Role of Effective Range in the Bulk Viscosity of Resonantly Interacting s- and p-Wave Fermi Gases

Jeff Maki and Shizhong Zhang[*]

Department of Physics and HKU-UCAS Joint Institute for Theoretical and Computational Physics at Hong Kong,
The University of Hong Kong, Hong Kong, China

We investigate the role of the effective range on the bulk viscosity of s- and p-wave Fermi gases. At resonance, the presence of the effective range breaks the scale invariance of the system, and hence results in a nonzero bulk viscosity. However, we show that the effective range plays a very different role in the two cases. In the s-wave case, the role of the effective range is perturbative, and its contribution to the bulk viscosity vanishes in the limit of zero effective range. On the other hand, the effective range in p-wave Fermi gases leads to a nonzero bulk viscosity, even in the zero-range limit. We employ a general diagrammatic approach to compute the bulk viscosity spectral function that includes the effects of the effective range. We then compute the analytic expressions for the spectral function in the high temperature limit, at low and high frequencies. We also derive the sum rules for the bulk viscosity spectral function for both s- and p-wave gases.

Introduction.—One of the most unique features of ultracold atomic gases is their universality [1]. At low energies, the details of the short-range interatomic potential are irrelevant, and only a few low-energy scattering parameters are needed to describe the low-energy properties of the system [2,3]. In the case of s-wave interactions, it is known that the scattering length, a_0, is the only relevant low-energy parameter for the so-called broad Feshbach resonance [4]. At resonance, when $a_0 = \pm\infty$, the system acquires scale symmetry, which imposes severe constraints on the thermodynamic [5,6], and transport properties of the system, the most amazing of which is the vanishing of the bulk viscosity [7–10].

In actual dilute gas systems, however, a nonzero effective range always exists which breaks the scale symmetry of the system at resonance. A similar situation also arises in neutron matter, where the presence of the effective range is even more pronounced [11–13]. It is then natural to expect that the thermodynamic and transport properties of these systems will also depend on the effective range. To be specific, let $\delta_\ell(k)$ be the phase shift for two colliding particles with relative wave number, k, and angular momentum, ℓ, then [14]

$$k^{2\ell+1} \cot \delta_\ell(k) = -\frac{1}{a_\ell} - R_\ell k^2 + \cdots. \qquad (1)$$

We will refer to a_ℓ as the scattering length, and R_ℓ as the effective range, noting that for p-wave scattering, a_1 has dimension of volume and R_1 has dimension of wave number. In actual experiments, a Feshbach resonance allows one to tune the scattering length, a_ℓ, while the effective range, R_ℓ, is approximately a constant. For any given system, the effective range, R_ℓ, will be determined by the range of the inter-atomic interaction, r_0. As a result, Eq. (1) is only valid at low energies $k \ll r_0^{-1}$. For the s wave, this means that $k \ll R_0^{-1}$ while for the p wave, $k \ll R_1$.

To see the effects of the effective range, let us consider, for example, the binding energy of the two-body bound state, E_b, close to threshold, when R_ℓ is small. In the s-wave case, $E_b = -(\hbar^2/m)(1/a_0^2 - 2R_0/a_0^3)$ and tends to the universal result $-\hbar^2/(ma_0^2)$ when $R_0 \to 0$; the effect of R_0 is perturbative. On the other hand, for the p-wave case [15,16], the existence of the shallow bound state depends crucially on the effective range: $E_b = -\hbar^2/(ma_1 R_1)$. It is thus impossible to set $R_1 = 0$ while maintaining a finite bound state energy. The role of the effective range on the energetic properties of the many-body system has been discussed for both s-wave [12,17–22] and p-wave cases [23–27].

