On the Synthesis of Static Output Feedback Controllers for Guaranteed RMS Gain of Switched Systems with Arbitrary Switching

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Abstract

This paper addresses the problem of determining static output feedback controllers for ensuring desired upper bounds of the root mean square (RMS) gain of continuous-time switched linear systems with arbitrary switching. The problem is addressed by searching for a homogeneous rational Lyapunov function (HRLF) parameterized rationally by the sought controller, and by introducing a polynomial for quantifying the feasibility of the Lyapunov inequalities. It is shown that such a controller exists if and only if a condition built solving three convex optimization problems with linear matrix inequalities (LMIs) holds for polynomials of degree sufficiently large.

I. Introduction

Switched systems are dynamical systems allowed to change with the time in a finite family under the selection of a signal called switching rule. Switched systems are generally classified into two main classes: switched systems with dwell time constraints and switched systems with arbitrary switching. In the former class, the changes among the mathematical models can occur only after a minimum time, called dwell time, which can be hard or average. In the latter class, the changes among the mathematical models can occur arbitrarily fast.

A fundamental problem in switched systems is stability analysis. See for instance the books [3], [17], [24] and the surveys [10], [18], [19] for general results. Other works include [4], [11], [15], [16], [25]. In particular, sufficient linear matrix inequality (LMI) conditions have been proposed in [12] based on quadratic Lyapunov functions, and necessary and sufficient LMI conditions have been proposed in [8], [9] based on homogeneous polynomial Lyapunov functions (HPLFs).

Another fundamental problem in switched systems is performance analysis, in particular concerning the root mean square (RMS) gain. The RMS gain has been studied for switched linear systems in [14], [19], [20] through techniques such as variational principles and worst-case control. LMI conditions have been proposed in order to determine upper bounds of the RMS gain through convex optimization. In particular, sufficient LMI conditions based on quadratic Lyapunov functions have been proposed in [13], and necessary and sufficient LMI conditions based on homogeneous rational Lyapunov functions (HRLFs) have been proposed in [6].

Unfortunately, the synthesis of output feedback controllers for ensuring desired upper bounds of the RMS gain of switched systems is still an open problem, and few contributions can be found, see for instance [1], [2] where sufficient LMI conditions have been proposed. Indeed, by letting the controller be a decision variable in the existing conditions for establishing upper bounds of the RMS gain, one obtains nonconvex optimization problems, in general due to the presence of products between the Lyapunov function and the controller.

This paper[1] addresses the problem of determining static output feedback controllers for ensuring desired upper bounds of the RMS gain of continuous-time switched linear systems with arbitrary switching. The problem is addressed by searching for an HRLF parameterized rationally by the sought controller, and by introducing a polynomial for quantifying the feasibility of the Lyapunov inequalities. It is shown that such a controller exists if and only if a condition built solving three convex optimization problems with LMIs holds for polynomials of degree sufficiently large. This paper extends our work [7] where the synthesis of stabilizing controllers for switched systems is addressed.

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II. PRELIMINARIES

The notation is as follows. $\mathbb{N}$, $\mathbb{R}$: sets of non-negative integers and real numbers. Unless specified otherwise, x_i denotes the i-th entry of a vector x, and $X_{i,j}$ denotes the (i,j)-th entry of a matrix X. 0, I: null matrix and identity matrix of size specified by the context. A': transpose of A. $\mathrm{he}(A)$: $A+A'$. $\mathrm{spec}(A)$: set of eigenvalues of A. $A\otimes B$: Kronecker's product between A and B. $A^{\otimes n}$: n-th Kronecker power. $\|A\|_2$, $\|A\|_\infty$ and $\|A\|_{Fro}$: 2-norm, ∞-norm and Frobenius' norm of A. $\|a(\cdot)\|_{\mathcal{L}_2}$: $\mathcal{L}_2$-norm of $a(t)$, i.e., $\|a\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty \|a(t)\|_2^2 dt}$. a^b: $a_1^{b_1}a_2^{b_2}\cdots$, where a and b are vectors. $A\geq 0$ (respectively, $A>0$): symmetric positive semidefinite (respectively, definite) matrix A. A function $V:\mathbb{R}^{m\times p}\to\mathbb{R}^{q\times r}$ is said to be a matrix polynomial if the entries of $V(K)$ are polynomials in the entries of K. $\star$: corresponding block in a symmetric matrix. s.t.: subject to.

