IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received March 22, 2019, accepted April 4, 2019, date of publication April 22, 2019, date of current version April 29, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2912354

A Novel Interpretation for Opinion Consensus in Social Networks With Antagonisms

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This work was supported by the High-Level Talent Start-Up Fund from SJTU Double First-Class University Plan through Project WF220509005.

ABSTRACT We take a new perspective for the consensus in DeGroot-type social networks with antagonistic interactions between some pairs of agents. We observe the analogies between social networks and electrical networks. A line with positive (or negative) conductance in the electrical network well corresponds to the cooperative (or antagonistic) interaction in the social network. Then, we introduce a refined definition of effective conductance (EC), which comes from electrical networks, into social networks as a characterization of the overall relationship between a pair of agents. The EC considers the effects of both direct and indirect interactions between the agents. Some EC-based consensus criteria are established by analytical and statistical approaches, showing that the sign of EC is a useful indicator of consensus. The opinion consensus can be generally interpreted as every pair of agents being overall cooperative despite antagonistic interactions, i.e., the corresponding EC being positive. The obtained results provide new insights into the consensus mechanism with clear intuition. Case study of a 15-agent network is provided as an illustration.

INDEX TERMS Consensus, effective conductance, signed graph, social network.

I. INTRODUCTION

The DeGroot model, first proposed by DeGroot in 1970s [1], is a representative social network model for a variety of sociological systems related to, e.g., politics [2], economics [3], [4] and education [5]. In this model, the agents in the society are connected via a certain interaction structure such that each agent iteratively updates its opinion according to its interactions with the others. It has now been developed into a family of DeGroot-type models, e.g., the consensus protocol considering time-varying interactions [6], higher-order interactions [7], [8], communication noise [9], stubborn agents [10], [11], and nonlinear opinion-making mechanisms [12]–[16]. The DeGroot-type social networks exhibit rich collective behaviors, among which the consensus problem is of the most fundamental importance and receiving sustained interest.

Commonly, social network structure is described by a graph with signed weighted lines, simply referred to as signed graph in the literature [17]–[19]. In the signed graph, a line with positive (or negative) weight represents a cooperative (or antagonistic) interaction between the corresponding pair of agents. The location of those antagonistic interactions plays an important role in reaching a consensus. So far extensive results have been developed based on a well-known graph concept namely structural balance. Structural balance characterizes the spatial distribution feature of antagonistic interactions in the network. A social network is said to be structurally balanced if there exists a bipartition of agents such that any two agents in the same subnetwork present cooperativeness and any two agents in distinct subnetworks present antagonism [20]. It has been revealed that structural balance property is a key factor in causing the failure of consensus, e.g., see [6], [14], [21]–[23]. The opinion disagreement could be in form of bipartite/polarized consensus [14], several clustered groups [15] or divergence [24] depending on different dynamics adopted. However, structural balance theory does not give a clear explanation of the opinion divergence in structurally unbalanced social networks.

In addition to structural balance, some recent works investigated the consensus problem from a new perspective by introducing the concept of effective conductance (EC) originated from electrical networks. An electrical network consists of a set of nodes interconnected by electric lines, the

The associate editor coordinating the review of this manuscript and approving it for publication was Feiqi Deng.

structure of which is highly similar to a social network. The EC is a common concept that measures the electrical coupling strength between a pair of nodes. In [25], [26], some consensus conditions for special-structured networks are established in terms of EC, which give a fresh viewpoint and deserves further exploration.

In this paper, we address the consensus problem of the DeGroot-type model with the presence of antagonistic interactions following the inspiration from electrical networks. We propose a refined definition of EC to measure the total coupling strength between a pair of nodes in electrical networks. This definition performs better than the traditional definition in describing the network with negative line conductances. Then, we introduce the EC to social networks as a proper characterization of the overall relationship between a pair of agents, which considers the two agents' direct interaction and indirect interactions via other agents. The EC applies to both structurally balanced and structurally unbalanced social networks. Novel consensus criteria in terms of EC are established by analytical and statistical approaches. It generally indicates that even if there are antagonistic interactions, the opinion consensus can be reached when every pair of agents has positive EC indicating overall cooperativeness. We also design an EC-based iterative algorithm that checks consensus quickly. These results provide a new and intuitive viewpoint for the mechanism of consensus.

The remainder of the paper is organized as follows. Section II formulates a DeGroot-type model with antagonistic interactions. In Section III, we discuss the analogies between social networks and electrical networks, and refine the definition of EC in the context of electrical networks. In Section IV, we apply the EC to social networks and establish some new results on consensus. Section V gives a case study to illustrate the obtained results, and Section VI makes a conclusion.

Notations: We introduce some notations that will be frequently used in the paper. For simplicity, we use $\boldsymbol{x} = [x_i] \in \mathbb{R}^n$ to denote an n-dimensional vector, and $\boldsymbol{A} = [A_{ij}] \in \mathbb{R}^{n\times n}$ to denote an n-by-n square matrix. The notation $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$ denotes an identity matrix of order n, and $\mathbf{1}_n \in \mathbb{R}^n$ denotes an n-dimensional vector with all entries being one. For a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, we use $\lambda_i(\boldsymbol{A})$ to denote its i-th eigenvalue.

II. FORMULATION OF OPINION DYNAMICS

Consider a connected social network with n agents. Let $x_i(t)$ be the opinion of agent i ($i = 1, 2, \ldots, n$) at time step t ($t = 0, 1, 2, \ldots$). We adopt the DeGroot-type model below to describe the opinion evolution process [27]

$$x_i(t+1) = x_i(t) + \kappa \sum_{j=1, j\neq i}^{n} w_{ij}(x_j(t) - x_i(t)) \tag{1}$$

where $\kappa > 0$ is the step size, and w_{ij} is the weight that agent i assigns to the opinion of agent j. In this model, each agent takes a weighted average opinion to be its opinion at the next time step. We consider a general case where the weight w_{ij} can be positive, negative or zero. A positive w_{ij} indicates that agent i tends to reach an agreement with agent j, and a larger w_{ij} describes a stronger cooperativeness. A negative w_{ij} indicates that agent i tends to be against agent j, and a smaller w_{ij} describes a stronger antagonism. A zero w_{ij} indicates that agent i has no direct interaction with agent j and hence does not take the opinion of agent j into account. In (1), we use a unified expression to describe the effects of cooperative and antagonistic interactions, which is similar to the models in [22], [25]. Also note that it has not yet reached an agreement on the modeling of antagonistic interactions. There are other types of models that use different functions to describe the cooperative and antagonistic interactions, e.g., see [6], [13], [14]. Moreover, we make the following assumption for the social network described by (1).

Assumption 1: $w_{ij} = w_{ji}, \forall i, j = 1, 2, \ldots, n, i \neq j$.