In this Letter, we investigate a natural extension of the above studies: what is the role of the effective range on the dynamics of s- and p-wave Fermi gases. In particular, we discuss the role of the effective range in the transport properties of both s- and p-wave Fermi gases. To our knowledge such an investigation has not been conducted, and *a priori* it is not obvious how the effective range will alter the dynamics. To this end, we calculate the bulk viscosity spectral function in the high temperature limit, and investigate how it depends on the effective range, particularly at resonance, i.e., when the scattering length is infinite: $a_\ell^{-1} = 0$. In this limit, the viscosity spectral functions do not depend on a_ℓ, and the role of the effective

range is most transparent. If the effective range is an irrelevant quantity in the low-energy limit, scale invariance will be restored when one sets R_ℓ to zero. As a result, the bulk viscosity will necessarily vanish. However, if the effective range is relevant in the low-energy limit, it will be impossible to truly set the effective range to zero; the scale symmetry will then remain broken, and there will be a finite bulk viscosity.

General setup.—To describe both the resonant *s*- and *p*-wave Fermi gases in a unified scheme, we use the following two-channel Hamiltonian describing a system of fermions with spin σ, and $2\ell + 1$ molecules interacting via the ℓth partial wave channel (we set $m = \hbar = 1$):

$$\hat{H}_\ell = \sum_{\mathbf{k},\sigma}\left(\frac{k^2}{2} - \mu\right)\psi_\sigma^\dagger(\mathbf{k})\psi_\sigma(\mathbf{k})$$

$$+ \sum_{\mathbf{Q}}\sum_{m=-\ell}^{\ell}\left(\frac{Q^2}{4} + \nu - 2\mu\right)d_m^\dagger(\mathbf{Q})d_m(\mathbf{Q})$$

$$+ g\sum_{m=-\ell}^{\ell}\sum_{\mathbf{Q},\mathbf{k},\sigma}\sqrt{\frac{4\pi}{V}}k^\ell$$

$$\left[Y_{\ell,m}(\hat{\mathbf{k}})d_m^\dagger(\mathbf{Q})\psi_\sigma\left(\frac{\mathbf{Q}}{2} + \mathbf{k}\right)\psi_{\bar\sigma}\left(\frac{\mathbf{Q}}{2} - \mathbf{k}\right) + \text{H.c.}\right]. \quad (2)$$

$\psi_\sigma(\mathbf{k})$ is the field operator for the fermions with spin projection σ and momentum $\mathbf{k}$, while $d_m(\mathbf{Q})$ is the field operator for the molecules with an azimuthal quantum number, m, and momentum $\mathbf{Q}$. V is the volume of the system, ν is the detuning of the molecular field, and g is the fermion-molecular coupling. We also let μ be the chemical potential, and $Y_{\ell,m}(\hat{\mathbf{k}})$ is simply the (ℓ, m)-spherical harmonic function. It is important to note that for *s*-wave interactions we assume the fermions have two spin projections: $\sigma = -\bar\sigma = 1/2$, while for the *p*-wave interactions, all the fermions have the same projection $\sigma = \bar\sigma$. Here we have assumed that the scattering potential is isotropic in space, and the coupling constants ν and g are independent of the azimuthal quantum number, m. By examining the two-body scattering of this model and comparing it to Eq. (1), the bare coupling constants ν and g can be related to the low-energy scattering parameters in the effective range expansion. For the *s*-wave case, we have $1/(4\pi a_0) = \Lambda/(2\pi^2) - \nu/g^2$ and $R_0 = 4\pi/g^2$, while for *p*-wave interactions $1/(2\pi a_1) = \Lambda^3/(3\pi^2) - \nu/g^2$ and $R_1/2\pi = \Lambda/\pi^2 + 1/g^2$, where Λ is an ultraviolet cutoff. The two-channel formulation has been successful in describing various thermodynamic properties of both *s*-wave [1,17] and *p*-wave gases [23,27,28].

