Ride-sharing with travel time uncertainty

by

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Abstract

Travel time uncertainty has significant effects on travel reliability and travelers' generalized trip cost. However, travel time uncertainty has not been considered in existing ride-sharing models, leading to an inaccurate estimation of the benefit from ride-sharing and irrational ride-sharing matches. To fill in the gap, this paper proposes a stochastic ride-sharing model, in which travel time is assumed to be stochastic and follow a time-independent general distribution that has a positive lower bound. Due to travel time uncertainty, travelers may not arrive at their destinations on time. Different from the traditional models taking time windows as hard constraints, the proposed ride-sharing system only requires each participant announcing a role and the desired arrival time window. In the model, the generalized trip cost consists of the cost of driving a vehicle, the cost of travel time, and the cost of schedule delay early and late. This study investigates the effect of the unit variable cost of driving, travelers' values of time (VOTs), and travel time uncertainty on the cost saving of ride-sharing trips compared to driving-alone trips. A bi-objective ride-sharing matching model is proposed to maximize both the total generalized trip cost saving and the number of matches. The proposed ride-sharing model is further extended to consider time-dependent travel time uncertainty, and the Monte Carlo simulation (MCS) method is developed to evaluate the mean generalized trip cost. Finally, numerical examples are provided to illustrate the properties of the two proposed models. The results show that the unit variable cost of driving, travelers' VOTs, travel time uncertainty, and the selection of the weight in the objective function have significant impacts on the performance of the proposed ride-sharing system with travel time uncertainty. The results also show that a feasible ride-sharing match based on deterministic travel time can become infeasible in a stochastic ride-sharing system. It is therefore important to consider travel time uncertainty when determining the matches.
Keywords: Ride-sharing; travel time uncertainty; generalized trip cost; feasible match.

1. Introduction

Ride-sharing involves a joint trip of at least two participants who share a vehicle and must coordinate their itineraries. With technological advances in GPS navigation devices, smartphones, and mobile internet, ride-sharing becomes a popular transportation mode by which travelers with similar schedules and itineraries can quickly match up, share a vehicle for a trip, and share travel costs such as gas, toll, and parking fees. From the viewpoint of participants (including both drivers and riders), ride-sharing can save travel cost and reduce travel time. From the viewpoint of society, ride-sharing can mitigate traffic congestion, conserve fuel, and reduce air pollution (Ferguson, 1997; Kelley, 2007; Morency, 2007; Chan and Shaheen, 2012; Liu et al., 2017). From the viewpoint of private companies, ride-sharing can generate a substantial profit. As a result, lots of ride-sharing platforms provided by private companies, such as Lyft, Uber, Sidecar, and Didi Chuxing, are established. The key problem in a ride-sharing system lies in how to coordinate itineraries and schedules between participants, which directly determines the performance of the whole system.

Ride-sharing problems can be roughly classified into two categories: static (e.g., Baldacci et al., 2004; Calvo et al., 2004; Yan et al., 2014; Naoum-Sawaya et al., 2015; Xu et al., 2015; Xiao et al., 2016; Bruck et al., 2017) and dynamic (e.g., Winter and Nittel, 2006; Agatz et al., 2011, 2012; Stiglic et al., 2015, 2016; Chen et al., 2017a, b; Ma and Zhang, 2017; Wang et al., 2018). In static ride-sharing problems, it is assumed that both riders and drivers must provide information on their origins, destinations, and time schedule preferences (i.e., desired departure times and/or arrival times) before the matching decision made by the platform. Different from static ride-sharing problems, dynamic ride-sharing problems assume that the trip information of both riders and drivers are sent to the platform in real time, and the platform matches up drivers and riders on very short notice or even en-route. In a dynamic ride-sharing system, time is usually discretized into many planning horizons, and the rolling horizon framework is adopted (e.g., Agatz et al., 2011, 2012; Stiglic et al., 2015, 2016; Masoud and Jayakrishnan, 2017). For each planning horizon, a matching problem should be solved to coordinate riders and drivers, which can be viewed as a static ride-sharing matching problem. Therefore, a dynamic ride-sharing problem can be viewed as a sequence of static ride-sharing problems.

Ride-sharing problems can be formulated as either flow-based models (e.g., Yan et al., 2014; Xu et al., 2015; Xiao et al., 2016; Ma and Zhang, 2017; Ordóñez and Dessouky, 2017; Di et al., 2018) or individual-based models (e.g., Baldacci et al., 2004; Calvo et al., 2004; Agatz et al., 2011, 2012; Naoum-Sawaya et al., 2015; Stiglic et al., 2015, 2016; Bruck et al., 2017; Masoud and Jayakrishnan, 2017; Ordóñez and Dessouky, 2017; Wang et al., 2018). The flow-based models treat travelers as flow, and usually formulate the ride-sharing problems as traffic assignment problems or equilibrium problems, in which travel times are assumed to be endogenized by travelers' choice (e.g. route choice, departure time choice, mode choice). Compared with the individual-based models, the flow-based models are easier to be solved. Therefore, the flow-based models are more suitable for large network planning and policy evaluations. As ride-sharing participants are discrete in nature, the individual-based models are more suitable for practical ride-sharing operations—matching drivers with riders. Different from the flow-based models, the individual-based models usually assume that the endogenous congestion caused by ride-sharing is not considered, and travel times are known or estimated before ride-sharing matching (e.g., Agatz et al., 2011, 2012; Stiglic et al., 2015, 2016; Masoud and Jayakrishnan, 2017; Wang et al., 2018).

Ride-sharing services on the market can usually offer automated matching (Furuhata et al., 2013). In the literature, various optimization models are developed to formulate the matching problem in ride-sharing systems (Baldacci et al., 2004; Agatz et al., 2011; Stiglic et al., 2015, 2016). Generally, there are three types of objectives in existing ride-sharing matching models: to minimize system-wide vehicle miles (e.g., Baldacci et al., 2004; Calvo et al., 2004; Agatz et al., 2011; Wang et al., 2018), to minimize system-wide travel time or cost (e.g., Winter and Nittel, 2006), and to maximize the number of participants (e.g., Baldacci et al., 2004; Calvo et al., 2004; Winter and Nittel, 2006; Stiglic et al., 2016; Masoud and Jayakrishnan, 2017). The first two objectives are compatible with each other, and help to reduce network-wide pollution and congestion. The third objective maximizes the number of satisfied drivers and riders in the system, which may be beneficial to a private ride-sharing provider whose revenue is linked to the number of successful ride-sharing arrangements (Agatz et al., 2012).

The major constraints in the matching problem for a ride-sharing system include the participants' time preferences, the availability of spare seats, and travel distance/time/cost reduction (Agatz et al., 2012). The participants' time preferences are usually captured by time windows (e.g., Agatz et al., 2011; Stiglic et al., 2015), each of which contains the earliest departure time and the latest arrival time. The trip of each potential participant should start within the time window announced by the participant. The availability of spare seats tends to be a softer constraint than the participants' time preferences. In the existing matching models, drivers who offer a ride can take either a single rider or multiple riders (e.g., Baldacci et al., 2004; Calvo et al., 2004; Winter and Nittel, 2006; Agatz et al., 2011; Ghoseiri et al., 2011; Stiglic et al., 2015; Chen et al., 2017b). Reducing travel cost is a primary motivation for the ride-sharing users choosing to participate, and hence most of the matching models include the constraints on feasible ride-sharing matches. Whether a match is feasible or not relies on trip cost saving from ride-sharing.

In the literature, the travel times of ride-sharing participants are assumed to be deterministic (e.g., Baldacci et al., 2004; Agatz et al., 2011; Stiglic et al., 2015; Xu et al., 2015; Xiao et al., 2016; Liu et al., 2017; Ma and Zhang, 2017). However, travel times are stochastic in reality, especially during peak hours. The uncertainty can be caused by stochastic road capacity (e.g., Li et al., 2009; Lindsey, 2009; Fosgerau, 2010; Peer et al., 2010; Xiao et al., 2015), stochastic traffic demand (e.g., Alpha and Minh, 1979; Clark and Watling, 2005; Fosgerau, 2010; Li et al., 2010; Fu et al., 2014), and unexpected driving behavior (e.g., Pattanamekar et al., 2003, Li et al., 2010). Due to travel time uncertainty, the participants' time preferences cannot always be satisfied. Travel time uncertainty can bring costs such as the costs of schedule delay late or early, which leads to higher travelers' generalized trip cost. As a result, feasible matching obtained by deterministic matching models may not be feasible anymore when travel times are stochastic. This implies that traditional deterministic matching models can output ineffective ride-sharing matches. Moreover, infeasible matching obtained by deterministic matching models may be feasible when travel times are stochastic. This implies that traditional deterministic matching model may also miss effective ride-sharing matches. Therefore, it is significant to consider travel time uncertainty in ride-sharing systems.

In this paper, we propose a static ride-sharing model in a road network with the consideration of time-independent travel time uncertainty. We mainly focus on the operational level of a ride-sharing platform, and hence we formulate the ride-sharing problem with travel time uncertainty as an individual-based model. Following existing individual-based models (e.g., Agatz et al., 2011, 2012; Stiglic et al., 2015, 2016; Masoud and Jayakrishnan, 2017; Wang et al., 2018), we assume that the endogenous congestion caused by ride-sharing is not considered. It is also assumed that the travel times of all travelers are uncertain, and the ride-sharing platform can collect enormous quantities of travel data so that the variability of travelers' travel times can be exactly estimated by the platform, and the users of the ride-sharing platform make departure time choice without any consideration of other users' choice. Different from the traditional concept of treating time windows as hard constraints, the proposed ride-sharing system only requires each participant announcing his/her role (e.g., be a driver, a rider, or either a driver or a rider) and the desired arrival time window. Due to travel time uncertainty, travelers may not arrive at their destinations on time. A penalty cost of schedule delay early or late is introduced to account for travel time uncertainty. The effects of the unit variable cost of driving, travelers' values of time (VOTs), and travel time uncertainty on the cost of driving alone and the cost saving of a ride-sharing trip are analyzed. Furthermore, a bi-objective ride-sharing matching model is proposed to maximize both the total mean generalized trip cost saving and the number of matches. Numerical examples are also developed to illustrate the properties of the proposed model.

The stochastic ride-sharing model with time-independent travel time uncertainty is further extended to formulate a stochastic ride-sharing model with time-dependent travel time uncertainty. For a general road network with time-dependent link travel time distributions, it is very difficult to exactly obtain the convolution of time-dependent link travel time distributions, i.e., the time-dependent trip (i.e., path) travel time distributions. Hence, it is also very difficult to exactly obtain the mean generalized trip cost, which relies on the distribution of time-dependent trip travel time. In this paper, the Monte Carlo Simulation (MCS) method is used to evaluate the mean generalized trip cost (e.g., Szeto et al., 2011; Meng and Liu, 2012; Liu and Meng, 2013; Long et al., 2018). The MCS method is determined by the sample size. The MCS method can guarantee a higher accuracy if a larger sample size is adopted. Theoretically, if the sample size is infinite, the MCS method is exact, and the MCS method can be more accurate than analytical approximation methods if the sample size is large enough. Another merit of the MCS method is that the stochastic variables can be defined by any distributions.

The main contributions of our research are as follows.

First, we propose a ride-sharing problem in road networks with time-independent travel time uncertainty. To the best of our knowledge, we are the pioneers to develop a stochastic ride-sharing model, in which travelers' travel times are stochastic and follow a general distribution that has a lower bound.

Second, we introduce the generalized trip cost functions for both driving-alone and ride-sharing trips, and analyze their mathematical properties.

Third, we extend the stochastic ride-sharing model with time-independent travel time uncertainty to a stochastic ride-sharing model with time-dependent travel time uncertainty. The MCS method is developed to evaluate the mean generalized trip cost.

Fourth, we illustrate the properties of the models. In particular, we show the effects of the unit variable cost of driving, travelers' VOTs, travel time uncertainty, and various objectives on the performance of the proposed ride-sharing models.

The remainder of this paper is organized as follows. The ride-sharing problem in road networks with time-independent travel time uncertainty is formulated in Section 2. In Section 3, the ride-sharing problem in road networks with time-dependent travel time uncertainty is proposed. Numerical examples are provided in Section 4. Section 5 presents the conclusions of this paper.

2. Stochastic ride-sharing model with time-independent travel time uncertainty

2.1. Problem description

In a road network, it is assumed that there is a ride-sharing platform for travelers. All users of the ride-sharing platform have private cars and can finish their trips by driving alone (e.g., Agatz et al., 2011; Stiglic et al., 2015, 2016; Lee et al., 2015). Each user can claim one of the three roles in the platform: (1) a driver, (2) a rider, and (3) either a driver or a rider. Let Φ, Φ_d, Φ_r, and $\Phi_{d/r}$ be the set of ride-sharing participants ($\Phi = \Phi_d \cup \Phi_r \cup \Phi_{d/r}$), the set of users who select to be a driver, the set of users who select to be a rider, and the set of users who select to be either a driver or a rider, respectively. We regard the operation of the ride-sharing platform as the business of a company for providing a service, and the total number of travelers to be served is limited. Hence, the number of travelers served takes a very small proportion to the total number of travelers in the whole network, and their travel choices (e.g., whether to join ride-sharing, departure time choice) have little impact on road congestion. Therefore, the endogenous congestion caused by ride-sharing is not considered. In addition, the number of zones which are taken as origins or destinations of travelers is also limited, and the company optimizes and suggests the departure times for all travelers served.

Traveler $i=(o_i,d_i)$ denotes the user who travels from origin o_i to destination d_i. We have the following two definitions:

Definition 1 (Driving-alone trip). If traveler *i* drives alone from origin o_i to destination d_i, then the trip of traveler *i* is defined as driving-alone trip *i*.

Definition 2 (Ride-sharing trip). Let travelers *i* and *j* ($i \neq j$) be a driver and a rider, respectively. If traveler *i* departs from origin o_i, picks up rider *j* at origin o_j, drops off rider *j* at destination d_j, and finally arrives at destination d_i, then the combined trip of travelers *i* and *j* is defined as ride-sharing trip (*i*, *j*).

In the considered road network, it is assumed that the travel times of all users are stochastic. In addition, the ride-sharing platform can collect the enormous quantities of travel data, and on this basis estimate the variability to travelers' travel times. The ride-sharing participants only need to announce their roles (i.e., be a driver, a rider, or either a driver or a rider) and desired arrival time windows. As shown in Fig. 1, we consider a driving-alone trip *i*, which has a stochastic travel time τ_i. Without loss of generality, we assume that the travel time τ_i follows a distribution with the range $[\tau_i^-,\tau_i^+]$, and has a mean value of $\overline{\tau}_i$, where τ_i^- and τ_i^+ are the minimum and maximum travel times of driving-alone trip *i*, respectively. Because travel times must be positive in reality, we define $\tau_i^->0$. Note that there is no restriction that τ_i^+ must be finite. Let $f_i(\tau)$ and $F_i(\tau)$ be the probability density function and cumulative distribution function of the travel time τ_i, respectively. $f_i(\tau)$ is assumed to be positive for all $\tau \in (\tau_i^-,\tau_i^+)$.

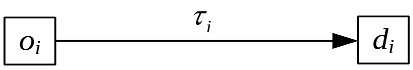

Fig. 1. A driving-alone trip.

According to the definitions, we have

$$F_i(\tau)=\int_{\tau_i^-}^{\tau} f_i(\omega)d\omega \quad \text{and} \tag{1}$$

$$\overline{\tau}_i=\int_{\tau_i^-}^{\tau_i^+} \tau f_i(\tau)d\tau . \tag{2}$$

We also define the following function:

$$G_i(\tau)=\int_{\tau_i^-}^{\tau} \omega f_i(\omega)d\omega . \tag{3}$$

Because $\tau_i^->0$ and $f_i(\tau)$ is positive for all $\tau \in \left[\tau_i^-,\tau_i^+\right]$, Eq. (3) implies that $G_i(\tau)$ is a strictly increasing function. Hence, we have

$$\overline{\tau}_i>G_i(\tau), \forall \tau \in [\tau_i^-,\tau_i^+) . \tag{4}$$

As shown in Fig. 2, we consider a ride-sharing trip (*i*, *j*). Let τ_{ij} be the total travel time of ride-sharing trip (*i*, *j*), and $\tau_{i\tilde{j}}$ be the travel time of driver *i* traveling from origin o_i to destination d_j. Without loss of generality, we assume that the travel time τ_{ij} ($\tau_{i\tilde{j}}$) follows a distribution with the range $[\tau_{ij}^-,\tau_{ij}^+]$ ($[\tau_{i\tilde{j}}^-,\tau_{i\tilde{j}}^+]$), and has a mean value of $\overline{\tau}_{ij}$ ($\overline{\tau}_{i\tilde{j}}$), where τ_{ij}^- ($\tau_{i\tilde{j}}^-$) and τ_{ij}^+ ($\tau_{i\tilde{j}}^+$) are the minimum and maximum travel times of the whole ride-sharing trip (*i*, *j*) (for driver *i* traveling from origin o_i, through origin o_j, and to destination d_j), respectively. Let $f_{ij}(\tau)$ ($f_{\tilde{ij}}(\tau)$) and $F_{ij}(\tau)$ ($F_{\tilde{ij}}(\tau)$) be the probability density function and cumulative distribution function of the stochastic travel time τ_{ij} ($\tau_{\tilde{ij}}$), respectively. We assume that $f_{ij}(\tau)$ ($f_{\tilde{ij}}(\tau)$) is positive for all $\tau \in (\tau_{ij}^-, \tau_{ij}^+)$ ($\tau \in (\tau_{\tilde{ij}}^-, \tau_{\tilde{ij}}^+)$).

According to the definitions, we have

$$F_{ij}(\tau) = \int_{\tau_{ij}^-}^{\tau} f_{ij}(w)dw, \tag{5}$$

$$F_{\tilde{ij}}(\tau) = \int_{\tau_{\tilde{ij}}^-}^{\tau} f_{\tilde{ij}}(w)dw, \tag{6}$$

$$\bar{\tau}_{ij} = \int_{\tau_{ij}^-}^{\tau_{ij}^+} \tau f_{ij}(\tau)d\tau, \text{ and} \tag{7}$$

$$\bar{\tau}_{\tilde{ij}} = \int_{\tau_{\tilde{ij}}^-}^{\tau_{\tilde{ij}}^+} \tau f_{\tilde{ij}}(\tau)d\tau. \tag{8}$$

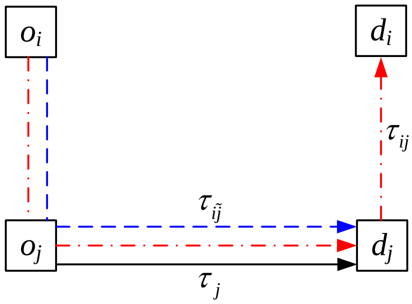

Fig. 2. A ride-sharing trip.

We also define the following functions:

$$G_{ij}(\tau) = \int_{\tau_{ij}^-}^{\tau} \omega f_{ij}(\omega)d\omega \quad \text{and} \tag{9}$$

$$G_{\tilde{ij}}(\tau) = \int_{\tau_{\tilde{ij}}^-}^{\tau} \omega f_{\tilde{ij}}(\omega)d\omega. \tag{10}$$

The following notations are adopted throughout this paper:

t_i^*	The middle point of the desired arrival time window of traveler *i*
Δ_i	The half interval of arrival time flexibility of traveler *i*
e_i^*	The optimal departure time of driving-alone trip *i*
e_{ij}^*	The optimal departure time of ride-sharing trip (*i*, *j*)
$C_i(t)$	The generalized cost of driving-alone trip *i* departing from origin o_i at time *t*
$\bar{C}_i^*$	The minimum mean generalized cost of driving-alone trip *i*
$C_{ij}(t)$	The total generalized cost of ride-sharing trip (*i*, *j*) departing from origin o_i at time *t*
$\bar{C}_{ij}^*$	The minimum mean total generalized cost of ride-sharing trip (*i*, *j*)
δ_{ij}	The generalized cost saving of ride-sharing trip (*i*, *j*)
x_{ij}	A 0-1 decision variable; if ride-sharing trip (*i*, *j*) is formed, then $x_{ij} = 1$; otherwise, $x_{ij} = 0$
Ω	The set of feasible ride-sharing trips

2.2. Generalized trip costs

2.2.1. The generalized cost of a driving-alone trip

The generalized cost of a driving-alone trip consists of three components: (1) driving cost (including fuel cost and vehicle wear and tear cost), (2) in-vehicle travel time cost, and (3) a "penalty" for reaching the destination early or late. Hence, the generalized cost of driving-alone trip *i* departing from origin o_i at time *t* can be formulated as follows:

$$C_i(t) = \mu_0 + \mu\tau_i + \alpha\tau_i + \beta\cdot\max\{0, t_i^* - \Delta_i - t - \tau_i\} + \gamma\cdot\max\{0, t + \tau_i - t_i^* - \Delta_i\}, \tag{11}$$

where μ_0 is the fixed driving cost, μ is the unit variable cost of driving, α is the unit cost of in-vehicle travel time, β and γ are the unit costs of early and late arrivals, respectively. According to the empirical results (Small, 1982), we have $0 < \beta < \alpha < \gamma$.