Let us consider the switched system

$$\begin{cases} \dot{x}(t) &=& A_{1,\sigma(t)}x(t)+B_{1,\sigma(t)}u(t)+B_{2,\sigma(t)}w(t)\\ y(t) &=& C_{1,\sigma(t)}x(t)+D_{1,\sigma(t)}u(t)+D_{2,\sigma(t)}w(t)\\ z(t) &=& C_{2,\sigma(t)}x(t)+D_{3,\sigma(t)}u(t)+D_{4,\sigma(t)}w(t)\\ \sigma(\cdot) &\in& \mathcal{D}_{arb}\end{cases} \tag{1}$$

where $t\in\mathbb{R}$ is the time, $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^{m_1}$ is the control input, $w(t)\in\mathbb{R}^{m_2}$ is the external input, $y(t)\in\mathbb{R}^{p_1}$ is the control output, $z(t)\in\mathbb{R}^{p_2}$ is the external output, $\sigma:\mathbb{R}\to\{1,\ldots,N\}$ is the switching rule, $\mathcal{D}_{arb}$ is the set of arbitrary switching rules

$$\mathcal{D}_{arb} = \{\sigma:\mathbb{R}\to\{1,\ldots,N\},\ \text{the state exists}\}, \tag{2}$$

and $A_{1,i}$, ..., $D_{4,i}$, $i=1,\ldots,N$, are real matrices of suitable sizes. The system (1) obtained for $\sigma(t)=i$ is called the i-th subsystem of the switched system (1).

Definition 1: The RMS gain of (1) is

$$\gamma_{RMS} = \sup_{\substack{u(\cdot)\\ \sigma(\cdot)\in\mathcal{D}_{arb}}} \frac{\|y(\cdot)\|_{\mathcal{L}_2}}{\|u(\cdot)\|_{\mathcal{L}_2}} \tag{3}$$

where $y(t)$ is the solution in (1) for $x(0^-)=0$. □

The switched system (1) is controlled by a mode-independent static output feedback controller, i.e.,

$$u(t) = Ky(t) \tag{4}$$

where $K \in \mathbb{R}^{m_1 \times p_1}$ has to be determined in the set

$$\mathcal{K} = \left\{ K \in \mathbb{R}^{m_1 \times p_1} : \ \|K\|_\infty \leq \rho \right\} \tag{5}$$

where $\rho \in \mathbb{R}$ is a given bound. Let us assume that the map between $u(t)$ and $y(t)$ is strictly proper, i.e.,

$$D_{1,i} = 0. \tag{6}$$

The closed-loop system is

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A_{\sigma(t)}(K)x(t) + B_{\sigma(t)}(K)w(t) \\ z(t) & = & C_{\sigma(t)}(K)x(t) + D_{\sigma(t)}(K)w(t) \\ \sigma(\cdot) & \in & \mathcal{D}_{arb} \end{array} \right. \tag{7}$$

where

$$\left\{ \begin{array}{rcl} A_i(K) & = & A_{1,i} + B_{1,i}KC_{1,i} \\ B_i(K) & = & B_{2,i} + B_{1,i}KD_{2,i} \\ C_i(K) & = & C_{2,i} + D_{3,i}KC_{1,i} \\ D_i(K) & = & D_{4,i} + D_{3,i}KD_{2,i}. \end{array} \right. \tag{8}$$

Problem 1: Determine $K \in \mathcal{K}$ such that the RMS gain of (7) is smaller than a desired value γ. □

Let us observe that, in order for Problem 1 to admit a solution, (1) must admit a static output feedback controller that stabilizes it with arbitrary switching. It is not required to know a priori such a controller, and it is not required that (1) is stable with arbitrary switching. The dependence on t of the various quantities will be omitted in the sequel of the paper for ease of notation unless specified otherwise.