Assumption 1 indicates that any pair of agents presents symmetric attitude to each other, which is a reasonable simplification of some real situations, e.g., see [4], [22], [25], [28]. With this property, the social network can be represented by an undirected signed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of agents, $\mathcal{E} = \{(i,j)|\ w_{ij} \neq 0\}$ is the set of interactions with (i,j) denoting an unordered pair of agents, and $\boldsymbol{w} = [w_{ij}] \in \mathbb{R}^l$, $\forall (i,j) \in \mathcal{E}$ is the vector of interaction weights between the agents (assuming the total number of interactions is l).

We now introduce the Laplacian matrix of graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$, say $\boldsymbol{L}_{\mathcal{G}} = [L_{ij}] \in \mathbb{R}^{n\times n}$, which is defined as $L_{ii} = \sum_{j=1, j\neq i}^{n} w_{ij}$, and $L_{ij} = L_{ji} = -w_{ij}$, $\forall (i,j) \in \mathcal{E}$ and $L_{ij} = 0$ otherwise. Then, (1) can be re-expressed as

$$\boldsymbol{x}(t+1) = (\boldsymbol{I}_n - \kappa \boldsymbol{L}_{\mathcal{G}})\boldsymbol{x}(t) \tag{2}$$

where $\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ collect the agent opinions at time step t.

System (2) reaches a consensus if $\lim_{t\to\infty} x_1(t) = x_2(t) = \cdots = x_n(t)$. Due to the symmetric interactions by Assumption 1, the consensus will be an average consensus [27], i.e., $\lim_{t\to\infty} x_1(t) = x_2(t) = \cdots = x_n(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$, where $x_i(0), i = 1, \ldots, n$ are the initial opinions. Also note that system (2) will not exhibit polarized consensus since it does not contain the nonlinear opinion-making mechanism as in [14]. System (2) has divergent opinions once it fails to reach a consensus. It is known that a consensus will be reached if all the interaction weights are positive [27], however, it is generally not the case if there are negative interaction weights. In the following we summarize the definition of consensus and have a basic result in terms of $\boldsymbol{L}_{\mathcal{G}}$.

Definition 1: The system (2) reaches an opinion consensus if $\lim_{t\to\infty} x_1(t) = x_2(t) = \cdots = x_n(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$.

Assumption 2: For $i = 1, 2, \ldots, n$, $\kappa \cdot \lambda_i(\boldsymbol{L}_{\mathcal{G}}) < 2$.

Lemma 1: The system (2) reaches an opinion consensus if and only if the Laplacian matrix $\boldsymbol{L}_{\mathcal{G}}$ is positive semi-definite and has only one zero eigenvalue.

Proof: Since $\boldsymbol{L}_{\mathcal{G}}$ is real symmetric, we have $\boldsymbol{u}_i^T\boldsymbol{u}_i = 1$ and $\boldsymbol{u}_i^T\boldsymbol{u}_j = 0$, $i \neq j$, where $\boldsymbol{u}_i$ denotes the eigenvector of $\boldsymbol{L}_{\mathcal{G}}$ with respect to the eigenvalue $\lambda_i(\boldsymbol{L}_{\mathcal{G}})$. It is trivial that $\boldsymbol{L}_{\mathcal{G}}$ has a zero eigenvalue and the corresponding eigenvector is $\frac{1}{\sqrt{n}}\boldsymbol{1}_n$. Without loss of generality, we suppose $\lambda_1(\boldsymbol{L}_{\mathcal{G}}) = 0$ and $\boldsymbol{u}_1 = \frac{1}{\sqrt{n}}\boldsymbol{1}_n$.

It follows from (2) that $\boldsymbol{x}(t) = (\boldsymbol{I}_n - \kappa\boldsymbol{L}_{\mathcal{G}})^t\boldsymbol{x}(0)$. Applying orthogonal decomposition to $(\boldsymbol{I}_n - \kappa\boldsymbol{L}_{\mathcal{G}})^t$ gives $(\boldsymbol{I}_n - \kappa\boldsymbol{L}_{\mathcal{G}})^t = \sum_{i=1}^{n}(1-\kappa\lambda_i(\boldsymbol{L}_{\mathcal{G}}))^t\boldsymbol{u}_i\boldsymbol{u}_i^T$ [29]. Hence, consensus is reached if and only if $0 < \kappa\lambda_i(\boldsymbol{L}_{\mathcal{G}}) < 2$, $i = 2, \ldots, n$ so that the terms $(1-\kappa\lambda_i(\boldsymbol{L}_{\mathcal{G}}))^t\boldsymbol{u}_i\boldsymbol{u}_i^T$, $i = 2, \ldots, n$ vanish as $t \to \infty$. Together with Assumption 2, we conclude that consensus is reached if and only if $\lambda_i(\boldsymbol{L}_{\mathcal{G}}) > 0$, $i = 2, \ldots, n$, i.e., $\boldsymbol{L}_{\mathcal{G}}$ is positive semi-definite and has only one zero eigenvalue. ■

Remark 1: Lemma 1 gives an algebraic condition for consensus. As shown in the proof, the inequality $0 < \kappa\lambda_i(\boldsymbol{L}_{\mathcal{G}}) < 2$, $i = 2, \ldots, n$ is required by reaching a consensus. We note that $\kappa\lambda_i(\boldsymbol{L}_{\mathcal{G}}) \geq 2$ is mainly caused by the step size being too large, while $\kappa\lambda_i(\boldsymbol{L}_{\mathcal{G}}) \leq 0$ is caused by the presence of antagonistic interactions. The aim of adopting Assumption 2 is to ensure that the step size is sufficiently small so that we focus on the impact of antagonistic interactions on consensus.

In the following sections, we will link the graph structure of social networks to electrical networks, and explore the consensus problem using the concept of EC from electrical networks. With the help of EC, we will translate Lemma 1 into some intuitive criteria that further reveal the role of antagonistic interactions.

III. ELECTRICAL NETWORKS & EFFECTIVE CONDUCTANCE

A. ANALOGIES BETWEEN SOCIAL NETWORKS AND ELECTRICAL NETWORKS

We first introduce some preliminaries for electrical networks. Consider an electrical network that has n nodes and l resistive lines. We slightly abuse the notations (i,j) and w_{ij} for social networks and electrical networks as it will be seen later that these two types of networks share many similarities. Let the unordered pair (i,j) denote the line connecting node i and node j, and $w_{ij} = w_{ji}$ denote the conductance of line (i,j), i.e., the reciprocal of line resistance. The line conductances are usually positive, while in special cases some lines may present a negative conductance feature [30]. Then, the electrical network can also be described by an undirected signed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$.

We observe close links between social networks and electrical networks that are both built over graph structures. Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$, the notations $\mathcal{V}$, $\mathcal{E}$ can be regarded as agents and interactions in the context of social networks, or nodes and lines in the context of electrical networks. In addition, the line conductance describes the coupling condition between a pair of nodes in an electrical network, e.g., line (i,j) with positive (or negative) conductance indicates that node i and node j are well connected (or abnormally connected). The function of conductance is highly similar to the function of weights in describing the interaction between a pair of agents in the social network. Thus, the concepts of "agent, interaction, weight" in social networks well correspond to the concepts of "node, line, conductance" in electrical networks. Henceforth we will use these two groups of terminologies interchangeably and interpret the notation $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$ as either a social network or the corresponding electrical network.