By definition, the mean generalized cost of driving-alone trip *i* can be formulated as follows:

$$\begin{aligned} E[C_i(t)] = \mu_0 + (\mu+\alpha)\overline{\tau}_i + \beta\int_{\tau_i^-}^{t_i^*-\Delta_i-t}(t_i^* - \Delta_i - t - \tau)f_i(\tau)d\tau \\ + \gamma\int_{t_i^*+\Delta_i-t}^{\tau_i^+}(t+\tau-t_i^*-\Delta_i)f_i(\tau)d\tau. \end{aligned} \tag{12}$$

Equivalently, we have

$$\begin{aligned} E[C_i(t)] = \mu_0 + (\mu+\alpha)\overline{\tau}_i + \beta(t_i^*-\Delta_i-t)F_i(t_i^*-\Delta_i-t) - \beta G_i(t_i^*-\Delta_i-t) \\ + \gamma(t+\overline{\tau}_i-t_i^*-\Delta_i) + \gamma(t_i^*+\Delta_i-t)F_i(t_i^*+\Delta_i-t) - \gamma G_i(t_i^*+\Delta_i-t). \end{aligned} \tag{13}$$

According to Eq. (13), the mean generalized trip cost $E[C_i(t)]$ is a function of departure time *t*. The first-order and second-order derivatives of $E[C_i(t)]$ with respect to *t* can be obtained as follows:

$$\frac{d}{dt}E[C_i(t)] = -\beta F_i(t_i^*-\Delta_i-t) + \gamma - \gamma F_i(t_i^*+\Delta_i-t) \quad \text{and} \tag{14}$$

$$\frac{d^2E[C_i(t)]}{dt} = \beta f_i(t_i^*-\Delta_i-t) + \gamma f_i(t_i^*+\Delta_i-t). \tag{15}$$

Based on these derivatives, we have the following assumptions:

Assumption 1: $\tau_i^+ - \tau_i^- \le 2\Delta_i$ is satisfied.

Assumption 2: $\tau_i^+ - \tau_i^- > 2\Delta_i$ is satisfied.

Proposition 1. Under Assumption 1, any time instants in $[t_i^*-\Delta_i-\tau_i^-, t_i^*+\Delta_i-\tau_i^+]$ are the optimal departure times of driving-alone trip *i*.

Proof: Under Assumption 1, we have $t_i^*-\Delta_i-\tau_i^- \le t_i^*+\Delta_i-\tau_i^+$. For any departure time $t \in [t_i^*-\Delta_i-\tau_i^-, t_i^*+\Delta_i-\tau_i^+]$, we have

$$t_i^*-\Delta_i-t \le t_i^*-\Delta_i-(t_i^*-\Delta_i-\tau_i^-) \le \tau_i^- \quad \text{and} \tag{16}$$

$$t_i^* + \Delta_i - t \geq t_i^* + \Delta_i - (t_i^* + \Delta_i - \tau_i^+) = \tau_i^+ . \tag{17}$$

According to inequalities (16) and (17), we have $F_i(t_i^* - \Delta_i - t) = F_i(\tau_i^-) = 0$, $G_i(t_i^* - \Delta_i - t) = G_i(\tau_i^-) = 0$, $F_i(t_i^* + \Delta_i - t) = F_i(\tau_i^+) = 1$, and $G_i(t_i^* + \Delta_i - t) = G_i(\tau_i^+) = \overline{\tau}_i$. Substituting these equations into Eq. (13), we have

$$E[C_i(t)] = \mu_0 + (\mu + \alpha)\overline{\tau}_i . \tag{18}$$

Eq. (18) implies that the traveler of driving-alone trip *i* does not experience any "penalty" for reaching the destination early or late. Hence, the traveler of driving-alone trip *i* arrives at his/her destination on time and bears the minimum mean generalized trip cost. □

Proposition 2. Under Assumption 2, the optimal departure time of driving-alone trip *i*, i.e., e_i^*, is unique, and we have

$$-\beta F_i(t_i^* - \Delta_i - e_i^*) + \gamma - \gamma F_i(t_i^* + \Delta_i - e_i^*) = 0 . \tag{19}$$

Proof: Under Assumption 2, we have $t_i^* + \Delta_i - \tau_i^+ < t_i^* - \Delta_i - \tau_i^-$. For all $t \leq t_i^* + \Delta_i - \tau_i^+$, according to Eq. (14), we have

$$\begin{aligned} & dE[C_i(t)]/dt = -\beta F_i(t_i^* - \Delta_i - t) + \gamma - \gamma F_i(t_i^* + \Delta_i - t) \\ & \leq -\beta F_i(t_i^* - \Delta_i - (t_i^* + \Delta_i - \tau_i^+)) + \gamma - \gamma F_i(t_i^* + \Delta_i - (t_i^* + \Delta_i - \tau_i^+)) \\ & < -\beta F_i(\tau_i^-) + \gamma - \gamma F_i(\tau_i^+) = 0. \end{aligned} \tag{20}$$

Similarly, for all $t \geq t_i^* - \Delta_i - \tau_i^-$, we have

$$\begin{aligned} & dE[C_i(t)]/dt \geq -\beta F_i(t_i^* - \Delta_i - (t_i^* - \Delta_i - \tau_i^-)) + \gamma - \gamma F_i(t_i^* + \Delta_i - (t_i^* - \Delta_i - \tau_i^-)) \\ & > -\beta F_i(\tau_i^-) + \gamma - \gamma F_i(\tau_i^+) = 0. \end{aligned} \tag{21}$$

Inequalities (20) and (21) imply that $E[C_i(t)]$ is monotonically decreasing when $t \leq t_i^* + \Delta_i - \tau_i^+$ and is monotonically increasing when $t \geq t_i^* - \Delta_i - \tau_i^-$. Hence, we have $e_i^* \in (t_i^* + \Delta_i - \tau_i^+, t_i^* - \Delta_i - \tau_i^-)$. Under Assumption 2, for all $t \in (t_i^* + \Delta_i - \tau_i^+, t_i^* - \Delta_i - \tau_i^-)$, we have $t > t_i^* + \Delta_i - \tau_i^+$, which implies $t_i^* - \Delta_i - t < t_i^* + \Delta_i - t < \tau_i^+$. We also have $t < t_i^* - \Delta_i - \tau_i^-$, which implies $t_i^* + \Delta_i - t > t_i^* - \Delta_i - t > \tau_i^-$. Hence, we have $t_i^* - \Delta_i - t \in (\tau_i^-, \tau_i^+)$ and $t_i^* + \Delta_i - t \in (\tau_i^-, \tau_i^+)$. This implies $d^2E[C_i(t)]/dt^2 > 0$ and $E[C_i(t)]$ is strictly convex for all $t \in (t_i^* - \Delta_i - \tau_i^-, t_i^* + \Delta_i - \tau_i^+)$. Therefore, the optimal departure time of each driving-alone trip is unique. According to the first-order optimal condition, we have $dE[C_i(t)]/dt = 0$ at $t = e_i^*$. Using Eq. (14), we can obtain Eq. (19). This completes the proof. □

According to Proposition 2, if $\Delta_i = 0$, the optimal departure time of driving-alone trip *i* is unique, and we have $-\beta F_i(t_i^* - e_i^*) + \gamma - \gamma F_i(t_i^* - e_i^*) = 0$, which implies $e_i^* = t_i^* - F_i^{-1}\left(\gamma / (\beta + \gamma)\right)$. This result is consistent with the result of Xiao et al. (2017).

Based on Eq. (19), we can prove the following two propositions.

Proposition 3. The optimal departure time of each driving-alone trip is independent of both the values of μ and α.

Proof: If $\tau_i^+ - \tau_i^- \le 2\Delta_i$ is satisfied, according to Proposition 1, any time instants in $[t_i^* - \Delta_i - \tau_i^-, t_i^* + \Delta_i - \tau_i^+]$ are optimal departure times of driving-alone trip *i*. This implies that the optimal departure time of each driving-alone trip is independent of both the values of μ and α. If $\tau_i^+ - \tau_i^- > 2\Delta_i$ is satisfied, based on Eq. (19), the partial derivative of e_i^* with respect to μ and α can be obtained as follows:

$$\frac{\partial e_i^*}{\partial \mu} = \frac{\partial e_i^*}{\partial \alpha} = 0. \tag{22}$$

Eq. (22) also implies that the optimal departure time of each driving-alone trip is independent of both the values of μ and α. This completes the proof. □

Proposition 4. Under Assumption 2, if the value of β is larger or the value of γ is smaller, then the optimal departure time of each driving-alone trip becomes later, and vice versa. If the value of β / γ is larger, then the optimal departure time of each driving-alone trip becomes later, and vice versa.

Proof: Based on Eq. (19), the partial derivative of e_i^* with respect to β, γ and β / γ can be obtained as follows:

$$\frac{\partial e_i^*}{\partial \beta} = \frac{F_i(t_i^* - \Delta_i - e_i^*)}{\beta f_i(t_i^* - \Delta_i - e_i^*) + \gamma f_i(t_i^* + \Delta_i - e_i^*)}, \tag{23}$$

$$\frac{\partial e_i^*}{\partial \gamma} = -\frac{1 - F_i(t_i^* + \Delta_i - e_i^*)}{\beta f_i(t_i^* - \Delta_i - e_i^*) + \gamma f_i(t_i^* + \Delta_i - e_i^*)}, \text{ and} \tag{24}$$

$$\frac{\partial e_i^*}{\partial (\beta / \gamma)} = \frac{\gamma F_i(t_i^* - \Delta_i - e_i^*)}{\beta f_i(t_i^* - \Delta_i - e_i^*) + \gamma f_i(t_i^* + \Delta_i - e_i^*)}. \tag{25}$$

According to the proof of Proposition 2, we have $e_i^* \in (t_i^* + \Delta_i - \tau_i^+, t_i^* - \Delta_i - \tau_i^-)$. This implies that $F_i(t_i^* - \Delta_i - e_i^*) > 0$ and $1 - F_i(t_i^* - \Delta_i - e_i^*) > 0$. Hence, we have $\partial e_i^* / \partial \beta > 0$, $\partial e_i^* / \partial \gamma < 0$, and $\partial e_i^* / \partial (\beta / \gamma) > 0$, which, respectively, imply that e_i^* increases as the value of β increases, or the value of γ decreases, or the value of β / γ grows up. This completes the proof. □

Substituting $t = e_i^*$ and Eq. (19) into Eq. (13), we can obtain the minimum mean generalized cost of driving-alone trip *i* as follows:

$$\begin{aligned} \overline{C}_i^* = E[C_i(e_i^*)] = \mu_0 + (\mu + \alpha)\overline{\tau}_i + \Delta_i[\gamma F_i(t_i^* + \Delta_i - e_i^*) - \beta F_i(t_i^* - \Delta_i - e_i^*)] \\ + \gamma(\overline{\tau}_i - \Delta_i) - \beta G_i(t_i^* - \Delta_i - e_i^*) - \gamma G_i(t_i^* + \Delta_i - e_i^*). \end{aligned} \tag{26}$$

Proposition 5. Under Assumption 2, if the values of μ, α, β, and γ increase, then the minimum mean generalized cost of each driving-alone trip increases, and vice versa.

Proof: Based on Eq. (26) and using Eqs. (23) and (24), the partial derivative of $\overline{C}_i^*$ with respect to μ, α, β, and γ can be obtained as follows:

$$\frac{\partial \overline{C}_i^*}{\partial \mu} = \frac{\partial \overline{C}_i^*}{\partial \alpha} = \overline{\tau}_i, \tag{27}$$

$$\frac{\partial \overline{C}_i^*}{\partial \beta} = (t_i^* - \Delta_i - e_i^*) F_i(t_i^* - \Delta_i - e_i^*) - G_i(t_i^* - \Delta_i - e_i^*), \text{ and} \tag{28}$$

$$\frac{\partial \overline{C}_i^*}{\partial \gamma} = \overline{\tau}_i + (t_i^* + \Delta_i - e_i^*)[F_i(t_i^* + \Delta_i - e_i^*) - 1] - G_i(t_i^* + \Delta_i - e_i^*). \tag{29}$$

By definition, we have $\overline{\tau}_i > 0$, and hence $\partial \overline{C}_i^* / \partial \mu = \partial \overline{C}_i^* / \partial \alpha > 0$. Let $g_1(\tau) = \tau F_i(\tau) - G_i(\tau)$. We have $g_1'(\tau) = F_i(\tau) > 0$ for all $\tau > \tau_i^-$. This implies that $g_1(\tau)$ is a strictly increasing function with respect to τ for all $\tau > \tau_i^-$. When $\tau \le \tau_i^-$, we have $g_1(\tau) = \tau F_i(\tau_i^-) - G_i(\tau_i^-) = 0$. With $t_i^* - \Delta_i - e_i^* > \tau_i^-$, we have $g_1(t_i^* - \Delta_i - e_i^*) > 0$. This implies $\partial \overline{C}_i^* / \partial \beta > 0$. Let $g_2(\tau) = \overline{\tau}_i + \tau[1 - F_i(\tau)] - G_i(\tau)$. We have $g_1'(\tau) = F_i(\tau) - 1 < 0$ for all $\tau < \tau_i^+$. This implies that $g_2(\tau)$ is a strictly decreasing function with respect to τ for all $\tau < \tau_i^+$. When $\tau \ge \tau_i^+$, we have $g_2(\tau_i^+) = \overline{\tau}_i + \tau[F_i(\tau_i^+) - 1] - G_i(\tau_i^+) = 0$. With $t_i^* + \Delta_i - e_i^* < \tau_i^+$, we have $g_2(t_i^* + \Delta_i - e_i^*) > 0$. This implies $\partial \overline{C}_i^* / \partial \gamma > 0$. In summary, $\overline{C}_i^*$ is strictly increasing with respect to the values of μ, α, β, and γ, respectively. This completes the proof. □

Proposition 5 implies that the values of μ, α, β, and γ can influence the minimum mean generalized cost of driving-alone trips when travel time uncertainty exists. However, if there is no uncertainty (i.e., $\tau_i^- = \tau_i^+ = \overline{\tau}_i$), according to the proof of Proposition 1, the traveler of driving-alone trip *i* arrives at his/her destination on time with the minimum generalized trip cost if the trip departs from origin o_i during the time period $[t_i^* - \Delta_i - \overline{\tau}_i, t_i^* + \Delta_i - \overline{\tau}_i]$. This implies that the values of β and γ have no influence on the minimum generalized cost of driving-alone trips if there is no uncertainty. By definition, we have $\overline{\tau}_i > 0$. According to Eq. (27), the minimum generalized cost of each driving-alone trip also increases as the values of μ and α increase if there is no uncertainty.

Definition 3. (*First-order stochastic dominance*). The distribution $F_\mathrm{b}(\cdot)$ first-order stochastically dominates $F_\mathrm{a}(\cdot)$ if and only if, for every non-decreasing function $u: R \to R$, we have

$$\int u(\tau) dF_\mathrm{a}(\tau) \le \int u(\tau) dF_\mathrm{b}(\tau). \tag{30}$$

Lemma 1. (Mas-Colell et al., 1995, Proposition 6.D.1). The distribution $F_\mathrm{b}(\cdot)$ first-order stochastically dominates the distribution $F_\mathrm{a}(\cdot)$ if and only if $F_\mathrm{a}(\tau) \ge F_\mathrm{b}(\tau)$ for every τ.

According to Definition 3 and Lemma 1, if $F_\mathrm{a}(\tau) \ge F_\mathrm{b}(\tau)$ is satisfied for every τ, we have Eq. (30) is satisfied for every non-decreasing function $u: R \to R$. Definition 3 and Lemma 1 will be used to prove the following proposition:

Proposition 6. Under Assumption 2, if the cumulative distribution function of the travel time of driving-alone trip *i*, i.e., τ_i, changes from $F_{i,\mathrm{a}}(\tau)$ to $F_{i,\mathrm{b}}(\tau)$ and $F_{i,\mathrm{a}}(\tau) \geq F_{i,\mathrm{b}}(\tau)$ is satisfied for every τ, then the optimal departure time of driving-alone trip *i* can become earlier or remain unchanged, and its minimum mean generalized trip cost can become larger or remain unchanged.

Proof: To simplify the notation, we add subscripts 'a' and 'b' to the functions and parameters corresponding to the cumulative distribution function of the travel time before and after the change, respectively. To prove this proposition, we need to prove that $e_{i,\mathrm{a}}^* \geq e_{i,\mathrm{b}}^*$ and $\bar{C}_{i,\mathrm{a}}^* \leq \bar{C}_{i,\mathrm{b}}^*$ are satisfied.

With $F_{i,\mathrm{a}}(\tau) \geq F_{i,\mathrm{b}}(\tau)$ for every τ, Eq. (2), Definition 3, and Lemma 1 imply $\bar{\tau}_{i,\mathrm{a}} \leq \bar{\tau}_{i,\mathrm{b}}$. Based on Eq. (19), we have

$$\gamma = \beta F_{i,\mathrm{a}}(t_i^* - \Delta_i - e_{i,\mathrm{a}}^*) + \gamma F_{i,\mathrm{a}}(t_i^* + \Delta_i - e_{i,\mathrm{a}}^*) \geq \beta F_{i,\mathrm{b}}(t_i^* - \Delta_i - e_{i,\mathrm{a}}^*) + \gamma F_{i,\mathrm{b}}(t_i^* + \Delta_i - e_{i,\mathrm{a}}^*). \quad (31)$$

According to Eq. (19) and inequality (31), we have

$$\gamma = \beta F_{i,\mathrm{b}}(t_i^* - \Delta_i - e_{i,\mathrm{b}}^*) + \gamma F_{i,\mathrm{b}}(t_i^* + \Delta_i - e_{i,\mathrm{b}}^*) \geq \beta F_{i,\mathrm{b}}(t_i^* - \Delta_i - e_{i,\mathrm{a}}^*) + \gamma F_{i,\mathrm{b}}(t_i^* + \Delta_i - e_{i,\mathrm{a}}^*). \quad (32)$$

$F_{i,\mathrm{b}}(\tau)$ is strictly increasing for all $\tau \in (\tau_{i,\mathrm{b}}^-, \tau_{i,\mathrm{b}}^+)$, so inequality (32) implies $e_{i,\mathrm{a}}^* \geq e_{i,\mathrm{b}}^*$.

