ARTICLE TEMPLATE

# Designing Parametric Linear Quadratic Regulators for Parametric LTI Systems via LMIs

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**ABSTRACT**
This paper addresses the problem of determining parametric linear quadratic regulators (LQRs) for continuous-time linear-time invariant (LTI) systems affected by parameters through rational functions. Three situations are considered, where the sought controller has to minimize the best cost, average cost, and worst cost, respectively, over the set of admissible parameters. It is shown that candidates for such controllers can be obtained by solving convex optimization problems with linear matrix inequality (LMI) constraints. These candidates are guaranteed to approximate arbitrarily well the sought controllers by sufficiently increasing the size of the LMIs. In particular, the candidate that minimizes the average cost approximates arbitrarily well the true LQR over the set of admissible parameters. Moreover, conditions for establishing the optimality of the found candidates are provided. Some numerical examples illustrate the proposed methodology.

**KEYWORDS**
Parametric system; Linear quadratic regulator; Linear matrix inequality.

## 1. Introduction

It is well-known that linear quadratic regulators (LQRs) play a key role in control systems. Specifically, these regulators are state-feedback linear controllers that minimize the sum of the weighted energies of the state and of the input. Given a linear-time invariant (LTI) system, the LQR can be found by solving the algebraic Riccati equation (ARE), which is a quadratic matrix equation. See for instance [12, 14, 3, 13].

LQRs have been studied in numerous contexts. For instance, [10, 1] have studied the case of singular control problems. Also, [21] has provided a parametrization of the solution set of the ARE and the algebraic Riccati inequality. Then, [17, 25] have addressed LQRs for time-varying systems. In [19], the design of LQRs has been studied for nonlinear systems. Recently, [5] has considered LQR with random input gains.

In the context of uncertain systems, the LQR has been studied from different viewpoints. In particular, in [15], it has been considered for pulse width modulation (PWM) converters modeled by a polytopic family. Also, in [24, 23], it has been addressed using

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reinforcement learning. More recently, in [4], it has been investigated with a modified Riccati equation.

A problem that remains open is finding a parametric LQR for systems affected by parameters. This problem is important because real systems often present parameters (e.g., for tuning purpose), and the LQR depends on these parameters. But how? What is the expression of the LQR as a function of the parameters?

Clearly, one may think to obtain the parametric LQR by solving the parametric ARE. Unfortunately, the parametric ARE cannot be solved because there does not exist an analytic solution of the ARE, and numerical methods cannot be used if the matrices of the systems are parameterized. Of course, a possibility would be to solve the ARE through numerical methods for frozen parameters for all admissible values of the parameters and store somewhere the solutions obtained, however, this is impossible because the number of admissible values is not finite. Also, another possibility would be to solve online (i.e., during the control of the process) the ARE, however, this is generally impossible because would require hardware and software for solving the ARE online, which are generally not allowed in real systems.

This paper addresses the problem of finding parametric LQRs for continuous-time LTI systems affected by parameters through rational functions. Three situations are considered, where the sought controller has to minimize the best cost, average cost, and worst cost, respectively, over the set of admissible parameters. It is shown that candidates for such controllers can be obtained by solving convex optimization problems with linear matrix inequality (LMI) constraints. These candidates are guaranteed to approximate arbitrarily well the sought controllers by sufficiently increasing the size of the LMIs. In particular, the candidate that minimizes the average cost approximates arbitrarily well the true LQR over the set of admissible parameters. Moreover, conditions for establishing the optimality of the found candidates are provided. Some numerical examples illustrate the proposed methodology.

The paper is organized as follows. Section 2 introduces the notation and problem formulation. Section 3 describes the proposed approach. Section 4 presents the examples. Lastly, Section 5 concludes the paper with some final remarks. A preliminary conference version of this paper (where only the worst cost is considered, and where the system is only a polynomial function of the parameters) appeared as reported in [26].

## 2. Preliminaries

The notation is as follows. The sets of nonnegative integers and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$. The symbols 0 and $I$ denote the null matrix and the identity matrix of size specified by the context. The symbol $\otimes$ denotes the Kronecker's product. The notation $A'$ denotes the transpose of $A$, and he(A) denotes $A + A'$. The notation $A \geq 0$ (respectively, $A > 0$) denotes that a symmetric matrix $A$ is positive semidefinite (respectively, definite). For a real number $x$, $\lceil x \rceil$ denotes the smallest integer not smaller than $x$. The notation $\|x\|_p$ denotes the $p$-norm of a vector $x$. For vectors $x$ and $y$ of same dimension, the notation $x^y$ denotes the quantity $x_1^{y_1} x_2^{y_2} \cdots$. Unless specified otherwise, $x_i$ denotes the $i$-th entry of $x$. The symbol $\star$ denotes a corresponding block in a symmetric matrix. The notation "s.t." stands for "subject to".

Let us consider the parametric LTI system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(p)x(t) + B(p)u(t) \\ x(0) & = & x_0 \\ p & \in & \mathcal{P} \end{array} \right. \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $x_0 \in \mathbb{R}^n$ is the initial condition, $p \in \mathbb{R}^q$ is the time-invariant parameter vector, $A(p)$ and $B(p)$ are given rational matrix functions, and $\mathcal{P}$ is the set of admissible parameter vectors given by

$$\mathcal{P} = \{p \in \mathbb{R}^q : \ \|p\|_\infty \leq 1\}. \tag{2}$$

It is supposed that $A(p)$ and $B(p)$ are continuous over $\mathcal{P}$. The LQR problem for (1) consists of solving

$$J^\#(p) = \inf_{u(t)} \int_0^\infty \left(x'(t)Qx(t) + u'(t)Ru(t)\right) dt \tag{3}$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices. The control input that achieves $J^\#(p)$ is given by

$$u(t) = K^\#(p)x(t) \tag{4}$$

where $K^\#(p) \in \mathbb{R}^{m\times n}$ is given by

$$K^\#(p) = -R^{-1}B(p)'V^\#(p) \tag{5}$$

and $V^\#(p) \in \mathbb{R}^{n\times n}$ is the solution of the ARE

$$0 = Q + A(p)'V^\#(p) + V^\#(p)A(p) - V^\#(p)B(p)R^{-1}B(p)'V^\#(p). \tag{6}$$

In order to estimate $J^\#(p)$ and $K^\#(p)$, we consider that the system (1) is controlled in closed-loop via the state-feedback control law

$$u(t) = K(p)x(t) \tag{7}$$

where $K(p) \in \mathbb{R}^{m\times n}$ is a controller to determine in order to minimize the cost

$$\begin{array}{rcl} J(K(p), p) & = & \displaystyle\int_0^\infty \left(x'(t)Qx(t) + u'(t)Ru(t)\right) dt \\ & = & \displaystyle\int_0^\infty x'(t)\left(Q + K(p)'RK(p)\right) x(t)dt. \end{array} \tag{8}$$

The problems addressed in this paper are as follows.