Bulk viscosity with finite effective range.—With the theory renormalized, we can now proceed to investigate the bulk viscosity for both the *s*- and *p*-wave gases. Using Kubo's formalism, the bulk viscosity is defined via the retarded correlation function of the stress-energy tensor [8,10]:

$$\zeta(\omega) = \frac{\text{Im}[\chi^R(\omega)]}{9\omega}, \quad (3)$$

$$\chi^R(\omega) = i\int_0^\infty dt e^{-i\omega t}\langle[\hat\Pi(t), \hat\Pi(0)]\rangle, \quad (4)$$

where the brackets denote the thermal average at temperature $T = 1/\beta$, and $\langle\hat\Pi(t)\rangle = 3p(t)V$ is the trace of the stress-energy tensor at time t, and $p(t)$ is the pressure. As discussed in a recent work [29], this formula, as it stands, requires modification since it neglects the contributions to the bulk viscosity from pressure fluctuations. However, in the regimes of interest to us here, we obtain the same results using Eqs. (3) and (4). By employing a scaling analysis, one can show that [30]

$$\hat\Pi(t) = 2\hat{H}_\ell + \frac{(2\ell+1)\hat{C}_{a_\ell}(t)}{a_\ell} + (2\ell-1)\hat{C}_{R_\ell}(t)R_\ell. \quad (5)$$

Here we have defined two thermodynamic contact operators as the derivatives of the Hamiltonian with respect to the scattering length and the effective range: $\hat{C}_{a_\ell} = -\partial\hat{H}/\partial a_\ell^{-1}$ and $\hat{C}_{R_\ell}(t) = -\partial\hat{H}/\partial R_\ell$. Explicitly, they are given by

$$\hat{C}_{a_\ell}(t) = \sum_{m=-\ell}^{\ell}\frac{g^2}{4\pi A_\ell}\int d\mathbf{R}\, d_m^\dagger(\mathbf{R}, t)d_m(\mathbf{R}, t) \quad (6)$$

$$\hat{C}_{R_\ell}(t) = \sum_{m=-\ell}^{\ell}\frac{g^2}{4\pi A_\ell}\int d\mathbf{R}\, d_m^\dagger(\mathbf{R}, t)\left(i\partial_t + \frac{\nabla^2}{4}\right)d_m(\mathbf{R}, t), \quad (7)$$

where $A_\ell = 1 - \delta_{\ell,1}/2$, is a symmetry factor that represents the indistinguishable nature of the fermions when $\ell = 1$. The thermodynamic contacts represent the change in the energy with respect to the scattering length, a_ℓ, and the effective range, R_ℓ, respectively [31–34]. We note that in order to obtain the current form of $\hat{C}_{R_\ell}$, we have integrated out the fermionic degrees of freedom using the Heisenberg equation of motion for the molecular field.

After substituting Eq. (5) into Eq. (4), one can see that the calculation of the bulk viscosity reduces to the evaluation of three retarded correlators [35]:

$$\chi_{a,a}^R(\omega) = i\frac{(2\ell+1)^2}{a_\ell^2}\int_0^\infty dt e^{-i\omega t}\langle[\hat{C}_{a_\ell}(t), \hat{C}_{a_\ell}(0)]\rangle \quad (8)$$

$$\chi_{a,R}^R(\omega) = i\frac{(2\ell-1)(2\ell+1)R_\ell}{a_\ell}\int_0^\infty dt e^{-i\omega t}\langle[\hat{C}_{a_\ell}(t), \hat{C}_{R_\ell}(0)]\rangle \quad (9)$$

$$\chi_{R,R}^R(\omega) = i(2\ell-1)^2 R_\ell^2\int_0^\infty dt e^{-i\omega t}\langle[\hat{C}_{R_\ell}(t), \hat{C}_{R_\ell}(0)]\rangle. \quad (10)$$

It is possible to evaluate the correlation functions in Eqs. (8)–(10) diagrammatically, as depicted in Fig. 1. In this scheme, the molecular propagator is dressed by the process of dissociation and association of two fermions. We neglect further interactions. For the bulk viscosity, these interactions will appear in four-body, or molecule-molecule scattering, processes. In the high temperature limit (see

below), these terms are higher order in the fugacity expansion, and are safely ignored.