We define the following function:

$$u_{i,\mathrm{b}}(\tau) = \begin{cases} \mu_0 + \mu\tau + \alpha\tau + \beta(t_i^* - \Delta_i - e_{i,\mathrm{b}}^* - \tau), & \text{if } \tau < t_i^* - \Delta_i - e_{i,\mathrm{b}}^*, \\ \mu_0 + \mu\tau + \alpha\tau + \gamma(e_{i,\mathrm{b}}^* + \tau - t_i^* - \Delta_i), & \text{if } \tau > t_i^* + \Delta_i - e_{i,\mathrm{b}}^*, \\ \mu_0 + \mu\tau + \alpha\tau, & \text{otherwise.} \end{cases} \quad (33)$$

With $0 < \beta < \alpha < \gamma$, we have $du_{i,\mathrm{b}}(\tau)/d\tau > 0$. Hence $u_{i,\mathrm{b}}(\tau)$ is a strictly increasing function with respect to τ. With $F_{i,\mathrm{a}}(\tau) \geq F_{i,\mathrm{b}}(\tau)$ for every τ, according to Lemma 1, we have

$$\bar{C}_{i,\mathrm{a}}^* = E[C_{i,\mathrm{a}}(e_{i,\mathrm{a}}^*)] \leq E[C_{i,\mathrm{a}}(e_{i,\mathrm{b}}^*)] = \int_{-\infty}^{+\infty} u_{i,\mathrm{b}}(\tau) dF_{i,\mathrm{a}}(\tau) \leq \int_{-\infty}^{+\infty} u_{i,\mathrm{b}}(\tau) dF_{i,\mathrm{b}}(\tau) = E[C_{i,\mathrm{b}}(e_{i,\mathrm{b}}^*)] = \bar{C}_{i,\mathrm{b}}^*. \quad (34)$$

Inequality (34) implies that the minimum mean generalized cost of driving-alone trip *i* can be larger or remain unchanged when the cumulative distribution function of the travel time τ_i changes from $F_{i,\mathrm{a}}(\tau)$ to $F_{i,\mathrm{b}}(\tau)$. This completes the proof. □

2.2.2. The generalized cost of a ride-sharing trip

The total generalized cost of a ride-sharing trip consists of four components: (1) the driving cost of the driver, (2) the total in-vehicle travel time cost of both the driver and the rider, (3) the "penalty" for the driver reaching the destination early or late, and (4) the "penalty" for the rider reaching the destination early or late. The total generalized cost of ride-sharing trip (*i*, *j*) departing from o_i at time *t* can be formulated as follows:

$$C_{ij}(t)=\mu_0+\mu\tau_{ij}+\alpha(\tau_{ij}+\overline{\tau}_{\tilde{ij}})+\beta\cdot\max\{0,t_i^*-\Delta_i-t-\tau_{ij}\}+\gamma\cdot\max\{0,t+\tau_{ij}-t_i^*-\Delta_i\}$$
$$+\beta\cdot\max\{0,t_j^*-\Delta_j-t-\tau_{\tilde{ij}}\}+\gamma\cdot\max\{0,t+\tau_{\tilde{ij}}-t_j^*-\Delta_j\}. \quad (35)$$

By definition, the mean total generalized cost of ride-sharing trip (i, j) can be formulated as follows:

$$E[C_{ij}(t)]=\mu_0+\mu\overline{\tau}_{ij}+\alpha(\overline{\tau}_{ij}+\overline{\tau}_{\tilde{ij}})+\beta\int_{\tau_{ij}^-}^{t_i^*-\Delta_i-t}(t_i^*-\Delta_i-t-\tau)f_{ij}(\tau)d\tau$$
$$+\gamma\int_{t_i^*+\Delta_i-t}^{\tau_{ij}^+}(t+\tau-t_i^*-\Delta_i)f_{ij}(\tau)d\tau+\beta\int_{\tau_{\tilde{ij}}^-}^{t_j^*-\Delta_j-t}(t_j^*-\Delta_j-t-\tau)f_{\tilde{ij}}(\tau)d\tau \quad (36)$$
$$+\gamma\int_{t_j^*+\Delta_j-t}^{\tau_{\tilde{ij}}^+}(t+\tau-t_j^*-\Delta_j)f_{\tilde{ij}}(\tau)d\tau.$$

Equivalently, we have

$$E[C_{ij}(t)]=\mu_0+\mu\overline{\tau}_{ij}+\alpha(\overline{\tau}_{ij}+\overline{\tau}_{\tilde{ij}})+\beta(t_i^*-\Delta_i-t)F_{ij}(t_i^*-\Delta_i-t)-\beta G_{ij}(t_i^*-\Delta_i-t)$$
$$+\gamma(t_i^*+\Delta_i-t)F_{ij}(t_i^*+\Delta_i-t)-\gamma G_{ij}(t_i^*+\Delta_i-t)+\gamma(t+\overline{\tau}_{ij}-t_i^*-\Delta_i)$$
$$+\beta(t_j^*-\Delta_j-t)F_{\tilde{ij}}(t_j^*-\Delta_j-t)-\beta G_{\tilde{ij}}(t_j^*-\Delta_j-t) \quad (37)$$
$$+\gamma(t_j^*+\Delta_j-t)F_{\tilde{ij}}(t_j^*+\Delta_j-t)-\gamma G_{\tilde{ij}}(t_j^*+\Delta_j-t)+\gamma(t+\overline{\tau}_{\tilde{ij}}-t_j^*-\Delta_j).$$

According to Eq. (37), the mean generalized trip cost $E[C_{ij}(t)]$ is a function of departure time t. The first-order and second-order derivatives of $E[C_{ij}(t)]$ with respect to t can be obtained as follows:

$$\frac{dE[C_{ij}(t)]}{dt}=-\beta F_{ij}(t_i^*-\Delta_i-t)-\gamma F_{ij}(t_i^*+\Delta_i-t)$$
$$-\beta F_{\tilde{ij}}(t_j^*-\Delta_j-t)-\gamma F_{\tilde{ij}}(t_j^*+\Delta_j-t)+2\gamma \quad \text{and} \quad (38)$$

$$\frac{d^2E[C_{ij}(t)]}{dt^2}=\beta f_{ij}(t_i^*-\Delta_i-t)+\gamma f_{ij}(t_i^*+\Delta_i-t)+\beta f_{\tilde{ij}}(t_j^*-\Delta_j-t)+\gamma f_{\tilde{ij}}(t_j^*+\Delta_j-t). \quad (39)$$

Let $\varphi_{ij}(t)=dE[C_{ij}(t)]/dt$, $\varphi'_{ij}(t)=d^2E[C_{ij}(t)]/dt^2$, $e_{ij}^-=\max\{t_j^*-\Delta_j-\tau_{\tilde{ij}}^-,t_i^*-\Delta_i-\tau_{ij}^-\}$, and $e_{ij}^+=\min\{t_j^*+\Delta_j-\tau_{\tilde{ij}}^+,t_i^*+\Delta_i-\tau_{ij}^+\}$. We have the following two assumptions:

Assumption 3: $e_{ij}^-\le e_{ij}^+$ is satisfied.

Assumption 4: $e_{ij}^->e_{ij}^+$ is satisfied.

We have the following propositions:

Proposition 7. Under Assumption 3, any time instants in $[e_{ij}^-,e_{ij}^+]$ are the optimal departure times of ride-sharing trip (i, j).

Proof: For any departure time $t\in[e_{ij}^-,e_{ij}^+]$, we have

$$\begin{cases} t_i^* - \Delta_i - t \le t_i^* - \Delta_i - e_{ij}^- \le t_i^* - \Delta_i - (t_i^* - \Delta_i - \tau_{ij}^-) = \tau_{ij}^-, \\ t_i^* + \Delta_i - t \ge t_i^* + \Delta_i - e_{ij}^+ \ge t_i^* + \Delta_i - (t_i^* + \Delta_i - \tau_{ij}^+) = \tau_{ij}^+, \\ t_j^* - \Delta_j - t \le t_j^* - \Delta_j - e_{ij}^- \le t_j^* - \Delta_j - (t_j^* - \Delta_j - \tau_{\tilde{i}j}^-) = \tau_{\tilde{i}j}^-, \\ t_j^* + \Delta_j - t \ge t_j^* + \Delta_j - e_{ij}^+ \ge t_j^* + \Delta_j - (t_j^* + \Delta_j - \tau_{\tilde{i}j}^+) = \tau_{\tilde{i}j}^+. \end{cases} \tag{40}$$

According to the system of inequalities (40), we have

$$\begin{cases} F_{ij}(t_i^* - \Delta_i - t) = F_{ij}(\tau_{ij}^-) = F_{\tilde{i}j}(t_j^* - \Delta_j - t) = F_{\tilde{i}j}(\tau_{\tilde{i}j}^-) = 0, \\ G_{ij}(t_i^* - \Delta_i - t) = G_{ij}(\tau_i^-) = G_{\tilde{i}j}(t_j^* - \Delta_j - t) = G_{\tilde{i}j}(\tau_{\tilde{i}j}^-) = 0, \\ F_{ij}(t_i^* + \Delta_i - t) = F_{ij}(\tau_{ij}^+) = F_{\tilde{i}j}(t_j^* + \Delta_j - t) = F_{\tilde{i}j}(\tau_{\tilde{i}j}^+) = 1, \\ G_{ij}(t_i^* + \Delta_i - t) = G_{ij}(\tau_{ij}^+) = \overline{\tau}_{ij}, \\ G_{\tilde{i}j}(t_j^* + \Delta_j - t) = G_{\tilde{i}j}(\tau_{\tilde{i}j}^+) = \overline{\tau}_{\tilde{i}j}. \end{cases} \tag{41}$$

Substituting Eq. (41) into Eq. (37), we have

$$E[C_{ij}(t)] = \mu_0 + \mu\overline{\tau}_{ij} + \alpha(\overline{\tau}_{ij} + \overline{\tau}_{\tilde{i}j}). \tag{42}$$

Eq. (42) implies that both the rider and the driver of ride-sharing trip (i, j) do not experience any "penalty" for reaching their destinations early or late. Hence, the travelers of ride-sharing trip (i, j) arrive at their destinations on time and have the minimum mean generalized trip cost. This completes the proof. □

According to Proposition 6, under Assumption 3, the optimal departure time of each ride-sharing trip is independent of the values of μ, α, β, and γ.

Proposition 8. Under Assumption 4, the optimal departure time of ride-sharing trip (i, j), i.e., e_{ij}^*, is unique, and we have $\varphi_{ij}(e_{ij}^*) = 0$ and $\varphi_{ij}'(e_{ij}^*) > 0$.

Proof: By definition, we have $f_{ij}(t_i^* - \Delta_i - t) \ge 0$, $f_{ij}(t_i^* + \Delta_i - t) \ge 0$, $f_{\tilde{i}j}(t_j^* - \Delta_j - t) \ge 0$, and $f_{\tilde{i}j}(t_j^* + \Delta_j - t) \ge 0$. Eq. (39) implies that $d^2E[C_{ij}(t)]/dt^2 \ge 0$, and hence $E[C_{ij}(t)]$ is convex with respect to t. Based on the necessary optimality condition $dE[C_{ij}(t)]/dt = 0$, $E[C_{ij}(t)]$ achieves its minimum at $t = e_{ij}^*$. Equivalently, $\varphi_{ij}(e_{ij}^*) = 0$ and we have

$$\varphi_{ij}(e_{ij}^*) = 2\gamma - \beta F_{ij}(t_i^* - \Delta_i - e_{ij}^*) - \gamma F_{ij}(t_i^* + \Delta_i - e_{ij}^*) - \beta F_{\tilde{i}j}(t_j^* - \Delta_j - e_{ij}^*) - \gamma F_{\tilde{i}j}(t_j^* + \Delta_j - e_{ij}^*) = 0. \tag{43}$$

For all $t \le e_{ij}^+ < e_{ij}^-$, by definition, we have $t \le \min\{t_j^* + \Delta_j - \tau_{\tilde{i}j}^+, t_i^* + \Delta_i - \tau_{ij}^+\}$ and $t < \max\{t_j^* - \Delta_j - \tau_{\tilde{i}j}^-, t_i^* - \Delta_i - \tau_{ij}^-\}$. Hence, we have (a) $t_j^* + \Delta_j - t \ge \tau_{\tilde{i}j}^+$ and $t_i^* + \Delta_i - t \ge \tau_{ij}^+$, and (b) $t_j^* - \Delta_j - t > \tau_{\tilde{i}j}^-$ or $t_i^* - \Delta_i - t > \tau_{ij}^-$. According to Eq. (38), we have

$$\begin{aligned} \frac{dE[C_{ij}(t)]}{dt} &= 2\gamma - \beta F_{ij}(t_i^* - \Delta_i - t) - \gamma F_{ij}(\tau_{ij}^+) - \beta F_{\tilde{i}j}(t_j^* - \Delta_j - t) - \gamma F_{\tilde{i}j}(\tau_{ij}^+) \\ &< -\beta F_{ij}(\tau_{ij}^-) - \beta F_{\tilde{i}j}(\tau_{\tilde{i}j}^-) = 0. \end{aligned} \tag{44}$$

For all $t \ge e_{ij}^- > e_{ij}^+$, we have (a) $t_j^* - \Delta_j - t \le \tau_{\tilde{i}j}^-$ and $t_i^* - \Delta_i - t \le \tau_{ij}^-$, and (b) $t_j^* + \Delta_j - t < \tau_{\tilde{i}j}^+$ or

$t_i^* + \Delta_i - t < \tau_{ij}^+$. According to Eq. (38), we have

$$\begin{aligned}\frac{dE[C_{ij}(t)]}{dt} &= 2\gamma - \beta F_{ij}(\tau_{ij}^-) - \gamma F_{ij}(t_i^* + \Delta_i - e_{ij}^-) - \beta F_{i\tilde{j}}(\tau_{i\tilde{j}}^-) - \gamma F_{i\tilde{j}}(t_j^* + \Delta_j - e_{ij}^-) \\ &> 2\gamma - \gamma F_{ij}(\tau_{ij}^+) - \gamma F_{i\tilde{j}}(\tau_{ij}^+) = 0.\end{aligned} \tag{45}$$

Inequalities (44) and (45) imply that $dE[C_i(t)]/dt < 0$ for all $t \le e_{ij}^+$, and $dE[C_i(t)]/dt > 0$ for all $t \ge e_{ij}^-$. These results imply $e_{ij}^* \in (e_{ij}^+, e_{ij}^-)$.

To prove the uniqueness of the solution to nonlinear equation (43), we have the following definitions:

$$\begin{cases} h_{i\tilde{j}}^- = t_j^* - \Delta_j - \tau_{i\tilde{j}}^-, h_{i\tilde{j}}^+ = t_j^* + \Delta_j - \tau_{i\tilde{j}}^+, \\ h_{ij}^- = t_i^* - \Delta_i - \tau_{ij}^-, h_{ij}^+ = t_i^* + \Delta_i - \tau_{ij}^+, \\ \mathrm{g}_{i\tilde{j}}^- = t_j^* + \Delta_j - \tau_{i\tilde{j}}^-, \mathrm{g}_{ij}^- = t_i^* + \Delta_i - \tau_{ij}^-. \end{cases} \tag{46}$$

By definition, we have $\mathrm{g}_{ij}^- > h_{ij}^+$, $\mathrm{g}_{i\tilde{j}}^- > h_{i\tilde{j}}^+$, $e_{ij}^- = \max\{h_{i\tilde{j}}^-, h_{ij}^-\}$ and $e_{ij}^+ = \min\{h_{i\tilde{j}}^+, h_{ij}^+\}$. Similar to the proof of Proposition 2, we have $t_j^* - \Delta_j - t \in (\tau_{i\tilde{j}}^-, \tau_{i\tilde{j}}^+)$ or $t_j^* + \Delta_j - t \in (\tau_{i\tilde{j}}^-, \tau_{i\tilde{j}}^+)$ for all $t \in [h_{i\tilde{j}}^+, h_{ij}^-]$, and $t_i^* - \Delta_i - t \in (\tau_{ij}^-, \tau_{ij}^+)$ or $t_i^* + \Delta_i - t \in (\tau_{ij}^-, \tau_{ij}^+)$ for all $t \in [h_{ij}^+, h_{ij}^-]$. If $t \in (h_{i\tilde{j}}^+, g_{i\tilde{j}}^-)$, we have $\tau_{i\tilde{j}}^- < t_j^* + \Delta_j - t < \tau_{i\tilde{j}}^+$. This implies $f_{i\tilde{j}}(t_j^* + \Delta_j - t) > 0$ for all $t \in (h_{i\tilde{j}}^+, g_{i\tilde{j}}^-)$. Similarly, we have $f_{ij}(t_i^* + \Delta_i - t) > 0$ for all $t \in (h_{ij}^+, g_{ij}^-)$.

If $t \ge \mathrm{g}_{i\tilde{j}}^-$, we have $t_j^* + \Delta_j - t \le \tau_{i\tilde{j}}^-$ and $t_j^* - \Delta_j - t \le \tau_{i\tilde{j}}^-$. Hence, we have

$$\begin{aligned}\frac{dE[C_{ij}(t)]}{dt} &= 2\gamma - \beta F_{ij}(t_i^* - \Delta_i - t) - \gamma F_{ij}(t_i^* + \Delta_i - t) - \beta F_{i\tilde{j}}(\tau_{i\tilde{j}}^-) - \gamma F_{i\tilde{j}}(\tau_{i\tilde{j}}^-) \\ &\ge 2\gamma - \beta F_{ij}(\tau_{ij}^+) - \gamma F_{ij}(\tau_{ij}^+) = \gamma - \beta > 0.\end{aligned} \tag{47}$$

Similarly, if $t \ge \mathrm{g}_{ij}^-$, we also have $dE[C_{ij}(t)]/dt > 0$. Hence, we have $dE[C_{ij}(t)]/dt > 0$ for all $t \ge \min\{\mathrm{g}_{ij}^-, \mathrm{g}_{i\tilde{j}}^-\}$. This implies that $e_{ij}^* < \min\{\mathrm{g}_{ij}^-, \mathrm{g}_{i\tilde{j}}^-\}$.

With $e_{ij}^- > e_{ij}^+$ and $e_{ij}^+ = \min\{h_{i\tilde{j}}^+, h_{ij}^+\}$, we have two cases: (i) $e_{ij}^+ = h_{i\tilde{j}}^+$ (i.e., $h_{i\tilde{j}}^+ \le h_{ij}^+$) and (ii) $e_{ij}^+ = h_{ij}^+$ (i.e., $h_{i\tilde{j}}^+ \ge h_{ij}^+$). In Case (i), with $e_{ij}^* \in (e_{ij}^+, e_{ij}^-)$ and $e_{ij}^* < \min\{\mathrm{g}_{ij}^-, \mathrm{g}_{i\tilde{j}}^-\}$, we have $e_{ij}^* \in (h_{i\tilde{j}}^+, \mathrm{g}_{i\tilde{j}}^-)$. With $f_{i\tilde{j}}(t_j^* + \Delta_j - t) > 0$ for all $t \in (h_{i\tilde{j}}^+, g_{i\tilde{j}}^-)$, we have $d^2E[C_i(t)]/dt^2 > 0$. This implies that $E[C_i(t)]$ is strictly convex for all $t \in (h_{i\tilde{j}}^+, g_{i\tilde{j}}^-)$. Therefore, the optimal departure time e_{ij}^* is unique and $\varphi_{ij}'(e_{ij}^*) > 0$. Similarly, in Case (ii), we have $e_{ij}^* \in (h_{ij}^+, \mathrm{g}_{ij}^-)$. With $f_{ij}(t_j^* + \Delta_j - t) > 0$ for all $t \in (h_{ij}^+, g_{ij}^-)$, we have $d^2E[C_i(t)]/dt^2 > 0$. This also implies that $E[C_i(t)]$ is strictly convex for all $t \in (h_{i\tilde{j}}^+, g_{i\tilde{j}}^-)$. Therefore, the optimal departure time e_{ij}^* is also unique and $\varphi_{ij}'(e_{ij}^*) > 0$. This completes the proof. □

Substituting $t = e_{ij}^*$ into Eq. (37), and rearranging the resultant equation, we have

$$\begin{aligned}\overline{C}_{ij}^* = {} & \mu_0 + \mu\overline{\tau}_{ij} + \alpha(\overline{\tau}_{ij} + \overline{\tau}_{i\tilde{j}}) + \beta(t_i^* - \Delta_i - e_{ij}^*)F_{ij}(t_i^* - \Delta_i - e_{ij}^*) - \beta G_{ij}(t_i^* - \Delta_i - e_{ij}^*) \\ & + \gamma(t_i^* + \Delta_i - e_{ij}^*)F_{ij}(t_i^* + \Delta_i - e_{ij}^*) - \gamma G_{ij}(t_i^* + \Delta_i - e_{ij}^*) + \gamma(e_{ij}^* + \overline{\tau}_{ij} - t_i^* - \Delta_i) \\ & + \beta(t_j^* - \Delta_j - e_{ij}^*)F_{i\tilde{j}}(t_j^* - \Delta_j - e_{ij}^*) - \beta G_{i\tilde{j}}(t_j^* - \Delta_j - e_{ij}^*) \\ & + \gamma(t_j^* + \Delta_j - e_{ij}^*)F_{i\tilde{j}}(t_j^* + \Delta_j - e_{ij}^*) - \gamma G_{i\tilde{j}}(t_j^* + \Delta_j - e_{ij}^*) + \gamma(e_{ij}^* + \overline{\tau}_{i\tilde{j}} - t_j^* - \Delta_j).\end{aligned} \tag{48}$$

Based on Eq. (48), we can obtain the following propositions.

Proposition 9. Under Assumption 4, the optimal departure time of each ride-sharing trip is independent of both the values of μ and α. However, if the value of β decreases or the value of γ increases, then the optimal departure time of each ride-sharing trip decreases, and vice versa.