**Problem 1.** Find $K(p)$ that minimizes the best cost $J(K(p), p)$ over $\mathcal{P}$, i.e., a minimizer of

$$J_{bst} = \inf_{K(p)} \inf_{p \in \mathcal{P}} J(K(p), p) \tag{9}$$

**Problem 2.** Find $K(p)$ that minimizes the average cost $J(K(p), p)$ over $\mathcal{P}$, i.e., a minimizer of

$$J_{avg} = \inf_{K(p)} \int_{p \in \mathcal{P}} J(K(p), p) dp. \tag{10}$$

**Problem 3.** Find $K(p)$ that minimizes the worst cost $J(K(p), p)$ over $\mathcal{P}$, i.e., a minimizer of

$$J_{wst} = \inf_{K(p)} \sup_{p \in \mathcal{P}} J(K(p), p). \tag{11}$$

Let us observe that the minimizer of Problem 2 is the parametric optimal controller $K^{\#}(p)$ in (5) independently on the initial condition of (1). Indeed, for any fixed $p$, $K^{\#}(p)$ is the controller that achieves the minimum cost and, hence, $K^{\#}(p)$ is also the controller that achieves the minimum average cost over $\mathcal{P}$. This means that, by solving Problem 2, one obtains the parametric optimal controller $K^{\#}(p)$. See also Section 4 where this property is shown graphically for some numerical examples.

Also, let us observe that the minimizers of Problems 1 and 3 are controllers that coincide with the parametric optimal controller $K^{\#}(p)$ for values of the parameters that determine the optimum in these problems. These values depend on the initial condition of (1) for definition of the cost $J^{\#}(p)$ in (3).

## 3. Proposed Approach

This section presents the proposed approach. Specifically, Section 3.1 describes the ideas and solutions for addressing Problem 1, while Sections 3.2–3.3 present their extensions for addressing Problems 2–3. The dependence on $t$ of the various quantities will be omitted in the sequel for ease of notation unless specified otherwise.

### 3.1. *Solution For Problem 1*

In order to solve the problems introduced in the previous section, we start by introducing a quadratic Lyapunov function candidate depending polynomially on the parameter vector $p$, specifically of the form

$$v(x, p) = x'V(p)x \tag{12}$$

where $V(p) \in \mathbb{R}^{n \times n}$ is a symmetric matrix polynomial to determine of degree not greater than $d_V$. It is worth mentioning that polynomially-dependent quadratic Lyapunov functions have been exploited for addressing various problems in control systems, in particular establishing robust stability of uncertain systems, see for instance [2, 9, 20, 16]. Let us express $A(p)$ and $B(p)$ as

$$\left\{\begin{array}{rcl} A(p) & = & \dfrac{\bar{A}(p)}{z_A(p)} \\ B(p) & = & \dfrac{\bar{B}(p)}{z_B(p)} \end{array}\right. \tag{13}$$

where $\bar{A}(p)$ and $\bar{B}(p)$ are matrix polynomials, and $z_A(p)$ and $z_B(p)$ are polynomials. Since $A(p)$ and $B(p)$ are continuous over $\mathcal{P}$, it is reasonable to suppose that $z_A(p)$ and $z_B(p)$ can be chosen positive over $\mathcal{P}$. Let $z(p)$ be any polynomial such that

$$\left\{\begin{array}{rcl} z(p) & = & z_A(p)z_1(p) \\ z(p) & = & z_B(p)z_2(p) \end{array}\right. \tag{14}$$

for some polynomials $z_1(p)$ and $z_2(p)$ positive over $\mathcal{P}$. For instance, one can simply choose $z(p) = z_A(p)z_B(p)$, $z_1(p) = z_B(p)$ and $z_2(p) = z_A(p)$. Let $\gamma \in \mathbb{R}$ be a scalar to determine, and let us define the symmetric matrix polynomial $G_i(p) \in \mathbb{R}^{l_i \times l_i}$, $i = 1, 2, 3$, given by

$$\left\{\begin{array}{rcl} G_1(p) & = & \left(\begin{array}{cc} z_1(p)\mathrm{he(V(p)\bar{A}(p)) + z(p)Q} & z_2(p)V(p)\bar{B}(p) \\ \star & z(p)R \end{array}\right) \\ G_2(p) & = & V(p) \\ G_3(p) & = & x_0'V(p)x_0 - \gamma \end{array}\right. \tag{15}$$

where

$$\left\{\begin{array}{rcl} l_1 & = & n+m \\ l_2 & = & n \\ l_3 & = & 1, \end{array}\right. \tag{16}$$

Let us define

$$\left\{\begin{array}{rcl} \delta_i & = & \left\lceil \dfrac{\deg(G_i(p))}{2} \right\rceil \\ d_i & = & \delta_i - 1 \end{array}\right. \quad \forall i = 1, 2, 3. \tag{17}$$

Let $H_{i,j}(p) \in \mathbb{R}^{l_i \times l_i}$, $i = 1, 2, 3$ and $j = 1, \ldots, q$, be symmetric matrix polynomials to determine of degree not greater than $2d_i$ (if $d_i < 0$, then we set $H_{i,j}(p) = 0$). Let us define the symmetric matrix polynomials $S_i(p) \in \mathbb{R}^{l_i \times l_i}$ given by

$$S_i(p) = G_i(p) - \sum_{j=1}^{q} (1 - p_j^2) H_{i,j}(p). \tag{18}$$

Let us observe that

$$\deg(S_i(p)) \leq 2\delta_i. \tag{19}$$

Hence, we can express $S_i(p)$ as

$$S_i(p) = (b(p, \delta_i) \otimes I)' \, \bar{S}_i \, (b(p, \delta_i) \otimes I) \tag{20}$$

where $b(p, \delta_i) \in \mathbb{R}^{c(q,\delta_i)}$ is a vector whose entries are monomials in $p$ of degree not greater than $\delta_i$, the quantity $c(q, \delta_i)$ is the number of these monomials given by

$$c(q, \delta_i) = \frac{(q + \delta_i)!}{q!\delta_i!}, \tag{21}$$

and $\bar{S}_i \in \mathbb{R}^{c(q,\delta_i)l_i \times c(q,\delta_i)l_i}$ is a symmetric matrix to determine. Similarly, let us express $H_{i,j}(p)$ as

$$H_{i,j}(p) = (b(p, d_i) \otimes I)' \, \bar{H}_{i,j} \, (b(p, d_i) \otimes I) \tag{22}$$

where $\bar{H}_{i,j} \in \mathbb{R}^{c(q,d_i)l_i \times c(q,d_i)l_i}$ is a symmetric matrix to determine. Lastly, let us express $V(p)$ as

$$V(p) = \bar{V} \, (b(p, d_V) \otimes I) \tag{23}$$

where $\bar{V} \in \mathbb{R}^{n \times c(q,d_V)n}$ is a symmetric matrix to determine under the constraint

$$V(p) = V(p)'. \tag{24}$$

Let us define

$$f = -\gamma \tag{25}$$

and the semidefinite program (SDP)

$$\begin{array}{ll} \inf_{\gamma, \bar{V}, \bar{S}_i, \bar{H}_{i,j}} & f \\ \text{s.t.} & \left\{ \begin{array}{l} (15)\text{–}(24) \text{ hold} \\ \bar{S}_i \geq 0 \\ \bar{H}_{i,j} \geq 0 \\ \forall i = 1, 2, 3 \ \forall j = 1, \ldots, q \end{array} \right. \end{array} \tag{26}$$