III. Proposed Methodology

The approach proposed in this paper for solving Problem 1 is based on the use of Lyapunov functions in the class of the homogeneous rational functions, i.e., functions that can be expressed as the ratio of homogeneous polynomials. Let us start by providing some basic definitions.

A function $v : \mathbb{R}^n \to \mathbb{R}$ is said to be a homogeneous polynomial of degree $2d$, $d \in \mathbb{N}$, if

$$v(x) = \sum_{\substack{c \in \mathbb{N}^n \\ c_1 + \ldots + c_n = 2d}} a_c x^c \tag{9}$$

for some $a_c \in \mathbb{R}$. The set of such functions is denoted by

$$\mathcal{P}_{2d} = \{v : \mathbb{R}^n \to \mathbb{R} : \text{ (9) holds}\} . \tag{10}$$

Homogeneous polynomials can be expressed in several ways. Hereafter, we adopt an expression based on symmetric matrices. Let us denote with $b\,(x,d)$ a vector whose entries are all the monomials in x of degree $d \in \mathbb{N}$ with unitary coefficient without repetition. The length of $b\,(x,d)$ is given by

$$c(n,d) = \frac{(n+d-1)!}{(n-1)!d!}. \tag{11}$$

By using the vector $b\,(x,d)$, a homogeneous polynomial $v(x)$ of degree $2d$ can be expressed as

$$v(x) = b\,(x,d)' \, V b\,(x,d) \tag{12}$$

for some $V = V' \in \mathbb{R}^{c(n,d)\times c(n,d)}$. The representation (12) is known as Gram matrix method or square matricial representation (SMR), see for instance [5] and references therein.

A function $v : \mathbb{R}^n \to \mathbb{R}$ is said to be a homogeneous rational function of total degree $r \in \mathbb{N}$ and relative degree $s \in \mathbb{N}$, with $r \geq s$, if

$$\begin{cases} v(x) & = & \dfrac{\phi(x)}{\psi(x)} \\ \phi & \in & \mathcal{P}_r \\ \psi & \in & \mathcal{P}_{r-s}. \end{cases} \tag{13}$$

The set of such functions is denoted by

$$\mathcal{Q}_{r,s} = \{v : \mathbb{R}^n \to \mathbb{R}, \text{ (13) holds}\} . \tag{14}$$

Lyapunov functions in the class of the homogeneous rational functions, i.e., HRLFs, have been introduced in [6] to derive upper bounds of the RMS gain of switched systems. In particular, these HRLFs are searched for in the set $\mathcal{Q}_{2d,2}$, and can be expressed as

$$\begin{cases} v(x) & = & \dfrac{\phi(x)}{\psi(x)} \\ \phi(x) & = & b\,(x,d)' \, \Phi b\,(x,d) \\ \psi(x) & = & b\,(x,d-1)' \, \Psi b\,(x,d-1) \end{cases} \tag{15}$$

where $\Phi = \Phi' \in \mathbb{R}^{c(n,d)\times c(n,d)}$ and $\Psi = \Psi' \in \mathbb{R}^{c(n,d-1)\times c(n,d-1)}$. Let us define

$$r(x,d,u) = \begin{pmatrix} b\,(x,2d-1) \\ b\,(x,2d-2) \otimes u \end{pmatrix} \tag{16}$$

and

$$\bar{\mathcal{L}} = \left\{ \tilde{L} = \tilde{L}' : \; r(x,d,u)'\tilde{L}r(x,d,u) = 0 \right\}. \tag{17}$$

The following result, proposed in [6], provides a necessary and sufficient LMI condition for establishing upper bounds of the RMS gain of (1).