Proof: Based on Eq. (43), we have

$$\varphi'_{ij}(e^*_{ij}) = \beta f_{ij}(t^*_i - \Delta_i - e^*_{ij}) + \gamma f_{ij}(t^*_i + \Delta_i - e^*_{ij}) + \beta f_{\tilde{i}\tilde{j}}(t^*_j - \Delta_j - e^*_{ij}) + \gamma f_{\tilde{i}\tilde{j}}(t^*_j + \Delta_j - e^*_{ij}), \quad (49)$$

$$\frac{\partial \varphi_{ij}(e^*_{ij})}{\partial \mu} = \frac{\partial \varphi_{ij}(e^*_{ij})}{\partial \alpha} = 0, \quad (50)$$

$$\frac{\partial \varphi_{ij}(e^*_{ij})}{\partial \beta} = -F_{ij}(t^*_i - \Delta_i - e^*_{ij}) - F_{\tilde{i}\tilde{j}}(t^*_j - \Delta_j - e^*_{ij}), \text{ and} \quad (51)$$

$$\frac{\partial \varphi_{ij}(e^*_{ij})}{\partial \gamma} = 2 - F_{ij}(t^*_i + \Delta_i - e^*_{ij}) - F_{\tilde{i}\tilde{j}}(t^*_j + \Delta_j - e^*_{ij}). \quad (52)$$

According to the proof of Proposition 6, under Assumption 4, we have $\varphi'_{ij}(e^*_{ij}) > 0$ and $e^*_{ij} \in (e^+_{ij}, e^-_{ij})$, i.e., $e^*_{ij} < \max\{t^*_j - \Delta_j - \tau^-_{\tilde{i}\tilde{j}}, t^*_i - \Delta_i - \tau^-_{ij}\}$ and $e^*_{ij} > \min\{t^*_j + \Delta_j - \tau^+_{\tilde{i}\tilde{j}}, t^*_i + \Delta_i - \tau^+_{ij}\}$. Thus, we have (a) $t^*_j - \Delta_j - e^*_{ij} > \tau^-_{\tilde{i}\tilde{j}}$ or $t^*_i - \Delta_i - e^*_{ij} > \tau^-_{ij}$, and (b) $t^*_j + \Delta_j - e^*_{ij} < \tau^+_{\tilde{i}\tilde{j}}$ or $t^*_i + \Delta_i - e^*_{ij} < \tau^+_{ij}$. Therefore, we have $\max\{F_{ij}(t^*_i - \Delta_i - e^*_{ij}), F_{\tilde{i}\tilde{j}}(t^*_j - \Delta_j - e^*_{ij})\} > 0$ and $\min\{F_{ij}(t^*_i + \Delta_i - e^*_{ij}), F_{\tilde{i}\tilde{j}}(t^*_j + \Delta_j - e^*_{ij})\} < 1$. Those imply that $F_{ij}(t^*_i - \Delta_i - e^*_{ij}) + F_{\tilde{i}\tilde{j}}(t^*_j - \Delta_j - e^*_{ij}) > 0$ and $F_{ij}(t^*_i + \Delta_i - e^*_{ij}) + F_{\tilde{i}\tilde{j}}(t^*_j + \Delta_j - e^*_{ij}) - 2 < 0$. Based on Eqs. (49)-(52), the partial derivative of e^*_{ij} with respect to μ, α, β, and γ can be obtained as follows:

$$\frac{\partial e^*_{ij}}{\partial \mu} = \frac{\partial e^*_{ij}}{\partial \alpha} = -\frac{1}{\varphi'_{ij}(e^*_{ij})} \cdot \frac{\partial \varphi_{ij}(e^*_{ij})}{\partial \mu} = -\frac{1}{\varphi'_{ij}(e^*_{ij})} \cdot \frac{\partial \varphi_{ij}(e^*_{ij})}{\partial \alpha} = -\frac{1}{\varphi'_{ij}(e^*_{ij})} \cdot 0 = 0, \quad (53)$$

$$\frac{\partial e^*_{ij}}{\partial \beta} = -\frac{1}{\varphi'_{ij}(e^*_{ij})} \cdot \frac{\partial \varphi_{ij}(e^*_{ij})}{\partial \beta} = \frac{1}{\varphi'_{ij}(e^*_{ij})} \cdot [F_{ij}(t^*_i - \Delta_i - e^*_{ij}) + F_{\tilde{i}\tilde{j}}(t^*_j - \Delta_j - e^*_{ij})] > 0, \text{ and} \quad (54)$$

$$\frac{\partial e^*_{ij}}{\partial \gamma} = -\frac{1}{\varphi'_{ij}(e^*_{ij})} \cdot \frac{\partial \varphi_{ij}(e^*_{ij})}{\partial \gamma} = -\frac{1}{\varphi'_{ij}(e^*_{ij})} \cdot [F_{ij}(t^*_i + \Delta_i - e^*_{ij}) + F_{\tilde{i}\tilde{j}}(t^*_j + \Delta_j - e^*_{ij}) - 2] < 0. \quad (55)$$

Eq. (53) implies that the optimal departure time of each ride-sharing trip is independent of both the values of μ and α. Inequalities (54) and (55) imply that the optimal departure time of each ride-sharing trip decreases if the value of β decreases or the value of γ increases. This completes the proof. □

Proposition 10. Under Assumption 4, if the value of β / γ is larger, then the optimal departure time of each ride-sharing trip becomes later, and vice versa.

Proof: Using Eq. (43), we define $\omega_{ij}(e^*_{ij}) = \varphi_{ij}(e^*_{ij}) / \gamma = 0$. We have $\omega'_{ij}(e^*_{ij}) = \varphi'_{ij}(e^*_{ij}) / \gamma > 0$ and

$$\frac{\partial \omega_{ij}(e_{ij}^*)}{\partial(\beta/\gamma)} = -F_{ij}(t_i^* - \Delta_i - e_{ij}^*) - F_{\tilde{ij}}(t_j^* - \Delta_j - e_{ij}^*). \tag{56}$$

The partial derivative of e_{ij}^* with respect to β/γ can be obtained as follows:

$$\frac{\partial e_{ij}^*}{\partial(\beta/\gamma)} = -\frac{1}{\omega_{ij}'(e_{ij}^*)} \cdot \frac{\partial \omega_{ij}(e_{ij}^*)}{\partial(\beta/\gamma)} = \frac{1}{\omega_{ij}'(e_{ij}^*)} \cdot [F_{ij}(t_i^* - \Delta_i - e_{ij}^*) + F_{\tilde{ij}}(t_j^* - \Delta_j - e_{ij}^*)] > 0. \tag{57}$$

Inequality (57) implies that e_{ij}^* increases as the value of β/γ grows up. This implies that the optimal departure time e_{ij}^* increases when the value of β/γ increases. This completes the proof. □

Proposition 11. Under Assumption 4, if the values of μ, α, β, and γ increase, then the minimum mean generalized cost of each ride-sharing trip increases, and vice versa.

Proof: Based on Eq. (48), the partial derivative of $\overline{C}_{ij}^*$ with respect to μ, α, β, and γ can be obtained as follows:

$$\frac{\partial \overline{C}_{ij}^*}{\partial \mu} = \overline{\tau}_{ij}, \tag{58}$$

$$\frac{\partial \overline{C}_{ij}^*}{\partial \alpha} = \overline{\tau}_{ij} + \overline{\tau}_{\tilde{ij}}, \tag{59}$$

$$\begin{aligned} \frac{\partial \overline{C}_{ij}^*}{\partial \beta} &= (t_i^* - \Delta_i - e_{ij}^*) F_{ij}(t_i^* - \Delta_i - e_{ij}^*) - G_{ij}(t_i^* - \Delta_i - e_{ij}^*) \\ &\quad + (t_j^* - \Delta_j - e_{ij}^*) F_{\tilde{ij}}(t_j^* - \Delta_j - e_{ij}^*) - G_{\tilde{ij}}(t_j^* - \Delta_j - e_{ij}^*), \end{aligned} \quad \text{and} \tag{60}$$

$$\begin{aligned} \frac{\partial \overline{C}_{ij}^*}{\partial \gamma} &= \overline{\tau}_{ij} + (t_i^* + \Delta_i - e_{ij}^*)[F_{ij}(t_i^* + \Delta_i - e_{ij}^*) - 1] - G_{ij}(t_i^* + \Delta_i - e_{ij}^*) \\ &\quad + \overline{\tau}_{\tilde{ij}} + (t_j^* + \Delta_j - e_{ij}^*)[F_{\tilde{ij}}(t_j^* + \Delta_j - e_{ij}^*) - 1] - G_{\tilde{ij}}(t_j^* + \Delta_j - e_{ij}^*). \end{aligned} \tag{61}$$

By definition, we have $\overline{\tau}_{\tilde{ij}} > 0$ and $\overline{\tau}_{ij} > 0$, and hence $\partial \overline{C}_{ij}^* / \partial \mu > 0$ and $\partial \overline{C}_{ij}^* / \partial \alpha > 0$. According to the proof of Proposition 8, we have (a) $t_j^* - \Delta_j - e_{ij}^* > \tau_{\tilde{ij}}^-$ or $t_i^* - \Delta_i - e_{ij}^* > \tau_{ij}^-$, and (b) $t_j^* + \Delta_j - e_{ij}^* < \tau_{\tilde{ij}}^+$ or $t_i^* + \Delta_i - e_{ij}^* < \tau_{ij}^+$. Similar to the proof of Proposition 4, we can prove that $(t_i^* - \Delta_i - e_{ij}^*) F_{ij}(t_i^* - \Delta_i - e_{ij}^*) - G_{ij}(t_i^* - \Delta_i - e_{ij}^*)$ and $(t_j^* - \Delta_j - e_{ij}^*) F_{\tilde{ij}}(t_j^* - \Delta_j - e_{ij}^*) - G_{\tilde{ij}}(t_j^* - \Delta_j - e_{ij}^*)$ are non-negative, and at least one of them is positive. Hence, we have $\partial \overline{C}_{ij}^* / \partial \beta > 0$. Similar to the proof of Proposition 4, we can prove that $\overline{\tau}_{ij} + (t_i^* + \Delta_i - e_{ij}^*)[F_{ij}(t_i^* + \Delta_i - e_{ij}^*) - 1] - G_{ij}(t_i^* + \Delta_i - e_{ij}^*)$ and $\overline{\tau}_{\tilde{ij}} + (t_j^* + \Delta_j - e_{ij}^*)[F_{\tilde{ij}}(t_j^* + \Delta_j - e_{ij}^*) - 1] - G_{\tilde{ij}}(t_j^* + \Delta_j - e_{ij}^*)$ are non-negative, and at least one of them is positive. Hence, we have $\partial \overline{C}_{ij}^* / \partial \gamma > 0$. Therefore, $\overline{C}_{ij}^*$ is strictly increasing with respect to μ, α, β, and γ, respectively. This completes the proof. □

Proposition 11 implies that the values of μ, α, β, and γ can influence the minimum mean generalized cost of ride-sharing trips when travel time uncertainty exists. If there is no uncertainty (i.e.,

$\tau_{ij}^- = \tau_{ij}^+ = \overline{\tau}_{ij}$ and $\tau_{\tilde{i}\tilde{j}}^- = \tau_{\tilde{i}\tilde{j}}^+ = \overline{\tau}_{\tilde{i}\tilde{j}}$), we can consider two cases: (i) $e_{ij}^- \le e_{ij}^+$, and (ii) $e_{ij}^- > e_{ij}^+$, where $e_{ij}^- = \max\{t_j^* - \Delta_j - \overline{\tau}_{\tilde{i}\tilde{j}}, t_i^* - \Delta_i - \overline{\tau}_{ij}\}$ and $e_{ij}^+ = \min\{t_j^* + \Delta_j - \overline{\tau}_{\tilde{i}\tilde{j}}, t_i^* + \Delta_i - \overline{\tau}_{ij}\}$. In Case (i), according to the proof of Proposition 7, both the rider and the driver of ride-sharing trip (*i*, *j*) arrive at their destinations on time with their minimum generalized trip costs if the trip departs from origin o_i during the time period $[e_{ij}^-, e_{ij}^+]$. This implies that the parameters β and γ have no influence on the minimum generalized cost of driving-alone trips if there is no uncertainty. In Case (ii), we have two sub-cases: (ii.i) $t_j^* - \Delta_j - \overline{\tau}_{\tilde{i}\tilde{j}} > t_i^* + \Delta_i - \overline{\tau}_{ij}$, and (ii.ii) $t_i^* - \Delta_i - \overline{\tau}_{ij} > t_j^* + \Delta_j - \overline{\tau}_{\tilde{i}\tilde{j}}$. In Case (ii.i), according to Eq. (35), we have

$$C_{ij}(t) = \mu_0 + \mu\tau_{ij} + \alpha(\tau_{ij} + \tau_{\tilde{i}\tilde{j}}) + \begin{cases} \beta(t_i^* - \Delta_i - t - \overline{\tau}_{ij}) + \beta(t_j^* - \Delta_j - t - \overline{\tau}_{\tilde{i}\tilde{j}}) & \text{if } t \le t_i^* - \Delta_i - \overline{\tau}_{ij}, \\ \beta(t_i^* - \Delta_i - t - \overline{\tau}_{ij}) & \text{if } t_i^* - \Delta_i - \overline{\tau}_{ij} < t \le t_i^* + \Delta_i - \overline{\tau}_{ij}, \\ \beta(t_i^* - \Delta_i - t - \overline{\tau}_{ij}) + \gamma(t + \overline{\tau}_{\tilde{i}\tilde{j}} - t_j^* - \Delta_j) & \text{if } t_i^* + \Delta_i - \overline{\tau}_{ij} < t \le t_j^* - \Delta_j - \overline{\tau}_{\tilde{i}\tilde{j}}, \\ \gamma(t + \overline{\tau}_{\tilde{i}\tilde{j}} - t_j^* - \Delta_j) & \text{if } t_j^* - \Delta_j - \overline{\tau}_{\tilde{i}\tilde{j}} < t \le t_j^* + \Delta_j - \overline{\tau}_{\tilde{i}\tilde{j}}, \\ \gamma(t + \overline{\tau}_{ij} - t_i^* - \Delta_i) + \gamma(t + \overline{\tau}_{\tilde{i}\tilde{j}} - t_j^* - \Delta_j) & \text{if } t_j^* + \Delta_j - \overline{\tau}_{\tilde{i}\tilde{j}} < t. \end{cases} \tag{62}$$

With $\gamma > \beta$, according to Eq. (62), we have $e_{ij}^* = t_i^* + \Delta_i - \overline{\tau}_{ij}$ and $\overline{C}_{ij}^* = \mu_0 + \mu\overline{\tau}_{ij} + \alpha(\overline{\tau}_{ij} + \overline{\tau}_{\tilde{i}\tilde{j}}) + \beta(e_{ij}^- - e_{ij}^+)$. Similarly, in Case (ii.ii), we have $e_{ij}^* = t_j^* + \Delta_j - \overline{\tau}_{\tilde{i}\tilde{j}}$ and $\overline{C}_{ij}^* = \mu_0 + \mu\overline{\tau}_{ij} + \alpha(\overline{\tau}_{ij} + \overline{\tau}_{\tilde{i}\tilde{j}}) + \beta(e_{ij}^- - e_{ij}^+)$. These results imply that the minimum generalized cost of ride-sharing trips is independent of the value of γ, and increases as the value of β grows up. By definition, we have $\overline{\tau}_{ij} > 0$ and $\overline{\tau}_{\tilde{i}\tilde{j}} > 0$. According to Eqs. (58) and (59), the minimum generalized cost of each ride-sharing trip also increases as the values of μ and α increase if there is no uncertainty.

Proposition 12. Under Assumption 4, if the cumulative distribution functions of the travel times of ride-sharing trip (*i*, *j*), i.e., τ_{ij} and $\tau_{\tilde{i}\tilde{j}}$, respectively, change from $F_{ij,\mathrm{a}}(\tau)$ to $F_{ij,\mathrm{b}}(\tau)$ and from $F_{\tilde{i}\tilde{j},\mathrm{a}}(\tau)$ to $F_{\tilde{i}\tilde{j},\mathrm{b}}(\tau)$, and $F_{ij,\mathrm{a}}(\tau) \ge F_{ij,\mathrm{b}}(\tau)$ and $F_{\tilde{i}\tilde{j},\mathrm{a}}(\tau) \ge F_{\tilde{i}\tilde{j},\mathrm{b}}(\tau)$ are satisfied for all τ, then the optimal departure time of ride-sharing trip (*i*, *j*) can become earlier or remain unchanged, and its minimum mean generalized trip cost can become larger or remain unchanged.

Proof: To simplify the notation, we add subscripts 'a' and 'b' to the functions and parameters corresponding to the cumulative distribution functions of travel times τ_{ij} and $\tau_{\tilde{i}\tilde{j}}$ before and after the changes, respectively. To prove this proposition, we need to prove that $e_{ij,\mathrm{a}}^* \ge e_{ij,\mathrm{b}}^*$ and $\overline{C}_{ij,\mathrm{a}}^* \le \overline{C}_{ij,\mathrm{b}}^*$ are satisfied.

With $F_{ij,\mathrm{a}}(\tau) \ge F_{ij,\mathrm{b}}(\tau)$ and $F_{\tilde{i}\tilde{j},\mathrm{a}}(\tau) \ge F_{\tilde{i}\tilde{j},\mathrm{b}}(\tau)$ for all τ, Eq. (38), and the definitions of $\varphi_{ij,\mathrm{a}}(t)$ and $\varphi_{ij,\mathrm{b}}(t)$, we have $\varphi_{ij,\mathrm{a}}(t) \ge \varphi_{ij,\mathrm{b}}(t)$. By the definitions of $e_{ij,\mathrm{a}}^*$ and $e_{ij,\mathrm{b}}^*$, we have $\varphi_{ij,\mathrm{a}}(e_{ij,\mathrm{a}}^*) = 0$ and $\varphi_{ij,\mathrm{b}}(e_{ij,\mathrm{b}}^*) = 0$. Hence, we have $\varphi_{ij,\mathrm{a}}(e_{ij,\mathrm{b}}^*) \ge \varphi_{ij,\mathrm{b}}(e_{ij,\mathrm{b}}^*) = 0 = \varphi_{ij,\mathrm{a}}(e_{ij,\mathrm{a}}^*)$. Meanwhile, based on Eq. (39) and the definition of $\varphi'_{ij,\mathrm{a}}(t)$, we have $\varphi'_{ij,\mathrm{a}}(t) \ge 0$ for all $t \in (-\infty, +\infty)$. This implies that $\varphi_{ij,\mathrm{a}}(t)$ is non-decreasing. According to Proposition 8, we have $\varphi'_{ij,\mathrm{a}}(e_{ij,\mathrm{a}}^*) > 0$. This together with

$\varphi_{ij,\mathrm{a}}(e^*_{ij,\mathrm{b}}) \ge \varphi_{ij,\mathrm{a}}(e^*_{ij,\mathrm{a}})$ gives $e^*_{ij,\mathrm{a}} \ge e^*_{ij,\mathrm{b}}$.

We define the following two functions:

$$u_{\tilde{ij},\mathrm{b}}(\tau) = \begin{cases} \mu_0 + \mu\tau + \alpha\tau + \beta(t^*_j - \Delta_j - e^*_{ij,\mathrm{b}} - \tau), & \text{if } \tau < t^*_j - \Delta_j - e^*_{ij,\mathrm{b}}, \\ \mu_0 + \mu\tau + \alpha\tau + \gamma(e^*_{ij,\mathrm{b}} + \tau - t^*_j - \Delta_j), & \text{if } \tau > t^*_j + \Delta_j - e^*_{ij,\mathrm{b}}, \text{ and} \\ \mu_0 + \mu\tau + \alpha\tau, & \text{otherwise,} \end{cases} \tag{63}$$

$$u_{ij,\mathrm{b}}(\tau) = \begin{cases} \mu_0 + \mu\tau + \alpha\tau + \beta(t^*_i - \Delta_i - e^*_{ij,\mathrm{b}} - \tau), & \text{if } \tau < t^*_i - \Delta_i - e^*_{ij,\mathrm{b}}, \\ \mu_0 + \mu\tau + \alpha\tau + \gamma(e^*_{ij,\mathrm{b}} + \tau - t^*_i - \Delta_i), & \text{if } \tau > t^*_i + \Delta_i - e^*_{ij,\mathrm{b}}, \\ \mu_0 + \mu\tau + \alpha\tau, & \text{otherwise.} \end{cases} \tag{64}$$

With $0 < \beta < \alpha < \gamma$, we have $du_{\tilde{ij},\mathrm{b}}(\tau)/d\tau > 0$ and $du_{ij,\mathrm{b}}(\tau)/d\tau > 0$. Hence $u_{\tilde{ij},\mathrm{b}}(\tau)$ and $u_{ij,\mathrm{b}}(\tau)$ are strictly increasing functions with respect to τ. With $F_{ij,\mathrm{a}}(\tau) \ge F_{ij,\mathrm{b}}(\tau)$ and $F_{\tilde{ij},\mathrm{a}}(\tau) \ge F_{\tilde{ij},\mathrm{b}}(\tau)$ for all τ, according to Lemma 1, we have

$$\int_{-\infty}^{+\infty} u_{\tilde{ij},\mathrm{b}}(\tau) dF_{\tilde{ij},\mathrm{a}}(\tau) + \int_{-\infty}^{+\infty} u_{ij,\mathrm{b}}(\tau) dF_{ij,\mathrm{a}}(\tau) \le \int_{-\infty}^{+\infty} u_{\tilde{ij},\mathrm{b}}(\tau) dF_{\tilde{ij},\mathrm{b}}(\tau) + \int_{-\infty}^{+\infty} u_{ij,\mathrm{b}}(\tau) dF_{ij,\mathrm{b}}(\tau). \tag{65}$$

By definition, we have

$$\bar{C}^*_{ij,\mathrm{a}} = E[C_{ij,\mathrm{a}}(e^*_{ij,\mathrm{a}})] \le E[C_{ij,\mathrm{a}}(e^*_{ij,\mathrm{b}})] = \int_{-\infty}^{+\infty} u_{\tilde{ij},\mathrm{b}}(\tau) dF_{\tilde{ij},\mathrm{a}}(\tau) + \int_{-\infty}^{+\infty} u_{ij,\mathrm{b}}(\tau) dF_{ij,\mathrm{a}}(\tau) \text{ and} \tag{66}$$

$$\bar{C}^*_{ij,\mathrm{b}} = E[C_{ij,\mathrm{b}}(e^*_{ij,\mathrm{b}})] = \int_{-\infty}^{+\infty} u_{\tilde{ij},\mathrm{b}}(\tau) dF_{\tilde{ij},\mathrm{b}}(\tau) + \int_{-\infty}^{+\infty} u_{ij,\mathrm{b}}(\tau) dF_{ij,\mathrm{b}}(\tau). \tag{67}$$

Inequalities (65) and (66) and Eq. (67) imply $\bar{C}^*_{ij,\mathrm{a}} \le \bar{C}^*_{ij,\mathrm{b}}$. This completes the proof. □

2.3. Feasible ride-sharing trips

The most important motivation for travelers using a ride-sharing trip is that both the driver and the rider can benefit from a ride-sharing trip. Therefore, a feasible ride-sharing matching can be defined as follows:

Definition 4 (Feasible ride-sharing trip). A ride-sharing trip (i, j) is defined as a feasible ride-sharing trip if the following condition is satisfied:

$$\bar{C}^*_{ij} \le \varepsilon \cdot (\bar{C}^*_i + \bar{C}^*_j), \tag{68}$$

where ε is the tolerance, and $0 < \varepsilon \le 1$.