(if $d_i < 0$, then we set $\bar{H}_{i,j} = 0$ in the SDP (26) since $H_{i,j}(p) = 0$). Let $\gamma^*$ and $\bar{V}^*$ be the optimal values of $\gamma$ and $\bar{V}$ in the SDP (26), and let $V^*(p)$ be $V(p)$ evaluated for $\bar{V} = \bar{V}^*$. Let us define the controller

$$K^*(p) = -R^{-1} B(p)' V^*(p). \tag{27}$$

**Theorem 3.1.** *For all $d_V$ one has*

$$\gamma^* \leq J_{bst} \leq \left( \inf_{p \in \mathcal{P}} J(K^*(p), p) \right). \tag{28}$$

*Moreover, for all $\varepsilon > 0$ there exists $d_V^*$ such that, for all $d_V \geq d_V^*$,*

$$\begin{cases} J_{bst} - \varepsilon \leq \gamma^* \leq J_{bst} \\ J_{bst} \leq \left( \inf_{p \in \mathcal{P}} J(K^*(p), p) \right) \leq J_{bst} + \varepsilon. \end{cases} \tag{29}$$

*Proof.* Let us start by considering the case where $J_{bst} < \infty$. First, let us prove (28). Let $\bar{S}_i^*$ and $\bar{H}_{i,j}^*$ be the optimal values of $\bar{S}_i$ and $\bar{H}_{i,j}$ in the SDP (26). From the LMIs in (26) one has $\bar{S}_i^* \geq 0$ and $\bar{H}_{i,j}^* \geq 0$. Let $S_i^*(p)$ and $H_{i,j}^*(p)$ be $S_i(p)$ and $H_{i,j}(p)$ evaluated for $\bar{S}_i = \bar{S}_i^*$ and $\bar{H}_{i,j} = \bar{H}_{i,j}^*$. From (20) and (22) it follows that $S_i^*(p) \geq 0$ and $H_{i,j}^*(p) \geq 0$ for all $p \in \mathbb{R}^q$. Since (20) holds, it follows that

$$S_i^*(p) = G_i^*(p) - \sum_{j=1}^{q} (1 - p_j^2) H_{i,j}^*(p)$$

where $G_i^*(p)$ is $G_i(p)$ evaluated for $\gamma = \gamma^*$ and $V(p) = V(p)^*$. Let $\bar{p} \in \mathcal{P}$. It follows

$$\begin{array}{rcl} 0 & \leq & S_i^*(\bar{p}) \\ & = & G_i^*(\bar{p}) - \displaystyle\sum_{j=1}^{q} (1 - \bar{p}_j^2) H_{i,j}^*(\bar{p}) \\ & \leq & G_i^*(\bar{p}) \end{array}$$

since $1 - \bar{p}_j^2 \geq 0$ and $H_{i,j}^*(\bar{p}) \geq 0$. Hence, $G_i^*(p) \geq 0$ for all $p \in \mathcal{P}$. This implies

$$\gamma^* \leq \inf_{p \in \mathcal{P}} x_0' V^*(p) x_0.$$

Let us observe that $G_1^*(p) \geq 0$ implies

$$0 \leq Q + \mathrm{he(V^*(p)A(p)) - V^*(p)B(p)R^{-1}B(p)'V^*(p)}.$$

From [3] one has

$$x_0' V^*(p) x_0 \leq J^{\#}(p)$$

and, hence,

$$\inf_{p \in \mathcal{P}} x_0' V^*(p) x_0 \leq J_{bst}.$$

Consequently,

$$\gamma^* \leq J_{bst}.$$

Also, let us observe from (9) that

$$J_{bst} \leq \left( \inf_{p \in \mathcal{P}} J(K^*(p), p) \right)$$

since $K(p)$ in (9) can be any matrix function. Therefore, (28) follows. Finally, let us observe that the solution of (26) does exist because its feasible set of (26) is closed and the objective is bounded from below since

$$f = -\gamma \geq -J_{bst}$$

and $J_{bst}$ is finite. Second, let us prove (29). Let $\varepsilon > 0$, and let $\gamma$ and $V(p)$ be such that

$$\begin{cases} G_i(p) & > & 0 \quad \forall p \in \mathcal{P} \\ J_{bst} - \varepsilon & \leq & \gamma \\ \left( \inf_{p \in \mathcal{P}} J(K(p), p) \right) & \leq & J_{bst} + \varepsilon \end{cases}$$

with $K(p) = -R^{-1}B(p)'V(p)$. Such $\gamma$ and $V(p)$ do exist, for instance one can choose $\gamma = J_{bst} - \varepsilon_1$ and $V(p) = (1-\varepsilon_2)V^{\#}(p)$ where $V^{\#}(p)$ is the solution of the parametric ARE (6), $\varepsilon_2$ is a sufficiently small positive scalar, and $\varepsilon_1 \in (0, \max\{\varepsilon, \varepsilon_2 x_0' x_0\})$. Since $\mathcal{P}$ is compact, $V(p)$ can be approximated arbitrarily well by a matrix polynomial over $\mathcal{P}$. In particular, there exists a matrix polynomial $V(p)$ such that $G_i(p) > 0$ over $\mathcal{P}$. Since $\mathcal{P}$ is compact and the polynomials $1-p_1^2, \ldots, 1-p_q^2$ have even degree and their highest degree forms are simultaneously zero only for $p = 0$, it follows from [18] that there exist $H_{i,j}(p)$ such that $S_i(p)$ and $H_{i,j}(p)$ are sums of squares of matrix polynomials. Hence, there exist $\bar{S}_i \geq 0$ and $\bar{H}_{i,j} \geq 0$ such that (20) and (22) hold, see for instance [6] and references therein. Therefore, (29) holds. Moreover, $d_V^*$ is given by the smallest integer $d_V$ such that $d_V \geq \deg(V(p))$ and $2d_i \geq \deg(H_{i,j}(p))$. Lastly, let us observe that, for all $d_V \geq d_V^*$, (29) still holds because the matrix polynomials $V(p)$ and $H_{i,j}(p)$ obtained for $d_V^*$ can be obtained for larger values of $d_V^*$ by introducing higher degree monomials with null coefficients. Lastly, let us consider the case where $J_{bst} \nless \infty$. It follows that (28) holds independently on $\gamma^*$. Moreover, (29) can be proved as done in the previous case by observing that, for all $\gamma > 0$, there exists $V(p)$ such that $G_i(p) \geq 0$. $\square$

Theorem 3.1 states that the SDP (26) provides a lower bound $\gamma^*$ of $J_{bst}$ for any chosen $d_V$. Moreover, for all $d_V$ sufficiently large, the lower bound $\gamma^*$ and the controller $K^*(p)$ approximate arbitrarily well the solution of Problem 1.