B. EFFECTIVE CONDUCTANCE: DEFINITION & BASIC PROPERTIES

For an electrical network $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$, let $i, j \in \mathcal{V}$, $i \neq j$ be a pair of nodes, and $\mathcal{V}_r = \mathcal{V}\backslash\{i,j\}$ be the set of the remaining nodes. Then, the Laplacian matrix of $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$ can be written into the block form

$$\boldsymbol{L}_{\mathcal{G}} = \begin{bmatrix} \boldsymbol{L}_{rr} & \boldsymbol{L}_{ri} & \boldsymbol{L}_{rj} \\ \boldsymbol{L}_{ri}^T & L_{ii} & L_{ij} \\ \boldsymbol{L}_{rj}^T & L_{ij} & L_{jj} \end{bmatrix} \tag{3}$$

where $\boldsymbol{L}_{ri} \in \mathbb{R}^{n-2}$ is the sub-matrix of $\boldsymbol{L}_{\mathcal{G}}$ whose rows and columns are indexed by node set $\mathcal{V}_r$ and node i, respectively. Similar interpretations apply to the other sub-matrices in (3). Note that $\boldsymbol{L}_{\mathcal{G}}$ is also called conductance matrix in the context of electrical networks [30], which is an important quantity in Kirchhoff's circuit law.

Assume node i is connected to a voltage source with unit potential, and node j is grounded with zero potential. Applying Kirchhoff's circuit law to the electrical network gives [30]

$$\begin{bmatrix} \boldsymbol{0} \\ I_i^{\text{inj}} \\ I_j^{\text{inj}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{rr} & \boldsymbol{L}_{ri} & \boldsymbol{L}_{rj} \\ \boldsymbol{L}_{ri}^T & L_{ii} & L_{ij} \\ \boldsymbol{L}_{rj}^T & L_{ij} & L_{jj} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_r \\ V_i \\ V_j \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{rr} & \boldsymbol{L}_{ri} & \boldsymbol{L}_{rj} \\ \boldsymbol{L}_{ri}^T & L_{ii} & L_{ij} \\ \boldsymbol{L}_{rj}^T & L_{ij} & L_{jj} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_r \\ 1 \\ 0 \end{bmatrix} \tag{4}$$

where I_i^{inj}, I_j^{inj} are the current injection at node i, j, respectively, $I_i^{\text{inj}} = -I_j^{\text{inj}}$ since the total current injection must be zero; and $\boldsymbol{V}_r \in \mathbb{R}^{n-2}$ is the voltage potential vector of the remaining nodes. By (4), the current injection at node i is

$$I_i^{\text{inj}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} I_i^{\text{inj}} \\ I_j^{\text{inj}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{L}_{ij}^{\text{red}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = L_{ii} - \boldsymbol{L}_{ri}^T\boldsymbol{L}_{rr}^{-1}\boldsymbol{L}_{ri} \tag{5}$$

where

$$\boldsymbol{L}_{ij}^{\text{red}} = \begin{bmatrix} L_{ii} - \boldsymbol{L}_{ri}^T\boldsymbol{L}_{rr}^{-1}\boldsymbol{L}_{ri} & L_{ij} - \boldsymbol{L}_{ri}^T\boldsymbol{L}_{rr}^{-1}\boldsymbol{L}_{rj} \\ L_{ij} - \boldsymbol{L}_{rj}^T\boldsymbol{L}_{rr}^{-1}\boldsymbol{L}_{ri} & L_{jj} - \boldsymbol{L}_{rj}^T\boldsymbol{L}_{rr}^{-1}\boldsymbol{L}_{rj} \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{6}$$

Thus, the circuit between node i and node j can be equivalent to a line with conductance being $L_{ii} - \boldsymbol{L}_{ri}^T\boldsymbol{L}_{rr}^{-1}\boldsymbol{L}_{ri}$, see Fig. 1. This conductance is referred to as the EC between node i and node j. We summarize it into the definition below.

Assumption 3: For any $i, j \in \mathcal{V}$, $i \neq j$, $\boldsymbol{L}_{rr}$ is nonsingular where $\mathcal{V}_r = \mathcal{V}\backslash\{i,j\}$.

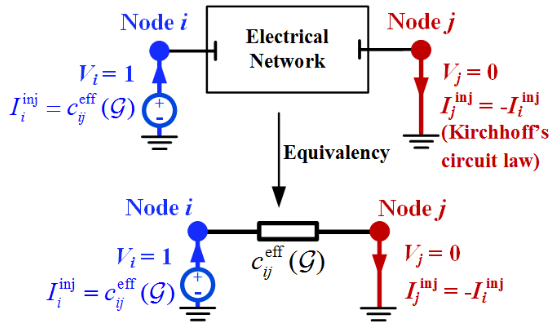

FIGURE 1. Illustration of the EC in an electrical network.

Definition 2: Let $i, j \in \mathcal{V}$, $i \neq j$ be a pair of nodes in the electrical network $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$, and $\mathcal{V}_r = \mathcal{V} \backslash \{i, j\}$. Define

$$c_{ij}^{\text{eff}}(\mathcal{G}) = L_{ii} - \boldsymbol{L}_{ri}^T \boldsymbol{L}_{rr}^{-1} \boldsymbol{L}_{ri} \tag{7}$$

as the effective conductance (EC) between node i and node j.

Assumption 3 is required by the expression of $c_{ij}^{\text{eff}}(\mathcal{G})$. If this is not true, i.e., $\boldsymbol{L}_{rr}$ is singular, then the equality $\boldsymbol{V}_r = -\boldsymbol{L}_{rr}^{-1}(\boldsymbol{L}_{ri}V_i + \boldsymbol{L}_{rj}V_j)$ (an equivalent form of the first row of (4)) becomes invalid. In this case, the circuit loses causality as $\boldsymbol{V}_r$ is no longer determined by V_i, V_j, which implies that the circuit is unrealistic and needs a remodeling [30]. From this viewpoint, we can safely apply Assumption 3 to realistic physical networks. This claim is also numerically confirmed by the statistical results in Section IV-B, where $\boldsymbol{L}_{rr}$ is nonsingular for a large amount of tests (8×10^6 different network scenarios in total).

From (5), a larger EC indicates that the voltage source with unit potential at node i induces larger current flow from node i to node j. In this sense, the EC concisely measures the overall coupling strength considering the effects of all possible connection paths between the two nodes. A large positive EC means the corresponding pair of nodes are tightly coupled, while a negative EC means the corresponding pair of nodes are electrically antagonistic, which may lead to undesirable consequences such as the electrical network losing passivity [30]. The appearance of negative EC will also be linked to consensus failure in social networks in the next section.

