equations, breathers are derived analytically and rogue waves are obtained through a long wave limit. Onset of modulation instability is related to the existence criterion of rogue waves. Although a few of these themes have also been covered in similar studies for other continuous and discrete dynamical systems, several new features are revealed here:

- Rogue waves and solitons for the single component Hirota equation[11,13,15,24] and coupled Ablowitz-Ladik systems[12,14] have indeed been generated analytically earlier in the literature. We demonstrate here that the maximum displacements of rogue waves can exhibit pulsation, oscillating from a position above the mean level to one below during the growth and decay cycle. This feature contrasts sharply with the Peregrine breather of the nonlinear Schrödinger equation, where the elevation portion always stays above the mean level.[18,19]
- Coupled waveguides can display an intriguing ‘reverse of polarity’ property, where a ‘depression’ rogue wave can be ‘flipped’ to an ‘elevation’ rogue wave after being affected by the other waveguide or vice versa.
- The transition from the Ablowitz-Ladik case to the Hirota system is elucidated, through a detailed study of the property of rogue waves as the parameter representing the ‘discrete’ third order dispersion changes from zero to a finite value. The signs of the self- phase modulation and cross-phase modulation play a critical role, with distinct dynamical patterns for different input parameters.
- Although modulation instability has been studied for Ablowitz-Ladik and Hirota systems,[5,13] coupling induced instability and thus probably new rogue wave modes are demonstrated for discrete equations here. This property has been well established for continuous nonlinear Schrödinger equations,[42] but apparently has not received adequate attention for discrete cases.
- Spatially periodic breathers are used to establish conservation laws, which are different from those of the Ablowitz-Ladik and the single component Hirota equations.[6,13,34]
- The evolution of rogue wave modes under perturbed initial conditions is intimately related to modulation instability, as strong instability might overwhelm the effect of rogue waves through violent growth of the background noise.[30,43] To demonstrate this connection, we study the evolution of rogue waves with a 1% noise in the new instability regime and confirm this property. Instead of focusing on the noise level, a direct correlation to the magnitude of modulation instability is made.[5,13]

We expect that further intriguing and elegant properties will be discovered or revealed for other exotic discrete nonlinear evolution equations. The present paper serves as illustrative example. Rogue waves for the single component Ablowitz-Ladik and the discrete Hirota equations have been derived by the bilinear scheme in the literature.[3,9,44,45] We extend these considerations to a two-component case with dispersion, SPM and XPM of various sign combinations. We should also mention that inverse scattering techniques and analysis on pole movements might also be applicable to this type of problems.[46,47] Discrete breathers are of immense importance in various physical applications.[26] The coupled Hirota system should provide a valuable addition by providing explicit analytical description of localized modes. Further works along these directions will definitely be fruitful.[48]

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