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Descriptor state-bounding observer design for positive Markov jump linear systems with sensor faults: simultaneous state and faults estimation

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Abstract

This paper presents a descriptor observer design approach for positive Markov jump linear systems subject to interval parameter uncertainties and sensor faults. First, by taking the sensor fault term as an auxiliary state, an augmented descriptor system is constructed. A pair of positive observers with state-bounding feature is then proposed, which enables simultaneous estimation of the system state and sensor faults. A necessary and sufficient condition on existence of the desired state-bounding observer is derived by considering positivity and robust mean exponential stability of corresponding observer error dynamics. An iterative optimization algorithm is developed for the computation of the optimized observer matrices. Finally, a numerical example is presented to show the validity of the proposed methods.

KEYWORDS:
descriptor system, fault estimation, Markov jump linear system, positive system, state-bounding observer

1 | INTRODUCTION

Systems in disciplines such as biology, ecology, and social sciences usually involve quantities with intrinsic nonnegative property.[1–3] These systems, with all their states and outputs evolved in the first quadrant whenever nonnegative initial conditions and inputs are given, are commonly referred to as (internally) positive systems, or nonnegative systems. During the last decades, considerable efforts have been devoted to the analysis and synthesis of positive systems,[4–7] which unraveled a host of elegant properties for these systems.

Owing to nonnegativity, the dynamic behavior of deterministic positive systems presents salient features which could generally simplify the performance analysis.[4] It has been shown that the L_1-gain[8,9] and L_∞-gain[10] are particularly suitable for positive systems in robustness and performance characterization. On the other hand, new challenges also arise in the analysis and synthesis of positive systems, since traditional methods designed for general systems, though applicable, are often handicapped when dealing with such systems as they are not defined on linear spaces but a convex polyhedral cone. Therefore, new issues concerning the realization,[11] reachability[12] and controllability[13] of positive systems have been studied. Despite much attention has been devoted to studying reachability and realization of positive system, it is worth noting that approaches to designing positive observers have received increasing attention due to their practical significance.[14,15]

Positive observers were first derived in terms of structural decomposition[14] for linear compartmental systems, which form a subclass of positive linear systems. The results were extended to general linear positive systems,[15,16] where Luenberger-type observers were considered and structural conditions were exploited with the prerequisite of accurate system models. When parameter uncertainties were considered, an extended Luenberger-type robust observer for positive systems was designed.[17]

Instead of widely used algebraic techniques in the aforementioned works, a unified linear matrix inequality (LMI) framework was used there and observer matrices were then constructed through the solution of LMIs. In addition, new observer structures were formulated in view of the limitation inherited by classical observers. For example, in the works of Li and Lam,[18,19] interval observers guaranteeing upper and lower bounds on the system states were introduced, and different LMI approaches were considered for observer construction and optimization. It is worth noting that the design of interval observers is highly relevant to the application of positive systems theory, based on which observer gains are determined to guarantee that the observer error dynamics are always positive.[20] There have been several approaches to designing interval observers for systems with disturbances or uncertainties,[21,22] time-varying systems,[23] delay systems,[19] nonlinear systems,[24] and discrete-time systems.[25] The techniques of interval observers can also been applied to tackle the problem of controller design and with guaranteed interval estimates, the interval observers often simplify the control of transition processes.[26–28] It was further shown that interval observers could enable a much easier realization of optimal peak-to-peak controllers/observers for linear systems due to the positivity of the error dynamics.[29] More recently, state-bounding observers for positive interval Markov jump systems have been designed[30] based on the ℓ_1 performance using linear programming (LP) approaches.

Beyond the difficulties of control synthesis, there is also the demand in reliability of practical systems in terms of ubiquitous faults. Hence the research into fault detection, diagnosis and estimation has long been recognized as an important aspect of system safety monitoring and reliable control, with various theoretical achievements[31–33] and engineering applications.[34,35] Generally speaking, a fault detection and diagnosis (FDD) module is adopted as the first step in fault accommodation to monitor the system by constructing appropriate indicators while fault estimation is utilized to depict the magnitude of faults on-line. A variety of fault estimation schemes have been developed by constructing sliding mode observers, descriptor observers, robust observers, as well as iterative observers.[36–40] Among these approaches, the descriptor observer approach possesses superiority in multiple faults and/or disturbance estimation. By introducing a descriptor augmentation transformation, faults and/or disturbance vectors can be decoupled from the system state. Based on the well-developed descriptor system methods,[41–43] descriptor observers with guaranteed performance can be constructed, thus leading to a simultaneous estimation of both the original system state and faults (and/or disturbances). Some relevant results on fault detection have been reported,[44,45] where the problem is transformed into an H_-/H_∞ or H_∞ filtering problem for positive systems. Robust filtering approaches or other types of positive observers can then be exploited to estimate the system state and faults with certain performance criteria. In the work of Oghbaee et al,[46] a special type of unknown input observer is introduced, where faults are converted as unknown inputs and then estimated through a positive filtering process of the output. However, to the best of our knowledge, the fault estimation problem of positive systems has not been fully investigated to date, which is the first motivation of the current study. In addition, due to inevitable random factors, such as random faults, abrupt environmental changes, and unexpected configuration changes, the system parameters are subject to abrupt jumps rather than constant. When the jumps are governed by an underlying Markov chain, the system dynamics can then be well described by Markov jump systems and be amenable to thorough theoretical analysis. Various results on Markov jump systems have been reported.[47–49] It is therefore natural to carry out studies related to the stochastic case of positive systems. For the rather limited research results, some can be found in the work of Bolzern et al,[50] where the stability analysis and stabilization for positive continuous-time Markov jump system are discussed with reference to different notions of stochastic stability; in the work of Zhu et al,[51] where the ℓ_1-gain performance-based positive filters are designed in a discrete-time setting; in the work of Zhang et al,[30] where the positive observer design problem is considered for positive Markov jump systems with time delay; and in the work of Zhu et al,[52] where necessary and sufficient conditions for L_1 stochastic stability and L_1-gain performance of continuous-time positive Markov jump system with time delay are established.

Motivated by the above issues, we will investigate the positive observer design problem for positive Markov jump linear systems with interval uncertainties and sensor faults. The main difficulty of the problem lies in the inherent constraint of nonnegativity on state estimation. The reason is that the sensor faults (whose components can be positive or negative) may be significantly amplified by the observer gain, making existing positive observer approaches unable to obtain nonnegative estimates, or nonnegative estimation errors. To cope with the aforementioned difficulty, a linear descriptor plant is constructed by introducing an augmented vector consisting of the state and sensor fault vectors. Thus the sensor faults are decoupled from the system state completely, after which the problem can be solved by dealing with an intersection of observer design for descriptor systems and positive Markov jump linear systems stability analysis. Specially, with interval uncertainties taken into consideration, the interval observer approach, which has been well adapted to the observer design for positive systems[18–20,29,30] and singular systems[24,53], is adopted in this paper.

To conclude, the objective is to extend these design tools to the class of positive Markov jump linear systems and main contributions of the work are the following. 1) As an extension of studies for positive systems, sensor faults are considered in this paper. By using the proposed descriptor plant construction, a simultaneous estimation of both system state and sensor faults can be obtained while the common assumptions of boundedness[54] or of low frequency[55] in fault estimation are relaxed as long as the descriptor system is detectable. 2) Different from previous studies on fault estimation where faults are estimated in an asymptotic way, here a pair of positive observers (called positive state-bounding observer) is designed so that the system state and sensor faults can be encapsulated at all times. Optimization is then performed with iterative algorithms to find the optimized state-bounding observer matrices. 3) Conditions on the existence of a desired observer are established in the form of LP, which is computationally more efficient compared with LMI techniques. Note that in recent papers, multiple descriptor observer approaches have been exploited for Markov jump systems,[56–58] in which the problem of observer design was usually solved through LMIs by employing conditions for mean-square stability of corresponding error systems. In this paper, owing to the nonnegative property of deterministic positive systems, stability analysis can be significantly simplified by investigating the descriptor observer design problem in the mean sense.[50,51] Accordingly, the corresponding conditions can be obtained in terms of the solutions of linear programming problems.

The rest of this paper is organized as follows. Section 2 presents notations and preliminaries about positive Markov jump linear systems. Section 3 introduces a descriptor system model and formulates the problem of positive state-bounding observers for the constructed descriptor plant. Section 4 is devoted to designing positive state-bounding observers and an illustrative example is provided in Section 5.

2 | NOTATION AND PRELIMINARIES

Throughout this paper, $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^{n\times m}$ denote the set of all n-dimensional real vectors, the nonnegative orthant of $\mathbb{R}^n$ and the set of all $n\times m$ real matrices, respectively. I_n denotes the n-dimensional identity matrix and $0_{m\times n}$ means the $m\times n$ matrix with all zero entries. $\mathbf{1}_n$ denotes the n-dimensional column vector of all ones. For a matrix $A\in\mathbb{R}^{n\times m}$, $[A]_{ij}$ denotes the element located at the ith row and the jth column. A real matrix $A\in\mathbb{R}^{n\times m}$ with all its entries greater than or equal to 0 (strictly greater than 0) is said to be nonnegative (positive) and is denoted by $A\geq 0$ ($A\succ 0$). Similar definitions and notation apply when $A\in\mathbb{R}^{n\times m}$ is either non-positive ($A\preceq 0$) or negative ($A\prec 0$). For two matrices $A,B\in\mathbb{R}^{n\times m}$, the expressions $A\succeq B$, $A\succ B$, $A\preceq B$, $A\prec B$ indicate that the difference $A-B$ is nonnegative, positive, nonpositive and negative, respectively. For matrices A, $\underline{A}$, $\overline{A}$ $\in\mathbb{R}^{n\times m}$, the notation $A\in[\underline{A},\overline{A}]$ means that $\underline{A}\preceq A\preceq\overline{A}$. To indicate that a real symmetric matrix $P\in\mathbb{R}^{n\times n}$ is positive definite (positive semi-definite) or negative definite (negative semi-definite), we will use the symbol $P>0$ ($P\geq 0$) or $P<0$ ($P\leq 0$). A square matrix $A\in\mathbb{R}^{n\times n}$ is called Metzler if all its off-diagonal elements are nonnegative, that is, $\forall(i,j), i\neq j, [A]_{ij}\geq 0$. The expectation of a stochastic variable v will be denoted as $\mathbb{E}[v]$. The symbol $\Pr\{\cdot\}$ will be used for the probability of an event.

