New Journal of Physics
The open access journal at the forefront of physics
Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

PAPER • **OPEN ACCESS**

Particle addition and subtraction channels and the behavior of composite particles

To cite this article: Pawe Kurzyski *et al* 2012 *New J. Phys.* **14** 093047

New Journal of Physics
The open-access journal for physics

Particle addition and subtraction channels and the behavior of composite particles

Paweł Kurzyński[1,2], **Ravishankar Ramanathan**[1], **Akihito Soeda**[1], **Tan Kok Chuan**[1] **and Dagomir Kaszlikowski**[1,3,4]

[1] Centre for Quantum Technologies, National University of Singapore, Singapore
[2] Faculty of Physics, Adam Mickiewicz University, Poznań, Poland
[3] Department of Physics, National University of Singapore, Singapore
E-mail: phykd@nus.edu.sg

New Journal of Physics **14** (2012) 093047 (12pp)
Received 22 May 2012
Published 26 September 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/9/093047

Abstract. Composite particles made of elementary fermions can exhibit a wide range of behavior ranging from fermionic to bosonic depending on the quantum state of the fermions and the experimental situation considered. This behavior is captured by the fundamental operations of single-particle addition and subtraction and two-particle interference. We analyze the quantum channels that implement the physical operations of addition and subtraction of indistinguishable particles. In particular, we construct optimal Kraus operators to implement these probabilistic operations for systems of a finite number of particles. We then use these to measure the quality of bosonic and fermionic behavior in terms of single-particle addition and subtraction and two-particle interference. For the specific case of composite particles made of two distinguishable fermions, we find a transition from fermionic to bosonic behavior as a function of the entanglement between the two constituents. We also apply these considerations to composite particles of two distinguishable bosons and identify the relation between entanglement and bosonic behavior for these systems.

[4] Author to whom any correspondence should be addressed.

Contents

1. Introduction

Most particles in Nature are not elementary and in fact are composed of elementary fermions and bosons. These composite particles can exhibit a variety of behavior ranging from fermionic to bosonic depending on the physical situation and the state of the system at hand. Fermionic and bosonic behavior is captured by the fundamental operations of addition and subtraction of single composite particles. For fermions, addition to an already occupied state is forbidden by the Pauli principle, whereas for bosons, it is easier to add a particle to an already occupied state than for distinguishable particles. The operations of single-particle addition and subtraction are known to be probabilistic, i.e. in general, one can never be certain that addition or subtraction of a single particle will be successful. These operations cannot be represented by unitary evolutions and are best described by the language of completely positive maps and Kraus operators, which, unlike unitary operations, allow one to represent irreversible physical evolutions such as relaxation [1]. The success probabilities of single-particle addition and subtraction are related to the quality of fermionic or bosonic behavior of the particles. Another physical situation that one may consider is two-particle interference, where bosonic behavior is captured by the tendency of particles to bunch (group), while fermionic behavior is related to their tendency to anti-bunch (stay apart). An interesting question is to quantify the quality of fermionic and bosonic behavior in composite particle systems in these scenarios.

Recently, it was shown [2–4] that the bosonic behavior of composite particles made of two distinguishable fermions (such as the hydrogen atom, exciton, positronium, etc) is related to the amount of entanglement between the two fermions. There, the quality of bosonic behavior was measured by the deviation from identity in number states of the commutator between the creation and annihilation operators. It was shown that as the entanglement increases, the commutation relation for the creation and annihilation operators of these composite particles approaches that for ideal bosons. However, in general, the behavior of these systems is more complicated and not entirely captured by the average value of the commutator in the number state. These particles can, in fact, exhibit a variety of behavior in two-particle interference and particle addition–subtraction experiments ranging from fermionic to bosonic, which is not detected by the commutator approach.

The main aim of this work is to construct a measure of bosonic and fermionic quality on the basis of single-particle addition followed by subtraction. We analyze the operations of single-particle addition and subtraction in terms of completely positive quantum channels. We construct optimal bosonic quantum channels to implement these operations and apply them to formulate a measure of bosonic and fermionic quality. This measure reflects the difficulty of adding a single particle to a mode that is already occupied by one particle. For composite particles made of two distinguishable fermions or two distinguishable bosons, the value of the measure depends on the entanglement between the constituents. We also apply addition and subtraction channels to construct a beam splitter for these composite particles and show that the ratio of anti-bunching to bunching probabilities in a two-particle interference experiment also depends on entanglement and that a transition point between fermionic and bosonic behavior exists.