Note that the conductance of a physical line (i, j) has the "non-directional" feature, i.e., the conductance keeps the same whether the current flows from node i to node j or from node j to node i. We show below that the defined EC preserves this feature. First, it follows from the definition of $\boldsymbol{L}_{\mathcal{G}}$ that

$$\begin{bmatrix} \boldsymbol{0} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{rr} & \boldsymbol{L}_{ri} & \boldsymbol{L}_{rj} \\ \boldsymbol{L}_{ri}^T & L_{ii} & L_{ij} \\ \boldsymbol{L}_{rj}^T & L_{ij} & L_{jj} \end{bmatrix} \begin{bmatrix} \boldsymbol{1}_{n-2} \\ 1 \\ 1 \end{bmatrix}.$$

Eliminating the first row of this equation gives $\boldsymbol{L}_{ij}^{\text{red}} \boldsymbol{1}_2 = \boldsymbol{0}$, where $\boldsymbol{L}_{ij}^{\text{red}}$ is defined in (6). Further, by $\boldsymbol{L}_{ij}^{\text{red}} \boldsymbol{1}_2 = \boldsymbol{0}$ and (7) we have

$$\boldsymbol{L}_{ij}^{\text{red}} = \begin{bmatrix} c_{ij}^{\text{eff}}(\mathcal{G}) & -c_{ij}^{\text{eff}}(\mathcal{G}) \\ -c_{ij}^{\text{eff}}(\mathcal{G}) & c_{ij}^{\text{eff}}(\mathcal{G}) \end{bmatrix} \tag{8}$$

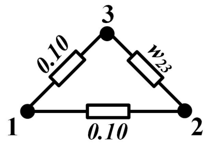

FIGURE 2. A simple electrical network.

which indicates that $c_{ij}^{\text{eff}}(\mathcal{G}) = L_{jj} - \boldsymbol{L}_{rj}^T \boldsymbol{L}_{rr}^{-1} \boldsymbol{L}_{rj}$. On the other hand, Definition 2 gives that $c_{ji}^{\text{eff}}(\mathcal{G}) = L_{jj} - \boldsymbol{L}_{rj}^T \boldsymbol{L}_{rr}^{-1} \boldsymbol{L}_{rj}$, and thus we have $c_{ji}^{\text{eff}}(\mathcal{G}) = c_{ij}^{\text{eff}}(\mathcal{G})$. So the EC between node i and node j is identical to the EC between node j and i, and we will not distinguish them.

Moreover, the proposed definition of EC makes an improvement over the traditional version shown below [25], [31]

$$g_{ij}^{\text{eff}} = \frac{1}{(\boldsymbol{e}_i - \boldsymbol{e}_j)^T \boldsymbol{L}_{\mathcal{G}}^{\dagger} (\boldsymbol{e}_i - \boldsymbol{e}_j)}$$

where the superscript $\dagger$ denotes Moore-Penrose inverse and $\boldsymbol{e}_i \in \mathbb{R}^n$ denotes a vector with the entry indexed by node i being one and the other entries being zero. The expressions of $c_{ij}^{\text{eff}}(\mathcal{G})$ and $g_{ij}^{\text{eff}}(\mathcal{G})$ are equivalent if all line conductances are positive. When there are negative line conductances, we show below that $c_{ij}^{\text{eff}}(\mathcal{G})$ has a better performance. Given the circuit in Fig. 2, it is trivial to check that $c_{12}^{\text{eff}} = g_{12}^{\text{eff}} = 0.133$ when $w_{ij} = 0.05$. When $w_{12} = -0.05$, we have $c_{12}^{\text{eff}} = 0$ but the traditional definition gives $g_{12}^{\text{eff}} = 0.2$. On the other hand, by simple circuit analysis, the EC between node 1 and node 2 should be $w_{12} + (w_{13}^{-1} + w_{23}^{-1})^{-1} = 0$, which coincides with the proposed definition. Hence, the proposed definition of EC is more appropriate than the traditional one in describing the network with negative line conductances.

IV. APPLICATION OF EFFECTIVE CONDUCTANCE TO SOCIAL NETWORK CONSENSUS

A. THEORETICAL RESULTS

By the analogies between social networks and electrical networks, we now introduce the EC into social networks as a proper characterization of the overall relationship between a pair of agents. A positive EC indicates an overall cooperative relationship between the two agents, while a negative EC indicates that the two agents are overall antagonistic to each other. We note that the EC takes into account both the possible direct interaction between the two agents and indirect interactions via other agents. The EC is applicable to any pair of agents whether they have direct interaction or not. In addition, a pair of agents with positive (or negative) EC does not necessarily mean their direction interaction is cooperative (or antagonistic), and vice versa.

With the introduction of EC into social networks, we will establish some new results for consensus. Let us begin with a special case that leads to some inspiring observations. Suppose all the agents except agent i, j have converged opinions, i.e., $x_k(t+1) = x_k(t)$, $k \in \mathcal{V}_r = \mathcal{V}\backslash\{i,j\}$, then (2) becomes

$$\begin{bmatrix} \mathbf{0} \\ x_i(t+1) - x_i(t) \\ x_j(t+1) - x_j(t) \end{bmatrix} = -\kappa \begin{bmatrix} \boldsymbol{L}_{rr} & \boldsymbol{L}_{ri} & \boldsymbol{L}_{rj} \\ \boldsymbol{L}_{ri}^T & L_{ii} & L_{ij} \\ \boldsymbol{L}_{rj}^T & L_{ij} & L_{jj} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_r(t) \\ x_i(t) \\ x_j(t) \end{bmatrix}.$$

Eliminating the first row of the above equation gives

$$\begin{bmatrix} x_i(t+1) - x_i(t) \\ x_j(t+1) - x_j(t) \end{bmatrix} = -\kappa \boldsymbol{L}_{ij}^{\mathrm{red}} \begin{bmatrix} x_i(t) \\ x_j(t) \end{bmatrix} = -\kappa \begin{bmatrix} c_{ij}^{\mathrm{eff}}(\mathcal{G}) & -c_{ij}^{\mathrm{eff}}(\mathcal{G}) \\ -c_{ij}^{\mathrm{eff}}(\mathcal{G}) & c_{ij}^{\mathrm{eff}}(\mathcal{G}) \end{bmatrix} \begin{bmatrix} x_i(t) \\ x_j(t) \end{bmatrix}$$

which is equivalent to

$$x_i(t+1) - x_j(t+1) = (1 - 2\kappa c_{ij}^{\mathrm{eff}}(\mathcal{G}))(x_i(t) - x_j(t)). \quad (9)$$

It can be seen that the EC is directly linked to opinion dynamics. The consensus of system (9) is determined by the sign of $c_{ij}^{\mathrm{eff}}(\mathcal{G})$—agent i, j will separate (or synchronize) their opinions if $c_{ij}^{\mathrm{eff}}(\mathcal{G}) < 0$ (or $c_{ij}^{\mathrm{eff}}(\mathcal{G}) > 0$). It clearly reveals that consensus is equivalent to the overall cooperativeness between the two agents. In the following, we extend the analysis to general cases and show that similar criteria based on the sign of EC still apply.