Consider the continuous-time Markov jump system as follows:

$$\begin{aligned}\dot{x}(t)&=A(\theta(t))x(t)+B(\theta(t))u(t),\\ y(t)&=C(\theta(t))x(t),\end{aligned}\tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the system input and $y(t)\in\mathbb{R}^p$ is the measured output; $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are mode-dependent real matrices with appropriate dimensions. $\{\theta(t),t\geq 0\}$ is the underlying continuous-time Markov stochastic process which takes values in a finite set $\mathcal{S}=\{1,2,\ldots,s\}$ with generator matrix $\Lambda=(\lambda_{ij})\in\mathbb{R}^{s\times s}$, $\quad i,j\in\mathcal{S}$. The mode transition is governed by

$$\Pr\{\theta(t+\Delta t)=j\mid\theta(t)=i\}=\begin{cases}\lambda_{ij}\Delta t+o(\Delta t) & j\neq i,\\ 1+\lambda_{ii}\Delta t+o(\Delta t) & j=i,\end{cases}\tag{2}$$

where $\Delta t>0$ is a small time increment, $\lim_{\Delta t\to 0^+}\frac{o(\Delta t)}{\Delta t}=0$, and $\lambda_{ij}\geq 0$ ($i\neq j$) is the transition probability rate from mode i at time t to mode j at time $t+\Delta t$. Also, we have $\lambda_{ii}=-\sum_{j=1,j\neq i}^{s}\lambda_{ij}$. Here the mode-dependent matrices $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are denoted as A_i, B_i and C_i, respectively, for each $\theta(t)=i$. In this paper the case in which all the subsystems of (1) belong to the class of linear positive systems is considered and it is assumed that the system mode $\theta(t)$ is accessible.

Definition 1. System (1) is said to be a positive Markov jump linear system (PMJLS) if for all $\theta(0)\in\mathcal{S}$, $x(0)\succeq 0$ and $u(t)\succeq 0$, we have $x(t)\succeq 0$ and $y(t)\succeq 0$ for $t>0$.

Lemma 1. [4] System (1) is a PMJLS if and only if A_i is Metzer, $B_i\succeq 0$ and $C_i\succeq 0$, $\forall i\in\mathcal{S}$.

Various notions of stochastic stability have been introduced for Markov jump systems among which the mean-square stability, implying asymptotic convergence to zero of the expected squared norm of state, has been widely investigated. However, it was reported in recent works[50,51] that for PMJLS, the convergence to zero of the expectation of state vector (mean stability) is particularly suitable due to the positivity. Such a result offers a viable alternative to ascertain almost-sure stability, which is recognized as being closer to the concerns of engineers in practice. Furthermore, the corresponding conditions can be obtained in the form of LP instead of LMIs, which is computationally more efficient than those in the sense of mean-square stability. Hence the stochastic stability for PMJLS (1) in this paper refers to mean stability and the convergence is of the exponential type.

Definition 2. The positive Markov jump linear system (1) with $u(t) = 0$ is said to be mean exponentially stable (MES) if there exist scalars $\alpha > 0$ and $\beta > 0$ such that

$$\mathbb{E}[x(t)] \preceq \alpha e^{-\beta t}\|x(0)\|\mathbf{1}_n, \tag{3}$$

for any initial condition $x(0) \succeq 0$ and any initial probability distribution.

Lemma 2. [50] The mean exponential stability of the positive Markov jump linear system in (1) with $u = 0$ is guaranteed if and only if there exist strictly positive vectors $q_i \in \mathbb{R}^n, i = 1, 2, \dots, s$, such that the following inequalities are satisfied:

$$q_i^T A_i + \sum_{j=1}^{s} \lambda_{ij} q_j^T \prec 0. \tag{4}$$

This paper addresses the problem of observer design for PMJLS in the presence of interval parameter uncertainties. Consider the case that the matrices A_i and B_i of the PMJLS in (1) are unknown matrices belonging to the uncertainty set

$$\Theta = \{(A_i, B_i) : A_i \in [\underline{A}_i, \overline{A}_i], B_i \in [\underline{B}_i, \overline{B}_i], i = 1, 2, \dots, s\}, \tag{5}$$

where $\underline{A}_i$, $\overline{A}_i$, $\underline{B}_i$ and $\overline{B}_i$ are given matrices. In relation to the definition of MES, we introduce the following definition of robust mean exponential stability for the uncertain PMJLS.

Definition 3. The positive Markov jump linear system (1) with $u(t) = 0$ is said to be robustly mean exponentially stable (RMES) if (1) is MES for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $i = 1, 2, \dots, s$.

3 | PROBLEM FORMULATION

Let $\{\theta(t), t \geq 0\}$ be a Markov stochastic process taking values in a finite set $\mathcal{S} = \{1, 2, \dots, s\}$ with transition probabilities given by (2) and consider the following positive Markov jump linear system with sensor faults:

$$\begin{aligned} \dot{x}(t) &= A_0(\theta(t))x(t) + B_0(\theta(t))u(t), \\ y(t) &= C_0(\theta(t))x(t) + f(t), \end{aligned} \tag{6}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the system state, input and measurement output, respectively; $f(t) \in \mathbb{R}^p$ represents the sensor fault vector and its components can be positive or negative; the system matrices of the ith mode are denoted by A_{0i}, B_{0i} and C_{0i}, respectively. Moreover, A_{0i} and B_{0i} are unknown matrices belonging to the uncertainty set

$$\Phi = \{(A_{0i}, B_{0i}) : A_{0i} \in [\underline{A}_{0i}, \overline{A}_{0i}], B_{0i} \in [\underline{B}_{0i}, \overline{B}_{0i}], i = 1, 2, \dots, s\}, \tag{7}$$

where $\underline{A}_{0i} \in \mathbb{R}^{n\times n}$ are Metzler matrices; $\underline{B}_{0i} \in \mathbb{R}_+^{n\times m}$, $C_{0i} \in \mathbb{R}_+^{p\times n}$. In this section, an observer is presented to estimate the system state and sensor faults simultaneously. Note that in this paper feedback will not be incorporated for system regulation, therefore, the original system (6) is assumed to be RMES (the system may have been suitably controlled). Denote

$$\begin{aligned} &E = \begin{bmatrix} I_n & 0_{n\times p} \\ 0_{p\times n} & 0_{p\times p} \end{bmatrix}, \quad A_i = \begin{bmatrix} A_{0i} & 0_{n\times p} \\ 0_{p\times n} & I_p \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{0i} \\ 0_{p\times m} \end{bmatrix}, \\ &B_{fi} = \begin{bmatrix} 0_{n\times p} \\ I_p \end{bmatrix}, \quad C_i = \begin{bmatrix} C_{0i} & -I_p \end{bmatrix}, \end{aligned} \tag{8}$$

then when (6) is in the ith mode, an augmented descriptor plant can be constructed as follows:

$$\begin{aligned} E\dot{\chi}(t) &= A_i\chi(t) + B_i u(t) + B_{fi} f(t), \\ y(t) &= C_i\chi(t), \end{aligned} \tag{9}$$

where $\chi(t) = [x^T(t) \quad -f^T(t)]^T$. That is, both $x(t)$ and $f(t)$ in (6) become the descriptor state vector of the augmented plant. If an observer exists for (9), the estimates of the state vector $x(t)$ and the sensor fault $f(t)$ will be obtained simultaneously.

Remark 1. Notice that for any $i \in \mathcal{S}$,

$$\text{rank}\begin{bmatrix} E \\ C_i \end{bmatrix} = \text{rank}\begin{bmatrix} I_n & 0 \\ 0 & 0 \\ \hline C_{0i} & -I_p \end{bmatrix} = n + p, \tag{10}$$

that is, there always exists a matrix L_i of appropriate dimension such that $\text{rank}(E + L_i C_i) = n + p$, that is, $(E + L_i C_i)$ is invertible. Moreover, the system (E, A_i, C_i) is completely detectable provided that the system is detectable.[38]

In the presence of interval parameter uncertainties, the Luenberger form may fail to construct positive observers for system (9). In linear system theory, the purpose of observer design is to provide an estimate of state $x(t)$ such that the estimation error converges to zero; while for positive linear systems, the observer is chosen in such a way that the estimate of $x(t)$ is positive, like the state itself. By resorting to the positivity property of positive systems, the interval or bounding observers provide an elegant solution for such problems. The underlying idea is to design an observer consisting of a couple of estimators such that the nonnegativity holds for the corresponding estimation errors. Thus, in this paper, a state-bounding observer is considered. A pair of observers named positive lower-bounding observer and positive upper-bounding observer, respectively, is introduced here to encapsulate the system state and sensor faults. The resultant observer provides information about the transient state while conventional observers usually work in an asymptotic way.