Unlike the previous approach to composite particles made of two distinguishable fermions [2–4], our measure allows one to identify a critical amount of entanglement for which the transition from fermionic to bosonic behavior occurs. Moreover, our treatment of single-particle addition and subtraction takes into account massive particle systems that, in general, are restricted by the superselection rules, whereas previous studies mainly concentrated on photonic systems [5–10] that, in principle, can be prepared in an arbitrary superposition of photon number states.

2. Addition and subtraction channels

In general, the processes of particle addition and subtraction are not deterministic [5–10]. Moreover, they cannot be formulated simply as Kraus operators K_j which describe non-deterministic evolutions in terms of completely positive quantum channels $\rho' = \sum_j K_j \rho K_j^\dagger$ [1]. The reason is that Kraus operators $\{K_0, K_1, \ldots\}$ which describe a quantum channel cannot increase the norm of the state, i.e. $\sum_j K_j^\dagger K_j \leqslant I$. Setting $K_0 = a^\dagger$ yields $K_0^\dagger K_0 = aa^\dagger = N+1$. The eigenvalues of the bosonic particle number operator N lie in the set of all non-negative integers, which together with the requirement that the norm cannot increase immediately implies the negativity of the remaining operators $K_j^\dagger K_j$ for $j \neq 0$. The case of the annihilation process is analogous. It is interesting that the probabilistic nature of the operators $a^\dagger$ and a can also be deduced from the fact that deterministic addition and subtraction could lead to an increase of entanglement via local operations. This can be seen, for example, by considering the state of a single particle in two modes A and B, $|\psi\rangle = \alpha|0_A 1_B\rangle + \beta|1_A 0_B\rangle$ with real parameters $\alpha^2 + \beta^2 = 1$ and $\alpha^2 > 4\beta^2$, that has entanglement measured by concurrence given as $2|\alpha\beta|$. It is clear that a local operation of addition followed by subtraction at mode A leads to the state $|\psi_{\mathrm{AS}}\rangle = \frac{1}{\sqrt{\alpha^2+4\beta^2}}(\alpha|0_A 1_B\rangle + 2\beta|1_A 0_B\rangle)$ with entanglement measured by concurrence given as $\frac{4\alpha\beta}{\alpha^2+4\beta^2}$, which is larger than the initial entanglement. The probabilistic nature of the operators $a^\dagger$ and a is needed to ensure that entanglement does not increase via local operations.

Any operator that effectively adds one particle to the system is of the form

$$a_{\mathrm{eff}}^\dagger = \sum_{n=0}^{\infty} f_n |n+1\rangle\langle n|. \tag{1}$$

An effective annihilation operator is the Hermitian conjugate of the above. This operator is a valid Kraus operator (satisfying $\sum_j K_j^\dagger K_j \leqslant I$) if $|f_n|^2 \leqslant 1$ for all n. Note that operationally, $\frac{|f_n|^2}{|f_{n-1}|^2}$ corresponds to the ratio of probabilities $\frac{p_{n\to n+1}}{p_{n-1\to n}}$, where $p_{n\to n+1}$ denotes the probability of adding a single particle to a mode in which there are already n particles. It is convenient to rewrite (1) in the following form:

$$a_{\text{eff}}^\dagger = g(N)a^\dagger = \sum_{n=0}^{\infty} g(n+1)\sqrt{n+1}|n+1\rangle\langle n|, \tag{2}$$

where $g(N)$ is a function of the particle number operator. The extreme case of an operator in which all multiplicative factors are equal to one corresponds to the creation operator of distinguishable particles $a_{\text{d}}^\dagger$, where $g(N) = 1/\sqrt{N}$.

To implement a perfect bosonic channel, we would like to have the ratios to be $\frac{f_n}{f_{n-1}} = \frac{\sqrt{n+1}}{\sqrt{n}}$. In this case, $g(N)$ is a constant; however, the normalization constraint and the fact that the sum over n goes to infinity imply that the only possible solution is $g(N) = 0$. This problem can be circumvented if the maximal number of particles is bounded. In this case, the optimal operator $a_{\text{eff}}^\dagger$ is state dependent, i.e. for the state supported on the subspace spanned by $\{|0\rangle, |1\rangle, \ldots, |n_{\max}\rangle\}$ the corresponding function is $g(N) = \frac{1}{\sqrt{n_{\max}+1}}$, a constant for $n \leqslant n_{\max}$, and $g(N) = \frac{1}{\sqrt{N}}$ for $n > n_{\max}$. The effective operator $a_{\text{eff}}^\dagger$ with this function is then the optimal operator to implement particle addition and its conjugate is the optimal particle subtraction operator. Finally, note that in the case of fermions the maximal number of particles in a single mode is naturally bounded by 1; therefore fermionic creation and annihilation operators are already optimal Kraus channels. We can now proceed to formulate a measure of bosonic and fermionic quality of particles based on these optimal addition and subtraction quantum channels.