After evaluating the diagrams shown in Fig. 1 in Matsubara space and by performing the standard analytical continuation, one can then show that the bulk viscosity in the normal phase is explicitly given by $\zeta_\ell(\omega) = \zeta_{a,a}(\omega) + 2\zeta_{a,R}(\omega) + \zeta_{R,R}(\omega)$ with

$$\zeta_{a,a}(\omega) = \frac{(2\ell + 1)^2}{9a_\ell^2 V} \sum_{m,\mathbf{Q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \mathrm{Im}\left[\frac{g^2 D(\mathbf{Q}, x + i\delta)}{4\pi A_\ell}\right] \mathrm{Im}\left[\frac{g^2 D(\mathbf{Q}, x + \omega + i\delta)}{4\pi A_\ell}\right] f(x, \omega) \tag{11}$$

$$\zeta_{a,R}(\omega) = \frac{(2\ell - 1)(2\ell + 1)R_\ell}{9a_\ell V} \sum_{m,\mathbf{Q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \mathrm{Im}\left[\frac{g^2 D(\mathbf{Q}, x + i\delta)}{4\pi A_l}\right] \mathrm{Im}\left[\frac{g^2 D(\mathbf{Q}, x + \omega + i\delta)}{4\pi A_l}\right] f(x, \omega)\left(x - \frac{Q^2}{4} + 2\mu + \frac{\omega}{2}\right) \tag{12}$$

$$\zeta_{R,R}(\omega) = \frac{(2\ell - 1)^2 R_\ell^2}{9V} \sum_{m,\mathbf{Q}} \int_{-\infty}^{\infty} \frac{dx}{\pi} \mathrm{Im}\left[\frac{g^2 D(\mathbf{Q}, x + i\delta)}{4\pi A_\ell}\right] \mathrm{Im}\left[\frac{g^2 D(\mathbf{Q}, x + \omega + i\delta)}{4\pi A_\ell}\right] f(x, \omega)\left(x - \frac{Q^2}{4} + 2\mu + \frac{\omega}{2}\right)^2, \tag{13}$$

where $f(x, \omega) = \omega^{-1}[n_B(x) - n_B(x + \omega)]$ and $n_B(x) = [\exp(\beta x) - 1]^{-1}$ is the Bose-Einstein distribution function. $D(\mathbf{Q}, x)$ is the molecular propagator with momentum $\mathbf{Q}$ and frequency x, evaluated in the presence of a thermal fermionic background [26]. Equations (11)–(13) provide the general expressions for the bulk viscosity for both s- and p-wave Fermi gases while including the effective range.

In the high temperature limit, i.e., for small fugacities, $z = e^{\beta\mu} \ll 1$, it is possible to neglect the presence of the thermal fermionic background, and use the two-body results. In this limit, Eq. (11) for s-wave interactions is consistent with the results obtained in Refs. [36–38]. A similar expression was also obtained in Ref. [39] for one-dimensional Fermi gases with three-body s-wave interactions. In addition to this term, the presence of the

effective range has generated two additional contributions to the bulk viscosity. We note that the bulk viscosity calculated in a similar limit is also of interest in nuclear physics in relation to the binary neutron star merger, and its subsequent viscous damping [40]. Similarly, the high temperature expansion is relevant to hot neutron matter, and is discussed in Refs [41,42].