Positive lower-bounding observer: Based on the obtained plant (9), the following descriptor observer is constructed:

$$\begin{aligned} (E + \underline{L}_i C_i)\dot{\check{z}}(t) &= (\underline{F}_i - \underline{L}_i C_{fi} - \underline{G}_i C_i)\check{z}(t) + \underline{K}_i u(t), \\ \check{\chi}(t) &= \check{z}(t) + (E + \underline{L}_i C_i)^{-1} \underline{L}_i y(t), \end{aligned} \tag{11}$$

where $\check{\chi}(t), \check{z}(t) \in \mathbb{R}^{n+p}$ and $\check{\chi}(t) = [\check{x}^T(t) \quad -\check{f}^T(t)]^T$. $\underline{L}_i$, $\underline{F}_i$, $\underline{G}_i$ and $\underline{K}_i$ are matrices with appropriate dimensions to be determined later. Besides, $\underline{F}_i \in \mathbb{R}^{(n+p)\times(n+p)}$, $i = 1, 2, \dots, s$, are the observer parameters which have the following structure:

$$\underline{F}_i = \begin{bmatrix} \underline{F}_{ai} & 0 \\ \underline{F}_{bi} & I_p \end{bmatrix}.$$

The observer matrix $\underline{L}_i$ is mainly used to make $(E + \underline{L}_i C_i)$ nonsingular. Using such an $\underline{L}_i = L = [0_{p\times n} \quad I_p]^T$, it can be shown that

$$(E + \underline{L}_i C_i)^{-1} = \begin{bmatrix} I_n & 0 \\ C_{0i} & -I_p \end{bmatrix}^{-1} = \begin{bmatrix} I_n & 0 \\ C_{0i} & -I_p \end{bmatrix}. \tag{12}$$

Denote $C_{fi} = \begin{bmatrix} C_{0i} & 0_{p\times p} \end{bmatrix}$. In terms of (8), we have

$$\underline{F}_i (E + LC_i)^{-1} L = -L, \tag{13}$$

$$C_{fi}(E + LC_i)^{-1} L = 0_{p\times p}, \tag{14}$$

$$C_i (E + LC_i)^{-1} L = I_p. \tag{15}$$

The observer in the form of (11) can then be rewritten as

$$(E + LC_i)\dot{\check{\chi}} = (\underline{F}_i - LC_{fi} - \underline{G}_i C_i)\check{\chi} + Ly + \underline{G}_i y + \underline{K}_i u + L\dot{y}. \tag{16}$$

Let $\check{e} = \chi - \check{\chi}$ denote the estimation error. Consider (9) and (16), the error dynamic equation can be obtained as

$$\begin{aligned} (E + LC_i)\dot{\check{e}} &= A_i \chi + B_i u + B_{fi} f + L\dot{y} - (E + LC_i)\dot{\check{\chi}} \\ &= (A_i - \underline{F}_i)\chi + (\underline{F}_i - LC_{fi} - \underline{G}_i C_i)\check{e} + (B_i - \underline{K}_i)u + B_{fi} f + L(C_{fi} - C_i)\chi \\ &= (A_i - \underline{F}_i)\chi + (\underline{F}_i - LC_{fi} - \underline{G}_i C_i)\check{e} + (B_i - \underline{K}_i)u \\ &= (A_i - \underline{F}_i)J_n x + (\underline{F}_i - LC_{fi} - \underline{G}_i C_i)\check{e} + (B_i - \underline{K}_i)u, \end{aligned} \tag{17}$$

where $J_n = [I_n \quad 0_{n\times p}]^T$. Define a new variable $\check{\xi} = [x^T(t) \quad \check{e}^T(t)]^T$; it follows from the above that

$$\dot{\check{\xi}} = \underline{\mathscr{A}}_i \check{\xi} + \underline{\mathscr{B}}_i u, \tag{18}$$

where

$$\begin{aligned}\underline{\mathscr{A}}_i &= \begin{bmatrix} A_{0i} & 0 \\ W_i^{-1}(A_i - \underline{F}_i)J_n & W_i^{-1}(\underline{F}_i - LC_{fi} - \underline{G}_i C_i)\end{bmatrix},\\ \underline{\mathscr{B}}_i &= \begin{bmatrix} B_{0i} \\ W_i^{-1}(B_i - \underline{K}_i)\end{bmatrix},\\ W_i &= E + LC_i.\end{aligned}$$

Definition 4. System (11) is a positive lower-bounding observer for PMJLS (6) if, for any $\check{z}(0) \geq 0$, $x(0) \geq \check{x}(0)$ and $u(t) \geq 0$, we have $0 \preceq \check{x}(t) \preceq x(t)$ for all $t > 0$.

Generally, the problem of observer design is to find the observer matrices in (11) such that the system in (18) is stable (and in this paper robust mean exponential stability is considered). However, according to Definition 4, the positivity of (18) is also required for the realization of the lower-bounding estimator. It is noted that the positivity of $x(t)$ should be taken into account since the interpretation in terms of a positive system in real applications requires that an estimate of $x(t)$ should always be nonnegative. With the specific structure of $(E + LC_i)^{-1}L$, the positivity of $\check{x}(t)$ will be guaranteed provided that $\check{z}(t)$ is nonnegative. These requirements lead to the basic conditions for the designing of a positive lower-bounding observer. In addition, since the augmented state $\chi(t) = [x^T(t) \quad -f^T(t)]^T$, the estimate of the sensor fault provided by the designed positive lower-bounding observer is actually a lower bound of $(-f(t))$.

One can formulate the positive upper-bounding observer design problem for the descriptor system in a similar way except for the fact that positivity of an upper estimate of $x(t)$ will be naturally satisfied due to the positivity of corresponding estimation error.

Positive upper-bounding observer: Construct a descriptor observer as follows:

$$\begin{aligned}(E + \overline{L}_i C_i)\dot{\hat{z}}(t) &= (\overline{F}_i - \overline{L}_i C_{fi} - \overline{G}_i C_i)\hat{z}(t) + \overline{K}_i u(t),\\ \hat{\chi}(t) &= \hat{z}(t) + (E + \overline{L}_i C_i)^{-1}\overline{L}_i y(t),\end{aligned} \tag{19}$$

where $\hat{\chi}(t), \hat{z}(t) \in \mathbb{R}^{n+p}$ and $\hat{\chi}(t) = [\hat{x}^T(t) \quad -\hat{f}^T(t)]^T$. $\overline{F}_i$, $\overline{G}_i$ and $\overline{K}_i$ are also matrices with appropriate dimensions to be determined while $\overline{L}_i = L = [0_{p\times n} \quad I_p]^T$ and

$$\overline{F}_i = \begin{bmatrix} \overline{F}_{ai} & 0 \\ \overline{F}_{bi} & I_p \end{bmatrix}.$$

Similarly, the observer in the form of (19) can be rewritten as

$$(E + LC_i)\dot{\hat{\chi}} = (\overline{F}_i - LC_{fi} - \overline{G}_i C_i)\hat{\chi} + Ly + \overline{G}_i y + \overline{K}_i u + L\dot{y}. \tag{20}$$

By defining $\hat{e} = \hat{\chi} - \chi$ and $\hat{\xi} = [x^T(t) \quad \hat{e}^T(t)]^T$, we may obtain an augmented system described by

$$\dot{\hat{\xi}} = \overline{\mathscr{A}}_i \hat{\xi} + \overline{\mathscr{B}}_i u, \tag{21}$$

where

$$\begin{aligned}\overline{\mathscr{A}}_i &= \begin{bmatrix} A_{0i} & 0 \\ W_i^{-1}(\overline{F}_i - A_i)J_n & W_i^{-1}(\overline{F}_i - LC_{fi} - \overline{G}_i C_i)\end{bmatrix},\\ \overline{\mathscr{B}}_i &= \begin{bmatrix} B_{0i} \\ W_i^{-1}(\overline{K}_i - B_i)\end{bmatrix},\\ W_i &= E + LC_i.\end{aligned}$$

Definition 5. System (19) is a positive upper-bounding observer for PMJLS (6) if, for any $\hat{z}(0) \geq 0$, $\hat{x}(0) \geq x(0)$ and $u(t) \geq 0$, we have $0 \preceq x(t) \preceq \hat{x}(t)$ for all $t > 0$.

With the constructed positive lower-bounding observer and positive upper-bounding observer, a state-bounding observer is obtained. That is, for any $\check{z}(0) \geq 0$, $\hat{z}(0) \geq 0$, $\hat{x}(0) \geq x(0) \geq \check{x}(0)$ and $u(t) \geq 0$, system (11) and (19) form a state-bounding observer for PMJLS (6) such that the inequality $\hat{x}(t) \geq x(t) \geq \check{x}(t) \geq 0$ holds for all $t > 0$.

Remark 2. Using the method developed by Gao and Wang,[38] a pair of descriptor observers with proportional and derivative structure has been developed in this paper to realize the estimation of the system state and sensor fault simultaneously. By using the proportional and derivative observers (11) and (19), we are able to decouple the sensor fault noise completely. The presented observers (11) and (19) are actually modified forms of (16) and (20), respectively. However, the differentiation of the output is avoided in the proposed observers, which makes it easy to implement the observers in the control system synthesis.

4 | MAIN RESULTS

In this section, the existence of a positive state-bounding observer for PMJLS without external excitation is investigated first by taking into account the positivity and robust mean exponential stability requirements. Necessary and sufficient conditions are established in terms of linear programming problems. When input signal $u(t)$ is present, due to the uncertainties and the excitation of $u(t)$, the estimation error may not converge to zero in general. The issue is then how to reduce the estimation error signal using optimization techniques while maintaining positivity and stability. To do so, an iterative optimization algorithm is proposed and the optimized state-bounding observer matrices are obtained.