At this point, a question arises: is a found lower bound $\gamma^*$ tight, i.e., $\gamma^* = J_{bst}$? In order to provide an answer to this question, let us define the set

$$\mathcal{Z} = \{p \in \mathbb{R}^q : \ \det(S_i^*(p)) = 0 \ \ \forall i = 1, 3\}. \tag{30}$$

**Theorem 3.2.** *Let us suppose $J_{bst} < \infty$. The upper bound $\gamma^*$ satisfies*

$$J_{bst} = \gamma^* \tag{31}$$

*if and only if there exists $p^* \in \mathbb{R}^q$ such that*

$$\begin{cases} p^* & \in & \mathcal{Z} \cap \mathcal{P} \\ \det(G_i^*(p^*)) & = & 0 \quad \forall i = 1, 3. \end{cases} \tag{32}$$

*Moreover, for such $p^*$,*

$$J_{bst} = J(K^*(p^*), p^*). \tag{33}$$

*Proof.* (Sufficiency) Suppose that there exists $p^* \in \mathbb{R}^q$ such that (32). Since $\det(G_1^*(p^*)) = 0$ and $\det(G_3^*(p^*)) = 0$, it follows that $G_1^*(p^*)$ and $G_3^*(p^*)$ are singular, and this implies that $V^*(p^*)$ is the maximizer of

$$\begin{array}{l} \sup\limits_{W} x_0' W x_0 \\ \text{s.t.} \left\{ \begin{array}{lcl} 0 & < & W \\ 0 & < & Q + \mathrm{he(WA(p^*)) - WB(p^*)R^{-1}B(p^*)'W.} \end{array} \right. \end{array}$$

From [3] this means that

$$\gamma^* = x_0' V^*(p^*) x_0 = J(K^*(p^*), p^*) = J^{\#}(p^*).$$

From (9) it follows that

$$J_{bst} \leq \gamma^*$$

and from Theorem 3.1 we also have

$$\gamma^* \leq J_{bst}.$$

(Necessity) Suppose that $J_{bst} = \gamma^*$. Let $p^*$ be the minimizer of $p$ in (9). We have

$$\left\{ \begin{array}{rcl} p^* & \in & \mathcal{P} \\ J^{\#}(p^*) & = & J_{bst}. \end{array} \right.$$

It follows that $G_1^*(p^*)$ and $G_3^*(p^*)$ are singular because the opposite would imply that $\gamma^* \neq J_{bst}$. Let $w \in \mathbb{R}^{l_1}$, $w \neq 0$, be such that $G_1^*(p^*)w = 0$. As proved in the proof of Theorem 3.1, one has $S_i^*(p) \geq 0$ and $H_{i,j}^*(p) \geq 0$ for all $p \in \mathbb{R}^q$. This implies that

$$\begin{array}{rcl} 0 & \leq & w' S_1^*(p^*) w \\ & = & w' \left( G_1^*(p^*) - \sum\limits_{j=1}^{q} (1 - (p_j^*)^2) H_{1,j}^*(p^*) \right) w \\ & \leq & w' G_1^*(p^*) w \\ & = & 0. \end{array}$$

Hence, $S_1^*(p^*)$ is singular, i.e., $p^* \in \mathcal{Z}$. Therefore, (32) holds. Let us observe that

$$\begin{array}{rcl} 0 & = & G_3^*(p^*) \\ & = & x_0' V^*(p^*) x_0 - \gamma^*. \end{array}$$

Since $G_1^*(p^*)$ is singular, it also follows that

$$x_0' V^*(p) x_0 = J(K^*(p^*), p^*).$$

Hence, (33) holds. $\square$

Theorem 3.2 provides a sufficient and necessary condition for establishing whether the found lower bound $\gamma^*$ and the found controller $K^*(p)$ solve Problem 1. In order to check this condition, one can firstly determine the set $\mathcal{Z}$ in (30). This can be done by observing that $\det(S_i^*(p)) = 0$ if and only if there exists $p$ such that

$$(b(p, \delta_i) \otimes I) \in \ker\left(\bar{S}_i^*\right) \tag{34}$$

where $\bar{S}_i^*$ is the optimal value of $\bar{S}_i$ in the SDP (26). As explained in [8, 7], determining the vectors $p$ that solve (34) can be done through linear algebra operations. Once that $\mathcal{Z}$ has been determined, one checks whether the vectors in this set satisfy (32).

### 3.2. *Solution For Problem 2*

Let us redefine $G_3(p)$ in (15) as

$$G_3(p) = \left(\int_P x_0' V(p) x_0 dp\right) - \gamma. \tag{35}$$

**Theorem 3.3.** *Let us replace $G_3(p)$ in (15) with (35). For all $d_V$ one has*

$$\gamma^* \le J_{avg} \le \left(\int_{\mathcal{P}} J(K^*(p), p) dp\right). \tag{36}$$

*Moreover, for all $\varepsilon > 0$ there exists $d_V^*$ such that, for all $d_V \ge d_V^*$,*

$$\begin{cases} J_{avg} - \varepsilon \le \gamma^* \le J_{avg} \\ J_{avg} \le \left(\int_{\mathcal{P}} J(K^*(p), p) dp\right) \le J_{avg} + \varepsilon. \end{cases} \tag{37}$$

*Proof.* Let us start by considering the case where $J_{avg} < \infty$. First, let us prove (36). As in the proof of Theorem 3.1, one gets $G_i^*(p) \ge 0$ for all $p \in \mathcal{P}$. This implies

$$\gamma^* \le \int_{p \in \mathcal{P}} x_0' V^*(p) x_0 dp.$$

From [3] one has

$$\int_{p \in \mathcal{P}} x_0' V^*(p) x_0 dp \le J_{avg}$$

and, hence,

$$\gamma^* \le J_{avg}.$$

Also, let us observe from (10) that

$$J_{avg} \leq \left( \int_{p \in \mathcal{P}} J(K^*(p), p) dp \right)$$

since $K(p)$ in (9) can be any matrix function. Hence, (36) follows. Finally, let us observe that the solution of (26) does exist because its feasible set of (26) is closed and the objective is bounded from below since

$$f = -\gamma \geq -J_{avg}$$

and $J_{avg}$ is finite. Second, let us prove (37). Let $\varepsilon > 0$, and let $\gamma$ and $V(p)$ be such that

$$\left\{ \begin{array}{rcl} G_i(p) & > & 0 \;\; \forall p \in \mathcal{P} \\ J_{avg} - \varepsilon & \leq & \gamma \\ \left( \int_{p \in \mathcal{P}} J(K(p), p) dp \right) & \leq & J_{avg} + \varepsilon \end{array} \right.$$

with $K(p) = -R^{-1} B(p)' V(p)$. As in the proof of Theorem 3.1, it follows that (37) holds for all $d_V \geq d_V^*$ where $d_V^*$ is a sufficiently large integer. Lastly, the case where $J_{avg} \not< \infty$ can be addressed as done in the proof of Theorem 3.1. □

Theorem 3.3 explains how one can modify the SDP (26) in order for $\gamma^*$ to be a lower bound of $J_{avg}$ for any chosen $d_V$. Moreover, for all $d_V$ sufficiently large, the lower bound $\gamma^*$ and the controller $K^*(p)$ approximate arbitrarily well the solution of Problem 2.