The above definition implies that the larger the tolerance is, the more the ride-sharing trip is a feasible one. Using inequality (68), we can define the set of feasible ride-sharing trips as follows:

Definition 5 (Set of feasible ride-sharing trips). The set of feasible ride-sharing trips is formulated as follows:

$$\Omega = \left\{(i, j) \mid \bar{C}^*_{ij} \le \varepsilon \cdot (\bar{C}^*_i + \bar{C}^*_j) \text{ and } i \ne j, \forall i \in \Phi_d \cup \Phi_{d/r}, j \in \Phi_r \cup \Phi_{d/r}\right\}. \tag{69}$$

Definition 6 (Mean generalized ride-sharing trip cost saving). The mean generalized cost saving of ride-sharing trip (i, j) is defined as follows:

$$\kappa_{ij} = (\bar{C}_i^* + \bar{C}_j^*) - \bar{C}_{ij}^*. \tag{70}$$

Proposition 13. If $\bar{\tau}_i - \bar{\tau}_{ij} < 0$ and $\bar{\tau}_j - \bar{\tau}_{\tilde{i}\tilde{j}} \le 0$, the mean generalized cost saving of ride-sharing trip (*i*, *j*) decreases as the value of α increases, and vice versa.

Proof: Based on Eq. (70), we have

$$\frac{\partial \kappa_{ij}}{\partial \alpha} = \frac{\partial \bar{C}_i^*}{\partial \alpha} + \frac{\partial \bar{C}_j^*}{\partial \alpha} - \frac{\partial \bar{C}_{ij}^*}{\partial \alpha}. \tag{71}$$

Substituting Eqs. (27) and (59) into Eq. (71) and using $\bar{\tau}_i - \bar{\tau}_{ij} < 0$ and $\bar{\tau}_j - \bar{\tau}_{\tilde{i}\tilde{j}} \le 0$, we have

$$\frac{\partial \kappa_{ij}}{\partial \alpha} = \bar{\tau}_i + \bar{\tau}_j - (\bar{\tau}_{ij} + \bar{\tau}_{\tilde{i}\tilde{j}}) < 0. \tag{72}$$

Eq. (72) implies that the mean generalized cost saving of ride-sharing trip (*i*, *j*) decreases as the value of α increases. This completes the proof. □

Remark 1: The condition $\bar{\tau}_i - \bar{\tau}_{ij} < 0$ means that the average travel time of driving-alone trip *i* is less than that of the whole ride-sharing trip (*i*, *j*). This condition usually holds except both driver *i* and rider *j* have the same origin and destination. The condition $\bar{\tau}_j - \bar{\tau}_{\tilde{i}\tilde{j}} \le 0$ means that the average travel time of driving-alone trip *j* is not greater than that of driver *i* traveling from origin o_i to origin o_j for picking up rider *j*, and then traveling to destination d_j. This condition must hold.

Proposition 14. If $\bar{\tau}_i + \bar{\tau}_j - \bar{\tau}_{ij} > 0$, the mean generalized trip cost saving of ride-sharing trip (*i*, *j*) increases as the value of μ increases, and vice versa.

Proof: According to Eq. (70), we have

$$\frac{\partial \kappa_{ij}}{\partial \mu} = \frac{\partial \bar{C}_i^*}{\partial \mu} + \frac{\partial \bar{C}_j^*}{\partial \mu} - \frac{\partial \bar{C}_{ij}^*}{\partial \mu}. \tag{73}$$

Substituting Eqs. (27) and (58) into Eq. (73), we have

$$\frac{\partial \kappa_{ij}}{\partial \mu} = \bar{\tau}_i + \bar{\tau}_j - \bar{\tau}_{ij} > 0. \tag{74}$$

Eq. (74) implies that the mean generalized trip cost saving of ride-sharing trip (*i*, *j*) increases as the value of μ increases. This completes the proof. □

Remark 2: The condition $\bar{\tau}_i + \bar{\tau}_j - \bar{\tau}_{ij} > 0$ means that compared with separately implementing driving-alone trips *i* and *j*, ride-sharing trip (*i*, *j*) can save the mean travel time. This condition cannot always hold because ride-sharing trip (*i*, *j*) cannot always save average travel time.

Based on the set of feasible ride-sharing trips, we define the set of candidate drivers and the set of candidate riders, respectively, as follows:

$$I = \{i \mid \exists j \in \Phi_r \cup \Phi_{d/r} \text{ such that } (i, j) \in \Omega\} \quad \text{and} \tag{75}$$

$$J = \{ j \mid \exists i \in \Phi_d \cup \Phi_{d/r} \text{ such that } (i, j) \in \Omega\}. \tag{76}$$

According to Eqs. (69) and (75), a candidate driver can be a ride-sharing participant who selects to be (1) *a driver* or (2) *either a driver or a rider*. According to Eqs. (69) and (76), a candidate rider can be a ride-sharing participant who selects to be (1) *a rider* or (2) *either a driver or a rider*.

We can define the set of candidate riders for each driver and the set of candidate drivers for each rider, respectively, as follows:

$$\Theta_i = \{ j \mid (i, j) \in \Omega\}, \forall i \in I \quad \text{and} \tag{77}$$

$$\Xi_j = \{i \mid (i, j) \in \Omega\}, \forall j \in J. \tag{78}$$

2.4. The static ride-sharing matching model

In this paper, we only consider a static ride-sharing matching problem. It is assumed that all users of the ride-sharing platform announce their trip schedule one day before, and the ride-sharing company makes money by providing the ride-sharing service. The company can deduct a percentage of the total benefit from all matched ride-sharing trips, and its revenue can also link to the number of successful ride-sharing arrangements (i.e., the number of satisfied drivers and riders in the system) (Agatz et al., 2012). Therefore, the platform optimizes users' matching to maximize the total benefit from all matched ride-sharing trips and the number of matches. Using the set of feasible ride-sharing trips, the set of candidate drivers, and the set of candidate riders, the matching problem can be formulated as a maximum weight bipartite matching problem, which is a mixed-integer linear programming (MILP) problem and is given as follows:

$$\max\ (1-\phi) \sum_{(i,j)\in\Omega} \kappa_{ij} x_{ij} + \phi\kappa_0 \sum_{(i,j)\in\Omega} x_{ij} \tag{79}$$

$$\text{s.t.} \quad \sum_{j\in\Theta_i} x_{ij} \le 1, \quad \forall i \in I, \tag{80}$$

$$\sum_{i\in\Xi_j} x_{ij} \le 1, \quad \forall j \in J, \text{ and} \tag{81}$$

$$x_{ij} = \{0,1\}, \forall (i, j) \in \Omega, \tag{82}$$

where ϕ is a non-negative parameter and $\phi \in [0,1]$, κ_0 is a positive parameter and denotes the monetary value of forming one more ride-sharing match, and κ_{ij} follows an earlier definition (see Eq. (70)). The objective (79) is to maximize a weighted combination of the mean total generalized trip cost saving and the number of matches. The objective is to maximize the total generalized trip cost saving (i.e., minimize the total generalized trip cost) if $\phi = 0$, and to maximize the number of matches if $\phi = 1$. Constraint (80) ensures that each driver is matched with no more than one rider. Constraint (81) ensures that each rider is matched with no more than one driver. Constraint (82) is the definitional constraint for ride-sharing trip matching decision variables.

2.5. The setting of travel time distributions

We assume that the travel times of all links are stochastic. For the ease of calculating the mean generalized cost saving κ_{ij} in the road network with time-independent travel time uncertainty, we assume that the travel times follow a distribution modified from a Gamma distribution. The definition of the Gamma distribution can be stated as follows (Hogg et al., 2013):

Definition 7 (Gamma distribution). The distribution with the following probability density function (PDF) is called the Gamma distribution with the positive parameters ϖ and θ, and it is denoted as $\Gamma(\varpi, \theta)$:

$$f(x \mid \varpi, \theta) = \begin{cases} \dfrac{x^{\varpi-1} e^{-x/\theta}}{\theta^{\varpi} \Gamma(\varpi)}, \text{ if } x \geq 0, \\ 0, \text{ otherwise,} \end{cases} \tag{83}$$

where $\Gamma(\varpi)$ is the Gamma function and defined by

$$\Gamma(\varpi) = \int_0^{\infty} x^{\varpi-1} e^{-x} dx . \tag{84}$$

The mean and variance of a random variable following the PDF defined by Eq. (83) are $\varpi\theta$ and $\varpi\theta^2$, respectively. According to the additivity property of Gamma distributions, we have the following lemma (Hogg et al., 2013):

Lemma 2. If we have a sequence of independent random variables

$$X_i \sim \Gamma(\varpi_i, \theta), \forall i \in \{1, 2, \cdots, n\}, \tag{85}$$

then $\sum_{i=1}^{n} X_i$ follows $\Gamma\left(\sum_{i=1}^{n} \varpi_i, \theta\right)$.

Definition 8. (Shifted Gamma distribution). Let X follows the Gamma distribution $\Gamma(\varpi, \theta)$. The distribution of $Y = X + \tau^-$ is called the shifted Gamma distribution with the parameters τ^-, $\overline{\tau}$, and θ, and is denoted as $\Gamma(\tau^-, \overline{\tau}, \theta)$, where τ^- is the minimum value of Y, and $\overline{\tau} = \theta\varpi + \tau^-$ is the mean of Y.

Note that the shifted Gamma distribution $\Gamma(\tau^-, \overline{\tau}, \theta)$ immediately follows the Gamma distribution $\Gamma(\overline{\tau} / \theta, \theta)$ if $\tau^- = 0$. If Y follows the shifted Gamma distribution $\Gamma(\tau^-, \overline{\tau}, \theta)$, then $X = Y - \tau^-$ follows the shifted Gamma distribution $\Gamma(0, \overline{\tau} - \tau^-, \theta)$, and also follows the Gamma distribution $\Gamma(\varpi, \theta)$, where $\varpi = (\overline{\tau} - \tau^-) / \theta$.

Proposition 15. If we have a sequence of independent random variables

$$Y_i \sim \Gamma(\tau_i^-, \overline{\tau}_i, \theta), \forall i \in \{1, 2, \cdots, n\}, \tag{86}$$

then $\sum_{i=1}^{n} Y_i$ follows $\Gamma\left(\sum_{i=1}^{n} \tau_i^-, \sum_{i=1}^{n} \overline{\tau}_i, \theta\right)$.

Proof: Let $X_i = Y_i - \tau_i^-$ for all $i \in \{1, 2, \cdots, n\}$. We have

$$X_i \sim \Gamma(\varpi_i, \theta), \forall i \in \{1, 2, \cdots, n\}, \tag{87}$$

where $\varpi_i = (\overline{\tau}_i - \tau_i^-)/\theta$.

According to Lemma 2, we have $\sum_{i=1}^{n} X_i$ follows the Gamma distribution $\Gamma\left(\sum_{i=1}^{n} \varpi_i, \theta\right)$ and hence the mean of the distribution $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \varpi_i \theta$. As $\varpi_i = (\overline{\tau}_i - \tau_i^-)/\theta$, $\sum_{i=1}^{n} \varpi_i \theta = \sum_{i=1}^{n} (\overline{\tau}_i - \tau_i^-)$. Hence, we have

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \varpi_i \theta = \sum_{i=1}^{n} (\overline{\tau}_i - \tau_i^-). \tag{88}$$

By definition, we also have

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} \tau_i^- \quad \text{and} \tag{89}$$

$$E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \tau_i^- + E\left[\sum_{i=1}^{n} X_i\right]. \tag{90}$$

Substituting Eq. (88) into Eq. (90), we get

$$E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \tau_i^- + \sum_{i=1}^{n} \varpi_i \theta = \sum_{i=1}^{n} \overline{\tau}_i. \tag{91}$$

Based on Definition 8, Eqs. (89) and (91), $\sum_{i=1}^{n} Y_i$ follows $\Gamma\left(\sum_{i=1}^{n} \tau_i^-, \sum_{i=1}^{n} \overline{\tau}_i, \theta\right)$. This completes the proof. □

Let t_a be the travel time on link *a*, which follows the shifted Gamma distribution $\Gamma(t_a^-, \overline{t}_a, \theta)$, where t_a^- and $\overline{t}_a$ are the minimum and mean travel times of link *a*, respectively. The uncertainty of the travel time of link *a* can increase in two ways: (1) fix the minimum value t_a^- but increase the mean value $\overline{t}_a$, and (2) fix both the minimum value t_a^- and the mean value $\overline{t}_a$ but increase the parameter value θ. In the first way, both the mean and variance of the link travel time increase. In the second way, the mean link travel time remains unchanged but only the variance of the link travel time increases. In order to better represent the changes of travel time uncertainty in the first way, we introduce a non-negative scale parameter λ such that $\overline{t}_a = t_a^- + \lambda(\overline{t}_a^0 - t_a^-)$, where $\overline{t}_a^0$ is the reference mean travel time of link *a*. We note that the value of λ can represent the level of network traffic congestion. When the value of λ is larger, the road network is more congested. Especially, if $\lambda = 0$, we have $\overline{t}_a = t_a^-$, which implies that the travel time of link *a* is deterministic;

if $\lambda = 1$, we have $\overline{t}_a = \overline{t}_a^0$, which implies that the mean travel time of link *a* equals its reference mean travel time. When the mean travel times of all links equal their reference mean travel times, the corresponding mean travel time of each trip is defined as the reference mean trip travel time.

According to the additivity property of shifted Gamma distributions (see Proposition 15), the travel time τ_i ($\forall i \in I$) follows the shifted Gamma distribution $\Gamma(\tau_i^-, \overline{\tau}_i, \theta)$, and the travel times τ_{ij} and $\tau_{i\tilde{j}}$ ($\forall (i,j) \in \Omega$) follow the shifted Gamma distributions $\Gamma(\tau_{i\tilde{j}}^-, \overline{\tau}_{i\tilde{j}}, \theta)$ and $\Gamma(\tau_{ij}^-, \overline{\tau}_{ij}, \theta)$, respectively.

Proposition 16. If the value of λ increases, then the distributions $\Gamma(\tau_i^-, \overline{\tau}_i, \theta)$, $\Gamma(\tau_{ij}^-, \overline{\tau}_{ij}, \theta)$ and $\Gamma(\tau_{i\tilde{j}}^-, \overline{\tau}_{i\tilde{j}}, \theta)$ after the increase first-order stochastically dominate those before the increase.

Proof: We consider that the value of λ increases from λ_{a} to λ_{b}. To simplify the notation, we add subscripts 'a' and 'b' to the functions and parameters corresponding to λ_{a} and λ_{b}. By definition, we have $\overline{\tau}_{i,\mathrm{a}} = \tau_i^- + \lambda_{\mathrm{a}}(\overline{\tau}_i^0 - \tau_i^-)$, $\overline{\tau}_{i,\mathrm{b}} = \tau_i^- + \lambda_{\mathrm{b}}(\overline{\tau}_i^0 - \tau_i^-)$, $\varpi_{i,\mathrm{a}} = \lambda_{\mathrm{a}}(\overline{\tau}_i^0 - \tau_i^-)/\theta$, and $\varpi_{i,\mathrm{b}} = \lambda_{\mathrm{b}}(\overline{\tau}_i^0 - \tau_i^-)/\theta$, where $\overline{\tau}_i^0$ is the reference mean travel time of driving-alone trip *i* and $\overline{\tau}_i^0 > \tau_i^-$. With $\lambda_{\mathrm{a}} < \lambda_{\mathrm{b}}$ and $\theta > 0$, we have $\varpi_{i,\mathrm{a}} < \varpi_{i,\mathrm{b}}$. Define $H_{\mathrm{a,b}}(\tau) = F_{i,\mathrm{a}}(\tau) - F_{i,\mathrm{b}}(\tau)$. By definition, we have

$$\begin{aligned} H_{\mathrm{a,b}}(\tau) &= F_{i,\mathrm{a}}(\tau) - F_{i,\mathrm{b}}(\tau) = \int_{t_i^-}^{\tau} \frac{(t-\tau_i^-)^{\varpi_{i,\mathrm{a}}-1} e^{-(t-\tau_i^-)/\theta}}{\theta^{\varpi_{i,\mathrm{a}}}\Gamma(\varpi_{i,\mathrm{a}})} dt - \int_{t_i^-}^{\tau} \frac{(t-\tau_i^-)^{\varpi_{i,\mathrm{b}}-1} e^{-(t-\tau_i^-)/\theta}}{\theta^{\varpi_{i,\mathrm{b}}}\Gamma(\varpi_{i,\mathrm{b}})} dt \\ &= \frac{1}{\theta^{\varpi_{i,\mathrm{a}}}\Gamma(\varpi_{i,\mathrm{a}})} \int_0^{\tau} \left[(t-\tau_i^-)^{\varpi_{i,\mathrm{a}}-\varpi_{i,\mathrm{b}}} - \frac{\Gamma(\varpi_{i,\mathrm{a}})}{\Gamma(\varpi_{i,\mathrm{b}})} \theta^{\varpi_{i,\mathrm{a}}-\varpi_{i,\mathrm{b}}} \right] (t-\tau_i^-)^{\varpi_{i,\mathrm{b}}-1} e^{-(t-\tau_i^-)/\theta} dt. \end{aligned} \tag{92}$$

By definition, we have $F_{i,\mathrm{a}}(\tau_i^-) = F_{i,\mathrm{b}}(\tau_i^-) = 0$ and $\lim_{\tau\to+\infty} F_{i,\mathrm{a}}(\tau) = \lim_{\tau\to+\infty} F_{i,\mathrm{b}}(\tau) = 1$. Those imply that $H_{\mathrm{a,b}}(\tau_i^-) = 0$ and $\lim_{\tau\to+\infty} H_{\mathrm{a,b}}(\tau) = 0$. Taking derivative for Eq. (92), we have

$$\frac{dH_{\mathrm{a,b}}(\tau)}{d\tau} = \frac{\left[(\tau-\tau_i^-)^{\varpi_{i,\mathrm{a}}-\varpi_{i,\mathrm{b}}} - \frac{\Gamma(\varpi_{i,\mathrm{a}})}{\Gamma(\varpi_{i,\mathrm{b}})} \theta^{\varpi_{i,\mathrm{a}}-\varpi_{i,\mathrm{b}}} \right] (\tau-\tau_a^-)^{\varpi_{i,\mathrm{b}}-1} e^{-(\tau-\tau_i^-)/\theta}}{\theta^{\varpi_{i,\mathrm{a}}}\Gamma(\varpi_{i,\mathrm{a}})}. \tag{93}$$

According to Eq. (93), we have $dH_{\mathrm{a,b}}(\tau)/d\tau < 0$ if $\tau_i^- < \tau < \tau_{\mathrm{a,b}}$ and $dH_{\mathrm{a,b}}(\tau)/d\tau > 0$ if $\tau_{\mathrm{a,b}} > \tau$, where

$$\tau_{\mathrm{a,b}} = \tau_i^- + \left\{ \frac{\Gamma(\varpi_{i,\mathrm{a}})}{\Gamma(\varpi_{i,\mathrm{b}})} \theta^{\varpi_{i,\mathrm{a}}-\varpi_{i,\mathrm{b}}} \right\}^{\frac{1}{\varpi_{i,\mathrm{a}}-\varpi_{i,\mathrm{b}}}} \quad \text{and} \quad H_{\mathrm{a,b}}(\tau_{\mathrm{a,b}}) = 0.$$

The above results imply that $H_{\mathrm{a,b}}(\tau) \geq 0$ for every τ. Equivalently, we have $F_{i,\mathrm{a}}(\tau) \geq F_{i,\mathrm{b}}(\tau)$ for every τ. Similarly, we can prove that $F_{ij,\mathrm{a}}(\tau) \geq F_{ij,\mathrm{b}}(\tau)$ and $F_{i\tilde{j},\mathrm{a}}(\tau) \geq F_{i\tilde{j},\mathrm{b}}(\tau)$ for every τ. This completes the proof. □

Proposition 16 implies that if the value of λ increases, the distributions of travel times that follow shifted Gamma distributions after the increase first-order stochastically dominate those before the increase.

However, as illustrated by the numerical examples in Section 4.1, the counterpart cannot be established if the value of θ increases.

Remark 3: Besides the shifted Gamma distribution, other distributions can be used in the proposed model. If other distributions are used, the convolution of distributions may be needed before determining the mean generalized ride-sharing trip cost saving.