4.1 | Design of positive state-bounding observer

Theorem 1. Given a RMES PMJLS (6) with $u(t) = 0$, a positive lower-bounding observer (11) exists such that the augmented system (18) is positive and mean exponentially stable for any $A_{0i} \in [\underline{A}_{0i}, \overline{A}_{0i}]$ (RMES) if and only if there exist strictly positive vectors $\underline{v}_i \in \mathbb{R}^n$, $(n+p)$-dimensional diagonal matrices $\underline{Q}_i \succ 0$, matrices $\underline{U}_i \in \mathbb{R}^{(n+p)\times n}$ and $\underline{V}_i \in \mathbb{R}^{(n+p)\times p}$, $i = 1, 2, \ldots, s$, satisfying

$$\underline{Q}_i W_i^{-1} J_n \underline{A}_{0i} - \underline{U}_i \geq 0, \tag{22}$$

$$\underline{U}_i J_n^T - \underline{Q}_i (LL^T + W_i^{-1} LC_{fi}) - \underline{V}_i C_i \quad \text{is Metzler,} \tag{23}$$

$$\underline{v}_i^T \overline{A}_{0i} + \mathbf{1}^T (\underline{Q}_i W_i^{-1} J_n \overline{A}_{0i} - \underline{U}_i) + \sum_{j=1}^{s} \lambda_{ij} \underline{v}_j^T \prec 0, \tag{24}$$

$$\mathbf{1}^T (\underline{U}_i J_n^T - \underline{Q}_i LL^T - \underline{Q}_i W_i^{-1} LC_{fi} - \underline{V}_i C_i + \sum_{j=1}^{s} \lambda_{ij} \underline{Q}_j) \prec 0, \tag{25}$$

where $\underline{F}_i$ and $\underline{G}_i$ are given by

$$\begin{aligned} &\underline{F}_i = W_i(\underline{M}_i J_n^T - LL^T), \quad \underline{M}_i = \underline{Q}_i^{-1} \underline{U}_i, \\ &\underline{G}_i = W_i \underline{R}_i, \quad \underline{R}_i = \underline{Q}_i^{-1} \underline{V}_i. \end{aligned} \tag{26}$$

Proof. (Sufficiency) It is easy to prove that the structure of $\underline{F}_i$ can be satisfied from (26) and we can further obtain that $W_i^{-1} \underline{F}_i J_n = \underline{M}_i$. Noticing that diagonal matrices $\underline{Q}_i \succ 0$, it follows from (22) that

$$W_i^{-1} J_n \underline{A}_{0i} - W_i^{-1} \underline{F}_i J_n \geq 0. \tag{27}$$

Moreover, since $A_{0i} \in [\underline{A}_{0i}, \overline{A}_{0i}]$, $i = 1, 2, \ldots, s$, it is clear that

$$W_i^{-1} J_n \underline{A}_{0i} \leq W_i^{-1} A_i J_n \leq W_i^{-1} J_n \overline{A}_{0i}, \tag{28}$$

which further implies that

$$W_i^{-1}(A_i - \underline{F}_i) J_n \geq W_i^{-1} J_n \underline{A}_{0i} - W_i^{-1} \underline{F}_i J_n \geq 0. \tag{29}$$

Similarly it follows from (23) that

$$W_i^{-1} \underline{F}_i J_n J_n^T - LL^T - W_i^{-1} LC_{fi} - W_i^{-1} \underline{G}_i C_i \quad \text{are Metzler .} \tag{30}$$

With the given structure of $\underline{F}_i$, we have $W_i^{-1} \underline{F}_i J_n J_n^T - LL^T = W_i^{-1} \underline{F}_i$, that is, $W_i^{-1}(\underline{F}_i - LC_{fi} - \underline{G}_i C_i)$ are Metzler. For the positive system in (6), A_{0i} are Metzler, hence the augmented system (18) is positive. Besides, considering that $W_i^{-1}(\underline{F}_i - LC_{fi} - \underline{G}_i C_i)$ are Metzler, it is easy to prove the positivity of $\check{z}$, that is, positivity of the estimate of $x(t)$.

For diagonal matrices $\underline{Q}_i$, $\underline{q}_i = \underline{Q}_i\mathbf{1}$ return strictly positive vectors of the main diagonal elements of $\underline{Q}_i$. By defining $\underline{\varrho}_i^T = [\underline{v}_i^T \quad \underline{q}_i^T]$, where $\underline{\varrho}_i, i = 1, 2, \dots, s$, are strictly positive vectors, we obtain from (24) and (25) that

$$\underline{\varrho}_i^T \begin{bmatrix} \overline{A}_{0i} & 0 \\ W_i^{-1}J_n\overline{A}_{0i} - W_i^{-1}\underline{F}_iJ_n & W_i^{-1}(\underline{F}_i - LC_{fi} - \underline{G}_iC_i) \end{bmatrix} + \sum_{j=1}^{s}\lambda_{ij}\underline{\varrho}_j^T \prec 0, \tag{31}$$

which indicates that

$$\underline{\varrho}_i^T\underline{\mathscr{A}}_i + \sum_{j=1}^{s}\lambda_{ij}\underline{\varrho}_j^T \prec 0, \tag{32}$$

since

$$\begin{aligned} \underline{\mathscr{A}}_i &= \begin{bmatrix} A_{0i} & 0 \\ W_i^{-1}(A_i - \underline{F}_i)J_n & W_i^{-1}(\underline{F}_i - LC_{fi} - \underline{G}_iC_i) \end{bmatrix} \\ &\le \begin{bmatrix} \overline{A}_{0i} & 0 \\ W_i^{-1}J_n\overline{A}_{0i} - W_i^{-1}\underline{F}_iJ_n & W_i^{-1}(\underline{F}_i - LC_{fi} - \underline{G}_iC_i) \end{bmatrix}. \end{aligned} \tag{33}$$

Then, from Lemma 2 and Definition 3, we can conclude that the lower-bounding observer (11) is positive and the augmented system (18) is positive and RMES. This completes the sufficiency.

(Necessity) Assume that the augmented Markov jump system (18) with $u = 0$ is RMES for any $A_{0i} \in [\underline{A}_{0i}, \overline{A}_{0i}]$. According to Lemma 2 and Definition 3, there exist $\underline{\varrho}_i \in \mathbb{R}^{2n+p}$ such that the inequalities in (32) hold for any $A_{0i} \in [\underline{A}_{0i}, \overline{A}_{0i}]$, $i \in \mathscr{S}$. Denoting $\underline{\varrho}_i^T = [\underline{v}_i^T \quad \underline{q}_i^T]$ where $\underline{v}_i \in \mathbb{R}^n$ and $\underline{q}_i \in \mathbb{R}^{n+p}$, we obtain

$$\underline{v}_i^T\overline{A}_{0i} + \underline{q}_i^T(W_i^{-1}J_n\overline{A}_{0i} - W_i^{-1}\underline{F}_iJ_n) + \sum_{j=1}^{s}\lambda_{ij}\underline{v}_j^T \prec 0, \tag{34}$$

$$\underline{q}_i^T W_i^{-1}(\underline{F}_i - LC_{fi} - \underline{G}_iC_i) + \sum_{j=1}^{s}\lambda_{ij}\underline{q}_j^T \prec 0. \tag{35}$$

Define diagonal matrices $\underline{Q}_i$, with the elements of $\underline{q}_i$ on the main diagonal. Based on the definitions of $\underline{M}_i$ and $\underline{R}_i$, it turns out that the change of variables in (26) linearizes the above constraints concerning stability and yields (24) and (25).

Since augmented system (18) and lower-bounding observer (11) are positive, we get that $\underline{\mathscr{A}}_i$ are Metzler matrices for any $i \in \mathscr{S}$, that is, $W_i^{-1}(A_i - \underline{F}_i)J_n$ are nonnegative and $W_i^{-1}(\underline{F}_i - LC_{fi} - \underline{G}_iC_i)$ are Metzler. With $\underline{M}_i = W_i^{-1}\underline{F}_iJ_n$, $\underline{R}_i = W_i^{-1}\underline{G}_i$ and $A_{0i} \in [\underline{A}_{0i}, \overline{A}_{0i}]$, the results in (24) and (25) can be obtained directly. □

As for positive upper-bounding observers, similar results can be readily obtained. A necessary and sufficient condition for the existence of such observers is presented in the following theorem with the proof omitted.

Theorem 2. Given a RMES PMJLS (6) with $u(t) = 0$, a positive upper-bounding observer (19) exists such that the augmented system (21) is positive and mean exponentially stable for any $A_{0i} \in [\underline{A}_{0i}, \overline{A}_{0i}]$ (RMES) if and only if there exist strictly positive vectors $\overline{v}_i \in \mathbb{R}^n$, $(n+p)$-dimensional diagonal matrices $\overline{Q}_i \succ 0$, matrices $\overline{U}_i \in \mathbb{R}^{(n+p)\times n}$ and $\overline{V}_i \in \mathbb{R}^{(n+p)\times p}$, $i = 1, 2, \dots, s$, satisfying

$$\overline{U}_i - \overline{Q}_iW_i^{-1}J_n\overline{A}_{0i} \ge 0, \tag{36}$$

$$\overline{U}_iJ_n^T - \overline{Q}_i(LL^T + W_i^{-1}LC_{fi}) - \overline{V}_iC_i \quad \text{is Metzler}, \tag{37}$$

$$\overline{v}_i^T A_{0i} + \mathbf{1}^T(\overline{U}_i - \overline{Q}_iW_i^{-1}J_n\underline{A}_{0i}) + \sum_{j=1}^{s}\lambda_{ij}\overline{v}_j^T \prec 0, \tag{38}$$

$$\mathbf{1}^T(\overline{U}_iJ_n^T - \overline{Q}_iLL^T - \overline{Q}_iW_i^{-1}LC_{fi} - \overline{V}_iC_i + \sum_{j=1}^{s}\lambda_{ij}\overline{Q}_j) \prec 0, \tag{39}$$

where $\overline{F}_i$ and $\overline{G}_i$ are given by

$$\begin{aligned} \overline{F}_i &= W_i(\overline{M}_iJ_n^T - LL^T), \quad \overline{M}_i = \overline{Q}_i^{-1}\overline{U}_i, \\ \overline{G}_i &= W_i\overline{R}_i, \quad \overline{R}_i = \overline{Q}_i^{-1}\overline{V}_i. \end{aligned} \tag{40}$$

4.2 | Optimization of observer design

In the following, we will deal with the case when $u(t) \neq 0$ and the aim is to reduce the error signals as far as possible for any $u(t) \succeq 0$ without violating positivity and stability.