Theorem 1 ([6]): Let $\Psi > 0$ be chosen. The RMS gain of (1) satisfies (for a fixed controller K)

$$\gamma_{RMS} < \gamma \tag{18}$$

if and only if, for some d, there exist Φ and Θ_i, $i = 1, \ldots, N$, satisfying the system of LMIs

$$\begin{cases} 0 & < & \Phi \\ 0 & > & F_i(K,\Phi) + G_i(K) + L(\Theta_i) \quad \forall i = 1, \ldots, N \end{cases} \tag{19}$$

where $L(\cdot)$ is a linear parametrization of $\bar{\mathcal{L}}$, and $F_i(K,\Phi)$ and $G_i(K)$ are linear matrix functions of Φ whose definitions are reported in the Appendix. □

Unfortunately, the condition provided by Theorem 1 cannot be used directly for solving Problem 1 because the second inequality of (19) would be nonlinear in the decision variables K and Φ.

The first idea for coping with this problem is to introduce a novel class of Lyapunov functions, specifically HRLFs depending rationally on the controller. These functions can be expressed as

$$v(x,K) = \frac{b\left(x,d\right)'\bar{\Phi}(K)b\left(x,d\right)}{b\left(x,d-1\right)'\Psi b\left(x,d-1\right)} \tag{20}$$

with

$$\bar{\Phi}(K) = \frac{\Phi(K)}{\zeta(K)} \tag{21}$$

where $\Phi(K) = \Phi(K)'$ and $\zeta(K)$ are matrix polynomials to be determined, and $\Psi = \Psi'$ is a chosen matrix. Let us define the matrix polynomials

$$\begin{cases} M_1(K) & = & \Phi(K) \\ M_{i,2}(K) & = & -F_i(K,\Phi(K)) - \zeta(K)G_i(K) \\ & & -L(\Theta_i(K)) - \xi(K)I \\ M_3(K) & = & \zeta(K) - 1 \end{cases} \tag{22}$$

where $L(\cdot)$ is a linear parametrization of $\bar{\mathcal{L}}$, and $\xi(K)$, $\Phi(K)$, $\Theta_i(K)$ and $\zeta(K)$ are matrix polynomials to be determined. The first optimization problem we define is

$$\begin{array}{rl} \sup\limits_{\xi(\cdot),\Phi(\cdot),\Theta_i(\cdot),\zeta(\cdot)} & h \\ \text{s.t.} & \left\{ \begin{array}{l} M(\cdot) \in \Sigma \ \forall M(\cdot) \in \mathcal{M} \\ h \leq 1 \end{array} \right. \end{array} \tag{23}$$

where $\mathcal{M}$ is a set containing the matrix polynomials in (22), Σ is the set of matrix polynomials that can be expressed as sums of squares of matrix polynomials, and

$$h = \int_{\mathcal{K}} \xi(K) dK. \tag{24}$$

The optimization problem (23) aims at maximizing the polynomial $\xi(K)$ over the set $\mathcal{K}$ and, hence, the positive semidefiniteness of $M_{i,2}(K)$ over this set. This optimization problem is an SDP because the cost function is linear and the constraints are equivalent to LMIs, see [5] and references therein for details.

The next step is to determine a candidate for the sought controller based on the solution of (23). To this end, let $\xi^*(K)$ be a maximizer of $\xi(K)$ in (23). Let us define

$$b(K) = a - \xi^*(K) - \sum_{\substack{i=1,\dots,m_1 \\ j=1,\dots,p_1}} (\rho^2 - K_{i,j}^2) s_{i,j}(K) \tag{25}$$

where $a \in \mathbb{R}$ and $s_{i,j}(K)$, $i = 1, \dots, m$, $j = 1, \dots, p$, are polynomials to determine. The second optimization problem we define is

$$\begin{array}{rl} \inf\limits_{a, s_{i,j}(\cdot)} & a \\ \text{s.t.} & b(K), s_{i,j}(K) \in \Sigma. \end{array} \tag{26}$$