At high frequencies, the bulk viscosity spectral function exhibits power law tails, while the magnitude of the tail is proportional to the contacts. In practice, the high frequency limit is determined by setting the frequency much larger than the bound state energy: $\omega \gg a_0^{-2}$ for the s wave and $\omega \gg 1/(a_1 R_1)$ for the p wave, but much smaller than the scale set by the effective range: $\omega \ll R_0^{-2}$ for s-wave interactions and $\omega \ll R_1^2$ for p-wave interactions. For frequencies larger than the scale set by the effective range, our effective field theory is inapplicable, as mentioned previously. Evaluating Eqs. (11)–(13) at high frequencies, and in the high temperature limit, yields the following divergent terms:
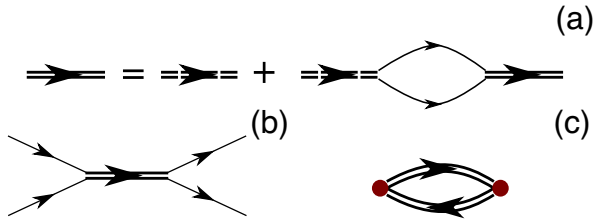
FIG. 1. Diagrammatic calculation for contact-contact correlation functions. (a) The full molecular propagator (solid double line) is given by the bare molecular propagator (dashed double line) dressed by fermionic pair propagation (single solid line). (b) The diagram for the scattering amplitude of two fermions. (c) The diagram for the calculation of bulk viscosity. Each vertex (red) represents an insertion of a contact operator $\hat{C}_{a_\ell}$ or $\hat{C}_{R_\ell}$, see Eqs. (8)–(10).

$$\zeta_0^d(\omega) \approx \frac{\sqrt{\omega}R_0^2}{36V}\left[C_{a_0}\left(1 - \frac{1}{a_0^2\omega} - \frac{4}{a_0 R_0 \omega}\right) + \frac{C_{R_0}}{\omega}\right], \tag{14}$$

$$\zeta_1^d(\omega) \approx \frac{\sqrt{\omega}}{36V}\left[C_{a_1}\left(1 + \frac{10}{a_1 R_1 \omega}\right) + \frac{7C_{R_1}}{2\omega}\right], \tag{15}$$

for the s wave (ζ_0^d), and the p wave (ζ_1^d), respectively. Here we note that the high temperature limit also requires $1/a_0^2 \ll k_B T \ll R_0^{-2}$, for s-wave gases, while for p-wave gases: $1/(a_1 R_1) \ll k_B T \ll R_1^2$. We also define C_{a_ℓ} as the thermal expectation value of the contact operator:

$C_{a_\ell} = \langle \hat{C}_{a_\ell} \rangle$. Equations (14), (15) represent the divergent high-frequency terms that enter the viscosity sum rule (discussed below), and arise solely from the contributions due to the effective range, $\zeta_{R,R}(\omega)$ and $\zeta_{a,R}(\omega)$. This is one of the main results of this work.

Upon examining Eqs. (14) and (15), it is possible to understand the relevancy of the effective range in the s- and p-wave cases. In the case of s-wave interactions, one can safely set the effective range $R_0 \to 0$ while still being consistent with the requirement: $\omega \ll R_0^{-2}$. Since the divergent term $\zeta_0^d(\omega)$ is proportional to R_0^2, the high frequency tails simply vanish when $R_0 \to 0$. From Eq. (11), one reproduces the standard result that

$$\zeta_0(\omega, R_0 \to 0) = \frac{1}{9a_0^2} \frac{1}{\omega^{3/2}} \frac{C_{a_0}}{V}. \qquad (16)$$

This expression is equivalent to the single channel model results obtained previously [10,36–38]. This result reinforces the common procedure of taking the zero-range limit for s-wave interactions. In the limit of a small, but finite, effective range, R_0, the correction to the zero-range bulk viscosity is perturbative in nature, and can be ignored. On the other hand, the role of the effective range, R_0, is most apparent at unitarity, when $a_0 \to \pm\infty$. In this limit, the presence of the effective range breaks the scale invariance of the unitary Fermi gas, and leads to a finite bulk viscosity. The bulk viscosity will entirely be due to $\zeta_{R,R}(\omega)$ in Eq. (13), and yields the following divergent term:

$$\zeta_0^d(\omega, a_0 \to \pm\infty) = \frac{R_0^2 C_{a_0}}{36V} \sqrt{\omega}. \qquad (17)$$

At resonance, the leading term in the contact C_{a_0} is a constant, independent of R_0, and therefore $\zeta_0^d(\omega, a_0 \to \pm\infty) \propto R_0^2$.