Theorem 3. For the RMES PMJLS (6), a state-bounding observer in (11), (19) exists with observer matrices $\underline{F}_i$, $\underline{G}_i$, $\overline{F}_i$, $\overline{G}_i$ given by Theorem 1, Theorem 2 and $\underline{K}_i$, $\overline{K}_i$ given by

$$\underline{K}_i = W_i \underline{\mathcal{K}}_i; \quad \overline{K}_i = W_i \overline{\mathcal{K}}_i, \tag{41}$$

where $\underline{\mathcal{K}}_i, \overline{\mathcal{K}}_i \in \mathbb{R}^{(n+p)\times m}$ satisfying

$$0 \le \underline{\mathcal{K}}_i \le W_i^{-1} J_n \underline{B}_{0i}; \quad \overline{\mathcal{K}}_i \ge W_i^{-1} J_n \overline{B}_{0i}. \tag{42}$$

Theorem 3 can be formulated directly using the dynamics of lower-bounding observer (11), upper-bounding observer (19) and the corresponding dynamics of $\check{e}$, $\hat{e}$. Based on the existence conditions in Theorem 3, optimization techniques are applied for the computation of the observer matrices. Introduce two auxiliary systems given by

$$\dot{\check{\mathbf{e}}} = W_i^{-1}(\underline{F}_i - LC_{fi} - \underline{G}_i C_i)\check{\mathbf{e}} + \underline{S}_i x + (W_i^{-1} J_n \overline{B}_{0i} - W_i^{-1} \underline{K}_i)u, \tag{43}$$

$$\dot{\hat{\mathbf{e}}} = W_i^{-1}(\overline{F}_i - LC_{fi} - \overline{G}_i C_i)\hat{\mathbf{e}} + \overline{S}_i x + (W_i^{-1} \overline{K}_i - W_i^{-1} J_n \underline{B}_{0i})u, \tag{44}$$

where $\underline{S}_i = W_i^{-1} J_n \overline{A}_{0i} - W_i^{-1} \underline{F}_i J_n$, $\overline{S}_i = W_i^{-1} \overline{F}_i J_n - W_i^{-1} J_n \underline{A}_{0i}$.

Proposition 1. For positive lower-bounding observer (11), if the initial conditions of the corresponding error dynamic system (17) and (43) satisfy $0 \preceq \check{e}(0) \preceq \check{\mathbf{e}}(0)$ for any $(A_{0i}, B_{0i}) \in \Phi$, then $0 \preceq \check{e}(t) \preceq \check{\mathbf{e}}(t)$ for any $t > 0$. Similarly, for positive upper-bounding observer (19), if the initial conditions satisfy $0 \preceq \hat{e}(0) \preceq \hat{\mathbf{e}}(0)$ for any $(A_{0i}, B_{0i}) \in \Phi$, then $0 \preceq \hat{e}(t) \preceq \hat{\mathbf{e}}(t)$ for any $t > 0$.

According to Proposition 1, (43) and (44) provide upper bounds for the observer error of lower-bounding and upper-bounding observer, respectively. An optimization approach is then proposed in the following by reducing the error signals $\check{e}$ and $\hat{e}$ through their upper bounds $\check{\mathbf{e}}$ and $\hat{\mathbf{e}}$, respectively. Different from classical optimization techniques such as the H_∞ and L_2-L_∞ optimization methods, which are applicable with the prerequisite of specific disturbance signals, the proposed approach can be used regardless of which class of signals $u(t)$ belongs to.

With the given structures of $\check{\mathbf{e}}$ and $\hat{\mathbf{e}}$ in (43) and (44) as well as Theorem 3, we have $\underline{K}_i = J_n \underline{B}_{0i}$ and $\overline{K}_i = J_n \overline{B}_{0i}$ so as to minimize the error signals. Moreover, considering that $\check{\mathbf{e}}$ and $\hat{\mathbf{e}}$ will be amplified by $x(t)$ through $\underline{S}_i$ and $\overline{S}_i$, we may minimize the norm of $\underline{S}_i$ and $\overline{S}_i$ to reduce the error signals. As for the minimization of the norm of $\underline{S}_i$, since $\underline{S}_i = W_i^{-1} J_n \overline{A}_{0i} - W_i^{-1} \underline{F}_i J_n$, and $\underline{F}_i$ is given by $\underline{F}_i = W_i(\underline{Q}_i^{-1} \underline{U}_i J_n^T - LL^T)$, an equivalent problem is as follows:

$$\min_{\underline{Q}_i, \underline{U}_i} \quad \epsilon_i \quad \text{subject to} \quad \|\underline{S}_i\|^2 < \epsilon_i \Longleftrightarrow \min_{\underline{Q}_i, \underline{U}_i} \quad \epsilon_i \quad \text{subject to} \quad \begin{bmatrix} -\epsilon_i I & \# \\ \underline{S}_i & -I \end{bmatrix} < 0. \tag{45}$$

With diagonal matrices $\underline{Q}_i \succ 0$, $i \in \mathcal{S}$, the problem in (45) can then be described by

$$\min_{\underline{Q}_i, \underline{U}_i} \quad \epsilon_i \quad \text{subject to} \quad \begin{bmatrix} -\epsilon_i I & \# \\ \underline{Q}_i \underline{S}_i & -\underline{Q}_i^2 \end{bmatrix} < 0. \tag{46}$$

Given $W_i^{-1} \underline{F}_i J_n = \underline{M}_i$, and $\underline{M}_i = \underline{Q}_i^{-1} \underline{U}_i$, we have $\underline{Q}_i \underline{S}_i = \underline{Q}_i W_i^{-1} J_n \overline{A}_{0i} - \underline{U}_i$. Since the problem (46) is a non-convex one, to find a solution numerically, we will propose an iterative algorithm which solves a series of convex optimization problems by fixing some of optimization variables. It follows from the inequality $-\underline{Q}_i^2 \le -\underline{Q}_i X_i - X_i^T \underline{Q}_i + X_i^T X_i$ that

$$\begin{bmatrix} -\epsilon_i I & \# \\ \underline{Q}_i \underline{S}_i & -\underline{Q}_i^2 \end{bmatrix} \le \begin{bmatrix} -\epsilon_i I & \# \\ \underline{Q}_i W_i^{-1} J_n \overline{A}_{0i} - \underline{U}_i & -\underline{Q}_i X_i - X_i^T \underline{Q}_i + X_i^T X_i \end{bmatrix}. \tag{47}$$

Now we have the optimization problem relaxed as

$$\min_{\underline{Q}_i, \underline{U}_i} \quad \epsilon_i \quad \text{subject to} \quad \begin{bmatrix} -\epsilon_i I & \# \\ \underline{Q}_i W_i^{-1} J_n \overline{A}_{0i} - \underline{U}_i & -\underline{Q}_i X_i - X_i^T \underline{Q}_i + X_i^T X_i \end{bmatrix} < 0. \tag{48}$$

Thus, the quadratic matrix inequality in (46) can be transformed into an LMI under a fixed X_i, $i \in \mathcal{S}$. On the other hand, the equality in (47) holds if and only if $X_i = \underline{Q}_i$. One considers a weighted combination of the objectives here. That is, for given

$\alpha_i \in (0,1)$, $i \in \mathscr{S}$, where $\sum_{i=1}^{s} \alpha_i = 1$, we will solve the following minimization problem:

$$\min_{\underline{v}_i,\underline{Q}_i,\underline{U}_i,\underline{V}_i} \quad \zeta \triangleq \sum_{i=1}^{s} \alpha_i \epsilon_i \quad \text{subject to} \quad \begin{bmatrix} -\epsilon_i I & \# \\ \underline{Q}_i W_i^{-1} J_n \overline{A}_{0i} - \underline{U}_i & -\underline{Q}_i X_i - X_i^T \underline{Q}_i + X_i^T X_i \end{bmatrix} < 0, \\ \text{and inequalities (22)–(25)}, \quad i = 1,2,\ldots,s. \tag{49}$$

The discussion above leads to the following iterative convex optimization algorithm.

Iterative optimization for positive lower-bounding observer:

- Step 1. Set $j = 1$. For the given robustly mean exponentially stable PMJLS (6) with $(A_{0i}, B_{0i}) \in \Phi$, solve the linear programming problems (22)–(25) with respect to positive vectors $\underline{v}_i$, diagonal matrices $\underline{Q}_i \succ 0$, matrices $\underline{U}_i$ and $\underline{V}_i$, $i = 1,2,\ldots,s$. If there does not exist a solution to the linear programming problems, then go to Step 5.
- Step 2. Set $X_i = \underline{Q}_i$, $i \in \mathscr{S}$. Solve the convex optimization problems (49) with respect to positive vectors $\underline{v}_i$, diagonal matrices $\underline{Q}_i \succ 0$, matrices $\underline{U}_i$ and $\underline{V}_i$. Denote ζ_j^* as the solution of the optimization problem in this iteration.
- Step 3. If $\zeta_j^* \le \bar{\zeta}$, where $\bar{\zeta} \ge 0$ is a prescribed bound, then a desired solution is obtained. STOP.
- Step 4. If $|\zeta_j - \zeta_{j-1}| \le \Delta\zeta$, where $\Delta\zeta$ is a prescribed tolerance bound, then go to Step 4. Otherwise, set $j = j+1$, go to Step 2.
- Step 5. A solution to the problem may not exist. STOP.