The optimization problem (26) is an SDP analogously to (23), and aims at determining the maximum of $\xi^*(K)$ over the set $\mathcal{K}$. Indeed, the maximizer of $\xi^*(K)$ over the set $\mathcal{K}$ is the best candidate for the sought controller since $\xi^*(K)$ quantifies the feasibility of the inequalities of (19). In order to determine such a maximizer, let a^* and $s_{i,j}^*(K)$ be minimizers of a and $s_{i,j}(K)$ in (26), and let $b^*(K)$ be $b(K)$ evaluated with such minimizers. Let us define the set

$$\mathcal{Z} = \{ K \in \mathcal{K} : \ b^*(K) = 0, \ \xi^*(K) = a^* \}. \tag{27}$$

The following result provides a necessary and sufficient condition for solving Problem 1.

Theorem 2: There exists K that solves Problem 1 if and only if there exists K^* in the set $\mathcal{Z}$ that solves this problem for sufficiently large degrees of the polynomials introduced.

Proof. The sufficiency is obvious, hence let us consider the necessity. Suppose there exists K that solves Problem 1, and let us indicate such a value as $K^{\#}$. For the chosen $\Psi > 0$, let $d \in \mathbb{N}$ be such that the LMI feasibility test (19) holds with $K = K^{\#}$ (observe that such a value of d does exist from Theorem 1). Let us define the function $\tilde{\xi}^* : \mathbb{R}^{m \times p} \to \mathbb{R}$ as

$$\tilde{\xi}^*(K) = \sup_{\tilde{\xi}, \tilde{\Phi}, \tilde{\Theta}_i} \tilde{\xi}$$

$$\text{s.t.} \left\{ \begin{array}{rcl} 0 & \leq & \tilde{\Phi} - \beta I \\ 0 & \leq & -F_i(K, \tilde{\Phi}) - G_i(K) - L(\tilde{\Theta}_i) \\ & & -\tilde{\xi} I \quad \forall i = 1, \ldots, N \end{array} \right.$$

where $\beta \in \mathbb{R}$ is an auxiliary quantity. Since the LMI feasibility test (19) holds with the changes mentioned above, it follows that $\tilde{\xi}^*(K^{\#}) > 0$ for some β positive and sufficiently small. Let us consider such a value of β hereafter. Since $\mathcal{K}$ is compact, the matrix functions $\tilde{\xi}^*(K)$, $\tilde{\Phi}^*(K)$ and $\tilde{\Theta}_i^*(K)$ can be approximated arbitrarily well over $\mathcal{K}$ through matrix polynomials. Since $\tilde{\xi}^*(K^{\#}) > 0$, it follows that there exist matrix polynomials $\hat{\xi}(K)$, $\hat{\Phi}(K)$ and $\hat{\Theta}_i(K)$ such that $\hat{\xi}(K^{\#}) > 0$ and the following matrix polynomials are positive definite for all $K \in \mathbb{R}^{m_1 \times p_1}$:

$$\left\{ \begin{array}{rcl} \hat{M}_1(K) & = & \hat{\Phi}(K) - \beta I \\ \hat{M}_{i,2}(K) & = & -F_i(K, \hat{\Phi}(K)) - G_i(K) \\ & & -L(\hat{\Theta}_i(K)) - \hat{\xi}(K) I. \end{array} \right.$$

Let us define the polynomial $f(K) = (1 + \|K\|_{Fro}^2)^\eta$ where $\eta \in \mathbb{N}$. From [22] it follows that there exists η such that the matrix polynomials above, multiplied by $f(K)$, are in Σ. Since the matrix polynomials in (22) are homogeneous in the set of variables $\xi(K)$, $\Phi(K)$, $\Theta_i(K)$ and $\zeta(K)$, and since $f(K) - 1$ is in Σ (being sum of powers of $\|K\|_{Fro}^2$ multiplied by positive coefficients), one has that the constraints of (23) can be satisfied by choosing

$$\left\{ \begin{array}{rcl} \xi(K) & = & f(K)\hat{\xi}(K) \\ \Phi(K) & = & f(K)\hat{\Phi}(K) \\ \Theta_i(K) & = & f(K)\hat{\Theta}_i(K) \\ \zeta(K) & = & f(K). \end{array} \right.$$