In the case of p-wave interactions, it is clear that we cannot take the zero-range limit, $R_1 \to 0$, while maintaining the high frequency regime: $1/(a_1 R_1) \ll \omega \ll R_1^2$. In practice, R_1 is finite, and a well-defined high frequency limit can always be achieved close to resonance. For instance, ^{40}K has $R_1 \approx 25 k_F$, which yields a low-energy window: $0 \ll \omega \ll 600\epsilon_F$. In this regime, the high frequency tails of $\zeta_1(\omega)$ will certainly be relevant in actual experiments. At high frequencies, one finds the same $\sqrt{\omega}$ dependence, as in the s-wave case, with a coefficient proportional to C_{a_1}:

$$\zeta_1^d(\omega, a_1 \to \pm\infty) = \frac{C_{a_1}}{36V} \sqrt{\omega}, \qquad (18)$$

where the contact $C_{a_1} \propto R_1^{-1}$, and so $\zeta_1^d(\omega, a_1 \to \pm\infty) \propto R_1^{-1}$. This expression is divergent in the *zero-range* limit, when $R_1 \to 0$, in contradistinction to the s-wave case. As in the s-wave case, the presence of the p-wave effective range breaks the scaling symmetry of the system, and

results in a nonzero bulk viscosity. However, its effect is dominant in the zero-range limit for the p-wave case.

The importance of the effective range can also be seen in the low-frequency limit. Here we report the bulk viscosities strictly at resonance when $a_\ell^{-1} = 0$. The relevant high temperature limit for the s-wave case is given by: $1/a_0^2 \ll k_B T \ll R_0^{-2}$, and for p-wave case: $1/(a_1 R_1) \ll k_B T \ll R_1^2$. One can again see that in the s-wave case, the high temperature limit can be safely taken in the zero range limit, while it is impossible for the p-wave case. From Eqs. (11)–(13), one obtains

$$\zeta_0(\omega \to 0) \approx \frac{2^{5/2} z^2}{9} \frac{R_0^2}{\lambda_T^5} \propto T^{5/2} R_0^2, \qquad (19)$$

$$\zeta_1(\omega \to 0) \approx \frac{2^{5/2} z^2}{3} \frac{1}{R_1^2 \lambda_T^5} \propto \frac{T^{5/2}}{R_1^2}, \qquad (20)$$

where $z = \exp(\beta\mu)$ is the fugacity, and $\lambda_T = \sqrt{2\pi\beta}$ is the thermal wavelength [43]. Although Eqs. (19) and (20) look similar, we note that due to the physical requirement of low energies, the effective range has opposite roles in the s- and p-wave case. When one takes the zero range limit, the bulk viscosity is vanishingly small for s-wave gases, but diverges for p-wave gases.

Sum rules for bulk viscosity spectral function.—Finally, we report the sum rules for the bulk viscosity spectral function. As noted from Eqs. (14) and (15), the spectral function is divergent. In order to have a well-defined sum rule, it is necessary to remove these divergent pieces, which we have labeled as $\zeta_\ell^d(\omega)$. Once that is done, it is possible to use the Kramers-Kronig relations to show that $\zeta_\ell(\omega)$ satisfies the following sum rules:

$$\int_{-\infty}^\infty \frac{d\omega}{\pi} [\zeta_\ell(\omega) - \zeta_\ell^d(\omega)]$$
$$= -\frac{1}{9V} \left[(2\ell+1)^2 \frac{\partial C_{a_\ell}}{\partial a_\ell} + (2\ell-1)^2 \frac{\partial C_{R_\ell}}{\partial R_\ell^{-1}} \right.$$
$$\left. + (2\ell-1)(2\ell+1) \left(\frac{1}{a_\ell R_\ell} \frac{\partial C_{a_\ell}}{\partial R_\ell^{-1}} + a_\ell R_\ell \frac{\partial C_{R_\ell}}{\partial a_\ell} \right) \right]. \qquad (21)$$

This relation generalizes the already well-known sum rule for the s-wave Fermi gases derived in Refs. [10,36–38] to include the presence of the effective range, as well as being applicable for p-wave gases.