Lemma 2 [32]: Let $\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix}$ be a real symmetric matrix where $\boldsymbol{A} \in \mathbb{R}^{p\times p}$ is nonsingular, $\boldsymbol{B} \in \mathbb{R}^{q\times p}$ and $\boldsymbol{C} \in \mathbb{R}^{q\times q}$. Then $i_-(\boldsymbol{H}) = i_-(\boldsymbol{H}') + i_-(\boldsymbol{A})$ and $i_0(\boldsymbol{H}) = i_0(\boldsymbol{H}')$, where $\boldsymbol{H}' = \boldsymbol{C} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^T$ and i_-, i_0 denote the number of negative eigenvalues and zero eigenvalues of a matrix, respectively.

Proposition 1: If the system (2) reaches an opinion consensus, then $c_{ij}^{\mathrm{eff}}(\mathcal{G}) > 0$ for any pair of agents $i, j \in \mathcal{V}$, $i \neq j$.

Proof: For any $i, j \in \mathcal{V}$, $i \neq j$ and $\mathcal{V}_r = \mathcal{V}\backslash\{i,j\}$, by Lemma 2 we have

$$i_0(\boldsymbol{L}_{\mathcal{G}}) = i_0(\boldsymbol{L}_{ij}^{\mathrm{red}})$$
$$i_-(\boldsymbol{L}_{\mathcal{G}}) = i_-(\boldsymbol{L}_{rr}) + i_-(\boldsymbol{L}_{ij}^{\mathrm{red}}).$$

If the system (2) reaches an opinion consensus, it follows from Lemma 1 that $i_-(\boldsymbol{L}_{\mathcal{G}}) = 0$ and $i_0(\boldsymbol{L}_{\mathcal{G}}) = 1$, and hence we have $i_-(\boldsymbol{L}_{ij}^{\mathrm{red}}) = 0$ and $i_0(\boldsymbol{L}_{ij}^{\mathrm{red}}) = 1$. So we can conclude $c_{ij}^{\mathrm{eff}}(\mathcal{G}) > 0$ by (8). ■

Let $\mathcal{E}_- = \{(i,j) |\ w_{ij} < 0,\ (i,j) \in \mathcal{E}\}$ be the set of antagonistic interactions. Let $\mathcal{G}^{ij}$ denote the subnetwork obtained by deleting the interaction (i,j) from the original network $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$, and the Laplacian matrix of $\mathcal{G}^{ij}$ is denoted by $\boldsymbol{L}_{\mathcal{G}^{ij}}$. Then, we have another condition below.

Proposition 2: Suppose there exists an antagonistic interaction $(i,j) \in \mathcal{E}_-$ such that $\boldsymbol{L}_{\mathcal{G}^{ij}}$ is positive semi-definite and has only one zero eigenvalue. Then the system (2) reaches an opinion consensus if and only if $c_{ij}^{\mathrm{eff}}(\mathcal{G}) > 0$.

Proof: The Laplacian matrices $\boldsymbol{L}_{\mathcal{G}}$ and $\boldsymbol{L}_{\mathcal{G}^{ij}}$ can be partitioned into

$$\boldsymbol{L}_{\mathcal{G}} = \begin{bmatrix} \boldsymbol{L}_{rr} & \boldsymbol{L}_{0i} & \boldsymbol{L}_{0j} \\ \boldsymbol{L}_{0i}^T & L_{ii} & -w_{ij} \\ \boldsymbol{L}_{0j}^T & -w_{ij} & L_{jj} \end{bmatrix}$$

$$\boldsymbol{L}_{\mathcal{G}^{ij}} = \begin{bmatrix} \boldsymbol{L}_{rr} & \boldsymbol{L}_{0i} & \boldsymbol{L}_{0j} \\ \boldsymbol{L}_{0i}^T & L_{ii} - w_{ij} & 0 \\ \boldsymbol{L}_{0j}^T & 0 & L_{jj} - w_{ij} \end{bmatrix}.$$

If $\boldsymbol{L}_{\mathcal{G}^{ij}}$ is positive semi-definite and has only one zero eigenvalue, its principal submatrix $\boldsymbol{L}_{rr}$ satisfies $i_-(\boldsymbol{L}_{rr}) = 0$ [29, Observation 7.1.2]. Then, by applying Lemma 2 to $\boldsymbol{L}_{\mathcal{G}}$ we have

$$i_0(\boldsymbol{L}_{\mathcal{G}}) = i_0(\boldsymbol{L}_{ij}^{\mathrm{red}})$$
$$i_-(\boldsymbol{L}_{\mathcal{G}}) = i_-(\boldsymbol{L}_{rr}) + i_-(\boldsymbol{L}_{ij}^{\mathrm{red}}) = i_-(\boldsymbol{L}_{ij}^{\mathrm{red}}).$$

So we can conclude from (8) that $i_-(\boldsymbol{L}_{\mathcal{G}}) = 0$ and $i_0(\boldsymbol{L}_{\mathcal{G}}) = 1$ if and only if $c_{ij}^{\mathrm{eff}}(\mathcal{G}) > 0$. We then complete the proof together with Lemma 1. ■

Remark 2: Proposition 1 is a necessary condition for consensus, and Proposition 2 further gives a necessary and sufficient condition for consensus under some precondition, which show the role of positive EC (indicating overall cooperativeness) in reaching a consensus. In addition, Proposition 2 inspires an easy-to-implement approach for checking consensus. Denote $\mathcal{G}_+(\mathcal{V}, \mathcal{E}_+, \boldsymbol{w}_+)$ as the positive subnetwork of $\mathcal{G}(\mathcal{V}, \mathcal{E}, \boldsymbol{w})$, where $\mathcal{E}_+ = \mathcal{E}\backslash\mathcal{E}_-$. If $\mathcal{G}_+$ is disconnected, $\boldsymbol{L}_{\mathcal{G}}$ must have negative eigenvalues [33] so that consensus cannot be reached. If $\mathcal{G}_+(\mathcal{V}, \mathcal{E}_+, \boldsymbol{w}_+)$ is connected, then the Laplacian matrix $\boldsymbol{L}_{\mathcal{G}_+}$ is positive semi-definite and has only one zero eigenvalue since $\mathcal{G}_+(\mathcal{V}, \mathcal{E}_+, \boldsymbol{w}_+)$ has no antagonistic interactions [27]. Hence, as inferred by Proposition 2, each time after adding an antagonistic interaction to $\mathcal{G}_+(\mathcal{V}, \mathcal{E}_+, \boldsymbol{w}_+)$, we can apply $c_{ij}^{\mathrm{eff}}(\mathcal{G}) > 0$ as a necessary and sufficient condition for consensus. We summarize this idea into Algorithm 1.