3. Measure of bosonic and fermionic quality

In this section, we propose a method to quantify the behavior of particles under the operations of one-particle addition and subtraction to a single mode. We compare the resulting state after particle addition followed by subtraction (AS) with the initial state of the particles which we assume to be in a mixed state $\rho = \sum_n p_n |n\rangle\langle n|$. The reason for this assumption is that, in general, the particles under consideration can be massive and superselection rules prevent us from preparing superpositions of different particle-number states. To detect the change caused by AS, it is sufficient to measure the probability distribution of the number of particles $\{p_0, p_1, \ldots\}$, where p_n denotes the probability of detecting n particles. In order to develop a measure that is independent of whether the particles are bosons or fermions, we restrict our considerations to p_0 and p_1 alone. Interestingly, although the number of particles involved in the measure is small, for composite particles made of two fermions such as hydrogen atoms, positronium atoms and excitons, we find that the measure provides us with information about the behavior of many-particle Fock states of these composite particles as well.

We begin by defining the following quantity:

$$M = p_0 - p_0^{\text{AS}}, \tag{3}$$

where p_0^{AS} denotes the vacuum occupancy after AS. The value of M is zero for distinguishable particles with the associated creation operators defined by $c_{\text{d}}^\dagger = \sum_n |n+1\rangle\langle n|$. This is because these operators do not alter the probability distribution in the state upon addition and subtraction.

We now argue that M is a measure of bosonic and fermionic quality in this scenario. For bosons the action of AS affects the probability distribution in the following manner: $p_n \rightarrow (n+1)^2 p_n$, which together with normalization implies a decrease in p_0. Due to the normalization the change in p_0 depends on the total probability distribution $\{p_n\}$. Note that

$$M = p_0 - \frac{p_0}{\sum_{k=0}^{n_{\max}} (k+1)^2 p_k} = p_0 - \frac{p_0}{\langle (N+1)^2 \rangle}, \tag{4}$$

where N denotes the particle number operator and $n_{\max}$ denotes the maximum number of particles in the system. M is maximized for $p_0 = \frac{n_{\max}+1}{n_{\max}+2}$ and $p_{n_{\max}} = 1 - p_0$. The greater the $n_{\max}$, the greater the change in p_0 after AS. Since we restrict ourselves to p_0 and p_1 only, the optimal probability distribution is $\{p_0 = \frac{2}{3}, p_1 = \frac{1}{3}\}$, for which $M = \frac{1}{3}$ in the case of perfect bosons. We therefore fix $p_0 = \frac{2}{3}$, and calculate *the measure with respect to the state*

$$\rho_{\mathcal{M}} = \tfrac{2}{3}|0\rangle\langle 0| + \tfrac{1}{3}|1\rangle\langle 1|. \tag{5}$$

For convenience, we now redefine the measure as $\mathcal{M} = 3M$

$$\mathcal{M} = 2 - 3p_0^{\mathrm{AS}}, \tag{6}$$

so that for ideal bosons $\mathcal{M} = 1$. For ideal fermions, successful addition to a state of the form (5) implies that there is no vacuum in the resulting state. Since at most one fermion can occupy a particular state, the only possible state is $|1\rangle$. It follows that subsequent particle subtraction leads to $p_0^{\mathrm{AS}} = 1$ and to $\mathcal{M} = -1$.

So far, we have shown that the three values of $\mathcal{M}$, namely 1, 0 and -1, correspond to bosons, distinguishable particles and fermions, respectively. However, the measure is not bounded to these values and, in general, depends on the probability of AS. Our considerations are restricted to the two probabilities of addition $p_{0\rightarrow 1}$ and $p_{1\rightarrow 2}$, and the two probabilities of subtraction $p_{2\rightarrow 1}$ and $p_{1\rightarrow 0}$. Since $p_{i\rightarrow j} = p_{j\rightarrow i}$ we are left with $p_{0\rightarrow 1}$ and $p_{1\rightarrow 2}$. For the state (5)

$$p_0^{\mathrm{AS}} = \frac{\frac{2}{3}p_{0\rightarrow 1}^2}{\frac{2}{3}p_{0\rightarrow 1}^2 + \frac{1}{3}p_{1\rightarrow 2}^2} = \frac{2}{2+\mathcal{R}^2}, \tag{7}$$

where

$$\mathcal{R} = \frac{p_{1\rightarrow 2}}{p_{0\rightarrow 1}} = \frac{|\langle 2|a^\dagger|1\rangle|^2}{|\langle 1|a^\dagger|0\rangle|^2}, \qquad 0 \leqslant \mathcal{R}. \tag{8}$$