Such an iterative optimization algorithm is similar in spirit to the D-K iterations used in robust control.[59] Step 3 is to guarantee the convergence of the algorithm. In spite of the fact the iterative algorithm is not guaranteed to converge to a global optimum, it has been observed that the solution sequence corresponding to the weighted combination of $\|\underline{S}_i\|$ can be suboptimal. The effectiveness of iterative optimization algorithms has also been discussed in many literature.[59,60] Similar analysis can be applied to the optimization of the positive upper-bounding observer design and is omitted for brevity here.

5 | ILLUSTRATIVE EXAMPLE

This section will present a numerical example to illustrate the effectiveness of the proposed observer approach. Consider a three-dimensional PMJLS of form (6) with $\theta(t) \in \mathscr{S} = \{1,2\}$, and its parameters are given by

$$A_{01} = \begin{bmatrix} -1.5 \pm 0.04 & 0.35 & 0.5 \\ 0.1 & -1.3 \pm 0.01 & 0.7 \pm 0.2 \\ 0.6 & 0.4 \pm 0.05 & -2 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0.2 \\ 0.5 \pm 0.05 \\ 0.3 \end{bmatrix}, \quad C_{01} = \begin{bmatrix} 0.9 & 0.8 & 1 \end{bmatrix},$$

$$A_{02} = \begin{bmatrix} -1.8 \pm 0.02 & 0.2 & 0.4 \pm 0.01 \\ 0.5 & -0.95 & 0.3 \\ 0.35 & 0.55 & -1.5 \pm 0.03 \end{bmatrix}, \quad B_{02} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \pm 0.06 \end{bmatrix}, \quad C_{02} = \begin{bmatrix} 0.8 & 0.9 & 0.9 \end{bmatrix},$$

The generator matrix is known and given as

$$\Lambda = \begin{bmatrix} -3.5 & 3.5 \\ 2.4 & -2.4 \end{bmatrix}$$

The positivity and robust mean exponential stability of the original system can be verified easily. Suppose the system input and the sensor fault are, respectively,

$$u(t) = 0.25|\sin(\pi t)| + 5e^{-3t}, \quad f(t) = \begin{cases} 0 & t < 2 \\ 0.2 - 0.4\sin(0.4\pi(-t+2)) & \text{otherwise} \end{cases},$$

a pair of state-bounding observers in the form of (11) and (19) is then constructed by solving the conditions in Theorems 1 and 2. Thus we obtain a set of observer matrices as follows:

$$\underline{F}_1 = \begin{bmatrix} -1.8414 & 0.2703 & 0.3823 & 0 \\ 0.0735 & -1.5363 & 0.3507 & 0 \\ 0.4454 & 0.2768 & -2.2917 & 0 \\ 0.4111 & 0.4999 & 0.3169 & 1.0000 \end{bmatrix}, \quad \underline{F}_2 = \begin{bmatrix} -2.0754 & 0.1810 & 0.3352 & 0 \\ 0.3895 & -1.0908 & 0.2713 & 0 \\ 0.2968 & 0.4733 & -1.7261 & 0 \\ 0.1630 & 0.3371 & 0.2202 & 1.0000 \end{bmatrix},$$

$$\overline{F}_1 = \begin{bmatrix} -1.2185 & 0.5580 & 0.7580 & 0 \\ 0.5395 & -1.1957 & 1.0480 & 0 \\ 0.8924 & 0.6724 & -1.7492 & 0 \\ 0.6256 & 0.3814 & 0.3553 & 1.0000 \end{bmatrix}, \quad \overline{F}_2 = \begin{bmatrix} -1.5889 & 0.3860 & 0.5946 & 0 \\ 0.6526 & -0.9050 & 0.4980 & 0 \\ 0.6535 & 0.6764 & -1.3150 & 0 \\ 0.3643 & 0.2796 & 0.3193 & 1.0000 \end{bmatrix},$$

$$\underline{G}_1 = \begin{bmatrix} 0.1122 & 0.0381 & 0.1313 & 3.1309 \end{bmatrix}^T, \quad \underline{G}_2 = \begin{bmatrix} 0.0754 & 0.1292 & 0.1495 & 2.3534 \end{bmatrix}^T,$$
$$\overline{G}_1 = \begin{bmatrix} 0.3378 & 0.3667 & 0.4286 & 1.0593 \end{bmatrix}^T, \quad \overline{G}_2 = \begin{bmatrix} 0.2118 & 0.2998 & 0.3506 & 0.8796 \end{bmatrix}^T.$$

Under the obtained observer matrices, the 50 realizations of responses of the state $x(t)$ and its bounding estimate when $u(t) = 0$ are simulated with initial conditions set as $x(0) = [0.5 \quad 0.4 \quad 0.3]^T$. $\check{x}(0)$ and $\hat{x}(0)$ are selected to ensure that $0 \preceq \check{x}(0) \preceq x(0) \preceq \hat{x}(0)$ always holds. Here $\check{x}(0)$ and $\hat{x}(0)$ are set as $[0.2 \quad 0.1 \quad 0.1]^T$ and $[1 \quad 1 \quad 1]^T$, respectively. The result for state estimation is plotted in Figure 1 and the simulation result for fault estimation is given in Figure 2. Therefore, the observer matrices given above are able to guarantee the existence of positive state-bounding observer in terms of positivity and stability.

For subsequent optimization of the observer design, the proposed iterative optimization algorithm in Section 4.2 is applied. Since optimizations involving the two modes are considered, there is a trade-off between the minimal values of $\|\underline{S}_1\|$ ($\|\overline{S}_1\|$) and $\|\underline{S}_2\|$ ($\|\overline{S}_2\|$). In accordance with the steady state probabilities of the Markov process,[61] the weighting parameters α_1 and α_2 are selected as $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$, respectively. With the selected parameters, we ran the algorithm presented in Section 4.2. Values related to the optimization for positive upper-bounding observer ($\|\overline{S}_1\|$, $\|\overline{S}_2\|$ and $\sum_{i=1}^{2} \alpha_i \|\overline{S}_i\|^2$) are plotted in Figure 3, where the monotonic non-increasing property of the actual optimization objective $\sum_{i=1}^{2} \alpha_i \|\overline{S}_i\|^2$ can be verified. Accordingly, a set of optimized observer matrices can be obtained:

$$\underline{F}_1 = \begin{bmatrix} -1.5400 & 0.3500 & 0.5000 & 0 \\ 0.1000 & -1.3100 & 0.5000 & 0 \\ 0.6000 & 0.3500 & -2.0000 & 0 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}, \quad \underline{F}_2 = \begin{bmatrix} -1.8200 & 0.1999 & 0.3900 & 0 \\ 0.4999 & -0.9764 & 0.3000 & 0 \\ 0.3500 & 0.5500 & -1.5300 & 0 \\ -0.0001 & 0.0238 & 0.0000 & 1.0000 \end{bmatrix},$$

$$\overline{F}_1 = \begin{bmatrix} -1.4600 & 0.3500 & 0.5000 & 0 \\ 0.2745 & -1.2900 & 0.9000 & 0 \\ 0.6220 & 0.5529 & -1.9604 & 0 \\ 0.1616 & 0.1029 & 0.0396 & 1.0000 \end{bmatrix}, \quad \overline{F}_2 = \begin{bmatrix} -1.7800 & 0.2000 & 0.4100 & 0 \\ 0.5000 & -0.9500 & 0.3788 & 0 \\ 0.4343 & 0.5500 & -1.4700 & 0 \\ 0.0707 & 0.0000 & 0.0710 & 1.0000 \end{bmatrix},$$

$$\underline{G}_1 = \begin{bmatrix} 0.2117 & 0.0582 & 0.2354 & 2.7456 \end{bmatrix}^T, \quad \underline{G}_2 = \begin{bmatrix} 0.1186 & 0.2076 & 0.2589 & 1.7116 \end{bmatrix}^T,$$
$$\overline{G}_1 = \begin{bmatrix} 0.4375 & 0.3050 & 0.6911 & 3.1588 \end{bmatrix}^T, \quad \overline{G}_2 = \begin{bmatrix} 0.2222 & 0.4209 & 0.5429 & 2.8626 \end{bmatrix}^T.$$

From Theorem 3 and Proposition 1, we have the observer matrices $\underline{K}_i$ and $\overline{K}_i$ as follows:

$$\underline{K}_1 = \begin{bmatrix} 0.2 & 0.45 & 0.3 & 0 \end{bmatrix}^T, \quad \underline{K}_2 = \begin{bmatrix} 0.1 & 0.2 & 0.24 & 0 \end{bmatrix}^T,$$
$$\overline{K}_1 = \begin{bmatrix} 0.2 & 0.55 & 0.3 & 0 \end{bmatrix}^T, \quad \overline{K}_2 = \begin{bmatrix} 0.1 & 0.2 & 0.36 & 0 \end{bmatrix}^T.$$

Simulation is carried out with initial conditions set as $x(0) = [0.5 \quad 0.4 \quad 0.3]^T$. For comparison, Figures 4, 5, and 6, which depict the trajectories of actual state, the optimized bounding estimate of system states and the bounding estimate without optimization, are presented to show the effectiveness of the iterative optimization algorithms in reducing error signals. Besides, it can be seen clearly that the states remain in the positive interval at all times for any $(A_i, B_i) \in \Phi$. Figure 7 illustrates the simulation result for fault estimation using the iterative optimization method. Asymptotic convergence of fault estimation error can be achieved in accordance with the above theoretical analysis.