This implies that matrix polynomials $\xi(K)$, $\Phi(K)$, $\Theta_i(K)$ and $\zeta(K)$ such that $M(\cdot) \in \Sigma$ for all $M(\cdot) \in \mathcal{M}$ and $\xi(K) > 0$ for some $K \in \mathcal{K}$ can be obtained by maximizing the integral of

$\xi(K)$ over $\mathcal{K}$ under the constraints of (23). Indeed, above we have shown that there exist such matrix polynomials without considering the constraint $h \leq 1$. If this constraint is satisfied by such matrix polynomials, then they can be considered, and $\xi(K) > 0$ for $K = K^{\#}$. Otherwise, we redefine $\xi(K)$ as

$$\xi(K) \ \rightarrow \ \xi(K) - \alpha$$

where

$$\alpha = \left(\int_{\mathcal{K}} dK \right)^{-1} \left(\int_{\mathcal{K}} \xi(K) dK - 1 \right),$$

and this ensures that $\xi(K)$ is still positive for some $K \in \mathcal{K}$ (since the integral of $\xi(K)$ over $\mathcal{K}$ is 1) and that the constraints of (23) hold (since $\alpha > 0$).

Next, let us define

$$\theta = \sup_{K \in \mathcal{K}} \xi^*(K).$$

Let us observe that $\rho^2 - K_{i,j}^2 \geq 0$ for all $i = 1, \ldots, m_1$ and $j = 1, \ldots, p_1$ if and only if $K \in \mathcal{K}$, moreover the polynomials $\rho^2 - K_{i,j}^2$ have even degree and the highest degree forms are zero if and only if $K = 0$. From Putinar's Positivstellensatz [21], it follows that, for all $a > \theta$, there exist polynomials $s_{i,j}(K)$ such that the constraints of (26) are satisfied. Hence, for polynomials $s_{i,j}(K)$ with sufficiently large degrees, one gets

$$a^* = \theta.$$

Let K^* be a maximizer of $\xi^*(K)$ over $\mathcal{K}$, i.e., $\xi^*(K^*) = \theta = a^*$. Since $b(K)$ is in Σ, it follows that

$$\begin{aligned} 0 \ &\leq \ b(K^*) \\ &= \ a^* - \xi^*(K^*) - \sum_{\substack{i=1,\ldots,m_1 \\ j=1,\ldots,p_1}} (\rho^2 - (K_{i,j}^*)^2) s_{i,j}^*(K^*) \\ &\leq \ 0. \end{aligned}$$

Hence, $b(K^*) = 0$ and, therefore, $K^* \in \mathcal{Z}$. Moreover, $\xi^*(K^*) > 0$ and, therefore, K^* solves the problem. □

Theorem 2 provides a strategy for solving Problem 1 based on (23) and (26). This strategy consists of narrowing the search space for the sought controller from the original set, i.e., the set $\mathcal{K}$, to a reduced one, i.e., the set $\mathcal{Z}$. Indeed, the set $\mathcal{Z}$ contains one element only in non-degenerate cases, specifically, the maximizer of $\xi^*(K)$ over the set $\mathcal{K}$. Once the set $\mathcal{Z}$ is found, one just checks if any of the controllers included in such a set solves Problem 1, for instance by using the LMI feasibility test (19).

The set $\mathcal{Z}$ can be determined from the Gram matrix of the polynomial $b^*(K)$ found when solving (26). Specifically, this determination involves the computation of the null space of this matrix, pivoting operations, and the computation of the roots of a polynomial in one variable. See [5] and references therein for details.