Therefore, the measure $\mathcal{M}$ reads

$$\mathcal{M} = \frac{2(\mathcal{R}^2 - 1)}{2+\mathcal{R}^2}, \qquad -1 \leqslant \mathcal{M} < 2. \tag{9}$$

We now discuss the various domains of validity of $\mathcal{M}$. Note that $\mathcal{M} < 0$ if $\mathcal{R} < 1$, which happens when $p_{1\rightarrow 2} < p_{0\rightarrow 1}$. Intuitively, in this regime it is harder to add a single particle to the mode when there is already one particle in it, which is an indication of fermionic behavior. The critical case when it is impossible to add a particle when there is already one other particle in the mode ($p_{1\rightarrow 2} = 0$) corresponds to true fermions. On the other hand, $\mathcal{M} > 0$ if $\mathcal{R} > 1$, which happens if $p_{1\rightarrow 2} > p_{0\rightarrow 1}$. This corresponds to the situation when it is easier to add a single particle to the mode when there is already one particle in it, an indication of bosonic behavior. Therefore, we can define domains $\mathcal{M} \in (-1, 0)$ and $\mathcal{M} \in (0, 1)$ as regions of *sub-fermionic* behavior and *sub-bosonic* behavior, respectively. Interestingly, if $\mathcal{R} > 2$, then $\mathcal{M} \in (1, 2)$. In this regime, it becomes easier to add a particle than in the case of true bosons, i.e. the probability of addition when there is already one particle in the system is larger than for ideal bosons. We call this the *super-bosonic* regime. In the following sections, we examine systems that can exhibit *sub-fermionic*, *sub-bosonic* and *super-bosonic* behavior.

3.1. Composite particles of two distinguishable fermions

Let us now examine situations for which $\mathcal{M} \in (-1, 1)$. Consider a composite particle made of two distinguishable elementary fermions (see [2–4]), such as excitons, hydrogen atoms, etc.

The general pure state of two distinguishable fermions can be written as

$$|\psi\rangle_{AB} = \sum_k \sqrt{\lambda_k} a_k^\dagger b_k^\dagger |0\rangle, \tag{10}$$

where $a_k^\dagger$ ($b_k^\dagger$) creates a fermion A (B) in mode k, and λ_k are probabilities that determine the structure of state (10). The above state is written in the Schmidt form, i.e. as a sum of tensor products of distinct orthogonal states. The modes k can refer, for instance, to energy levels of a confining potential, or to the position of the center of mass of A and B. The state can be considered as a single boson state if the operator $c^\dagger = \sum_k \sqrt{\lambda_k} a_k^\dagger b_k^\dagger$ behaves as a proper bosonic creation operator [2, 3]. The corresponding commutation relation reads $[c, c^\dagger] = I - \Delta$, where $\Delta = \sum_k \lambda_k (a_k^\dagger (a)_k + b_k^\dagger b_k)$. The state of n composite bosons is

$$|n\rangle = \chi_n^{-1/2} \frac{c^{\dagger n}}{\sqrt{n!}} |0\rangle, \tag{11}$$

on which the action of the annihilation operator gives $c|n\rangle = \alpha_n \sqrt{n} |n-1\rangle + |\varepsilon_n\rangle$. The parameters χ_n and $\alpha_n = \sqrt{\chi_n / \chi_{n-1}}$ are normalization constants and $|\varepsilon_n\rangle$ is a vector orthogonal to $|n-1\rangle$ that does not correspond to any state of the tested composite particles (for details see [2]).

Now, let us examine the operation of particle addition for composite particles. The corresponding Kraus channel is written as

$$K_0 = c_{\text{eff}}^\dagger = \sum_{n=0}^{n_{\max}} g(n+1) \alpha_{n+1} \sqrt{n+1} |n+1\rangle\langle n|, \tag{12}$$

where the terms related to $|\varepsilon_n\rangle$ are incorporated into other Kraus operators. The particle subtraction operation is given by taking Hermitian conjugate of the above. The optimal function corresponding to realistic implementation of the addition operator is a constant $g(n+1) = g$.