As said at the end of Section 2, the minimizer of Problem 2 is the parametric optimal controller $K^{\#}(p)$ in (5) independently on the initial condition of (1). This means that Theorem 3.3 allows one to approximate arbitrarily well the parametric optimal controller $K^{\#}(p)$ with $K^*(p)$, and, similarly, the solution of the parametric ARE $V^{\#}(p)$ with $V^*(p)$. See also Section 4 where this property is shown graphically for some numerical examples.

### 3.3. ***Solution For Problem 3***

Let $\bar{z}(p)$ be any polynomial such that

$$\left\{ \begin{array}{rcl} \bar{z}(p) & = & z_A(p) z_3(p) \\ \bar{z}(p) & = & z_B(p)^2 z_4(p) \end{array} \right. \tag{38}$$

for some polynomials $z_3(p), z_4(p)$. Let us redefine $G_i(p)$ in (15) as

$$\begin{cases} G_1(p) &=& \begin{pmatrix} g_1(p) & -\bar{z}(p)V(p) \\ \star & \bar{z}(p)Q^{-1} \end{pmatrix} \\ G_2(p) &=& V(p) \\ G_3(p) &=& \begin{pmatrix} \gamma & x_0' \\ \star & V(p) \end{pmatrix} \\ g_1(p) &=& z_4(p)\bar{B}(p)R^{-1}\bar{B}(p)' - z_3(p)\mathrm{he}(\bar{\mathrm{A}}(\mathrm{p})\mathrm{V}(\mathrm{p})), \end{cases} \tag{39}$$

$l_i$ in (16) as

$$\begin{cases} l_1 &=& 2n \\ l_2 &=& n \\ l_3 &=& n+1, \end{cases} \tag{40}$$

and $f$ in (25) as

$$f = \gamma. \tag{41}$$

Also, let us redefine the controller $K^*(p)$ in (27) as

$$K^*(p) = -R^{-1}B(p)'V^*(p)^{-1}. \tag{42}$$

**Theorem 3.4.** *Let us replace* (15), (16), (25) *and* (27) *with* (39), (40), (41) *and* (42). *For all $d_V$ one has*

$$J_{wst} \leq \left( \sup_{p\in\mathcal{P}} J(K^*(p), p) \right) \leq \gamma^*. \tag{43}$$

*Moreover, for all $\varepsilon > 0$ there exists $d_V^*$ such that, for all $d_V \geq d_V^*$,*

$$J_{wst} \leq \left( \sup_{p\in\mathcal{P}} J(K^*(p), p) \right) \leq \gamma^* \leq J_{wst} + \varepsilon. \tag{44}$$

*Proof.* Let us start by considering the case where $J_{wst} < \infty$. First, let us prove (43). As in the proof of Theorem 3.1, one gets $G_i^*(p) \geq 0$ for all $p \in \mathcal{P}$. This implies

$$\gamma^* \geq \sup_{p\in\mathcal{P}} x_0' V^*(p)^{-1} x_0.$$

Let us observe that $G_1^*(p) \geq 0$ implies

$$0 \leq B(p)R^{-1}B(p)' - \mathrm{he}(\mathrm{V}^*(\mathrm{p})\mathrm{A}(\mathrm{p})') - \mathrm{V}^*(\mathrm{p})\mathrm{Q}\mathrm{V}^*(\mathrm{p}).$$

From [3] one has

$$x_0' V^*(p)^{-1} x_0 \geq J^\#(p)$$

and, hence,

$$\sup_{p\in\mathcal{P}} x_0' V^*(p)^{-1} x_0 \geq J_{wst}.$$

Consequently,

$$\gamma^* \geq J_{wst}.$$

Also, let us observe from (11) that

$$J_{wst} \leq \left( \sup_{p\in\mathcal{P}} J(K^*(p), p) \right)$$

since $K(p)$ in (11) can be any matrix function. Moreover,

$$x_0' V^*(p)^{-1} x_0 \geq J(K^*(p), p).$$

Therefore, (43) follows. Finally, let us observe that the solution of (26) does exist because its feasible set of (26) is closed and the objective is bounded from below since

$$f = \gamma \geq J_{wst}$$

and $J_{wst}$ is finite. Second, let us prove (44). Let $\varepsilon > 0$, and let $\gamma$ and $V(p)$ be such that

$$\begin{cases} G_i(p) > 0 \quad \forall p \in \mathcal{P} \\ J_{wst} \leq \left( \sup_{p\in\mathcal{P}} J(K(p), p) \right) \leq \gamma \leq J_{wst} + \varepsilon \end{cases}$$

with $K(p) = -R^{-1}B(p)'V(p)^{-1}$. Such $\gamma$ and $V(p)$ do exist, for instance one can choose $\gamma = J_{wst} + \varepsilon_1$ and $V(p) = (1-\varepsilon_2)V^{\#}(p)$ where $V^{\#}(p)$ is the solution of the parametric ARE (6), $\varepsilon_2$ is a sufficiently small positive scalar, and $\varepsilon_1 \in (J_{wst}\varepsilon_2/(1-\varepsilon_2), \varepsilon)$. As in the proof of Theorem 3.1, it follows that (44) holds for all $d_V \geq d_V^*$ where $d_V^*$ is a sufficiently large integer. Lastly, let us consider the case where $J_{wst} \not< \infty$. From the previous case it directly follows that the feasible set of (26) is empty. Hence, (43)–(44) hold. □

Theorem 3.4 explains how one can modify the SDP (26) in order for $\gamma^*$ to be an upper bound of $J_{wst}$ for any chosen $d_V$. Moreover, for all $d_V$ sufficiently large, the upper bound $\gamma^*$ and the controller $K^*(p)$ approximate arbitrarily well the solution of Problem 3.