Conclusions.—In this work we illustrated how the bulk viscosity spectral function depends on the effective range for both s- and p-wave Fermi gases. We focused on the case of resonantly interacting Fermi gases in the high-temperature limit, where the effects of the effective-range are most transparent. We show that the presence of the effective range introduces a second thermodynamic contact, and as a result, new terms to the bulk viscosity. For s-wave Fermi

gases, the effective range is an irrelevant parameter; the low-energy and zero-range limits can be simultaneously taken. The bulk viscosity will then vanish alongside the effective range at resonance. While for p-wave gases the situation is different, the effective range is a relevant parameter; the low-energy and zero-range limits can not be simultaneously taken. Therefore, even at resonance, the presence of the effective range will lead to a finite bulk viscosity. This work clearly reinforces the traditional prescription of taking the zero-range limit for s-wave gases, for both the energetics and transport dynamics, and shows how the p-wave gas is fundamentally different.

In principle, one could extend this calculation to gases interacting via higher partial waves, and to arbitrary order in the effective range expansion. However, the calculation becomes increasingly complex, as one needs to consider the correlations between the different contact operators. A simple scaling analysis shows that for a given partial wave, ℓ, the first $\ell + 1$ terms in the effective range expansion are relevant; with the higher order terms irrelevant. This scaling analysis is consistent with the results obtained in this Letter. Such physics could potentially be studied in atomic gases, thanks to the discovery of higher partial-wave Feshbach resonances [44–46].