We are interested in the states $|0\rangle$, $|1\rangle$ and $|2\rangle$ and in parameter χ_2. Note that $\chi_1 = 1$, $c^\dagger|0\rangle = |1\rangle$ and $c|1\rangle = |0\rangle$ follow from definitions (10) and (11). Moreover, we do not consider vectors $|\varepsilon_n\rangle$, which are interpreted as states resulting from an unsuccessful subtraction. Effectively, we describe successful addition to the one-particle state and successful subtraction from the two-particle state as

$$c^\dagger|1\rangle = \sqrt{2\chi_2}|2\rangle, \qquad c|2\rangle = \sqrt{2\chi_2}|1\rangle. \tag{13}$$

The optimal Kraus channel for the addition of these composite particles is then given by

$$K_0 = c_{\text{eff}}^\dagger = g|1\rangle\langle 0| + g\sqrt{2\chi_2}|2\rangle\langle 1|. \tag{14}$$

The parameter χ_2 is related to the entanglement between the two constituent fermions as

$$\chi_2 = 2\sum_{k>l} \lambda_k \lambda_l = \sum_{k,l} \lambda_k \lambda_l - \sum_k \lambda_k^2 = 1 - P, \tag{15}$$

where $0 < P \leqslant 1$ denotes purity. The purity is an entanglement measure for pure bipartite systems and is defined as $P = \mathrm{Tr}\,(\mathrm{Tr}_B(\rho_{AB}))^2$ [1]. For $P = 1$ there is no entanglement between A and B, whereas for $P \to 0$ the entanglement between A and B goes to infinity (for the singlet state of two qubits $P = \frac{1}{2}$). Hence P (and as a consequence χ_2) measures the amount of entanglement between the constituent fermions.

Although we consider only the case of a vacuum, a single particle and two particles, the value of $\mathcal{M}$ reveals also the properties of many-particle Fock states. For composite particles made of two distinguishable fermions, it was shown in [3] that purity can be used to bound the ratios of χ-parameters, i.e.

$$1 - NP \leqslant \frac{\chi_{N+1}}{\chi_N} \leqslant 1 - P. \tag{16}$$

As a consequence, using χ_2 one can estimate other χ-parameters and put constraints on structural parameters λ_k. Note that the value of $\mathcal{M}$ is related to the condensate fraction $F = \langle N|c^\dagger c|N\rangle$ via the relation $F = N\frac{\chi_N}{\chi_{N-1}}$.

We now show that the measure $\mathcal{M}$ is related to the entanglement between the two constituent fermions of the composite particle. Firstly, we note that in order to evaluate $\mathcal{M}$ for composite particles, one does not have to specify g. Since $\mathcal{R} = 2\chi_2 = 2 - 2P$ the measure $\mathcal{M}$ is related to the purity as

$$\mathcal{M}(P) = 2 - \frac{3}{3 + 2P(P - 2)}. \tag{17}$$

It is a continuous monotonically decreasing function of P. In the limit of infinite entanglement the two fermions behave like a boson $\mathcal{M}(0) = 1$; this behavior was also observed in [2–4]. On the other hand, when there is no entanglement the two free fermions evidently exhibit fermionic behavior $\mathcal{M}(1) = -1$. For $0 < P < 1$ the two fermions exhibit either *sub-fermionic* or *sub-bosonic* behavior, depending on the value of the purity. The transition between the two types of behavior for the case of such composite particles in a single mode occurs for $P = \frac{1}{2}$, i.e. for exactly 1 ebit of entanglement, see figure 1. The existence of a critical value of entanglement for the transition between fermionic and bosonic behavior is an important and intuitive result in contrast to the results derived so far in [2–4].

3.2. Composite particles of two distinguishable bosons

Let us now discuss the regime $\mathcal{M} > 1$. Consider a system composed of two distinguishable bosons, such as two photons created in a parametric down-conversion process. It is described by equations (10)–(13), with $a_k^\dagger$ and $b_k^\dagger$ in (10) now being bosonic creation operators, the commutation relation being $[c, c^\dagger] = I + \Delta$. The optimal channel for the addition of these composite particles is also given by (14) with the parameter χ_2 defined as

$$\chi_2 = 2\sum_{k\geqslant l} \lambda_k \lambda_l = \sum_{k,l} \lambda_k \lambda_l + \sum_k \lambda_k^2 = 1 + P. \tag{18}$$