**Theorem 3.5.** *Let us suppose $J_{wst} < \infty$. Let us replace* (15), (16) *and* (25) *with* (39), (40) *and* (41). *The lower bound $\gamma^*$ satisfies*

$$J_{wst} = \gamma^* \tag{45}$$

*if and only if there exists $p^* \in \mathbb{R}^q$ such that* (32) *holds. Moreover, for such $p^*$,*

$$J_{wst} = J(K^*(p^*), p^*). \tag{46}$$

*Proof.* (Sufficiency) Suppose that there exists $p^* \in \mathbb{R}^q$ such that (32). It follows that $G_1^*(p^*)$ and $G_3^*(p^*)$ are singular, and this implies that $V^*(p^*)$ is the minimizer of

$$\begin{array}{l} \inf_{W} x_0' W^{-1} x_0 \\ \text{s.t.} \left\{ \begin{array}{lcl} 0 & < & W \\ 0 & < & B(p^*)R^{-1}B(p^*)' - \mathrm{he}(\mathrm{WA(p^*)'}) - \mathrm{WQW}. \end{array} \right. \end{array}$$

From [3] this means that

$$\gamma^* = x_0' V^*(p^*)^{-1} x_0 = J(K^*(p^*), p^*) = J^{\#}(p^*).$$

From (11) it follows that

$$J_{wst} \geq \gamma^*$$

and from Theorem 3.1 we also have

$$\gamma^* \geq J_{wst}.$$

(Necessity) Suppose that $J_{wst} = \gamma^*$. Let $p^*$ be the maximizer of $p$ in (11). We have

$$\left\{ \begin{array}{rcl} p^* & \in & \mathcal{P} \\ J^{\#}(p^*) & = & J_{wst}. \end{array} \right.$$

It follows that $G_1^*(p^*)$ and $G_3^*(p^*)$ are singular because the opposite would imply that $\gamma^* \neq J_{wst}$. As proved in the proof of Theorem 3.2, one has that $S_1^*(p^*)$ is singular, i.e., $p^* \in \mathcal{Z}$. Therefore, (32) holds. Let us observe that the singularity of $G_3^*(p^*)$ implies

$$\gamma^* = x_0' V^*(p^*)^{-1} x_0.$$

Moreover, the singularity of $G_1^*(p^*)$ implies

$$x_0' V^*(p)^{-1} x_0 = J(K^*(p^*), p^*).$$

Hence, (46) holds. □

Theorem 3.5 provides a sufficient and necessary condition for establishing whether the upper bound $\gamma^*$ and the controller $K^*(p)$ found with Theorem 3.2 solve Problem 3.

## 4. Examples

In this section we present some illustrative examples of the proposed methodology. The SDP (26) is solved with the toolbox SeDuMi [22] for Matlab on a standard computer with Windows 10, Intel Core i7, 3.4 GHz, 8 GB RAM.

For comparison, the method in [15] is considered, which addresses the problem of finding a robust LQR (i.e., an LQR independent on the parameters that achieve a guaranteed upper bound on the cost for all parameters) in the case of polytopic systems (i.e., systems affected linearly by parameters constrained into a polytope). This problem is a subproblem of our Problem 3, since in the latter the LQR is allowed to be parametric.

### 4.1. *Example 1*

In this first example we consider a simple model for illustrating the proposed results, in particular

$$\begin{cases} \dot{x}(t) & = & \begin{pmatrix} 0 & 1 \\ -1-2p & -1+p \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ p & \in & [-1,1]. \end{cases}$$

The goal is to solve Problems 1–3 with the choice

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1/2, \quad x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

#### *4.1.1. Solving Problem 1*

Using $d_V = 0$ we find the lower bound $\gamma^* = 2.233$ and the controller

$$K^*(p) = (-0.539, -0.491)$$

which are guaranteed to satisfy (28) from Theorem 3.1. We also obtain $\mathcal{Z} = \{0\}$, however $p^* = 0$ does not satisfy (32). From Theorem 3.2 we conclude that the found lower bound is not tight.

In order to get less conservative results, we increase $d_V$. Using $d_V = 1$ we find the lower bound $\gamma^* = 3.759$ and the controller

$$K^*(p) = (0.615p - 1.135, -0.271p - 1.294).$$

We also obtain $\mathcal{Z} = \{0.681\}$, however $p^* = 0.681$ does not satisfy (32).

Hence, we use $d_V = 2$, finding the lower bound $\gamma^* = 3.791$ and the controller

$$K^*(p) = (0.087p^2 + 0.681p - 1.173, 0.11p^2 - 0.39p - 1.293).$$

We also obtain $\mathcal{Z} = \{0.316\}$, and $p^* = 0.316$ satisfies (32). From Theorem 3.2 we conclude that the found lower bound is tight, i.e., $J_{bst} = 3.791$.

Figure 1 shows, for $d_V = 0, 1, 2$, the lower bound $\gamma^*$ and the cost $J(K^*(p), p)$ for some values of the parameter. The figure also shows the true cost $J^{\#}(p)$ in (3) found by brute force. The number of LMI scalar variables with $d_V = 2$ is 51 and the computational time is less than 1 second.

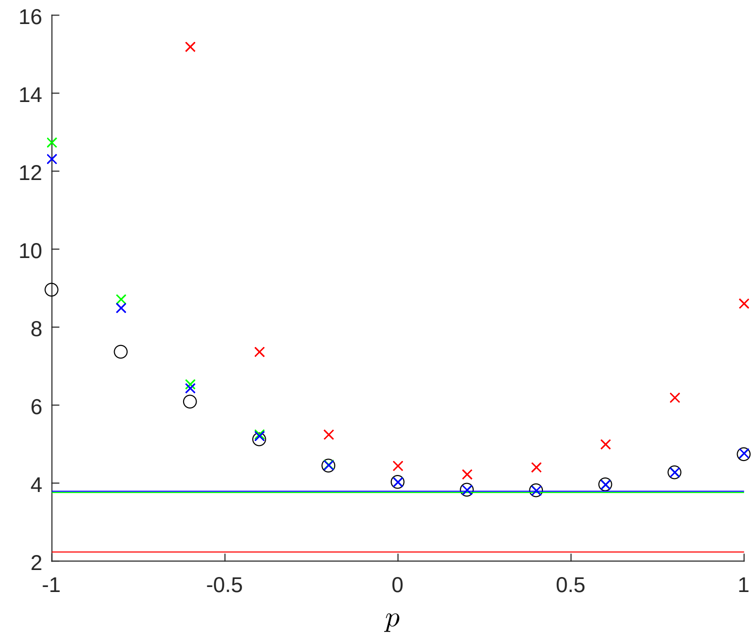


**Figure 1.** Example 1, Problem 1. Lower bound $\gamma^*$ (solid line) and cost $J(K^*(p), p)$ ("$\times$" mark) for $d_V = 0$ (red color), $d_V = 1$ (green color) and $d_V = 2$ (blue color). The figure also shows the true cost $J^\#(p)$ (black "o" mark).

### *4.1.2. Solving Problem 2*

Using $d_V = 0$ we find the lower bound $\gamma^* = 4.467$ and the controller

$$K^*(p) = (-0.539, -0.491)$$

which are guaranteed to satisfy (36) from Theorem 3.3.