———————————

[*] shizhong@hku.hk
[1] *The BCS-BEC Crossover and the Unitary Fermi Gas*, edited by W. Zwerger (Springer, New York, 2011).
[2] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University Press, Cambridge, England, 2008).
[3] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity*, 2nd ed. (Oxford University Press, 7New York, 2016).
[4] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. **82**, 1225 (2010).
[5] T.-L. Ho, Phys. Rev. Lett. **92**, 090402 (2004).
[6] J. E. Thomas, J. Kinast, and A. Turlapov, Phys. Rev. Lett. **95**, 120402 (2005).
[7] D. T. Son, Phys. Rev. Lett. **98**, 020604 (2007).
[8] Y. Nishida and D. T. Son, Phys. Rev. D **76**, 086004 (2007).
[9] T. Enss, R. Haussmann, and W. Zwerger, Ann. Phys. (Amsterdam) **326**, 770 (2011).
[10] E. Taylor and M. Randeria, Phys. Rev. Lett. **109**, 135301 (2012).
[11] A. Gezerlis and J. Carlson, Phys. Rev. C **77**, 032801(R) (2008).
[12] M. M. Forbes, S. Gandolfi, and A. Gezerlis, Phys. Rev. A **86**, 053603 (2012).
[13] A. Boulet and D. Lacroix, Phys. Rev. C **97**, 014301 (2018).
[14] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon Press, Toronto, Canada, 1991).
[15] C. A. Bertulani, H.-W.Hammer, and U.van Kolck, Nucl. Phys. **A712**, 37 (2002).
[16] E. Braaten, P. Hagen, H.-W. Hammer, and L. Platter, Phys. Rev. A **86**, 012711 (2012).
[17] A. Schwenk and C. J. Pethick, Phys. Rev. Lett. **95**, 160401 (2005).
[18] F. Werner and Y. Castin, Phys. Rev. A **86**, 013626 (2012).
[19] D. Lacroix, Phys. Rev. A **94**, 043614 (2016).
[20] L. M. Schonenberg and G. J. Conduit, Phys. Rev. A **95**, 013633 (2017).
[21] G. A. Miller, Phys. Lett. B **777**, 442 (2018).
[22] C. Wellenhofer, C. Drischler, and A. Schwenk, Phys. Lett. B **802**, 135247 (2020).
[23] V. Gurarie and L. Radzihovsky, Ann. Phys. (Amsterdam) **322**, 2-119 (2007).
[24] C.-H. Cheng and S.-K. Yip, Phys. Rev. Lett. **95**, 070404 (2005).
[25] C.-H. Cheng and S.-K. Yip, Phys. Rev. B **73**, 064517 (2006).
[26] J. Yao and S. Zhang, Phys. Rev. A **97**, 043612 (2018).
[27] S. Ding and S. Zhang, Phys. Rev. Lett. **123**, 070404 (2019).
[28] D. Inotani, P. van Wyk, and Y. Ohashi, J. Phys. Soc. Jpn. **86**, 024302 (2017).
[29] K. Fujii and Y. Nishida, Phys. Rev. A **102**, 023310 (2020).
[30] To obtain this result we note from dimensional analysis $p(t) = \beta^{-5/2} G(\beta\mu, \beta^{(2\ell+1)/2} a_\ell^{-1}, \beta^{(2\ell-1)/2} R_\ell)$, where G is some dimensionless function. Since $\exp(\beta p(t) V) = \mathrm{Tr}[\exp(-\beta H)]$, one can obtain the pressure relation by considering the derivative of $p(t)$ with respect to β.
[31] S. M. Yoshida and M.Ueda, Phys. Rev. Lett. **115**, 135303 (2015).
[32] Z. Yu, J. H. Thywissen, and S. Zhang, Phys. Rev. Lett. **115**, 135304 (2015); **117**, 019901(E) (2016).
[33] M. He, S. Zhang, H. M. Chan, and Q. Zhou, Phys. Rev. Lett. **116**, 045301 (2016).
[34] S.-G. Peng, X.-J. Liu, and H. Hu, Phys. Rev. A **94**, 063651 (2016).
[35] The retarded correlators involving the Hamiltonian and a single contact are defined to vanish in thermal equilibrium, as it is related to the time derivative of the contact. As described in Ref. [29], this is appropriate if one considers finite frequencies.
[36] Y. Nishida, Ann. Phys. (Amsterdam) **410**, 167949 (2019).
[37] T. Enss, Phys. Rev. Lett. **123**, 205301 (2019).
[38] J. Hofmann, Phys. Rev. A **101**, 013620 (2020).
[39] J. Maki and F. Zhou, Phys. Rev. A **100**, 023601 (2019).
[40] M. Alford, A. Harutyunyan, and A. Sedrakian, Phys. Rev. D **100**, 103021 (2019).
[41] A. Rios, A. Polls, and I. Vidana, Phys. Rev. C **79**, 025802 (2009).
[42] C. J. Horowitz, O. L. Caballero, Z. Lin, E. O'Connor, and A. Schwenk, Phys. Rev. C **95**, 025801 (2017).
[43] We note that if one deviates from resonance the expression for $\zeta_{a,a}(\omega)$ is nonanalytical when one considers the limits of resonance and zero frequency. These two limits do not

commute due to a logarithmic singularity. This issue is most relevant in the zero-range limit, where the dominant contribution to the bulk viscosity is from $\zeta_{a,a}(\omega)$. See Refs. [36–38], and more recently Ref. [29] for more details.

[44] Q. Beaufils, A. Crubellier, T. Zanon, B. Laburthe-Tolra, E. Marechal, L. Vernac, and O. Gorceix, Phys. Rev. A **79,** 032706 (2009).

[45] Y. Cui, C. Shen, M. Deng, S. Dong, C. Chen, R. Lu, B. Gao, M. K. Tey, and L. You, Phys. Rev. Lett. **119,** 203402 (2017).

[46] X. C. Yao, R. Qi, X. P. Liu, X. Q. Wang, Y. X. Wang, Y. P. Wu, H. Z. Chen, P. Zhang, H. Zhai, Y. A. Chen, and J. W. Pan, Nat. Phys. **15,** 570 (2019).