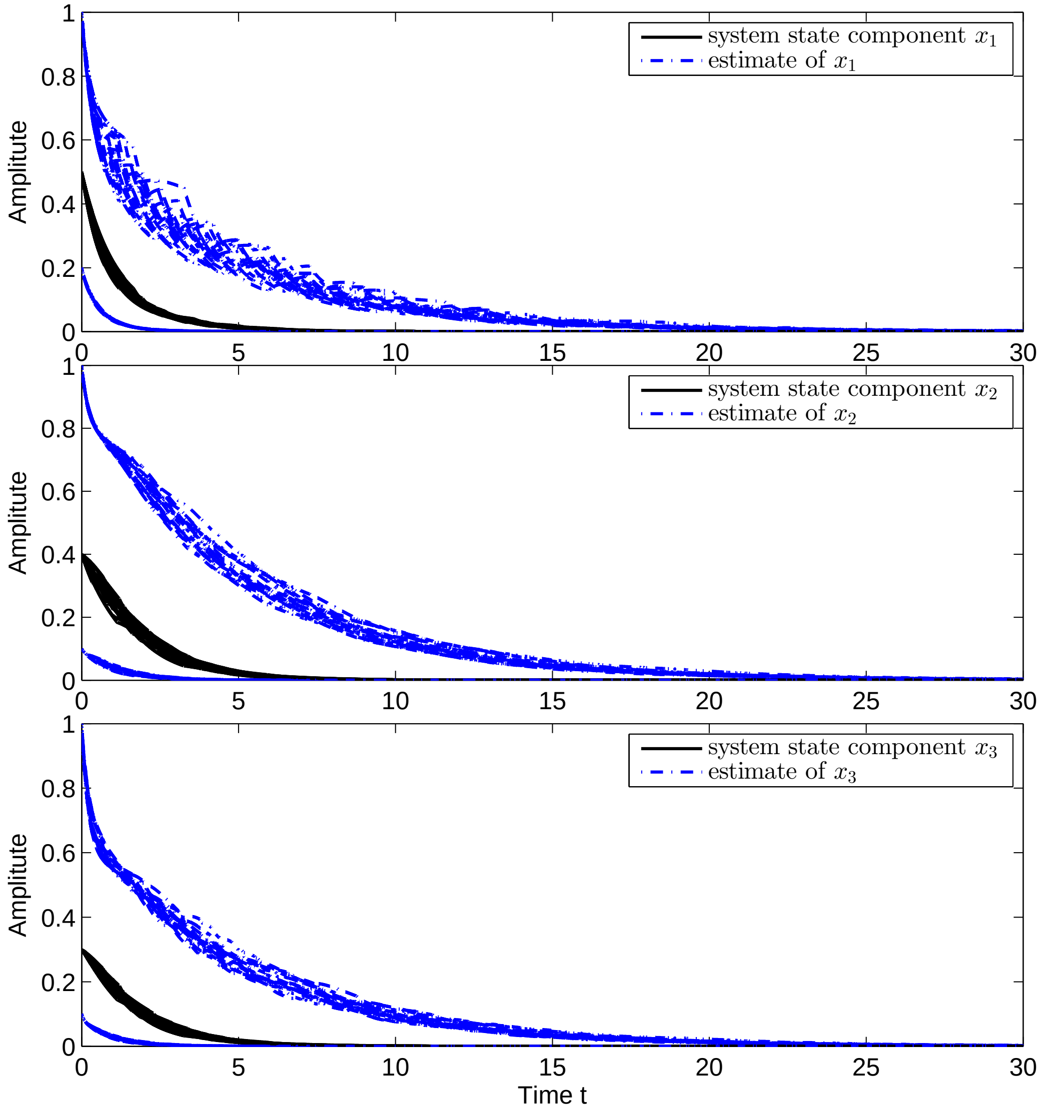

FIGURE 1 System state components and their upper- and lower-bounding estimates

6 | CONCLUSION

The problem of observer design for positive Markov jump systems with uncertainties and sensor faults has been investigated in this paper. By transforming the system to an augmented descriptor system, new observer design techniques have been formulated. With system positivity and robust mean exponential stability taken into consideration, a state-bounding observer approach has been proposed to provide the upper and lower estimates of the system state and sensor faults simultaneously. An iterative optimization algorithm has also been proposed to compute the optimized state-bounding observer matrices. Further research work should also consider actuator and component faults so as to realize a simultaneous fault estimation and compensation for positive Markov jump systems in real applications. On the other hand, in recent years, the effects of time delays on stochastic stability and performance of positive Markov jump systems have been discussed. It would be interesting to extend the state-bounding observer approach to positive Markov jump systems with time delays in the future.

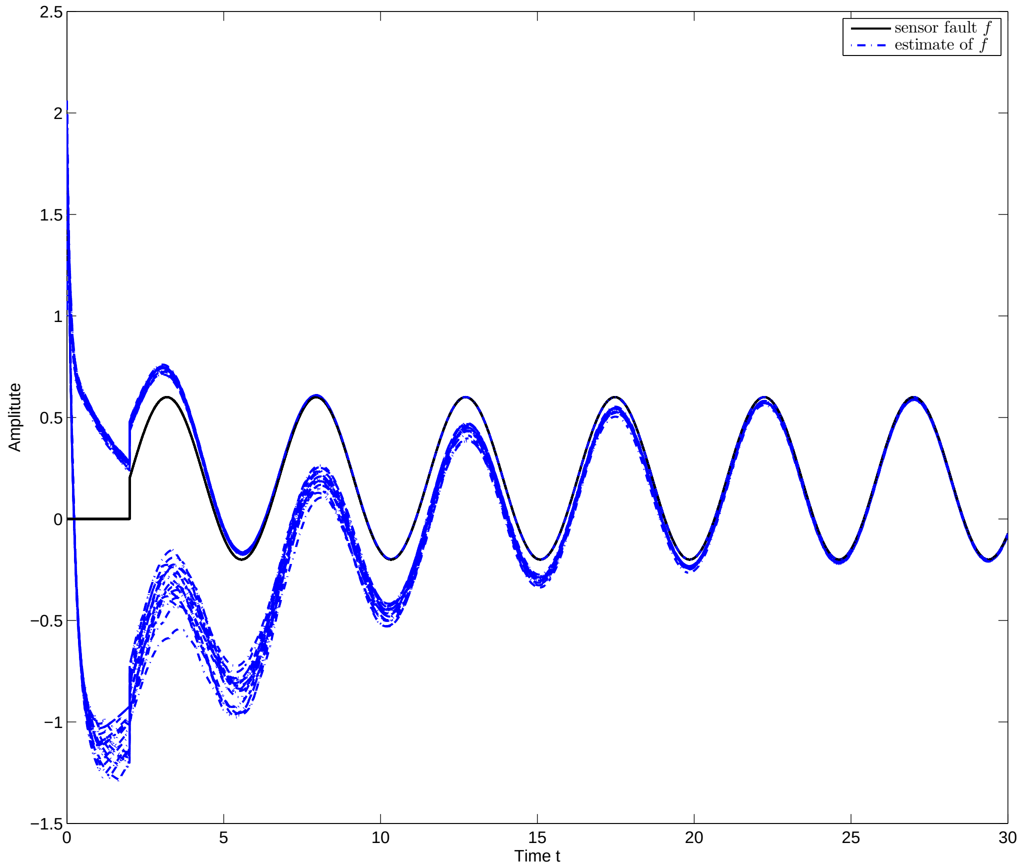

FIGURE 2 Sensor fault $f(t)$ and its upper- and lower-bounding estimates

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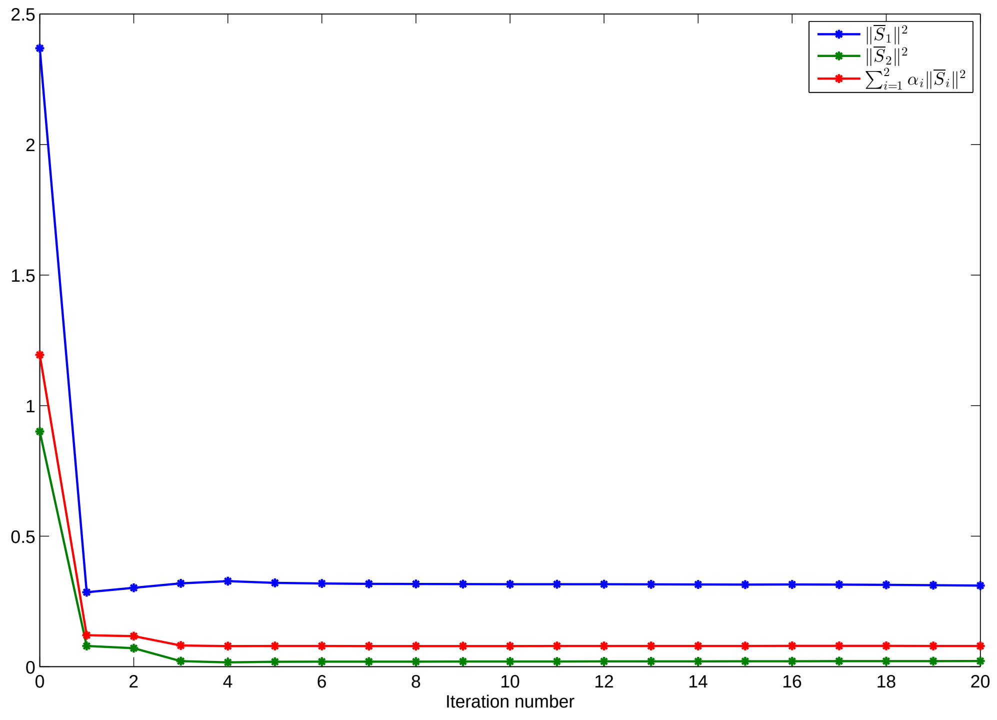