The second constraint in (23) is introduced in order to ensure that the solution of the optimization problem is bounded. The constant 1 on the right hand side of this constraint can be replaced with any other positive number, and the constraint itself is unnecessary in typical cases.

Let us observe that the sufficiency of Theorem 2 is achieved for any degrees of the polynomials introduced. The necessity, instead, is achieved for sufficiently large degrees of these polynomials. Some guidelines for choosing these degrees are as follows. First, choose d (which defines the degree of the HRLF), the degree of $\Phi(K)$ (denoted by d_Φ) and the degree of $\zeta(K)$ (denoted by d_ζ). Second, set the degrees of $\zeta(K)$, $\Theta_i(K)$ and $\xi(K)$ equal to the degree of $F_i(K, \Phi(K))$. Third, set the degrees of the polynomials $s_{i,j}(K)$ as the largest degrees for which $b(K)$ has its minimum degree. Summarizing, one chooses d, d_Φ and d_ζ, and the other degrees are automatically selected.

IV. Example

In this section we present an illustrative example of the proposed methodology. The LMI problems are solved by using the toolbox SeDuMi [23] for Matlab on a personal computer with Windows 10, Intel Core i7, 3.4 GHz, 8 GB RAM. The matrix Ψ is simply chosen as the diagonal matrix such that $\psi(x) = \|x\|_2^{2d-2}$. The matrix polynomials $\Theta_i(K)$ and $\zeta(K)$ are chosen of degree 0. For brevity of description, it is assumed $B_{1,i} = B_{2,i}$, $C_{1,i} = C_{2,i}$ and $D_{j,i} = 0$ in (1).

Let us consider (1) with $N = 2$ and

$$\begin{cases} A_1 = \begin{pmatrix} 3 & 7 & 0 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix}, & A_2 = \begin{pmatrix} 3 & 4 & -2 \\ -1 & -1 & -1 \\ 1 & 0 & -2 \end{pmatrix} \\ B_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, & B_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & C_2 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \end{cases}$$

The problem is to determine $K = (k_1, k_2)$ in the set $\mathcal{K} = \{K \in \mathbb{R}^2 : \ \|K\|_\infty \leq 10\}$ such that the RMS gain of (7) is smaller than $\gamma = 10$.

First of all, let us observe that the RMS gain of the open loop switched system is unbounded. Indeed, the matrices A_1 and A_2 are not Hurwitz, since

$$\begin{cases} \mathrm{spec}(A_1) & = & \{-0.158, 1.079 \pm j2.275\} \\ \mathrm{spec}(A_2) & = & \{-2.166, 1.083 \pm j1.587\}. \end{cases}$$

Let us use the methodology proposed in Section III. First, we solve (23). We choose to search for HRLFs of total degree $2d = 2$ with linear dependence in the controller (i.e., $d_V = 1$). We find

$$\begin{array}{rcl} \xi^*(K) & = & 10^{-3}\left(-0.898k_1^2 + 0.036k_1k_2 - 3.814k_1\right. \\ & & \left. -0.240k_2^2 - 0.389k_2 - 3898.461\right). \end{array}$$

The number of LMI scalar variables is 69, and the computational time is less than 1 second. Second, we solve (26), finding

$$\mathcal{Z} = \{(-2.143, -0.972)\}.$$

The number of LMI scalar variables is 19, and the computational time is less than 1 second. Third, we test the LMI feasibility test (19) for $K = K^*$ with $K^* = (-2.143, -0.972)$ being the element of $\mathcal{Z}$, finding that this test holds. The number of LMI scalar variables is 3, and the computational time is less than 1 second. Hence, the found controller K^* solves the problem.

It is worth remarking that even the simpler problem of finding a stabilizing static output feedback controller for any of the subsystems in this example is a nonconvex optimization problem and, hence, hard to be solved. On the other hand, the proposed approach is able to find, with convex optimization, a static output feedback controller that not only stabilizes all the subsystems but also ensures a guaranteed RMS gain for all possible switching rules.

