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Article

On Improvements of Kantorovich Type Inequalities

Chang-Jian Zhao [1,*] and Wing-Sum Cheung [2]

1 Department of Mathematics, China Jiliang University, Hangzhou 310018, China
2 Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong, China; wscheung@hku.hk
* Correspondence: chjzhao@163.com

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Abstract: In the paper, we give some new improvements of the Kantorovich type inequalities by using Popoviciu's, Hölder's, Bellman's and Minkowski's inequalities. These results in special case yield Hao's, reverse Cauchy's and Minkowski's inequalities.

Keywords: Popoviciu's inequality; Bellman's inequality; Hölder's weighted inequality; Minkowski's inequality

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1. Introduction

The Pólya–Szegö's inequality can be stated as follows ([1] or ([2], p. 62)).

If u_k and v_k are non-negative real sequences, and $0 < m_1 \leq u_k \leq M_1$, and $0 < m_2 \leq v_k \leq M_2$ for $k = 1, 2, \ldots, n$, then

$$\sum_{k=1}^{n} u_k^2 \sum_{k=1}^{n} v_k^2 \leq \frac{(M_1M_2 + m_1m_2)^2}{4m_1m_2M_1M_2} \left(\sum_{k=1}^{n} u_k v_k \right)^2. \tag{1}$$

The Pólya–Szegö's inequality was studied extensively and numerous variants, generalizations, and extensions appeared in the literature (see [3–6] and the references cited therein). The integral forms of Pólya–Szegö's inequality were recently established in [7–10]. The weighted version of inequality (1) was proved in papers of Watson [11] and Greub and Rheinboldt [12]:

$$\sum_{k=1}^{n} \omega_k u_k^2 \cdot \sum_{k=1}^{n} \omega_k v_k^2 \leq \frac{(M_1M_2 + m_1m_2)^2}{4m_1m_2M_1M_2} \left(\sum_{k=1}^{n} \omega_k u_k v_k \right)^2, \tag{2}$$

where ω_k is a nonnegative n-tuple.

An interesting generalization of Kantorovich type inequality was given by Hao ([13], p. 122), so we shall give his result:

$$\left(\sum_{k=1}^{n} \omega_k u_k^2 \right)^{1/p} \left(\sum_{k=1}^{n} \omega_k v_k^2 \right)^{1/q} \leq \ell \left(\sum_{k=1}^{n} \omega_k u_k v_k \right), \tag{3}$$

where $0 < \frac{1}{q} \leq \frac{1}{p} < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\ell = \frac{qM_1M_2 + pm_1m_2}{pq(m_1M_1)^{1/q}(m_2M_2)^{1/p}}. \tag{4}$$

We recall that, with the name "Kantorovich", we also usually refer to some integral-type extension of classical inequalities, classical pointwise operators, and other mathematical tools—see, e.g., [14–17].

The first aim of this paper is to give a new improvement of the Kantorovich type inequality (3). We combine organically Popoviciu's, Hölder's, and Hao's inequalities to derive a new inequality, which is a generalization of Label (3).

Corresponding to (3), we can obtain a reverse Minkowski's inequality as follows:

$$\ell\left(\sum_{k=1}^{n}\omega_k(u_k+v_k)^2\right)^{1/p} \geq \left(\sum_{k=1}^{n}\omega_k u_k^2\right)^{1/p} + \left(\sum_{k=1}^{n}\omega_k v_k^2\right)^{1/p}, \tag{5}$$

where p, q, ω_k, u_k, v_k are as in (3), and ℓ is definied in (4).

Another aim of this paper is to give a new reverse Minkowski's inequality. We combine organically Bellman's and Minkowski's inequalities to derive a new inequality, which is generalization of the reverse Minkowski's inequality (5).

2. Results

We need the following Lemmas to prove our main results.

Lemma 1. *(Popoviciu's inequality) ([18], p. 58) Let* $p > 0, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $a = \{a_1, \ldots, a_n\}$ *and* $b = \{b_1, \ldots, b_n\}$ *be two series of positive real numbers and such that* $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ *and* $b_1^q - \sum_{i=2}^{n} b_i^q > 0$. *Then,*

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p}\left(b_1^q - \sum_{i=2}^{n} b_i^q\right)^{1/q} \leq a_1 b_1 - \sum_{i=2}^{n} a_i b_i, \tag{6}$$

with equality if and only if $a = \mu b$, *where* μ *is a constant.*

Lemma 2. *(Bellman's inequality) ([19], p. 38) Let* $a = \{a_1, \ldots, a_n\}$ *and* $b = \{b_1, \ldots, b_n\}$ *be two series of positive real numbers and* $p > 1$ *such that* $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ *and* $b_1^p - \sum_{i=2}^{n} b_i^p > 0$, *then*