FIGURE 3 Iterative optimization for positive upper-bounding observer

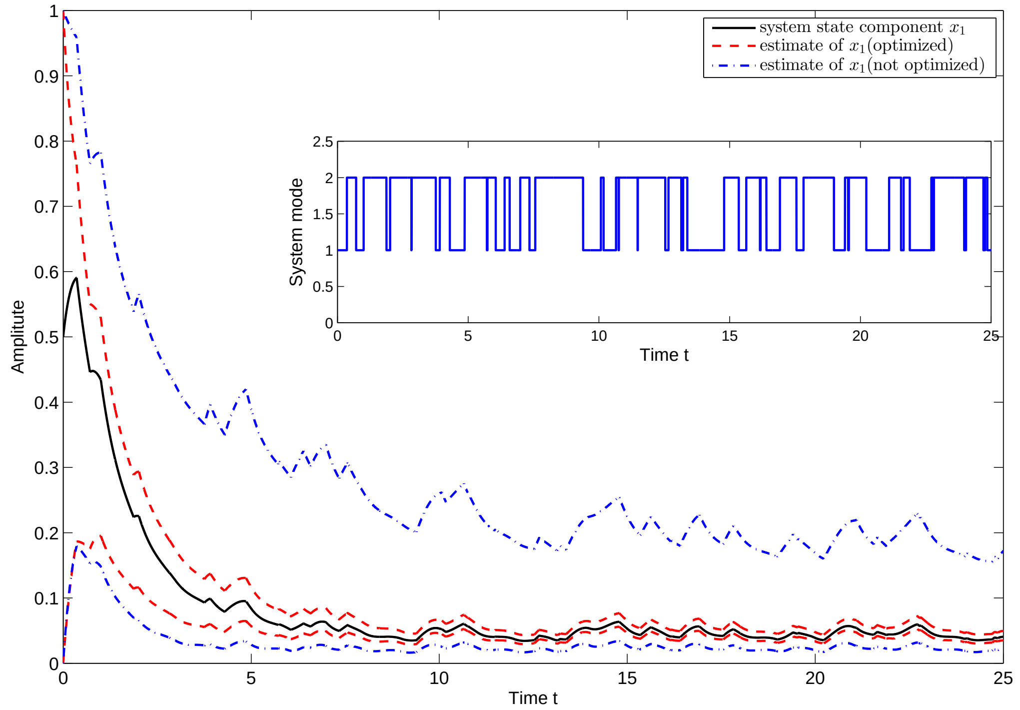

FIGURE 4 System state component $x_1(t)$ and its upper- and lower-bounding estimates

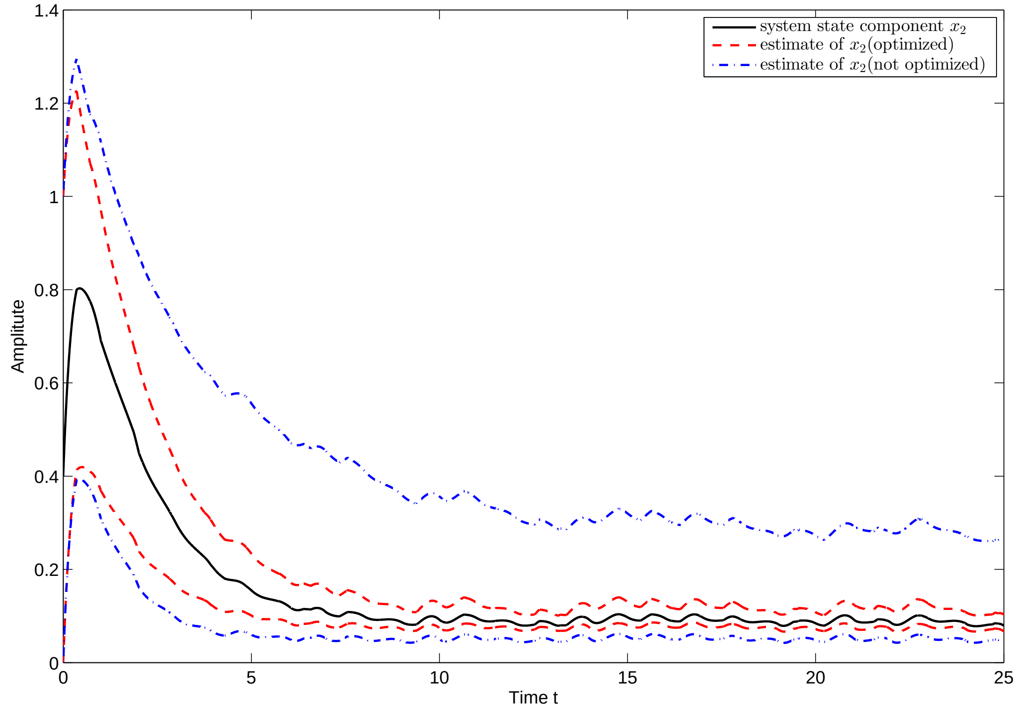

FIGURE 5 System state component $x_2(t)$ and its upper- and lower-bounding estimates

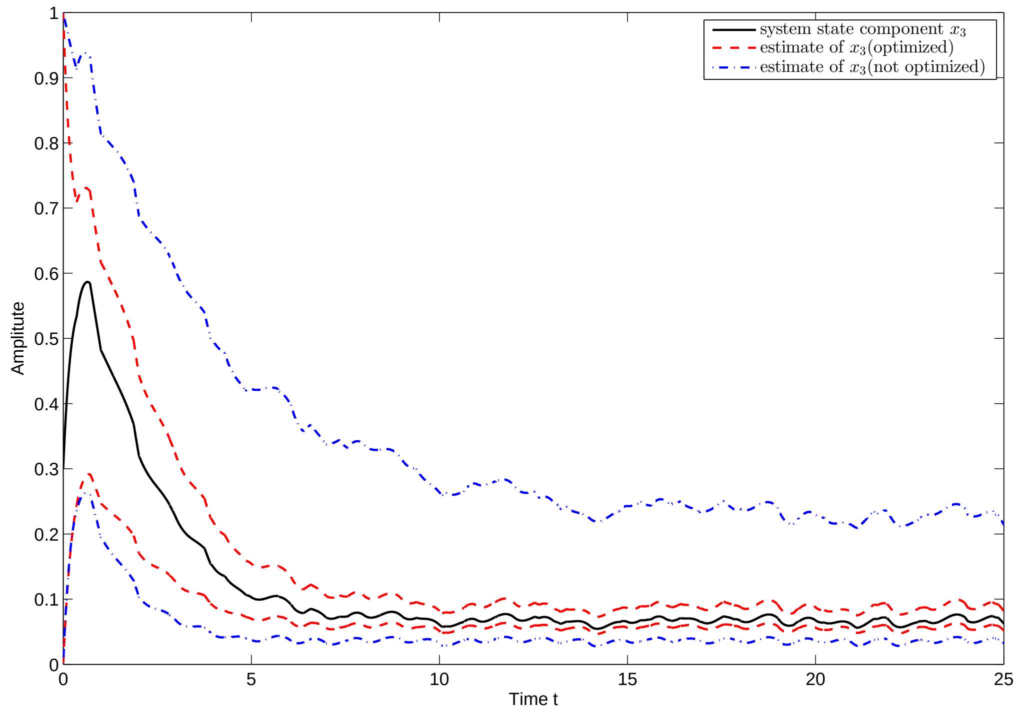

FIGURE 6 System state component $x_3(t)$ and its upper- and lower-bounding estimates

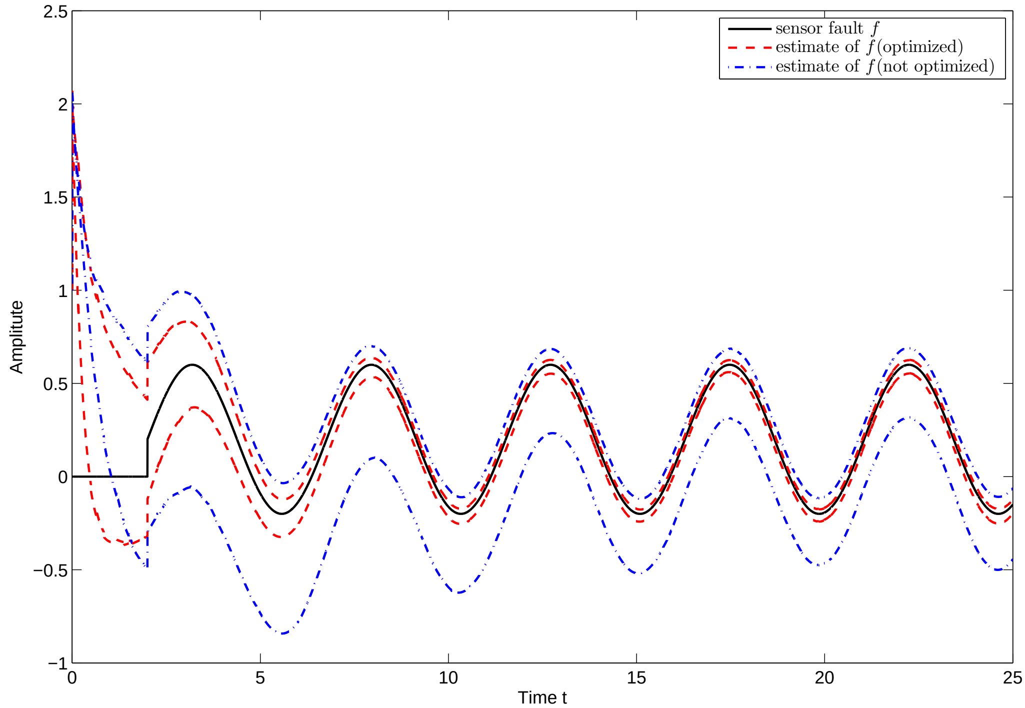

FIGURE 7 Sensor fault $f(t)$ and its upper- and lower-bounding estimates

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