In this case, the measure $\mathcal{M}$ is related to the entanglement between the two constituent bosons and is given by

$$\mathcal{M}(P) = 2 - \frac{3}{3 + 2P(P + 2)}. \tag{19}$$

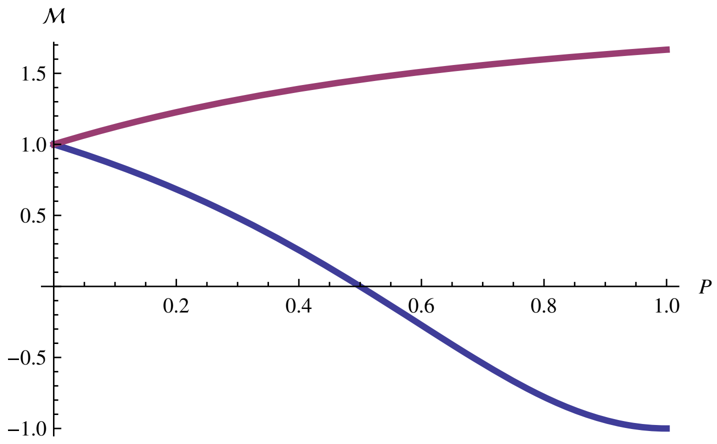

Figure 1. The plot of $\mathcal{M}$ as a function of P. Top curve—the measure $\mathcal{M}$ for composite particles made of two distinguishable bosons; bottom curve—composite particles made of two distinguishable fermions.

As in the case of composite particles of two fermions, the value of $\mathcal{M}$ can be used to detect entanglement in the system and to learn structural properties via χ_2. The plot of $\mathcal{M}$ is presented in figure 1. In the limit of infinite entanglement ($P = 0$) between the two bosons, the system behaves like a true boson $\mathcal{M}(0) = 1$. However, for intermediate values of entanglement ($0 < P \leqslant 1$), the system exhibits enhanced bosonic behavior, which we term *super-bosonic*. In this regime, the probability of addition of a single composite particle to an already occupied mode is larger than for ideal bosons. The maximal value of $\mathcal{M}(1) = \frac{5}{3}$ occurs for free unentangled bosons.

3.3. The meaning of entanglement

We now propose an intuitive explanation for the fact that in the limit of large entanglement, the value of $\mathcal{M}$ converges to the same point for both the system composed of two bosons and the system composed of two fermions. We start by analyzing the reduced state of the subsystem A in equation (10). At the moment we do not specify whether we deal with bosons or fermions. The reduced state is given by

$$\rho_A = \sum_k \lambda_k a_k^\dagger |0\rangle\langle 0| a_k. \tag{20}$$

In the limit of small entanglement, the distribution $\{\lambda_k\}$ is localized around some k and the subsystem is in a nearly pure state. Its properties are therefore well defined and it can exhibit either fermionic or bosonic behavior, depending on the type of particle. On the other hand, in the limit of large entanglement the distribution $\{\lambda_k\}$ is almost uniform, the state of the subsystem is almost completely mixed and its properties are undefined. Since it can be anywhere in the state space spanned by all $a_k^\dagger|0\rangle$ it is of little consequence to the system behavior whether the particle is a boson or a fermion. This phenomenon is, for example, observed in the Hong–Ou–Mandel experiment [11], where one does not observe bunching when the initial state of the two input photons to the beam splitter is fully random. This explains why in the limit of infinite entanglement, systems composed of two bosons and two fermions behave in a similar

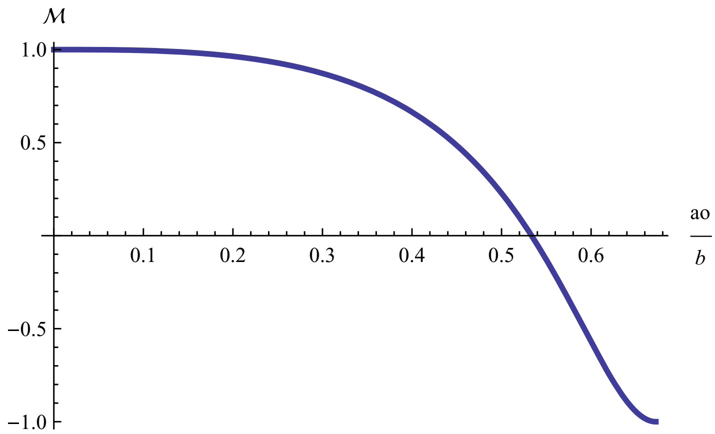

Figure 2. The plot of $\mathcal{M}$ as a function of $\frac{a_0}{b}$ for a single hydrogen atom in a harmonic trap.

way (both yielding $\mathcal{M} = 1$). When the entanglement between the constituents is not infinite, the anti-bunching or bunching of subsystems starts to play a role in the behavior of the total system, resulting in the two regimes $-1 \leqslant M < 1$ and $M > 1$.

For composite particles made of two distinguishable bosons, as the entanglement between the bosons decreases, the value of $\mathcal{M}$ increases up to a maximal value of $\frac{5}{3}$. For systems composed of infinitely many bosons $\mathcal{M}$ could reach its maximal value of 2. It would be interesting to investigate if there are any effects in physical phenomena of bosons linked to this regime.