3. Stochastic ride-sharing model with time-dependent travel time uncertainty

3.1. Representation of time-dependent travel time uncertainty

We consider a road network $G(N, A)$ with multiple origins and destinations, where N and A are defined as the set of nodes and the set of arcs (links), respectively. We discretize the time period *T* of interest into a finite set of time intervals $K = \{k = 1, 2, \cdots, \underline{K}\}$. Let t_{k-1}, t_k, and $\overline{t_k}$ be the start time, the end time and the middle time of time interval *k*. It is assumed that each link $a \in A$ has a stochastic travel time $\tau_{a,k}$ during interval $k \in K$. Without loss of generality, we assume that the travel time $\tau_{a,k}$ follows a distribution with the range $[\tau^-_{a,k}, \tau^+_{a,k}]$, and has a mean value of $\overline{\tau}_{a,k}$, where $\tau^-_{a,k}$ and $\tau^+_{a,k}$ are the minimum and maximum travel times of travelers traveling through link *a* during interval *k*, respectively. Because travel times must be positive in reality, we define $\tau^-_{a,k} > 0$. Let $f_{a,k}(\tau)$ and $F_{a,k}(\tau)$ be the probability density function and cumulative distribution function of the travel time $\tau_{a,k}$, respectively. We use the mean value of the travel time $\tau_{a,k}$ to reflect the congestion level of link *a* during interval *k*. It is assumed that all trips use the paths with the minimum mean travel times. Let $P_i(k)$ be the path of driving alone trip *i* that departs origin o_i during interval *k*. Let $P_{ij}(k)$ be the path of ride-sharing trip (*i*, *j*) that departs origin o_i during interval *k*.

3.2. Evaluating the mean generalized trip cost

We consider a given driving-alone trip *i* departing at time instant t, where $t \in (t_{k_1-1}, t_{k_1}]$. By assumption, path $P_i(k_1)$ is used by driving-alone trip *i*. Let w_i be the number of links on path $P_i(k_1)$, and a_v be the *v*th link on path $P_i(k_1)$. We have $P_i(k_1) = \{a_1, a_2, \cdots, a_{w_i}\}$. Let t_{a_v} and k_v be the time instant and time interval of driving-alone trip *i* entering link a_v. The following MCS method with a sample size of *M* can be used to obtain the mean generalized cost of driving-alone trip *i* departing at time instant t:

Procedure 1: MONTECARLO_ DRIVING-ALONETRIPCOST (*i*, *t*, *M*)

Set $m = 1$.
while $m \le M$ **do**
    Set $v = 1$, $t_{a_1}{}^{(m)} = t$, and $k_1{}^{(m)} = k_1$.
    while $v \le w_i$ **do**
        Sample the link travel time $\tau_{a_v}{}^{(m)}$ according to the travel time density function $f_{a_v, k_v{}^{(m)}}(\tau)$.
        Calculate the link entry time $t_{a_{v+1}}{}^{(m)} = t_{a_v}{}^{(m)} + \tau_{a_v}{}^{(m)}$.

Find $k_{v+1}{}^{(m)}$ such that $t_{a_{v+1}}{}^{(m)} \in (t_{k_{v+1}{}^{(m)}-1}, t_{k_{v+1}{}^{(m)}}]$.

$v = v+1$

end do

Update the trip travel time $\tau_i{}^{(m)} = t_{a_{w_i+1}}{}^{(m)} - t$

Calculate the trip travel cost $C_i^{(m)}(t)$ according to Eq. (11), where $\tau_i = \tau_i{}^{(m)}$.

Estimate the mean generalized trip cost $\overline{C}_i^{(m)}(t) = [(m-1)\overline{C}_i^{(m-1)}(t) + C_i^{(m)}(t)]/m$.

$m = m+1$

end do

Output the mean generalized trip cost $\overline{C}_i^{(M)}(t)$.

The inputs to Procedure 1 are driving-along trip *i*, departure time *t*, and the sample size M. In this procedure, firstly, the link travel time of each link on path $P_i(k_1)$ was sampled according to its density function. Then, the sampled link travel times were used to calculate the path travel time of driving-along trip *i*. The path travel time and the departure time were then used to update its generalized trip cost and the updated generalized trip cost was further used to determine the mean generalized trip cost. The above steps were repeated by M times and finally, the mean generalized trip cost $\overline{C}_i^{(M)}(t)$ was outputted.

We consider a given ride-sharing trip (*i*, *j*) departing at time instant t during interval k_1, where $t \in (t_{k_1-1}, t_{k_1}]$. By assumption, path $P_{ij}(k_1)$ is used by ride-sharing trip (*i*, *j*) departing at time instant t. Let w_{ij} and $w_{\tilde{i}\tilde{j}}$ be the numbers of links on path $P_{ij}(k_1)$ from origin o_i to destinations d_i and d_j, respectively. Let a_v be the *v*th link on path $P_{ij}(k_1)$. By definition, w_{ij} is the number of links on path $P_{ij}(k_1)$, and $P_{ij}(k_1) = \{a_1, a_2, \cdots, a_{w_{\tilde{i}\tilde{j}}}, \cdots, a_{w_{ij}}\}$. The following MCS method with a sample size of *M* can be used to obtain the mean generalized cost of ride-sharing trip (*i*, *j*) departing at time instant t:

Procedure 2: MONTECARLO_ RIDE-SHARINGTRIPCOST (*i*, *j*, *t*, *M*)

Set $m = 1$.

while $m \le M$ **do**

Set $v = 1$, $t_{a_1}{}^{(m)} = t$, and $k_1{}^{(m)} = k_1$.

while $v \le w_i$ **do**

Sample the link travel time $\tau_{a_v}{}^{(m)}$ according to the travel time density function $f_{a_v, k_v{}^{(m)}}(\tau)$.

Calculate link entry time $t_{a_{v+1}}{}^{(m)} = t_{a_v}{}^{(m)} + \tau_{a_v}{}^{(m)}$.

Find $k_{v+1}{}^{(m)}$ such that $t_{a_{v+1}}{}^{(m)} \in (t_{k_{v+1}{}^{(m)}-1}, t_{k_{v+1}{}^{(m)}}]$.

$v = v+1$

end do

Update the trip travel time $\tau_{\tilde{i}\tilde{j}}{}^{(m)} = t_{a_{w_j+1}}{}^{(m)} - t$ and $\tau_{ij}{}^{(m)} = t_{a_{w_i+1}}{}^{(m)} - t$.

Calculate the generalized trip cost $C_{ij}^{(m)}(t)$ according to Eq. (35), where $\tau_{\tilde{i}\tilde{j}} = \tau_{\tilde{i}\tilde{j}}{}^{(m)}(t)$ and $\tau_{ij} = \tau_{ij}{}^{(m)}(t)$.

Estimate the mean generalized trip cost $\overline{C}_{ij}^{(m)}(t) = [(m-1)\overline{C}_{ij}^{(m-1)}(t) + C_{ij}^{(m)}(t)]/m$.

$m = m+1$

end do

Output the mean generalized trip cost $\bar{C}_{ij}^{(M)}(t)$.

The inputs to Procedure 2 are ride-sharing trip (*i*, *j*), departure time *t*, and the sample size M. In this procedure, firstly, the link travel time of each link on path $P_{ij}(k_1)$ was sampled according to its density function. Then, the sampled link travel times were used to calculate the path travel time of ride-sharing trip (*i*, *j*). The path travel time and the departure time were then used to update its generalized trip cost and the updated generalized trip cost was further used to determine the mean generalized trip cost. The above steps were repeated by M times and finally, the mean generalized trip cost $\bar{C}_{ij}^{(M)}(t)$ was outputted.

3.3. Ride-sharing match schemes by Monte Carlo Simulation methods

Before using the ride-sharing matching model (79)-(82) to determine an optimal ride-sharing match scheme, the ride-sharing platform should firstly estimate the optimal departure time and the minimum trip cost of each driving-alone trip and ride-sharing trip. Our method is that each time interval is divided into smaller sub-intervals. Let H_k be the set of sub-intervals of time interval *k*, and H be the set of all sub-intervals. Let t_h be the middle time instant of sub-interval $h \in H$. By using Procedures 1 and 2, we can obtain the values of $\bar{C}_i(t_h)$ ($\forall i \in \Phi, h \in H$) and $\bar{C}_{ij}(t_h)$ ($\forall i \in \Phi, j \in \Phi, h \in H$), respectively. The optimal departure time of a driving-alone trip *i* can be approximated by $e_i^* = t_{h_i^*}$, where $h_i^* = \arg\min_{h\in H}\{\bar{C}_i(t_h)\}$, and the optimal departure time of a ride-sharing trip (*i*, *j*) can be approximated by $e_{ij}^* = t_{h_{ij}^*}$, where $h_{ij}^* = \arg\min_{h\in H}\{\bar{C}_{ij}(t_h)\}$. The minimum mean generalized costs of driving-alone trips and ride-sharing trips can be, respectively, estimated as follows:

$$\bar{C}_i^* = \bar{C}_i(t_{h_i^*}) = \min_{h\in H}\{\bar{C}_i(t_h)\}, \forall i \in \Phi \quad \text{and} \tag{94}$$

$$\bar{C}_{ij}^* = \bar{C}_{ij}(t_{h_{ij}^*}) = \min_{h\in H}\{\bar{C}_{ij}(t_h)\}, \forall i \in \Phi, j \in \Phi \; \bar{C}_{ij}^* = \bar{C}_{ij}(t_{h_{ij}^*}) = \min_{h\in H}\{\bar{C}_{ij}(t_h)\}, \forall i \in \Phi, j \in \Phi, h \in H. \tag{95}$$

Substituting Eqs. (96) and (97) into Eqs. (69) and (70), we can obtain the set of feasible ride-sharing trips Ω and the mean generalized ride-sharing trip cost saving κ_{ij}, respectively. Then, we can solve the MILP problem (79)-(82) to obtain the optimal ride-sharing match scheme.

4. Numerical examples

To demonstrate the properties of the proposed models, we present four numerical examples in this section. All experiments were run on a computer with an Intel (R) Xeon(R) E5-2420 2.20GHz CPU and a 32.0GB RAM. The ride-sharing matching model in this paper was solved by a commercial software package, IBM ILOG CPLEX (version 12.5). Unless otherwise stated, the following parameters are set as follows: α = 0.8 HK$/min, β = 0.5 HK$/min, γ = 3.0 HK$/min, μ = 1.2 HK$/min, μ_0 = 0 HK$, and $\varepsilon = 0$.

4.1. A driving-alone trip

In this example, we consider a single driving-alone trip. The trip travel time follows a shifted Gamma distribution. Unless otherwise stated, we have $\theta = 10.0$ min and $\lambda = 1$. The minimum and reference mean travel times of this driving-alone trip are set to be 30 min and 50 min, respectively. The preferred arrival time of the driver is 9:00 am.

4.1.1. The effect of the unit variable cost of driving and traveler's VOTs

We set different values to the unit variable cost of driving (μ), and traveler's VOTs (α, β, and γ) and obtained the optimal departure time and the mean generalized cost of the driving-alone trip as illustrated in Fig. 3. We can observe from Figs. 3(a) and 3(b) that the optimal departure time remains unchanged as the values of μ and α increase. This result is consistent with Proposition 2 that the optimal departure time of a driving-alone trip is independent of both the values of μ and α. The results presented in Figs. 3(c) and 3(d) show that the optimal departure time of the driving-alone trip becomes earlier when the value of β is smaller or the value of γ is larger, which is consistent with Proposition 3. One can also observe from Fig. 3 that the mean generalized cost of the driving-alone trip uniformly increases as the values of μ, α, β, and γ increase. This result is consistent with Proposition 4.

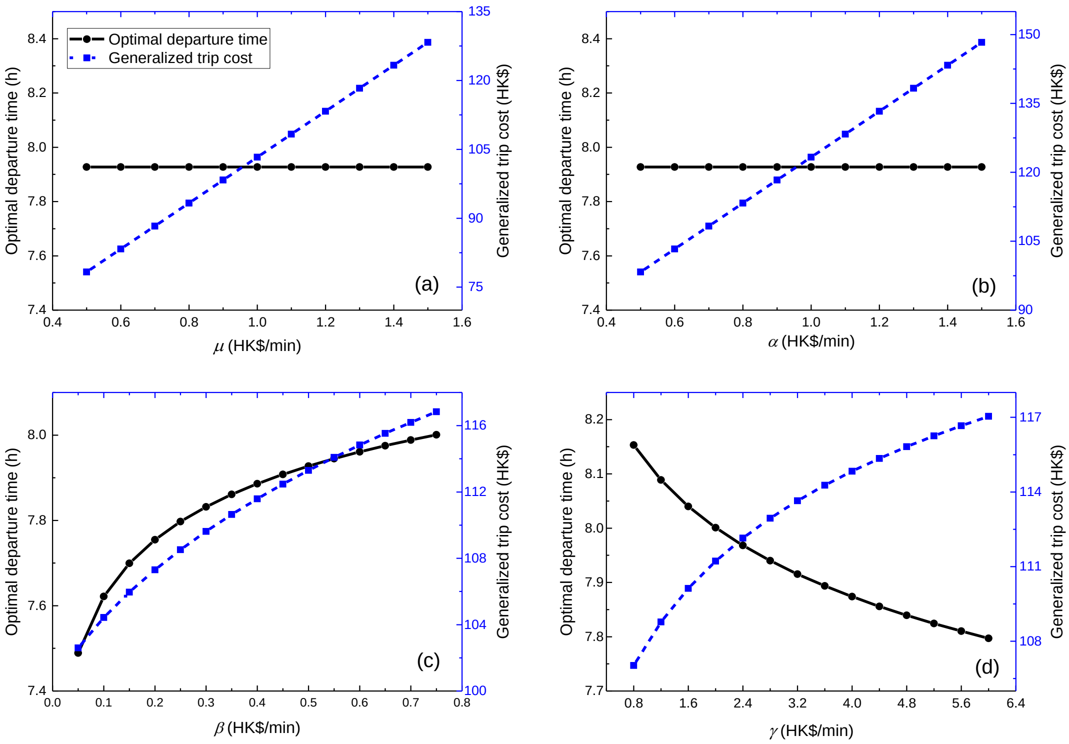

Fig. 3. The effect of the unit variable cost of driving and traveler's VOTs on the optimal departure time and the mean generalized cost of a driving-alone trip.

4.1.2. The effect of travel time uncertainty

With different values of λ, we obtained the corresponding cumulative distribution function of the driving-alone trip and its curves is shown in Fig. 4. One can observe that the distribution of the travel time of the driving-alone trip after the increase of the value of λ first-order stochastically dominates that before the increase. This result is consistent with Proposition 16. We also calculated the corresponding optimal departure time and mean generalized cost of the driving-alone trip and the results are shown in Fig. 5. It is observed that the optimal departure time of the driving-alone trip becomes earlier and the mean generalized cost of the driving-alone trip becomes larger when the value of λ is larger (i.e., the uncertainty of the driving-alone trip travel time is larger). This result is consistent with Proposition 5 that the optimal departure time of a driving-alone trip becomes earlier and its mean generalized cost is larger when its travel time uncertainty is larger. Besides, Fig. 5 indicates that the cost of driving (including both the fixed and variable driving cost) and the in-vehicle travel time cost have a larger increase than the costs of early and late arrivals. This is because an increase in λ leads to an increase in the mean trip travel time, and a traveler's driving cost and in-vehicle travel time cost are very sensitive to the mean trip travel time. However, the driver can avoid a large increase in the costs of early and late arrivals by adjusting his/her departure time choice.

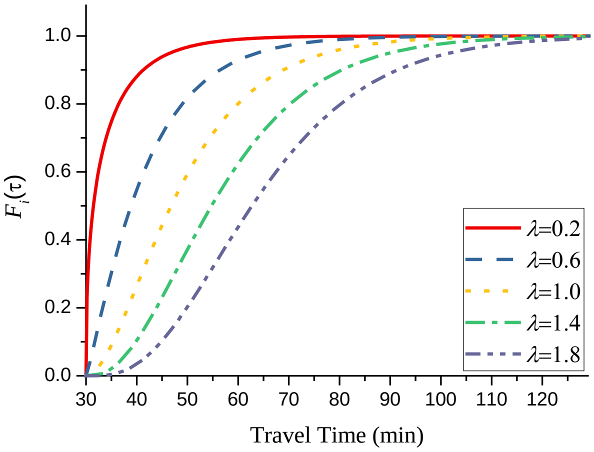

Fig. 4. The effect of the value of λ on the cumulative distribution function of a driving-alone trip.

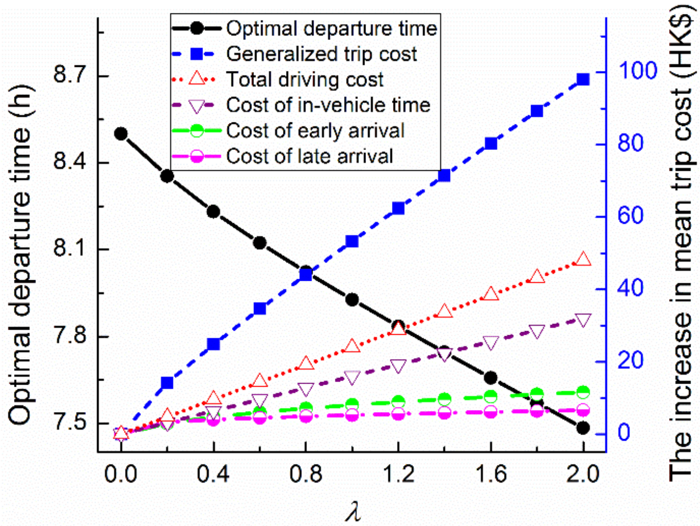

Fig. 5. The effect of the value of λ on the optimal departure time and the trip cost of a driving-alone trip.

We increased the value of θ from 0 min to 20 min with other parameters fixed, and obtained the cumulative distribution function of travel time, the optimal departure time, and the mean generalized cost of the driving-alone trip as revealed in Figs. 6 and 7. One can observe from Fig. 6 that the distribution of the travel time of the driving-alone trip after the increase of the value of θ does not always first-order stochastically dominates that before the increase. It is also observed from Fig. 7 that the optimal departure time of the driving-alone trip becomes earlier and the generalized cost of the driving-alone trip becomes larger when the value of θ is larger. This result implies that the optimal departure time of a driving-alone trip becomes earlier and its mean generalized cost is larger when its travel time uncertainty is larger. We can further observe that the total driving cost and the cost of in-vehicle time do not change when the value of θ grows up. This is because the mean trip travel time remains unchanged although the variance of the link travel time increases as the value of θ changes. However, an increase in travel time uncertainty can lead to an increase in the costs of early and late arrivals, leading to an increase in the mean generalized cost.

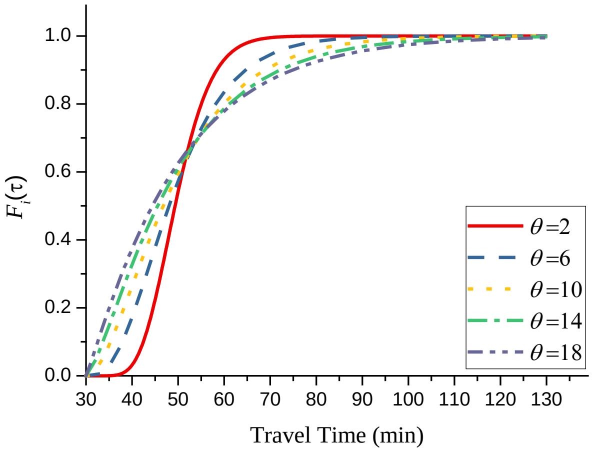

Fig. 6. The effect of the value of θ on the cumulative distribution function of a driving-alone trip.

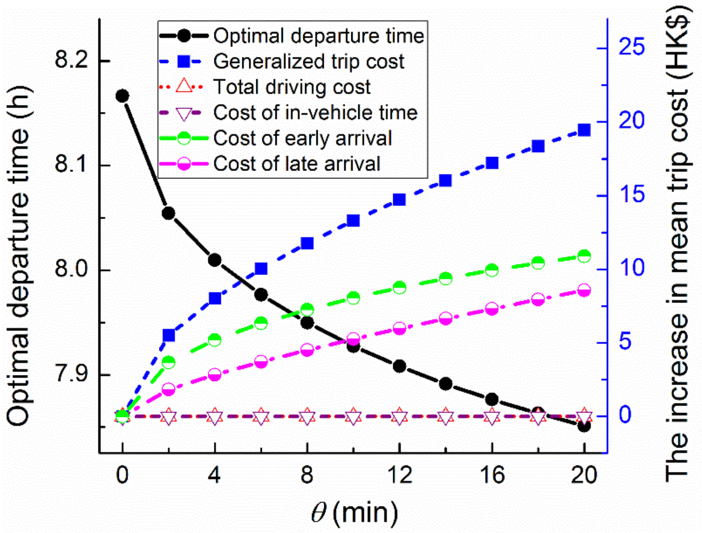

Fig. 7. The effect of the value of θ on the optimal departure time and the trip cost of a driving-alone trip.

4.2. A ride-sharing trip

In this example, we consider a ride-sharing trip (i, j) with two scenarios. The preferred arrival times for rider i and driver j are both 9:00 am in Scenario 1, and are 8:45 am and 9:00 am in Scenario 2, respectively. In both scenarios, rider i and driver j have the same destination. The trip travel times follow shifted Gamma distributions. Unless otherwise stated, we have $\theta = 10.0$ min and $\lambda = 1$. The trip travel times related to the ride-sharing trip for both scenarios are provided in Table 1.