V. Conclusions

This paper has addressed the problem of determining static output feedback controllers for ensuring desired upper bounds of the RMS gain of continuous-time switched linear systems with arbitrary switching. The problem has been addressed by searching for an HRLF parameterized rationally by the sought controller, and by introducing a polynomial for quantifying the feasibility of the Lyapunov inequalities. It has been shown that such a controller exists if and only if a condition built solving three convex optimization problems with LMIs holds for polynomials of degree sufficiently large.

Several directions can be investigated in future works, such as the extension to the presence of dwell-time constraints on the switching rule, and the extension to the synthesis of dynamic output feedback controllers.

Appendix

Hereafter we report the definition of the quantities exploited in Theorem 1. Let us define

$$q(x,d,u) = b\left(x,d-1\right) \otimes u,$$

and let $J_1,\ldots,J_5$ be the matrices satisfying

$$\left\{\begin{array}{rcl} b\left(x,d\right) \otimes b\left(x,d-1\right) & = & J_1 b\left(x,2d-1\right) \\ b\left(x,d-1\right)^{\otimes 2} \otimes x & = & J_2 b\left(x,2d-1\right) \\ b\left(x,d-1\right) \otimes u \otimes b\left(x,d-1\right) & = & J_3 q(x,2d-1,u) \\ b\left(x,d\right) \otimes b\left(x,d-2\right) \otimes u & = & J_4 q(x,2d-1,u) \\ b\left(x,d-1\right)^{\otimes 2} \otimes u & = & J_5 q(x,2d-1,u). \end{array}\right.$$

For $s = 0,1$, let $A_{i,s}(K)$ and $B_{i,s}(K)$ be the matrices satisfying

$$\frac{db\left(x,d-s\right)}{dx} A_i(K)x = A_{i,s}(K) b\left(x,d-s\right)$$

and

$$\frac{db\left(x,d-s\right)}{dx} B_i(K)u = B_{i,s}(K) q(x,d-s,u).$$

Let us define

$$F_i(K,\Phi)=\begin{pmatrix} E_{i,1}(K,\Phi)-E_{i,2}(K,\Phi) & \star \\ F_{i,1}(K,\Phi)'-F_{i,2}(K,\Phi)' & -\gamma^2 F_{i,3}\end{pmatrix}$$

where

$$\begin{cases} E_{i,1}(K,\Phi) &= J_1'\left(\mathrm{he}\left(\Phi A_{i,0}(K)\right)\otimes\Psi\right)J_1 \\ E_{i,2}(K,\Phi) &= J_1'\left(\Phi\otimes\mathrm{he}\left(\Psi A_{i,1}(K)\right)\right)J_1, \end{cases}$$

and

$$\begin{cases} F_{i,1}(K,\Phi) &= J_1'\left(\Phi B_{i,0}(K)\otimes\Psi\right)J_3 \\ F_{i,2}(K,\Phi) &= J_1'\left(\Phi\otimes\Psi B_{i,1}(K)\right)J_4 \\ F_{i,3} &= J_5'\left(\Psi^{\otimes 2}\otimes I\right)J_5. \end{cases}$$

Lastly, let us define

$$G_i(K)=\begin{pmatrix} G_{i,1}(K) & G_{i,2}(K) \\ \star & G_{i,3}(K)\end{pmatrix}$$

where

$$\begin{cases} G_{i,1}(K) &= J_2'\left(\Psi^{\otimes 2}\otimes C_i(K)'C_i(K)\right)J_2 \\ G_{i,2}(K) &= J_2'\left(\Psi^{\otimes 2}\otimes C_i(K)'D_i(K)\right)J_5 \\ G_{i,3}(K) &= J_5'\left(\Psi^{\otimes 2}\otimes D_i(K)'D_i(K)\right)J_5. \end{cases}$$

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