Algorithm 1 (EC-Based Consensus Check)

1: Form the positive subnetwork $\mathcal{G}_+(\mathcal{V}, \mathcal{E}_+, \boldsymbol{w}_+)$.
2: Stop the algorithm if $\mathcal{G}_+$ is disconnected, the social network fails to reach a consensus.
3: Find the most antagonistic interaction in $\mathcal{E}_-$, i.e., $(i,j) = \arg\min_{(i,j)\in\mathcal{E}_-} w_{ij}$.
4: Add the interaction w_{ij} into $\mathcal{G}_+(\mathcal{V}, \mathcal{E}_+, \boldsymbol{w}_+)$, i.e., $\mathcal{E}_+ \leftarrow \mathcal{E}_+ \cup \{(i,j)\}$ and $\boldsymbol{w}_+ \leftarrow \begin{bmatrix}\boldsymbol{w}_+^T & w_{ij}\end{bmatrix}^T$. Update $\mathcal{E}_- \leftarrow \mathcal{E}_-\backslash\{(i,j)\}$.
5: Calculate $c_{ij}^{\mathrm{eff}}(\mathcal{G}_+)$ and make judgment:
   Case A: $c_{ij}^{\mathrm{eff}}(\mathcal{G}_+) \leq 0$. Stop the algorithm, the social network fails to reach a consensus.
   Case B: $c_{ij}^{\mathrm{eff}}(\mathcal{G}_+) > 0$ and $\mathcal{E}_- = \phi$. Stop the algorithm, the social network will reach a consensus.
   Case C: $c_{ij}^{\mathrm{eff}}(\mathcal{G}_+) > 0$ and $\mathcal{E}_- \neq \phi$. Go back to step 3.

We further explain Case A in the algorithm. In this case, it follows from the proof of Proposition 2 that the current Laplacian matrix $\boldsymbol{L}_{\mathcal{G}_+}$ has a negative eigenvalue (if $c_{ij}^{\mathrm{eff}}(\mathcal{G}_+) < 0$) or has two zero eigenvalues (if $c_{ij}^{\mathrm{eff}}(\mathcal{G}_+) = 0$). If $\mathcal{E}_- = \phi$, the current network $\mathcal{G}_+$ is identical to the original network $\mathcal{G}$, so the social network fails to reach a consensus by Lemma 1. If $\mathcal{E}_- \neq \phi$, by eigenvalue sensitivity [34], the negative

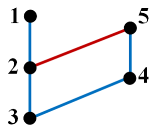

FIGURE 3. An example of structurally unbalanced network.

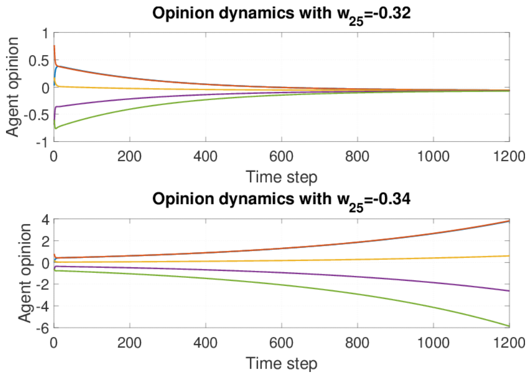

FIGURE 4. Opinion dynamics under different weights of antagonistic interaction.

eigenvalue or the two zero eigenvalues of $\boldsymbol{L}_{\mathcal{G}_+}$ will be non-increasing after adding the remaining negative interactions to $\mathcal{G}_+$ to form the original network $\mathcal{G}$. So $\boldsymbol{L}_{\mathcal{G}}$ has negative eigenvalues or more than one zero eigenvalue, and the social network fails to reach a consensus.

Moreover, we note that these results require no assumption on structural balance and hence apply to structurally balanced or unbalanced networks. A structurally balanced networks has such a property that the agents are divided into two separate groups after deleting all antagonistic interactions [20], i.e., the corresponding $\mathcal{G}_+$ is disconnected. So a structurally balanced social network fails to reach a consensus by Algorithm 1.

For structurally unbalanced networks, consensus is determined by the placements and weights of antagonistic interactions, where the obtained results can provide new viewpoints. Let us consider a typical structurally unbalanced network in Fig. 3 as an example, which is a undirected version of the one studied in [35]. The blue and red lines in this figure refer to cooperative and antagonistic interactions, respectively. Then, we can analyze the impact of the antagonistic interaction using Proposition 2. For simplicity, suppose the weights of all cooperative interactions equal to one. By Proposition 2, we only need to check the EC between node 2 and node 5, leading to the conclusion that the network reaches a consensus if and only if $w_{25} > -\frac{1}{3}$. For illustration, we choose two values around the threshold $-\frac{1}{3}$, say $w_{25} = -0.32$ or $w_{25} = -0.34$, and depict the opinion dynamics under the two settings (step size $\kappa = 0.2$ in both cases) in Fig. 4. The opinions get synchronized in case of $w_{25} = -0.32$ and go divergent in case of $w_{25} = -0.34$, which coincides with the expectation. In addition, we will see a comprehensive case study on another structurally unbalanced network with more complex topology in Section V.

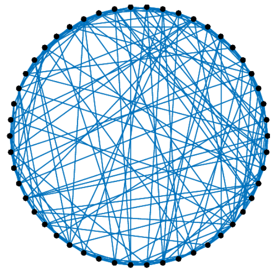

FIGURE 5. An example small-world network used in the test.

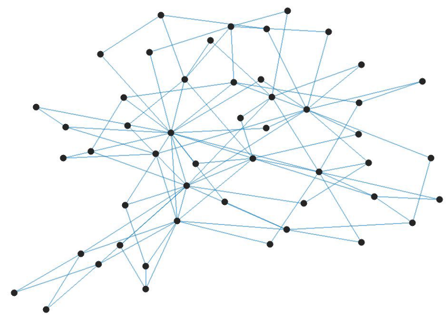

FIGURE 6. An example scale-free network used in the test.

B. STATISTICAL RESULTS

In this subsection, we focus on the following hypothesis, which is the inverse proposition of Proposition 1.

Hypothesis 1: If the system (2) fails to reach an opinion consensus, then there exists a pair of nodes $i, j \in \mathcal{V}$, $i \neq j$ such that $c_{ij}^{\text{eff}}(\mathcal{G}) \leq 0$.

It should be noted that Hypothesis 1 is not rigorously true. Nevertheless, we will show by Monte Carlo tests that it is statistically valid.

Two representative types of social networks, namely small-world network and scale-free network, are selected for test. The network scenarios are constructed by the rules below.

1) Network topology: Assume each network to be tested have 50 agents. We randomly generate 50 small-world network topologies with the mean degree being 8 and rewiring probability being 0.5, which are obtained from the Watts-Strogatz model [36]. Also, we randomly generate 50 scale-free network topologies by using the Barabási-Albert model [36]. The example small-world network and scale-free network adopted in the test are shown in Fig. 5 and Fig. 6, respectively.