Table 1. The trip travel times related to ride-sharing trip (i, j).

Scenario	Travel time	τ_i (min)	τ_j (min)	$\tau_{i\tilde{j}}$ (min)	τ_{ij} (min)
Scenario 1	Minimum travel time	30	20	40.5	40.5
	Reference mean travel time	50	33.3	67.5	67.5
Scenario 2	Minimum travel time	30	20	39.5	39.5
	Reference mean travel time	40	26.7	52.7	52.7

4.2.1. The effect of the unit variable cost of driving and traveler's VOTs

We set different values of the unit variable cost of driving and traveler's VOTs for the ride-sharing participants in Scenario 1, and got the optimal departure times and the mean generalized cost savings of the ride-driving trip as shown in Fig. 8. We can observe from Figs. 8(a) and 8(b) that the optimal departure time remains unchanged as the values of μ and α increase. However, a larger value of μ, i.e., a larger unit variable cost of driving, can lead to a larger mean generalized ride-sharing trip cost saving, and a larger value of α, i.e., a larger unit cost of in-vehicle travel time, can lead to a smaller mean generalized ride-sharing trip cost saving. This is because a ride-sharing trip increases total travel time while decreases total driving time compared to a driving-alone trip. The results presented in Figs. 8(c) and 8(d) show that the optimal departure time of the ride-sharing trip becomes earlier when the value of β is smaller or the value of γ is larger. This result is consistent with Proposition 7. We can also observe from Fig. 8 that the mean generalized ride-sharing trip cost saving uniformly increases when the value of the unit variable cost of driving grows up. This result is consistent with Proposition 8. We also set different values of the unit variable cost of driving and traveler's VOTs for the ride-sharing participants in Scenario 2, and the results are similar to those in Scenario 1.

4.2.2. The effect of travel time uncertainty

We conducted the sensitivity analysis of solutions to the travel time uncertainty of the ride-sharing trip (i.e., changing the mean or the variance of trip travel times for the ride-sharing trip), and obtained the optimal departure time of each ride-sharing trip and the optimal departure time of each driving-alone trip for both scenarios for different values of λ and θ as shown in Fig. 9. It is seen from Figs. 9(a) and 9(b) that the optimal departure time of the ride-sharing trip becomes earlier in both scenarios as travel time uncertainty is larger, i.e., the mean or the variance of the travel time of the ride-sharing trip is larger. This result is consistent with Proposition 10. Besides, the optimal departure time of the ride-sharing trip in each scenario is earlier than that of each driving-alone trip. This is because the ride-sharing trips in both scenarios adopt earlier departure times to coordinate the driver and the rider.

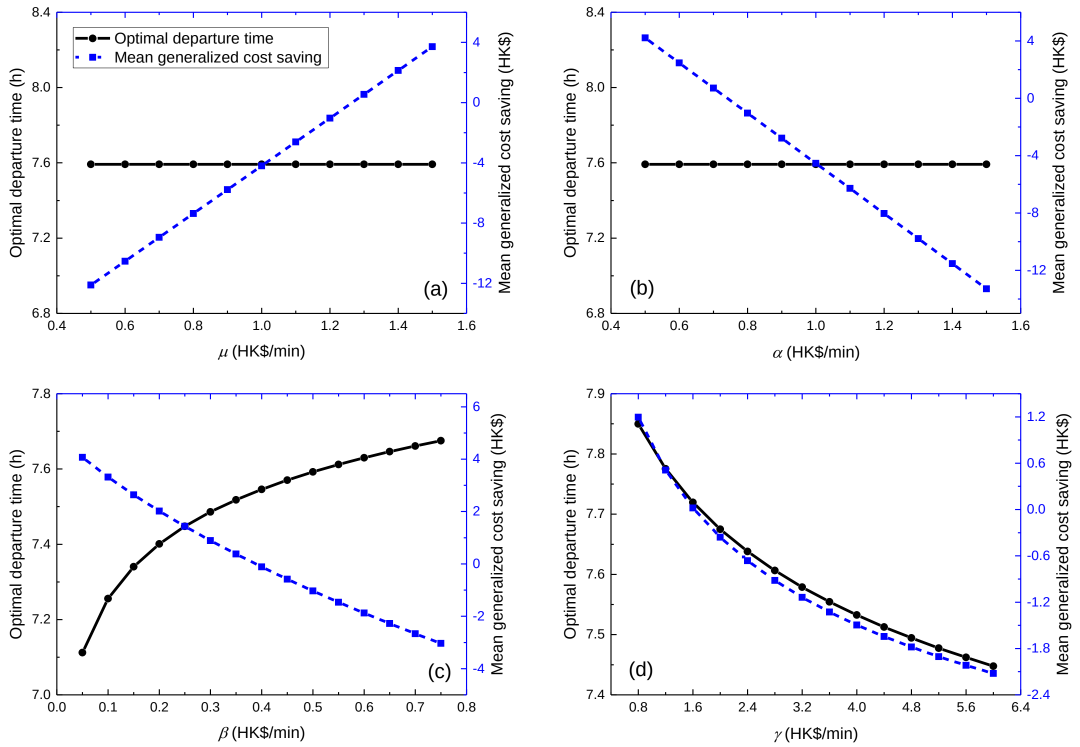

Fig. 8. The effect of the unit variable cost of driving and traveler's VOTs on the optimal departure time and the mean generalized cost saving of the ride-sharing trip in Scenario 1.

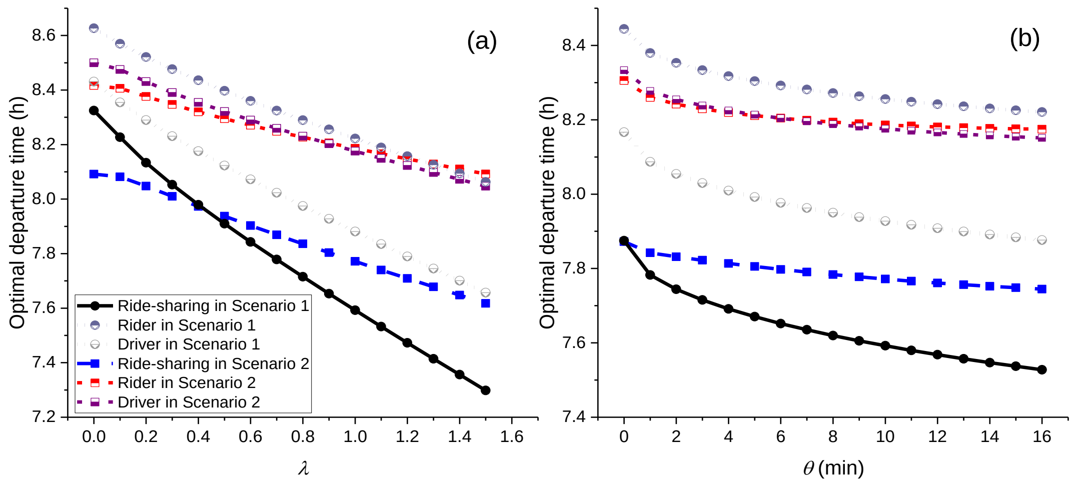

Fig. 9. The effect of travel time uncertainty on the optimal departure times.

The mean generalized cost savings of the driving-alone trip for both scenarios are presented in Fig. 10. One can observe from Figs. 10(a) and 10(b) that the mean generalized cost saving of the ride-sharing trip is initially positive in Scenario 1 and decreases and becomes negative as the values of λ and θ increase. The travel time of the ride-sharing trip becomes deterministic when $\lambda = 0$ or $\theta =$ 0 min, and travel time uncertainty increases as the values of λ and θ increase. Hence, the results presented in Fig. 10 associated with Scenario 1 indicate that a feasible ride-sharing match based on deterministic travel times can become infeasible when travel time uncertainty increases. Fig. 10(a) indicates that as the value of λ increases in Scenario 2, the mean generalized cost saving of the ride-sharing trip increases and becomes positive when $\lambda \geq 1$. Fig. 10(b) indicates that the mean generalized cost saving of the ride-sharing trip increases θ from 0 to 6.0 min and initially the mean generalized cost saving is negative and then becomes positive when $\theta \in [2,10]$ min. Hence, the results presented in Fig. 10 associated with Scenario 2 indicate that an infeasible ride-sharing match with deterministic travel times can become a feasible one when travel time uncertainty increases.

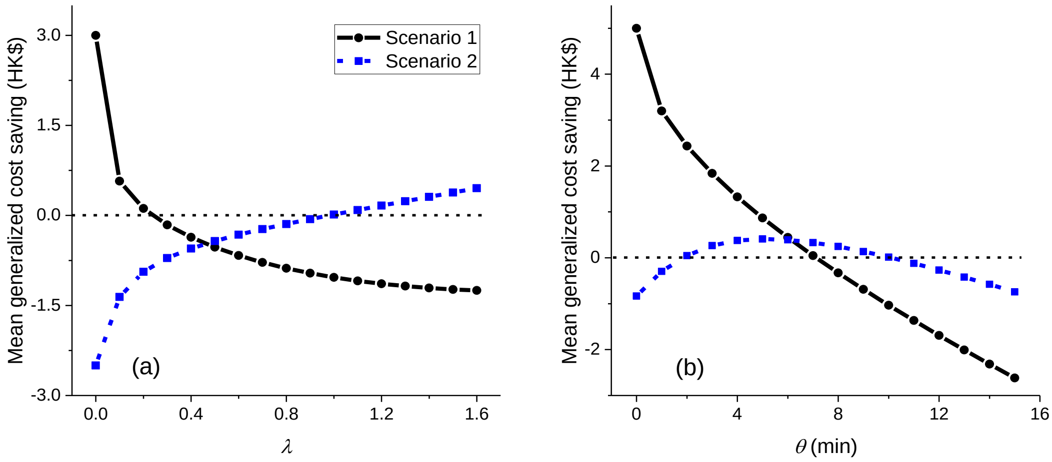

Fig. 10. The effect of travel time uncertainty on the mean generalized cost saving of the driving-alone trip.

4.3. Ride-sharing in the Chicago sketch network with time-independent travel time uncertainty

In this example, the Chicago sketch network is adopted to illustrate the properties of the proposed stochastic ride-sharing model with time-independent travel time uncertainty. The network has 387 zones, 933 nodes, and 2950 links. The relevant network data were taken from the transportation network datasets maintained by Bar-Gera (2018) (http://www.bgu.ac.il/~bargera/tntp/). The free flow travel time of each link is adopted as its minimum link travel time. The travel time of each link follows a shifted Gamma distribution. Unless otherwise stated, we set $\theta = 10.0$ min and $\lambda = 1$. Let ρ be the proportion of ride-sharing participants who select to be either a driver or a rider. The proportions of ride-sharing participants who select to be a ride-sharing driver and who select to be a ride-sharing rider are the same. Each participant randomly took a zone as his/her origin and another zone as his/her destination. It is assumed that all trips use the paths with the minimum mean travel times. The mean travel time of each link was randomly generated from 1.5 times to 2.5 times of its minimum link travel time. The preferred arrival time of each traveler follows a truncated normal distribution with a lower bound value of 7:30, an upper bound value of 9:30, a mean value of 8:30, and a variance value of 30 min. All ride-sharing participants have the same arrival time flexibility of 5 min. Unless otherwise stated, we use the following parameters: N = 7500, $\rho = 0.5$, and $\phi = 0.5$.

The proposed ride-sharing system aims to maximize the number of feasible matches (i.e., the ride-sharing match rate) and the total mean generalized trip cost saving from all matched ride-sharing trips. The ride-sharing match rate and the mean generalized cost saving rate are defined as follows, respectively:

$$\omega_M = \frac{\sum_{(i,j)\in\Omega} 2x_{ij}^*}{|\Phi|}\times 100\% \quad \text{and}$$

$$\omega_C = \frac{\sum_{(i,j)\in\Omega} \kappa_{ij} x_{ij}^*}{\sum_{i\in\Phi} \bar{C}_i^*}\times 100\%,$$

where $|\Phi|$ is the number of ride-sharing participants and the numerator in the first equation is the number of people involved in all feasible matches.

4.3.1. The effect of the unit variable cost of driving and travelers' VOTs

We set different values of the unit variable cost of driving and travelers' VOTs for the ride-sharing participants, and obtained the ride-sharing match rates and the mean generalized cost saving rates as shown in Fig. 11. It is observed that both the ride-sharing match rate and the mean generalized cost saving rate monotonically increase as the unit variable cost of driving increases or travelers' VOTs decrease. This is because an increase in the value of the unit variable cost of driving or a decrease in travelers' VOTs can lead to an increase in the mean generalized cost saving of each feasible ride-sharing trip. This eventually increases the participants of ride-sharing in the system.

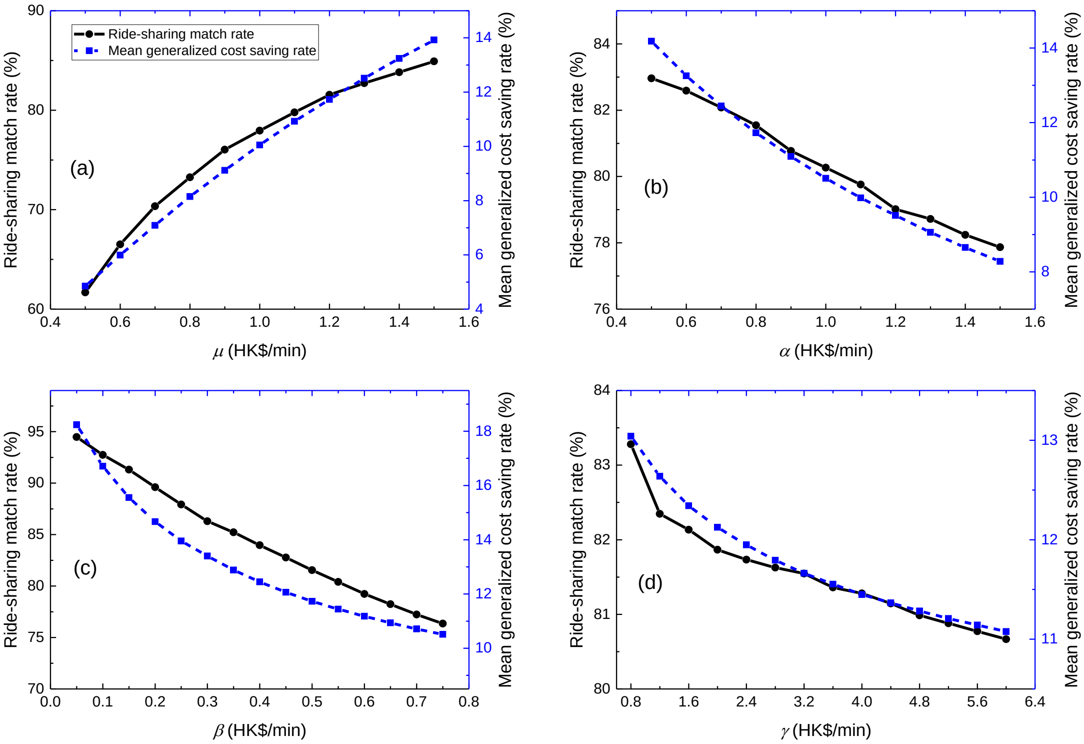

Fig. 11. The effect of the unit variable cost of driving and travelers' VOTs on the ride-sharing match rate and the mean generalized cost saving rate.

4.3.2. The effect of trip travel time uncertainty and the number of ride-sharing participants

We set different values of λ and different numbers of ride-sharing participants, and obtained the corresponding ride-sharing match rate and mean generalized cost saving rate as illustrated in Fig. 12. It is observed that both the ride-sharing match rate and the cost saving rate monotonically increase as the value of λ increases. If the value of λ grows up, the average link travel times increase. Hence, the results presented in Fig. 12 imply that ride-sharing participants can get more generalized trip cost saving in a more congested network, which encourages more ride-sharing participation. Besides, it is illustrated that both the ride-sharing match rate and the cost saving rate monotonically increase as the number of ride-sharing participants increases. This is because the ride-sharing system can provide more feasible and better matches when there are more ride-sharing participants.

We set different values of θ, and obtained the corresponding ride-sharing match rate and mean generalized cost saving rate as shown in Fig. 13. We can notice that the ride-sharing match rate slightly increases and the mean generalized cost saving rate monotonically decreases as the value of θ increases. This implies that travelers can get less generalized trip cost saving from ride-sharing participation if the network has a higher uncertainty of travel time. The results presented in Fig. 13 also confirm that both the ride-sharing match rate and the cost saving rate monotonically increase as the number of ride-sharing participants increases.

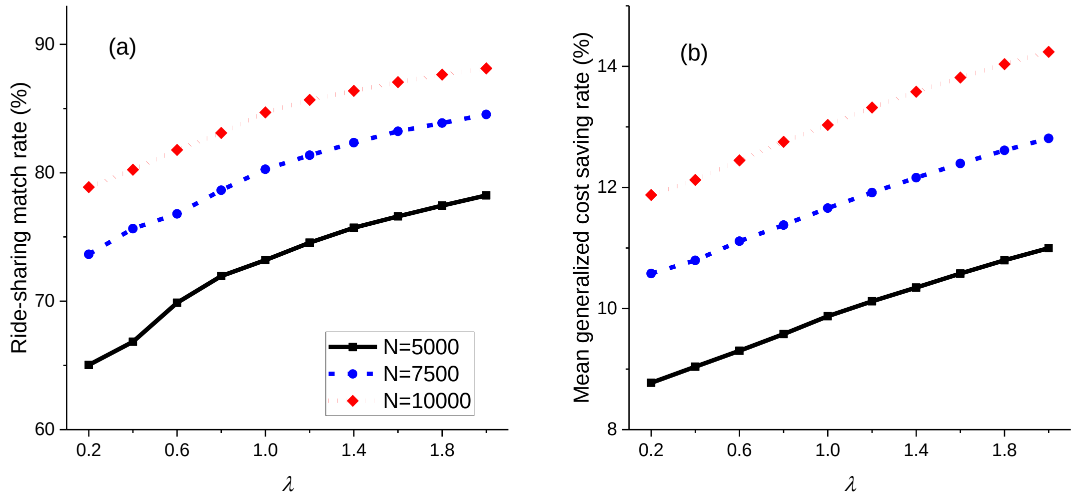

Fig. 12. The effect of the value of λ and the number of ride-sharing participants on the ride-sharing match rate and the mean generalized cost saving rate.

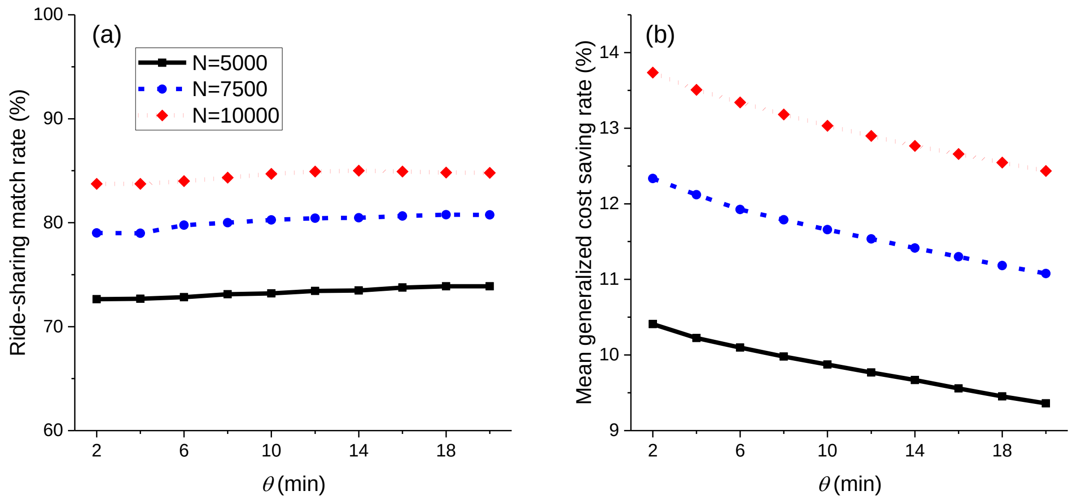

Fig. 13. The effect of the value of θ and the number of ride-sharing participants on the ride-sharing match rate and the mean generalized cost saving rate.

4.3.3. The effect of travelers' role tendency

We set different values of ρ and different numbers of ride-sharing participants, and got the ride-sharing match rates and the mean generalized cost saving rates as shown in Fig. 14. We can observe that both the ride-sharing match rate and the mean generalized cost saving rate monotonically increase as the value of ρ increases. This is because more ride-sharing participants selecting to be either a driver or a rider can bring more feasible ride-sharing matches, which can lead to a larger feasible solution set for the ride-sharing match model. The results presented in Fig. 14 also confirm that both the ride-sharing match rate and the cost saving rate monotonically increase as the number of ride-sharing participants increases.