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^{n} b_i^p\right)^{1/p} \leq \left((a_1+b_1)^p - \sum_{i=2}^{n}(a_i+b_i)^p\right)^{1/p}, \tag{7}$$

with equality if and only if $a = vb$, *where* v *is a constant.*

Lemma 3. *(Hölder's weighted inequality) ([13], p. 100) Let* $p > 0, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* a_k, b_k *and* ω_k *be non-negative real numbers, then*

$$\sum_{k=1}^{n}\omega_k a_k b_k \leq \left(\sum_{k=1}^{n}\omega_k a_k^p\right)^{1/p}\left(\sum_{k=1}^{n}\omega_k b_k^q\right)^{1/q}. \tag{8}$$

Lemma 4. *Let* $0 < \frac{1}{q} \leq \frac{1}{p} < 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* u_k, $v(k)$ *and* ω_k *are non-negative real sequences, and* $0 < m_1 \leq u_k \leq M_1$, *and* $0 < m_2 \leq v_k \leq M_2$ *for* $k = 1, 2, \ldots, n$, *then*

$$\ell\left(\sum_{k=1}^{n}\omega_k(u_k+v_k)^2\right)^{1/p} \geq \left(\sum_{k=1}^{n}\omega_k u_k^2\right)^{1/p} + \left(\sum_{k=1}^{n}\omega_k v_k^2\right)^{1/p}, \tag{9}$$

where ℓ *is as in Label (4).*

Proof. From (3), we have

$$\begin{aligned}\ell \sum_{k=1}^{n} \omega_k (u_k+v_k)^2 &= \ell \sum_{k=1}^{n} \omega_k u_k (u_k+v_k) + \ell \sum_{k=1}^{n} \omega_k v_k (u_k+v_k) \\ &\geq \left(\sum_{k=1}^{n} \omega_k u_k^2\right)^{1/p} \left(\sum_{k=1}^{n} \omega_k (u_k+v_k)^2\right)^{1/q} + \left(\sum_{k=1}^{n} \omega_k v_k^2\right)^{1/p} \left(\sum_{k=1}^{n} \omega_k (u_k+v_k)^2\right)^{1/q}.\end{aligned}$$

Hence,

$$\ell \left(\sum_{k=1}^{n} \omega_k (u_k+v_k)^2\right)^{1/p} \geq \left(\sum_{k=1}^{n} \omega_k u_k^2\right)^{1/p} + \left(\sum_{k=1}^{n} \omega_k v_k^2\right)^{1/p}.$$

This proof is complete. □

Our main results are given in the following theorems.

Theorem 1. *Let $m, n \in \mathbb{N}^+$, $0 < \frac{1}{q} \leq \frac{1}{p} < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let u_k, v_k, a_k, b_k, ω_k and μ_k be non-negative real sequences such as $\omega_k u_k^2 > m\mu_k a_k^p$ and $\omega_k v_k^2 > m\mu_k b_k^q$, where $k = 1, 2, \ldots, n$. If $0 < m_1 \leq u_k \leq M_1$ and $0 < m_2 \leq v_k \leq M_2$, then*

$$\sum_{k=1}^{n} (\ell \omega_k u_k v_k - m\mu_k a_k b_k) \geq \left(\sum_{k=1}^{n} \left(\omega_k u_k^2 - m\mu_k a_k^p\right)\right)^{1/p} \left(\sum_{k=1}^{n} \left(\omega_k v_k^2 - m\mu_k b_k^q\right)\right)^{1/q}, \quad (10)$$

where ℓ is as in (4).

Proof. Let's prove this theorem by mathematical induction for m. First, we prove that (10) holds for $m = 1$. From (3) and (8), we obtain

$$\ell \left(\sum_{k=1}^{n} \omega_k u_k v_k\right) \geq \left(\sum_{k=1}^{n} \omega_k u_k^2\right)^{1/p} \left(\sum_{k=1}^{n} \omega_k v_k^2\right)^{1/q}, \quad (11)$$

and

$$\left(\sum_{k=1}^{n} \mu_k a_k b_k\right) \leq \left(\sum_{k=1}^{n} \mu_k a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} \mu_k b_k^q\right)^{1/q}. \quad (12)$$