2) Percentage of antagonistic interactions: For each of the generated network topologies, a certain percentage of interactions, say p_{ant}, are set to be antagonistic. We set p_{ant} ranges from 1% to 40% with step size 1%. The cases where $p_{ant} > 40\%$ are not the major concern in this paper. The system with a majority of antagonisms almost surely diverges, while the system behavior with a minority of antagonisms is of more interest.

3) Interaction weights: For each of those generated network topologies with a certain percentage of antagonistic interactions p_{ant}, we create 2000 profiles of interaction weights such that the Laplacian matrix $\boldsymbol{L}_{\mathcal{G}}$ in each profile has negative

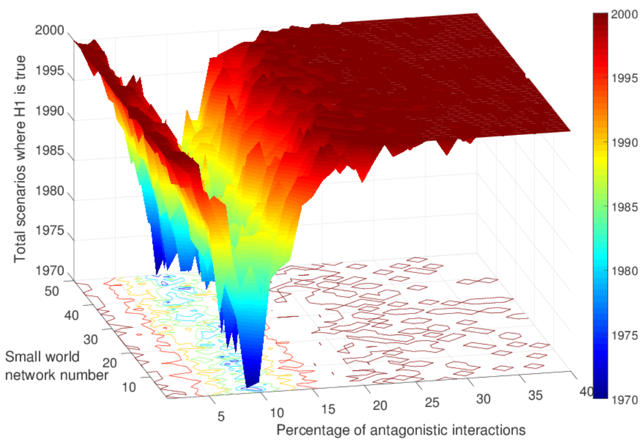

FIGURE 7. **Statistics of Hypothesis 1 in small-world networks.**

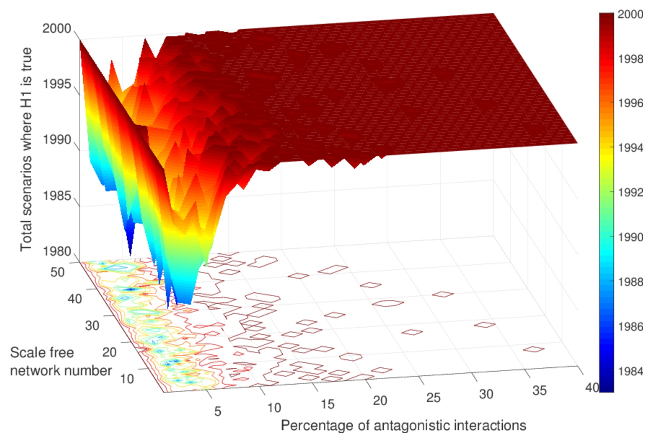

FIGURE 8. **Statistics of Hypothesis 1 in scale-free networks.**

eigenvalues or more than one zero eigenvalue (implying the failure of reaching a consensus by Lemma 1). This can be done by randomly choosing round($l \cdot p_{ant}$) interactions to be antagonistic and the remaining interactions to be cooperative, where l denotes the total number of interactions in the network and round() denotes the rounding function to the nearest integer. The weight of each cooperative and antagonistic interaction independently follows a uniform distribution over the interval [0,1] and [-1,0], respectively. If the Laplacian matrix $\boldsymbol{L}_{\mathcal{G}}$ of a profile is positive semi-definite and has only one zero eigenvalue, then this profile is discarded and replaced by a newly generated one. The random generator keeps working until we obtain 2000 desired profiles of interaction weights.

Hence, we obtain 2000 groups of scenarios for small-world networks and scale-free networks, respectively. Each group of scenarios have the same network topology and percentage of antagonistic interactions but different interaction weight profiles (2000 profiles per group) where the Laplacian matrices all have negative eigenvalues or more than one zero eigenvalue. For each group, we count the number of scenarios with the existence of nodes $i, j \in \mathcal{V}$ such that $c_{ij}^{\text{eff}}(\mathcal{G}) \leq 0$, which indicates Hypothesis 1 is true. The numerical accuracy is set to be 10^{-6}, i.e., we regard $c_{ij}^{\text{eff}}(\mathcal{G})$ (or $\lambda_i(\boldsymbol{L}_{\mathcal{G}})$) to be non-positive if $c_{ij}^{\text{eff}}(\mathcal{G}) < 10^{-6}$ (or $\lambda_i(\boldsymbol{L}_{\mathcal{G}}) < 10^{-6}$).

The statistical results are shown in Fig. 7 and Fig. 8 for small-world networks and scale-free networks, respectively. We observe that Hypothesis 1 fails in a very small number of scenarios for a low percentage of antagonistic interactions ($p_{ant} < 10\%$) in small-world or scale-free networks. In addition, Hypothesis 1 is almost always true for a higher percentage of antagonistic interactions ($p_{ant} > 10\%$). In total, there are 7815 and 2055 scenarios where Hypothesis 1 fails in the generated 4×10^6 small-world networks and scale-free networks, respectively. Accordingly, the empirical probabilities of Hypothesis 1 being true for small-world networks and scale-free networks are 99.80% and 99.95%, respectively, which are sufficiently high for practical use.

Next we further analyze the accuracy and confidence level of the obtained empirical probabilities. By the Chernoff-Hoeffding bound [37], we need $N \geq \frac{1}{2\epsilon^2} \lg \frac{2}{\delta}$ scenarios to achieve the following level of accuracy and confidence

$$\text{Prob}(|P_{actual} - P_{emp}| < \epsilon) > 1 - \delta \tag{10}$$

where ϵ is the desired accuracy level, δ is the desired confidence level, P_{actual} is the actual probability of Hypothesis 1 being true and P_{emp} is the empirical one. If we set $\epsilon = \delta = 0.10\%$ for (10), then $N \geq 3.8 \times 10^6$ scenarios are needed, which is satisfied by our tests. Therefore, we conclude the following statement:

With a 99.90% confidence level, there is at least 99.90% accuracy such that Hypothesis 1 is true with a probability of 99.80% and 99.95% for small-world networks and scale-free networks, respectively.

Remark 3: Combining Proposition 1 and Hypothesis 1 leads to a practical criterion for consensus, that is, the opinion consensus is almost equivalent to $c_{ij}^{\text{eff}}(\mathcal{G}) > 0$ for any agent pair $i, j \in \mathcal{V}$, which provides new insights into the mechanism of consensus. Reaching a consensus can be interpreted as every pair of agents presenting overall cooperativeness to each other despite of the presence of antagonistic interactions. In other words, the failure of consensus can be explained by some antagonistic interactions inducing non-cooperative relationship between some pairs of agents in the society.