3.4. The hydrogen atom

Finally, let us apply the measure $\mathcal{M}$ to a simple composite particle made of two fermions, namely a single hydrogen atom in a harmonic trap. It is assumed that the atom is both in the ground electronic state and in the ground state of the trap. We use the formula for the purity of a state of the electron (or equivalently proton) that was derived in [3]:

$$P = \frac{33}{4\sqrt{2\pi}} \left(\frac{a_0}{b}\right)^3, \tag{21}$$

where a_0 is the Bohr radius and b is the size of the trap. The purity depends on the ratio between the volume of the atom and the volume of the trap. The range of the value of P is between 1 and 0 for the ratio $\left(\frac{a_0}{b}\right)^3$ being in between $\frac{4\sqrt{2\pi}}{33} \approx 3.21$ and 0, respectively.

In order to calculate $\mathcal{M}$ for the hydrogen atom, we have to apply formula (17). One can easily check that the greater the size of the trap, the more bosonic the system behaves. On the other hand, for small traps the system behaves in a fermionic way ($\mathcal{M} < 0$). The transition between fermionic and bosonic behavior occurs for $b \approx 1.9a_0$. The value of $\mathcal{M}$ increases very fast and for $b > 10a_0$ one has $\mathcal{M} > 0.99$ (see figure 2). It would be interesting to test this theoretical result via direct experimental implementation of a single hydrogen atom addition and subtraction to and from the trap.

4. Two-particle interference

We now investigate the properties of composite particles with respect to two-particle interference under the action of beam splitter-like Hamiltonians [11].

4.1. Composite particles of two distinguishable fermions

The operator (14) can be used to construct a beam splitter-like Hamiltonian for composite particles made of two distinguishable fermions

$$H_{\mathrm{BS}} = c_{\mathrm{eff}}^{\dagger(1)} c_{\mathrm{eff}}^{(2)} + c_{\mathrm{eff}}^{\dagger(2)} c_{\mathrm{eff}}^{(1)}, \tag{22}$$

where superscripts (1) and (2) denote two beam splitter modes. It is easy to find that a single composite particle in one of the two modes under the action of this Hamiltonian evolves into an even superposition of the two modes in time $t = \frac{\pi}{4}$, irrespective of the χ_2 factor. On the other hand, the evolution of a two-particle state depends on χ_2

$$|\psi(t)\rangle = \cos\left(2\sqrt{\chi_2}t\right)|11\rangle + \mathrm{i}\sin\left(2\sqrt{\chi_2}t\right)\left(\frac{|20\rangle + |02\rangle}{\sqrt{2}}\right), \tag{23}$$

where the bunched state $(|20\rangle + |02\rangle)/\sqrt{2}$ denotes two composite particles in one mode and the anti-bunched $|11\rangle$ denotes one composite particle in each mode. The probabilities of bunching (p_{B}) and anti-bunching (p_{AB}) after time $t = \frac{\pi}{4}$ given as a function of purity $P = 1 - \chi_2$ are

$$p_{\mathrm{B}} = \sin^2\left(\frac{\pi\sqrt{1-P}}{2}\right), \qquad p_{\mathrm{AB}} = \cos^2\left(\frac{\pi\sqrt{1-P}}{2}\right). \tag{24}$$

For $P = 1$ one observes perfect anti-bunching, whereas for $P = 0$ one observes perfect bunching. The transition between bosonic and fermionic behavior, i.e. $p_{\mathrm{B}} = p_{\mathrm{AB}}$, occurs for $P = \frac{3}{4}$.

The above example demonstrates that the notion of bosonic and fermionic quality is not absolute, unlike previous considerations [2–4]. In fact, this quality must be defined with respect to specific physical scenarios. For situations where particles are added and subtracted to a single mode, only the transition from fermionic to bosonic behavior occurs at $P = \frac{1}{2}$, whereas for beam splitter-like situations where particles are added and subtracted to two modes simultaneously, the transition occurs at $P = \frac{3}{4}$. It is possible that for physical situations where an infinite number of modes can be occupied, there is no transition, i.e. the composite particle made of two distinguishable fermions would always behave like a boson.

4.2. Composite particles of two distinguishable bosons

One can also consider a beam splitter-like Hamiltonian for composite particles made of two distinguishable bosons. In this case one finds that the probabilities of bunching (p_{B}) and anti-bunching (p_{AB}) as a function of the entanglement between the two constituent bosons are given by

$$p_{\mathrm{B}} = \sin^2\left(\frac{\pi\sqrt{1+P}}{2}\right), \qquad p_{\mathrm{AB}} = \cos^2\left(\frac{\pi\sqrt{1+P}}{2}\right). \tag{25}$$

As expected, for all values of P, bunching dominates anti-bunching with pure bunching observed at $P = 0$. Moreover, for a given entanglement, one finds that the composite particle made of two bosons exhibits higher probability of bunching than the composite particle made of two fermions. However, as P increases the probability of anti-bunching also increases; therefore the particle exhibits sub-bosonic behavior in this test rather than *super-bosonic*.