In order to get less conservative results, we increase $d_V$. Using $d_V = 1$ we find the lower bound $\gamma^* = 7.956$ and the controller

$$K^*(p) = (1.024p - 1.225, -0.125p - 1.305).$$

Also, using $d_V = 2$ we find the lower bound $\gamma^* = 9.521$ and the controller

$$K^*(p) = (-0.592p^2 + 1.212p - 1.215, -0.299p^2 - 0.13p - 1.3).$$

Figure 2 shows, for $d_V = 0, 1, 2$, the cost $J(K^*(p), p)$ for some values of the parameter. The figure also shows the true cost $J^\#(p)$ in (3) found by brute force. The number of LMI scalar variables with $d_V = 2$ is 51 and the computational time is less than 1 second.

Figure 3 shows, for $d_V = 0, \ldots, 4$, the controller $K^*(p)$ for some values of the parameter. The figure also shows the true controller $K^\#(p)$ in (5) found by brute force.

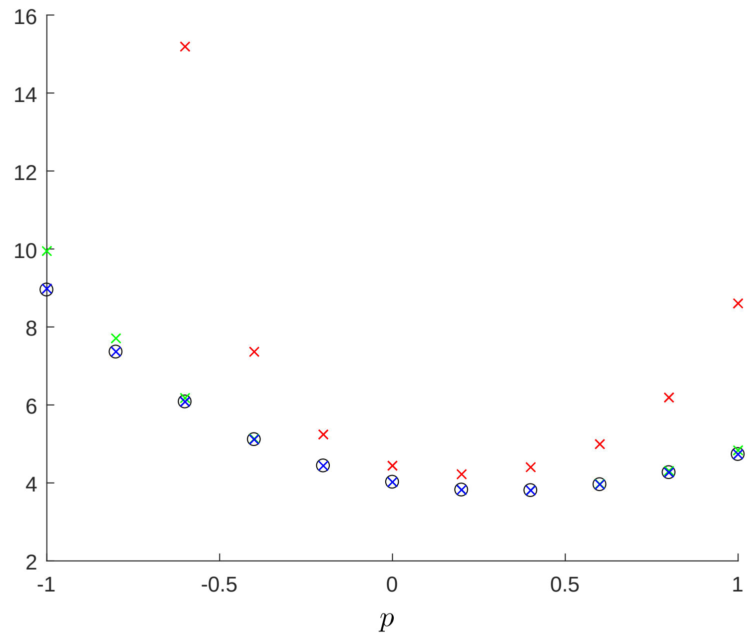


**Figure 2.** Example 1, Problem 2. Cost $J(K^*(p), p)$ ("$\times$" mark) for $d_V = 0$ (red color), $d_V = 1$ (green color) and $d_V = 2$ (blue color). The figure also shows the true cost $J^\#(p)$ (black "$\circ$" mark).

As we can see, the controller $K^*(p)$ approximates arbitrary well $K^\#(p)$ as $d_V$ increases.

### *4.1.3. Solving Problem 3*

Using $d_V = 0$ we find the upper bound $\gamma^* = 11.176$ and the controller

$$K^*(p) = (-3.332, -3.557)$$

which are guaranteed to satisfy (43) from Theorem 3.4. We also obtain $\mathcal{Z} = \emptyset$. From Theorem 3.5 we conclude that the found upper bound is not tight.

In order to get less conservative results, we increase $d_V$. Using $d_V = 1$ we find the upper bound $\gamma^* = 9.105$ and the controller

$$K^*(p) = \frac{(9.632p - 11.056, 3.259p - 7.472)}{-0.085p^2 - 2.744p + 3.693}.$$

As in the previous case, we also obtain $\mathcal{Z} = \emptyset$.

Hence, we use $d_V = 2$, finding the upper bound $\gamma^* = 8.950$ and the controller

$$K^*(p) = \frac{\begin{pmatrix} -5.153p^2 + 12.689p - 9.783 \\ -0.263p^2 + 4.065p - 8.746 \end{pmatrix}'}{-0.506p^4 + 0.626p^3 + 2.191p^2 - 3.784p + 3.693}.$$

We also obtain $\mathcal{Z} = \{-1\}$, and $p^* = -1$ satisfies (32). From Theorem 3.5 we conclude that the found upper bound is tight, i.e., $J_{wst} = 8.950$.

Figure 4 shows, for $d_V = 0, 1, 2$, the upper bound $\gamma^*$ and the cost $J(K^*(p), p)$ for some values of the parameter. The figure also shows the true cost $J^\#(p)$ in (3)

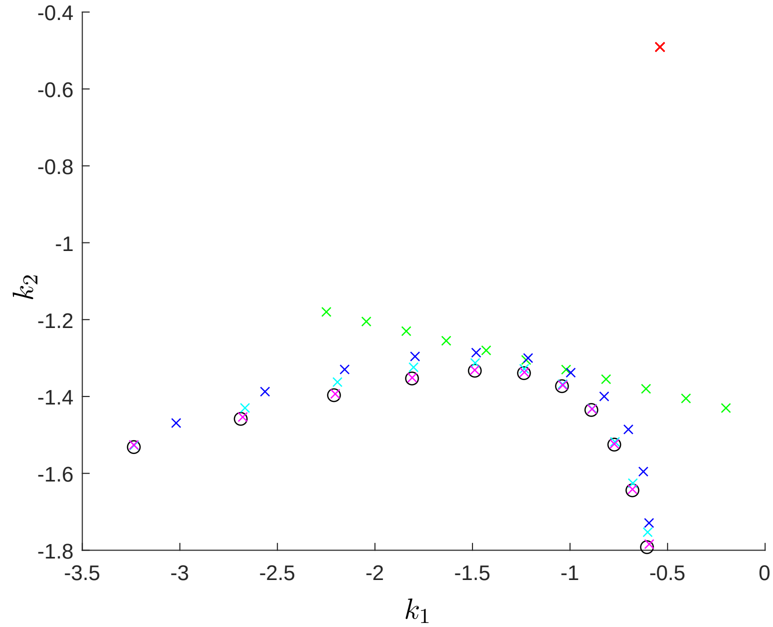


**Figure 3.** Example 1, Problem 2. Controller $K^*(p) = (k_1, k_2)$ for some values of the parameter ("×" mark) for $d_V = 0$ (red color), $d_V = 1$ (green color), $d_V = 2$ (blue color), $d_V = 3$ (cyan color) and $d_V = 4$ (magenta color). The figure also shows the true controller $K^{\#}(p) = (k_1, k_2)$ (black "o" mark).

found by brute force. The number of LMI scalar variables with $d_V = 2$ is 87 and the computational time is less than 1 second.

For comparison, we consider the method in [15] as said at the beginning of this section, which looks for a robust LQR (i.e., an LQR independent on the parameters that achieve a guaranteed upper bound on the cost for all parameters). We find that the cost guaranteed by the robust LQR found by this method is 17.901. This means that the proposed approach provides a less conservative robust LQR in this case, indeed, as we have seen above, for $d_V = 0$ the cost guaranteed by the found LQR (which is robust since independent on the parameters) is 11.176.