V. CASE STUDY

We illustrate the application of EC to a 15-agent social network in Fig. 9, where the blue and red lines refer to the cooperative and antagonistic interactions, respectively. Note that this network is structurally unbalanced as it is still connected after deleting all red lines. It originates from an ever-existed society in Chinese academic history and the agents refer to the members of Academia Sinica in 1940s. The following network settings coincide with the qualitative features of these academicians' interpersonal relationships that are confirmed by textual research [38]. Let $\mathcal{V}_1 = \{\text{Hu, Zhao, Fu, Dong, Li, Xiao}\}$ and $\mathcal{V}_2$ be the set consisting of the remaining agents. For cooperative interactions,

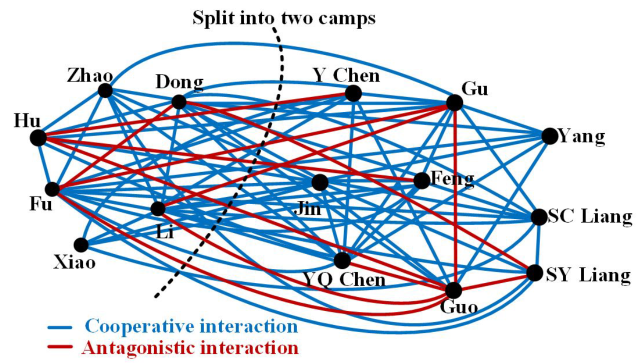

FIGURE 9. The 15-agent network.

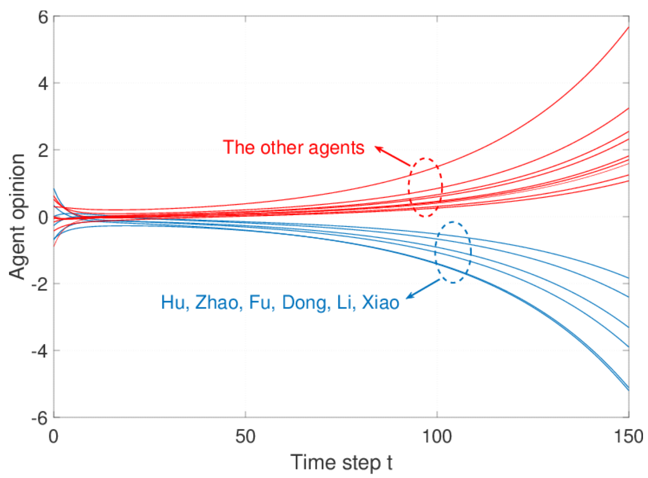

FIGURE 10. The opinion dynamics.

we set $w_{ij} = 0.25$ if $i, j \in \mathcal{V}_1$ or $i, j \in \mathcal{V}_2$, $w_{ij} = 0.02$ if $i \in \mathcal{V}_1$, $j \in \mathcal{V}_2$. For antagonistic interactions, we set $w_{ij} = -0.2$ if $i, j \in \mathcal{V}_1$, $w_{ij} = -0.02$ if $i, j \in \mathcal{V}_2$, and $w_{ij} = -0.1$ if $i \in \mathcal{V}_1$, $j \in \mathcal{V}_2$. In addition, we set $\kappa = 0.2$ as the step size.

Under these settings, the corresponding Laplacian matrix $\boldsymbol{L}_{\mathcal{G}}$ has a negative eigenvalue $\lambda_- = -0.133$. The opinion dynamics is plotted in Fig. 10, where the opinions diverge and present the bipartite feature. In the camp consisting of $\mathcal{V}_1$, the opinion values tend to increase, while the opinion values in the other camp consisting of $\mathcal{V}_2$ tend to decrease.

We now look into the case from the perspective of EC. We implement Algorithm 1 to check consensus. The associated calculations are given in Table 1. A negative EC occurs after iteratively adding two antagonistic interactions, which quickly tells that the opinion consensus will not be reached. In addition, we list the pairs of agents that have negative ECs in Table 2. These negative ECs all appear between the agents in different camps, which coincides with the fact that the two camps present opposite opinions.

We have another interesting observation that the ECs between some pairs of agents have the opposite signs to the weights of their direct interactions. For instance, the EC between Fu and Dong is positive (equals to 0.110) though their direct interaction is highly antagonistic (the corresponding $w_{ij} = -0.2$), indicating this antagonistic interaction does

TABLE 1. Iterative process to check consensus.

Iteration	Agent i	Agent j	Added interaction	$c_{ij}^{\text{eff}}(\mathcal{G}_+)$ after addition
1	Fu	Dong	-0.2	0.375
2	Hu	Feng	-0.1	-4.456

TABLE 2. Negative ECs between the agents.

Agent i	Agent j	$c_{ij}^{\text{eff}}(\mathcal{G})$	Agent i	Agent j	$c_{ij}^{\text{eff}}(\mathcal{G})$
Hu	Jin	-1.202	Fu	Y Chen	-1.306
Hu	Gu	-0.797	Fu	Guo	-0.388
Hu	Feng	-4.656	Fu	Yang	-3.855
Hu	Y Chen	-0.674	Dong	Guo	-5.428
Hu	YQ Chen	-1.547	Li	Jin	-4.797
Hu	Guo	-0.260	Li	Gu	-2.980
Hu	Yang	-0.836	Li	Y Chen	-1.293
Zhao	Y Chen	-4.031	Li	YQ Chen	-30.671
Zhao	Guo	-0.498	Li	Guo	-0.399
Fu	Jin	-6.479	Li	Yang	-2.790
Fu	Gu	-7.557			

not lead to opinion divergence. The positive EC is largely due to Fu and Dong having many indirect cooperative links via other agents. Also it provides a new viewpoint to explain why these two agents stand in the same camp despite of high antagonism in their direction interaction. In addition, Hu and Jin have cooperative direct interaction, but their indirect links via other agents induce the negative EC (equals to -1.202). This negative EC coincides with that Hu and Jin eventually stand in different camps, which is one of the causes for opinion divergence according to Proposition 1.

The above discussion concludes that the proposed EC is a more essential concept than the interaction weight in revealing the consensus mechanism.

VI. CONCLUDING REMARKS

We have investigated the consensus problem in social networks with antagonisms following the inspiration from electrical networks. A refined definition of EC has been proposed to measure the coupling strength between any pair of nodes in electrical networks. The concept of EC is then introduced to social networks to describe the overall relationship between a pair of agents, which applies to both structurally balanced and structurally unbalanced networks. Novel EC-based consensus criteria have been established, showing that the EC between any pair of agents being positive is the key to reaching a consensus. From these criteria, the consensus can interpreted as any pair of agents still presenting overall cooperativeness even with antagonistic interactions. Also an EC-based algorithm has been proposed that quickly checks consensus in an iterative manner. The obtained results have twofold merits over the existing ones in the literature. First, the proposed EC works more properly than the traditional one in those cases with negative line conductances. Second, the existing EC-based consensus criteria work under some special cases, e.g., when there is only one antagonistic interaction in the network or there is no cycle containing two antagonistic interactions [25]. By comparison, it has been shown by analytical and statistical approaches that the proposed consensus criteria have more general applicability, which leads to a more comprehensive and intuitive understanding of the consensus mechanism.

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