5. Conclusions

In this work, we have analyzed the quality of bosonic and fermionic behavior in composite particle systems made of two distinguishable fermions or bosons. We found Kraus operators that optimally implement addition and subtraction channels and applied them to formulate measures of bosonic and fermionic quality in terms of single-particle addition–subtraction and two-particle interference. We find that the bosonic or fermionic quality of these particles depends on their quantum state as well as on the experimental scenario considered. Finally, we apply our measure based on single-particle addition–subtraction to a hydrogen atom in a harmonic trap and show that its bosonic and fermionic quality depends on the size of the trap.

Contrary to previous considerations, we find more complex behavior and identify transitions between fermionic and bosonic behavior for composite particles made of two distinguishable fermions in terms of the entanglement between the constituent fermions. For the single-particle addition–subtraction scenario, the transition from fermionic to bosonic behavior occurs for exactly 1 ebit of entanglement between the constituent particles. For composite particles made of two distinguishable bosons, we relate the quality of bosonic behavior to the entanglement between the constituent bosons. We find that these systems can exhibit superbosonic behavior ($\mathcal{M} > 1$) with respect to the measure $\mathcal{M}$ (see figure 1). In the limit of large entanglement, both systems, made of two fermions and of two bosons, behave in a similar (bosonic) way. The intuitive reason behind this phenomenon is that in this case the states of individual constituent particles are highly mixed; therefore their properties are undefined. The particles are still indistinguishable; however, due to the size of the state space they occupy it does not matter whether they are bosons or fermions.

Note that the quality of the bosonic behavior can be determined also in other ways; see, for example [12–14]. One can also study the same entanglement characteristics by using the deformed creation and annihilation operators over the so-called algebra of fermionic and bosonic q-deformed oscillators [15]. Finally, an interesting open problem is to find the relation between the approach to composite particles via entanglement that was undertaken in this work and in [2–4], and the quantum field theoretical one [16]. The problem is that given a (pure) wave function of a two-particle system, one can easily evaluate the entanglement and purity of the subsystem; it is not clear how this entanglement is related to our measure $\mathcal{M}$. It was shown in [17] that relativistic entanglement changes under Lorentz transformations, which suggests that extension of our approach to relativistic systems may not be straightforward.

Acknowledgments

This research was supported by the National Research Foundation and the Ministry of Education in Singapore. PK was also supported by the Foundation for Polish Science. We acknowledge useful discussions with J Bandhyopadhyay, R Fazio, B Hessmo, K Modi, T Paterek and M Paternostro.

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Law C K 2005 *Phys. Rev.* A **71** 034306
[3] Chudzicki C, Oke O and Wootters W K 2010 *Phys. Rev. Lett.* **104** 070402
[4] Ramanathan R, Kurzyński P, Chuan T K, Santos M F and Kaszlikowski D 2011 *Phys. Rev.* A **84** 034304
[5] Zavatta A, Viciani S and Bellini M 2004 *Science* **306** 660
[6] Parigi V, Zavatta A, Kim M S and Bellini M 2007 *Science* **317** 1890
[7] Zavatta A, Parigi V, Kim M, Jeong H and Bellini M 2009 *Phys. Rev. Lett.* **103** 140406
[8] Kim M S 2008 *J. Phys. B: At. Mol. Opt. Phys.* **41** 133001
[9] Lee S Y and Nha H 2010 *Phys. Rev.* A **82** 053812
[10] Lee S Y, Ji S W, Kim H J and Nha H 2011 *Phys. Rev.* A **84** 012302
[11] Hong C K, Ou Z Y and Mandel L 1987 *Phys. Rev. Lett.* **59** 2044
[12] Marek P and Filip R 2010 *Phys. Rev.* A **81** 022302
[13] Usuga M A *et al* 2010 *Nature Phys.* **6** 767
[14] Zavatta A, Fiurasek J and Bellini M 2011 *Nature Photon.* **5** 52
[15] Gavrilik A M and Mishchenko Y U 2012 *Phys. Lett.* A **376** 1596
[16] Salpeter E E and Bethe H A 1951 *Phys. Rev.* **84** 1232
[17] Peres A, Scudo P F and Terno D R 2002 *Phys. Rev. Lett.* **88** 230402