### 4.2. *Example 2*

In this second example we consider the model of an electric motor, specifically [11]

$$\left\{ \begin{array}{rcl} J\ddot{\theta} + b\dot{\theta} + s\theta & = & Ci_a \\ L_a\dot{i}_a + R_a i_a & = & v_a - D\dot{\theta} \end{array} \right.$$

where $\theta$ is the angle, $i_a$ is the current, $J$ is the inertia, $b$ is the friction, $s$ is the stiffness, $C$ and $D$ are coupling coefficients, $L_a$ is the inductance, $R_a$ is the resistance, and $v_a$ is the input voltage. Let us choose the plausible values

$$b = 0.3, \quad s = 0.5, \quad C = 0.6, \quad L_a = 0.5, \quad R_a = 2$$

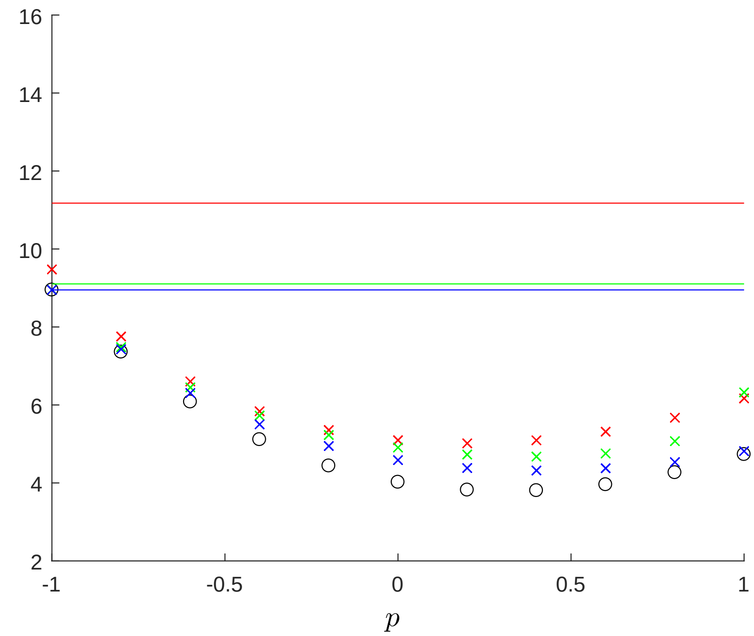


**Figure 4.** Example 1, Problem 3. Upper bound $\gamma^*$ (solid line) and cost $J(K^*(p), p)$ ("×" mark) for $d_V = 0$ (red color), $d_V = 1$ (green color) and $d_V = 2$ (blue color). The figure also shows the true cost $J^\#(p)$ (black "○" mark).

and let us consider $J$ and $D$ parameters in some intervals, specifically

$$J \in [3, 7], \quad D \in [0.5, 1.5].$$

By defining

$$\begin{cases} x_1 = \theta, \quad x_2 = \dot{\theta}, \quad x_3 = i_a, \quad u = v_a \\ p_1 = 0.5(J - 5), \quad p_2 = 2(D - 1), \end{cases}$$

the electric motor can be described as in (1) with

$$A(p) = \frac{\begin{pmatrix} 0 & p_1 + 2.5 & 0 \\ -0.25 & -0.15 & 0.3 \\ 0 & -p_1 p_2 - 2p_1 - 2.5p_2 - 5 & -4p_1 - 10 \end{pmatrix}}{p_1 + 2.5}, \quad B(p) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

We choose

$$Q = I, \quad R = 1, \quad x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let us start by considering Problem 2. Using $d_V = 0, \ldots, 3$ we find the lower bounds $\gamma^* = 12.253, 17.067, 18.602, 18.686$. Figure 5 shows, for $d_V = 0, \ldots, 3$, the controller $K^*(p)$ for some values of the parameters. The figure also shows the true controller $K^\#(p)$ in (5) found by brute force. As we can see, the controller $K^*(p)$

approximates arbitrary well $K^{\#}(p)$ as $d_V$ increases.

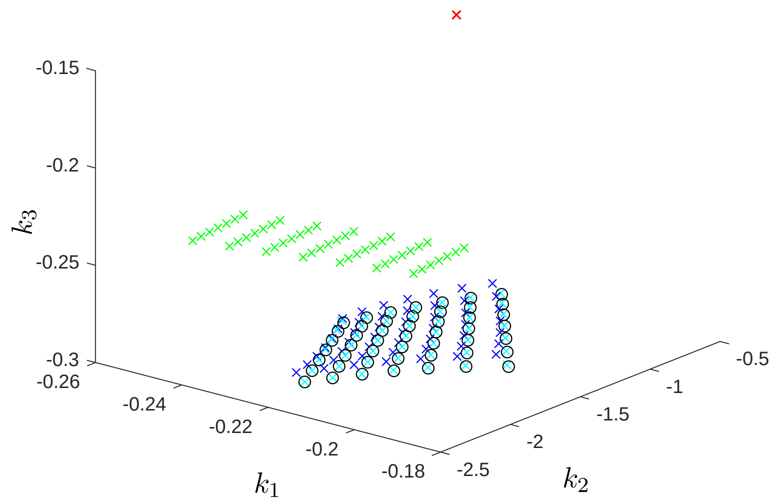


**Figure 5.** Example 2, Problem 2. Controller $K^*(p) = (k_1, k_2, k_3)$ for some values of the parameter ("$\times$" mark) for $d_V = 0$ (red color), $d_V = 1$ (green color), $d_V = 2$ (blue color) and $d_V = 3$ (cyan color). The figure also shows the true controller $K^{\#}(p) = (k_1, k_2, k_3)$ (black "$\circ$" mark).

Next, let us consider Problem 3. Using $d_V = 0, 1$ we find the upper bounds $\gamma^* = 9.016, 6.062$. For $d_V = 1$ we also obtain $\mathcal{Z} = \{(1, -1)\}$, and $p^* = (1, -1)$ satisfies (32). From Theorem 3.5 we conclude that the found upper bound is tight, i.e., $J_{wst} = 6.062$.

In this example, we do not consider the method in [15] because it can be used only in the case of polytopic systems, and the electric motor under investigation is not a polytopic system since $A(p)$ is not a linear function of $p$.

## 5. Conclusions

The problem of determining parametric LQRs for continuous-time LTI systems affected by parameters has been addressed. Three situations of interest have been considered, where the sought controller has to minimize the best cost, average cost, and worst cost, respectively, over the set of admissible parameters. It has been shown that candidates for such controllers can be obtained by solving convex optimization problems with LMI constraints. These candidates are guaranteed to approximate arbitrarily well the sought controllers by sufficiently increasing the size of the LMIs. In particular, the candidate that minimizes the average cost approximates arbitrarily well the true LQR over the set of admissible parameters. Moreover, conditions for establishing the optimality of the found candidates have been provided.

Several directions could be explored in future work. For instance, one could investigate the extension of the proposed methodology to discrete-time LTI systems or nonlinear systems.

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