From (11), (12) and, in view of the Popoviciu's inequality, we have

$$\begin{aligned}\sum_{k=1}^{n} (\ell \omega_k u_k v_k - \mu_k a_k b_k) &\geq \left(\sum_{k=1}^{n} \omega_k u_k^2\right)^{1/p} \left(\sum_{k=1}^{n} \omega_k v_k^2\right)^{1/q} - \left(\sum_{k=1}^{n} \mu_k a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} \mu_k b_k^q\right)^{1/q} \\ &\geq \left(\sum_{k=1}^{n} (\omega_k u_k^2 - \mu_k a_k^p)\right)^{1/p} \left(\sum_{k=1}^{n} (\omega_k v_k^2 - \mu_k b_k^q)\right)^{1/q}.\end{aligned}$$

This shows (10) right for $m = 1$.

Suppose that (10) holds when $m = r - 1$; we have

$$\sum_{k=1}^{n} (\ell \omega_k u_k v_k - (r-1)\mu_k a_k b_k) \geq \left(\sum_{k=1}^{n} \left(\omega_k u_k^2 - (r-1)\mu_k a_k^p\right)\right)^{1/p} \left(\sum_{k=1}^{n} \left(\omega_k v_k^2 - (r-1)\mu_k b_k^q\right)\right)^{1/q}. \quad (13)$$

From (6), (12) and (13), we obtain

$$\begin{aligned}\sum_{k=1}^{n}(\ell\omega_k u_k v_k - r\mu_k a_k b_k) \quad &\geq \quad \left(\sum_{k=1}^{n}\left(\omega_k u_k^2-(r-1)\mu_k a_k^p\right)\right)^{1/p}\left(\sum_{k=1}^{n}\left(\omega_k v_k^2-(r-1)\mu_k b_k^q\right)\right)^{1/q}\\ &- \quad \left(\sum_{k=1}^{n}\mu_k a_k^p\right)^{1/p}\left(\sum_{k=1}^{n}\mu_k b_k^q\right)^{1/q}\\ &\geq \quad \left(\sum_{k=1}^{n}\left(\omega_k u_k^2-r\mu_k a_k^p\right)\right)^{1/p}\left(\sum_{k=1}^{n}\left(\omega_k v_k^2-r\mu_k b_k^q\right)\right)^{1/q}.\end{aligned}$$

This shows that (10) is correct if $m = r-1$, then $m = r$ is also correct. Hence, (10) is right for any $m \in \mathbb{N}^+$.

This proof is complete. □

Taking $m = 1$ and $\omega_k = \mu_k$ in Theorem 1, we have the following result.

Corollary 1. *Let p, q, u_k, v_k, a_k, b_k and ω_k are as in Theorem 1, then*

$$\sum_{k=1}^{n}(\omega_k(\ell u_k v_k - a_k b_k)) \geq \left(\sum_{k=1}^{n}\omega_k(u_k^2-a_k^p)\right)^{1/p}\left(\sum_{k=1}^{n}\omega_k(v_k^2-b_k^q)\right)^{1/q},$$

where ℓ is as in (4).

Taking $m = 1$, $p = q = 2$ and $\omega_k = \mu_k = 1$ in Theorem 1, we have the following result.

Corollary 2. *Let u_k, v_k, a_k and b_k are as in Theorem 1, then*

$$\sum_{k=1}^{n}\left(\frac{M_1M_2+m_1m_2}{2\sqrt{m_1m_2M_1M_2}}u_k v_k - a_k b_k\right) \geq \left(\sum_{k=1}^{n}(u_k^2-a_k^2)\right)^{1/2}\left(\sum_{k=1}^{n}(v_k^2-b_k^2)\right)^{1/2}. \tag{14}$$

Taking for $a_k = 0$ and $b_k = 0$ in (14), we get the following interesting reverse Cauchy's inequality.

$$\frac{M_1M_2+m_1m_2}{2\sqrt{m_1m_2M_1M_2}}\cdot\sum_{k=1}^{n}u_k v_k \geq \left(\sum_{k=1}^{n}u_k^2\right)^{1/2}\left(\sum_{k=1}^{n}v_k^2\right)^{1/2}.$$

Theorem 2. *Let $m, n \in \mathbb{N}^+$, $0 < \frac{1}{q} \leq \frac{1}{p} < 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. Let u_k, v_k, a_k, b_k, ω_k and μ_k be non-negative real sequences such as $\omega_k u_k^2 > m a_k^p$ and $\omega_k v_k > m b_k^q$, where $k = 1, 2, \ldots, n$. If $0 < m_1 \leq u_k \leq M_1$ and $0 < m_2 \leq v_k \leq M_2$, then*

$$\left(\sum_{k=1}^{n}\left(\ell^p\omega_k(u_k+v_k)^2 - m(a_k+b_k)^p\right)\right)^{1/p} \geq \left(\sum_{k=1}^{n}(\omega_k u_k^2 - m a_k^p)\right)^{1/p} + \left(\sum_{k=1}^{n}(\omega_k v_k^2 - m b_k^p)\right)^{1/p}, \tag{15}$$

where ℓ is as in (4).