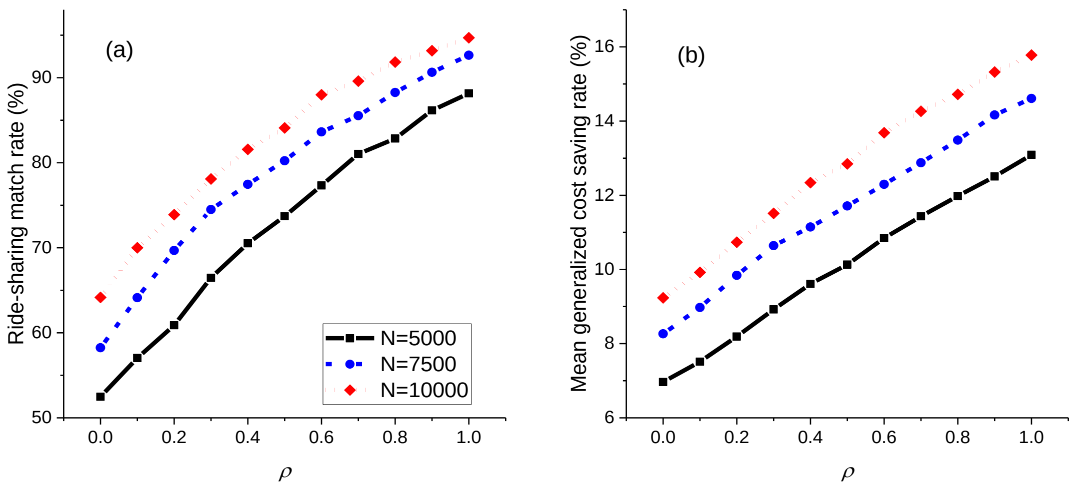

Fig. 14. The effect of the value of ρ and the number of ride-sharing participants on the ride-sharing match rate and the mean generalized cost saving rate.

4.3.4. A comparison with the ride-sharing system without capturing uncertainty

A given ride-sharing trip is defined as a missed feasible match if the ride-sharing trip is feasible in the stochastic ride-sharing system but infeasible in the deterministic ride-sharing system. A given ride-sharing trip is defined as an improper match if the trip is infeasible in the stochastic ride-sharing system but feasible in the deterministic ride-sharing system. The percentage of missed feasible matches ϖ_m and the percentage of improper matches ϖ_i are defined as follows:

$$\varpi_m = \frac{\left\| \Omega(\theta) \right| - \left| \Omega(\theta) \bigcap \Omega(0) \right\|}{\left| \Omega(\theta) \right|} \times 100\% \quad \text{and}$$

$$\varpi_i = \frac{\left\| \Omega(0) \right| - \left| \Omega(\theta) \bigcap \Omega(0) \right\|}{\left| \Omega(\theta) \right|} \times 100\%,$$

where $\Omega(\theta)$ is the set of feasible matches associated with the parameter θ and $|\Omega(\theta)|$ is the number of feasible matches associated with the parameter θ. In particular, $\theta = 0$ implies no travel time uncertainty and $\Omega(0)$ is the set of feasible matches in the deterministic ride-sharing system. $\Omega(\theta) \bigcap \Omega(0)$ is the set of feasible matches in both the deterministic and stochastic ride-sharing systems.

We varied the value of θ and the number of ride-sharing participants, and obtained the sets of feasible matches for the deterministic and stochastic ride-sharing systems. Fig. 15 shows the percentage of missed feasible matches and the percentage of improper matches. It is observed that both percentages grow up as travel time uncertainty increases. This implies that the deterministic ride-sharing match model considers more improper ride-sharing matches and misses more feasible ride-sharing matches when travel time uncertainty is larger. It can also observe from Fig. 15 that the number of ride-sharing participants has little influence on the percentage of missed feasible matches and the percentage of improper matches.

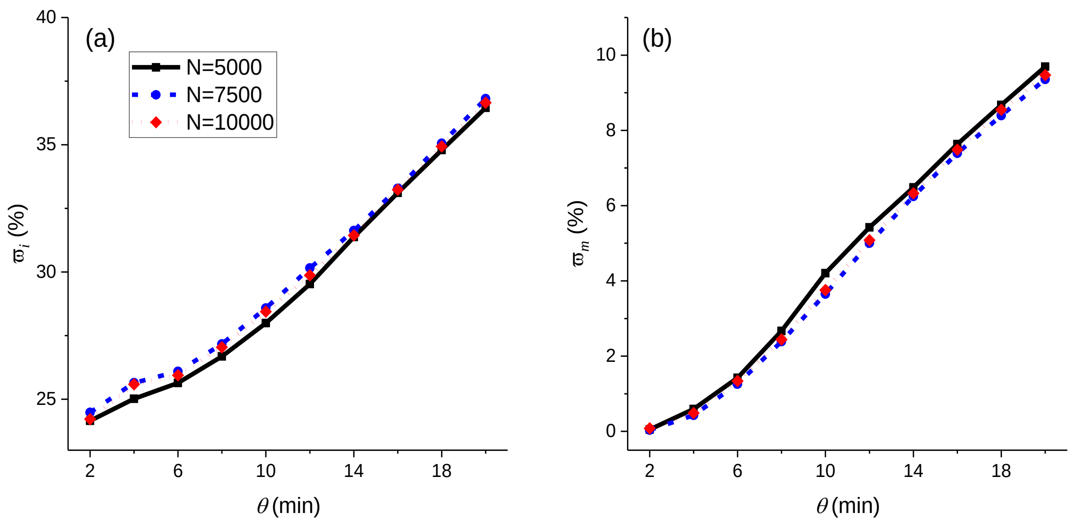

Fig. 15. The effect of travel time uncertainty on the percentage of missed feasible matches and the percentage of improper matches.

To illustrate the importance of capturing uncertainty, we compared the performance of the ride-sharing systems with and without capturing uncertainty. The mean total generalized cost of each ride-sharing trip in each obtained ride-sharing system was evaluated by Eq. (37). We adopted the feasible ride-sharing match rate and the mean generalized cost saving rate to illustrate the performance of the two ride-sharing systems.

We varied the value of ϕ in the objective function (79) and solved the static ride-sharing matching model for both the deterministic and stochastic ride-sharing systems. Fig. 16 shows the feasible ride-sharing match rate and the mean generalized cost saving rate for both the deterministic and stochastic ride-sharing systems. Compared with the ride-sharing system without capturing uncertainty, the ride-sharing system that captures uncertainty can significantly improve the ride-sharing match rate and increase the mean generalized cost saving rate. This result implies that travel time uncertainty should be captured in the ride-sharing system.

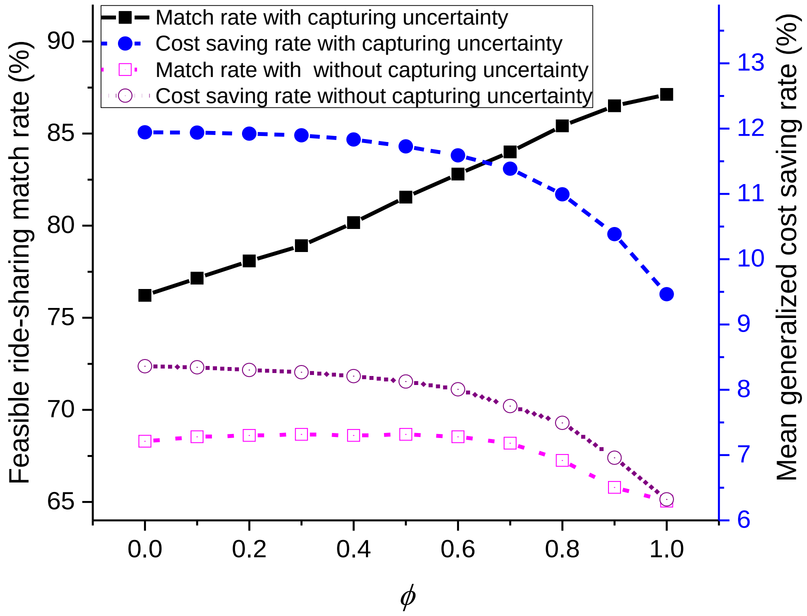

Fig. 16. The effect of the value of ϕ on the feasible ride-sharing match rate and the mean generalized cost saving rate of the ride-sharing system with and without capturing uncertainty.

4.4. Ride-sharing in the Chicago sketch network with time-dependent travel time uncertainty

In this example, the Chicago sketch network is also adopted to illustrate the properties of the proposed stochastic ride-sharing model with time-dependent travel time uncertainty. It is assumed that the travel time of each link during each time interval follows has a uniform distribution. The mean link travel time is set as follows:

$$\overline{\tau}_{a,k} = \tau_a^0 + \max\left\{0, \xi\tau_a^U - \frac{\xi\tau_a^U}{R^2}(\overline{t}_k - t_a^C)^2\right\}, \tag{96}$$

where τ_a^0 is the free flow travel time of link *a*, t_a^C and R are the middle time and the half-width of morning rush hours of link *a*, ξ is a non-negative parameter, and τ_a^U is the reference maximum mean travel time minus the free flow travel time of link *a* during the morning rush hours. We note that the value of ξ can reflect the level of network traffic congestion. When the value of ξ is larger, the road network is more congested. The minimum travel time of travelers traveling through link *a* during interval *k* is set as $\tau_{a,k}^{-} = \zeta\tau_a^0 + (1-\zeta)\overline{\tau}_{a,k}$, where ζ is a non-negative parameter and $\zeta \in [0,1]$. When the value of ζ is larger, the road network has larger travel time uncertainty. Especially, if $\zeta = 0$, the minimum travel time $\tau^{-}_{a,k}$ equals the mean travel time $\overline{\tau}_{a,k}$, and there is no uncertainty. If $\zeta = 1$, the minimum travel time $\tau^{-}_{a,k}$ equals the free-flow travel time τ^{0}_{a}. The parameter t^{C}_{a} was randomly generated from a truncated normal distribution with a lower bound value of 7:30, an upper bound value of 9:30, a mean value of 8:30, and a variance value of 15 min. The value of R is two hours. τ^{U}_{a} was randomly generated from 1.5 times to 2.5 times of the free flow travel time of link *a*. The length of each time interval is 10 min. Unless otherwise stated, we use the following parameters: $N = 7500$, $\xi = 1$, $\zeta = 0.5$, $\phi = 0.5$, and $M = 10000$.

4.4.1. The effect of sample sizes

We set different sample sizes (M) to obtain the optimal ride-sharing match schemes. The obtained optimal ride-sharing match schemes were reevaluated by the MCS method with a sample size of $M = 10000$. The resultant ride-sharing match rate and the mean generalized cost saving rate are shown in Fig. 17. We can observe that both the ride-sharing match rate and the mean generalized cost saving rate monotonically increase as the value of M increases, and the two curves in Fig. 17 are concave. This implies that a further increase in the value of M leads to almost no change in the two values if the value of M is large enough.

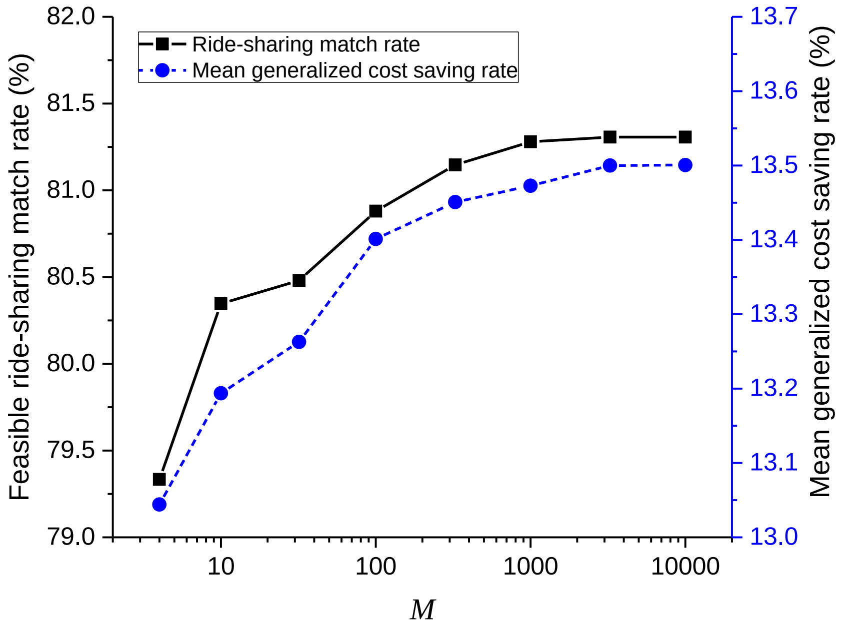

Fig. 17. The effect of the sample size on the ride-sharing match rate and the mean generalized cost saving rate.

4.4.2. The effect of travel time uncertainty

We set different values of ξ and different numbers of ride-sharing participants, and obtained the ride-sharing match rates and the mean generalized cost saving rates as illustrated in Fig. 18. It is observed that both the ride-sharing match rate and the cost saving rate monotonically increase as the value of ξ increases. If the value of ξ grows up, the average link travel times increase. Hence, the results presented in

Fig. 18 imply that travelers can get more generalized trip cost saving from ride-sharing participation in a more congested network, which encourages more ride-sharing participation. This result is consistent with that in Section 4.3.2. Moreover, the results presented in Fig. 18 confirm that both the ride-sharing match rate and the cost saving rate monotonically increase as the number of ride-sharing participants increases.

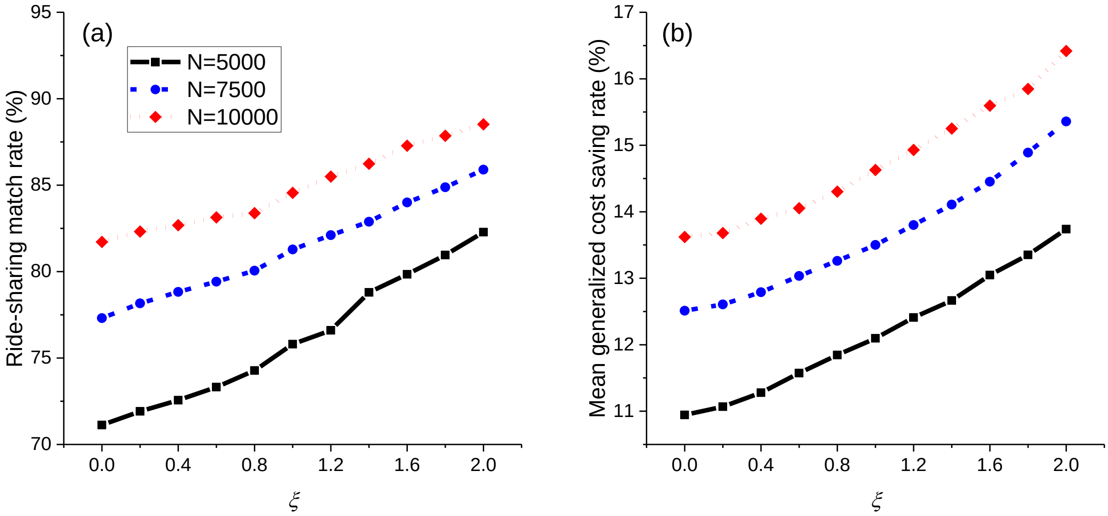

Fig. 18. The effect of the value of ξ on the feasible ride-sharing match rate and the mean generalized cost.

We set different values of ζ, and obtained the ride-sharing match rates and the mean generalized cost saving rates as shown in Fig. 19. We can see that both the ride-sharing match rate and the mean generalized cost saving rate monotonically decrease as the value of ζ increases. This implies that travelers can get less generalized trip cost saving from ride-sharing participation if the network has a higher uncertainty of travel time. We can also observe from Fig. 19 that the ride-sharing system that captures uncertainty can significantly improve the ride-sharing match rate and increase the mean generalized cost saving rate than the counterpart without capturing uncertainty, especially when the travel time uncertainty increases. This result confirms that travel time uncertainty should be captured in the ride-sharing system.

4.4.3. Model size and computational efficiency of the static ride-sharing matching models

We set different numbers of ride-sharing participants, and obtained the corresponding number of decision variables in the static ride-sharing matching model and CPU time for solving the model as illustrated in Table 2. It is observed that both the number of variables and the CPU time monotonically increase as the number of ride-sharing participants increases. This is because an increase in the number of ride-sharing participants can lead to an increase in the number of feasible ride-sharing matches. This eventually increases the CPU time for solving the matching model. The results presented in Table 2 also show that the number of feasible ride-sharing trips is less than ten times of the number of ride-sharing participants and the CPU time is less than 50 seconds when the number of ride-sharing participants is no more than 10000. Moreover, we can see from Table 2 that the ride-sharing matching model that captures uncertainty has less number of variables and can be solved with less CPU time than the counterpart without capturing uncertainty.

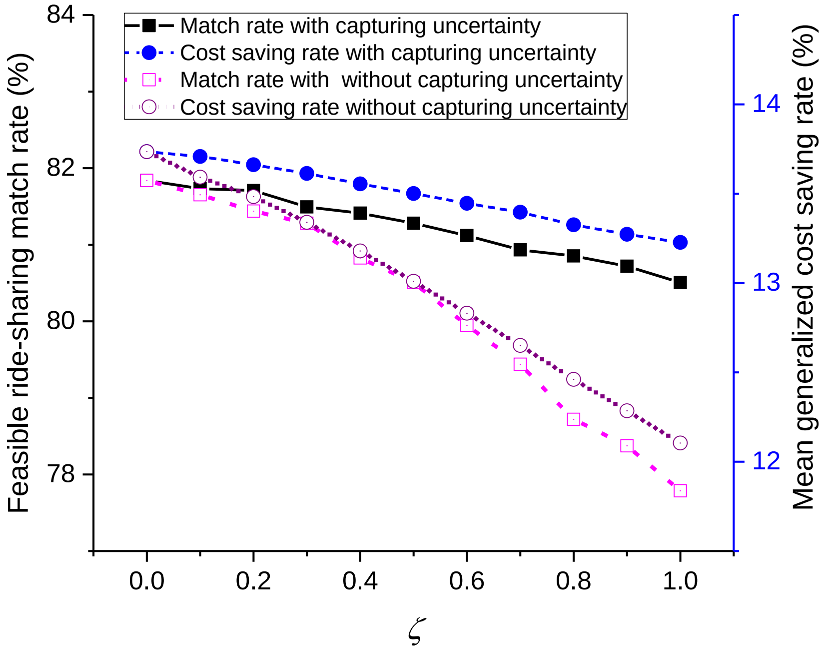

Fig. 19. The effect of the value of ζ on the feasible ride-sharing match rate and the mean generalized cost saving rate of the ride-sharing system with and without capturing uncertainty.

Table 2. The model size and computational efficiency of the static ride-sharing matching model

Number of ride-sharing participants	Without capturing uncertainty		With capturing uncertainty	
	Number of variables	CPU Time(s)	Number of variables	CPU Time(s)
1000	1111	0.15	941	0.05
2000	3173	0.15	2794	0.11
3000	8826	0.32	7666	0.37
4000	13074	0.44	11851	0.38
5000	23698	1.11	21179	1.63
6000	35623	3.07	30863	3.01
7000	45694	3.84	40692	2.30
8000	60800	5.70	53884	4.07
9000	72968	6.60	63849	7.14
10000	93995	40.53	84331	7.24

5. Conclusions

In this paper, we propose a ride-sharing system with the consideration of travel time uncertainty. In the proposed ride-sharing system, a traveler's generalized trip cost consists of the cost of driving a vehicle, the cost of travel time, and the costs of schedule delay early and late. The effects of the unit variable cost of driving, traveler's VOTs, and travel time uncertainty on the cost of a driving-alone trip and the cost saving of a ride-sharing trip are analyzed. Furthermore, a bi-objective ride-sharing matching model is proposed to maximize both the total generalized trip cost saving and the number of matches. The proposed ride-sharing system is further extended to consider time-dependent travel time uncertainty, and the MCS method is developed to evaluate the mean generalized trip cost. Finally, numerical examples are provided to illustrate the properties of the proposed models. The results show that the unit variable cost of driving, travelers' VOTs, travel time uncertainty, and the selection of the weights in the objective function have significant impacts on the performance of the ride-sharing systems. In particular, a feasible ride-sharing match based on deterministic travel time can become infeasible in a stochastic ride-sharing system. It is therefore important to consider travel time uncertainty when determining the matches.

This study assumes that travelers' VOTs are identical. Moreover, a ride-sharing driver only takes a single rider and travelers in the ride-sharing platform announce their trip schedule one day before. However, the modeling framework in the paper can be easily extended to consider factors such as one driver taking multiple riders (e.g., Baldacci et al., 2004; Calvo et al., 2004; Ghoseiri et al. 2011), dynamic ride-sharing (e.g., Agatz et al., 2011, 2012; Stiglic et al., 2015, 2016; Masoud and Jayakrishnan, 2017), and different VOTs (e.g., Xiao et al., 2014; Wu and Huang, 2015). In other words, this study lays a good foundation for future research about developing a more realistic ride-sharing matching model.

This study assumes that the distributions of link travel times are independent of the choice of the users of the ride-sharing platform (i.e., endogenous congestion-free). In reality, the interaction and competition among users in the whole network may not only endogenize the travel time, but also impact the match and price of ride-sharing, which is one of the main revenue sources of the private platform. In the future, the interaction and competition among users in a network will be considered into our model.

Acknowledgments

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