Proof. First, we prove that (15) holds for $m = 1$. From (9) and in view of Minkowski's inequality, it is easy to obtain

$$\ell\left(\sum_{k=1}^{n}\omega_k(u_k+v_k)^2\right)^{1/p} \geq \left(\sum_{k=1}^{n}\omega_k u_k^2\right)^{1/p} + \left(\sum_{k=1}^{n}\omega_k v_k^2\right)^{1/p}, \tag{16}$$

and
$$\left(\sum_{k=1}^{n}(a_k+b_k)^p dx\right)^{1/p} \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}. \tag{17}$$

From (16), (17) and the Bellman's inequality, we have

$$\begin{aligned}
&\left(\sum_{k=1}^{n}\left(\ell^p\omega_k(u_k+v_k)^2-(a_k+b_k)^p\right)\right)^{1/p}\\
&\ge \left\{\left[\left(\sum_{k=1}^{n}\omega_k u_k^2\right)^{1/p}+\left(\sum_{k=1}^{n}\omega_k v_k^2\right)^{1/p}\right]^p-\left[\left(\sum_{k=1}^{n}a_k^p\right)^{1/p}+\left(\sum_{k=1}^{n}b_k^p\right)^{1/p}\right]^p\right\}^{1/p}\\
&\ge \left(\sum_{k=1}^{n}(\omega_k u_k^2-a_k^p)\right)^{1/p}+\left(\sum_{k=1}^{n}(\omega_k v_k^2-b_k^p)\right)^{1/p}.
\end{aligned}$$

This shows that (15) holds for $m=1$

Supposing that (15) holds when $m=r-1$, we have

$$\begin{aligned}
\left(\sum_{k=1}^{n}\left(\ell^p\omega_k(u_k+v_k)^2-(r-1)(a_k+b_k)^2\right)\right)^{1/p} &\ge \left(\sum_{k=1}^{n}(\omega_k u_k^2-(r-1)a_k^p)\right)^{1/p}\\
&+\left(\sum_{k=1}^{n}(\omega_k v_k^2-(r-1)b_k^p)\right)^{1/p}.
\end{aligned} \tag{18}$$

From (17), (18) and by using the Bellman's inequality again, we obtain

$$\begin{aligned}
\left(\sum_{k=1}^{n}\left(\ell^p\omega_k(u_k+v_k)^2-r(a_k+b_k)^2\right)\right)^{1/p} &\ge \left\{\left[\left(\sum_{k=1}^{n}(\omega_k u_k^2-(r-1)a_k^p)\right)^{1/p}\right.\right.\\
&+\left.\left(\sum_{k=1}^{n}(\omega_k v_k^2-(r-1)b_k^p)\right)^{1/p}\right]^p\\
&-\left.\left[\left(\sum_{k=1}^{n}a_k^p\right)^{1/p}+\left(\sum_{k=1}^{n}b_k^p\right)^{1/p}\right]^p\right\}^{1/p}\\
&\ge \left(\sum_{k=1}^{n}(\omega_k u_k^2-ra_k^p)\right)^{1/p}+\left(\sum_{k=1}^{n}(\omega_k v_k^2-rb_k^p)\right)^{1/p}.
\end{aligned}$$

This shows that (15) is correct if $m=r-1$, then $m=r$ is also correct. Hence, (15) is right for any $m\in\mathbb{N}^+$.

This proof is complete. □

Taking for $m=1$, $p=2$ and $\omega_k=1$, we have the following result.

Corollary 3. *Let u_k, v_k, a_k, b_k, m_1, m_2, M_1, and M_2 be as in Theorem 2, then*

$$\left[\left(\sum_{k=1}^{n}\left(\hbar(u_k+v_k)^2-(a_k+b_k)^2\right)\right)\right]^{1/2} \ge \left(\sum_{k=1}^{n}(u_k^2-a_k^2)\right)^{1/2}+\left(\sum_{k=1}^{n}(v_k^2-b_k^2)\right)^{1/2},$$

where

$$\hbar=\frac{(M_1M_2+m_1m_2)^2}{4m_1m_2M_1M_2}.$$

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