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HERMITE–HADAMARD INEQUALITY FOR SEMICONVEX FUNCTIONS OF RATE (k_1, k_2) ON THE COORDINATES AND OPTIMAL MASS TRANSPORTATION

PING CHEN AND WING-SUM CHEUNG

We give a new Hermite–Hadamard inequality for a function $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ which is semiconvex of rate (k_1, k_2) on the coordinates. This generalizes some existing results on Hermite–Hadamard inequalities of S. S. Dragomir. In addition, we explain the Hermite–Hadamard inequality from the point of view of optimal mass transportation with cost function $c(x, y) := f(y-x)+\frac{k_1}{2}|x_1-y_1|^2+\frac{k_2}{2}|x_2-y_2|^2$, where $f(\cdot) : [a, b] \times [c, d] \to [0, \infty)$ is semiconvex of rate (k_1, k_2) on the coordinates and $x = (x_1, x_2)$, $y = (y_1, y_2) \in [a, b] \times [c, d]$.

1. Introduction

The classical Hermite–Hadamard inequality, first published in [6], gives an estimate of the mean value of a convex function f on $[a, b]$:

$$f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1-1}$$

A simple proof of inequality (1-1) is given in [3]. In the two-dimensional situation, for any function $f : [a, b] \times [c, d] \to \mathbb{R}$ which is convex on the coordinates on $[a, b] \times [c, d]$, Dragomir proved in 2001 the following two-dimensional Hermite–Hadamard inequality (Theorem 1 in [1]):

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x_1, \frac{c+d}{2}\right)dx_1 + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, x_2\right)dx_2\right] \\ &\le \frac{1}{(b-a)(c-d)}\int_a^b\int_c^d f(x_1, x_2)\,dx_1\,dx_2 \\ &\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x_1, c)\,dx_1 + \frac{1}{b-a}\int_a^b f(x_1, d)\,dx_1 \right.\\ &\qquad\left. + \frac{1}{d-c}\int_c^d f(a, x_2)\,dx_2 + \frac{1}{d-c}\int_c^d f(b, x_2)\,dx_2\right] \\ &\le \frac{f(a, c)+f(a, d)+f(b, c)+f(b, d)}{4} \end{aligned} \tag{1-2}$$

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Interested readers are also referred to [2] for more details. On the other hand, Hermite–Hadamard type inequalities involving two functions, and Hermite–Hadamard's inequality for log-convex functions are established in [7] and [11]. We also refer to [4] for Hadamard type inequalities for twice differentiable functions.

Inspired from the Hermite–Hadamard inequality for semiconvex functions $f:[a,b]\to\mathbb{R}$ in [8], we establish a refinement of the Hermite–Hadamard inequality (1-2) for a function $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ which is semiconvex of rate (k_1,k_2) on the coordinates. Dragomir's result (1-2), the Hermite–Hadamard inequality for one- and two-dimensional semiconvex functions (see Theorem 2.2 and Theorem 2.9, respectively) can all be seen as special cases. Finally, we interpret the meaning of the new Hermite–Hadamard inequality obtained in the previous section from the point of view of optimal mass transportation problems by studying and comparing the transportation costs of various transportation plans of a Kantorovich problem.

2. Two-dimensional Hermite–Hadamard inequality for semiconvex functions of rate (k_1, k_2) on the coordinates

We first recall some preliminaries on semiconvexity and the one-dimensional Hermite–Hadamard inequality for semiconvex functions $f:[a,b]\to\mathbb{R}$ of rate k.

Definition 2.1 [5; 8]. A function $f:[a,b]\to\mathbb{R}$ is said to be *semiconvex of rate* $k\in\mathbb{R}$ if the function

$$h(\cdot):=f(\cdot)+\frac{k}{2}|\cdot|^2$$

is convex in $[a,b]$, that is,

$$f((1-\lambda)x+\lambda y)\le(1-\lambda)f(x)+\lambda f(y)+\frac{k}{2}\lambda(1-\lambda)|x-y|^2$$

for all $x, y\in[a,b]$ and all $\lambda\in[0,1]$.

Theorem 2.2 [8]. *If μ is a Borel probability measure on an interval $[a,b]$ with barycenter*

$$b_\mu=\int_a^b x\,d\mu(x), \tag{2-1}$$

then for every semiconvex function $f:[a,b]\to\mathbb{R}$ of rate k, we have

$$f(b_\mu)\le\int_a^b f(x)\,d\mu(x)+\frac{k}{2}\int_a^b|x-b_\mu|^2\,d\mu(x), \tag{2-2}$$

$$\le\frac{b-b_\mu}{b-a}f(a)+\frac{b_\mu-a}{b-a}f(b)+\frac{k}{2}(b_\mu-a)(b-b_\mu). \tag{2-3}$$

Definition 2.3. (i) A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be semiconvex of rate $k\in\mathbb{R}$ if the function

$$h(\cdot):=f(\cdot)+\frac{k}{2}|\cdot|^2,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$, is convex in $[a,b]\times[c,d]$.

(ii) If $k_1, k_2 \in \mathbb{R}$, $f : [a, b] \times [c, d] \to \mathbb{R}$ is said to be *semiconvex of rate* (k_1, k_2) *on the coordinates* if for all $x_1 \in [a, b]$ and $x_2 \in [c, d]$, the partial map

$$f_{x_1}(u) : [c, d] \to \mathbb{R}, \quad f_{x_1}(u) := f(x_1, u)$$

is semiconvex of rate k_2, and the partial map

$$f_{x_2}(v) : [a, b] \to \mathbb{R}, \quad f_{x_2}(v) := f(v, x_2)$$

is semiconvex of rate k_1.

Now the main two-dimensional Hermite–Hadamard inequality is given as follows.

Theorem 2.4. *If μ_1, μ_2 are Borel probability measures on an interval $[a, b]$ and $[c, d]$ respectively, then for every semiconvex function $f : [a, b] \times [c, d] \to \mathbb{R}$ of rate (k_1, k_2) on the coordinates, we have*

$$f(b_{\mu_1}, b_{\mu_2}) + \left(\frac{k_1}{2}b_{\mu_1}^2 + \frac{k_2}{2}b_{\mu_2}^2\right) \tag{2-4}$$

$$\le \frac{1}{2}\left[\int_a^b f(x_1, b_{\mu_2})\, d\mu_1(x_1) + \int_c^d f(b_{\mu_1}, y_1)\, d\mu_2(y_1)\right] + \left[\frac{k_1}{4}\int_a^b x_1^2\, d\mu_1(x_1) + \frac{k_2}{4}\int_c^d x_2^2\, d\mu_2(x_2)\right] + \left(\frac{k_1}{4}b_{\mu_1}^2 + \frac{k_2}{4}b_{\mu_2}^2\right) \tag{2-5}$$

$$\le \int_a^b \int_c^d f(x_1, x_2)\, d\mu_1(x_1) \otimes \mu_2(x_2) + \left[\frac{k_1}{2}\int_a^b x_1^2\, d\mu_1(x_1) + \frac{k_2}{2}\int_c^d x_2^2\, d\mu_2(x_2)\right] \tag{2-6}$$

$$\le \int_c^d \left[\frac{b - b_{\mu_1}}{2(b-a)} f_{x_2}(a) + \frac{b_{\mu_1} - a}{2(b-a)} f_{x_2}(b)\right] d\mu_2(x_2) + \int_a^b \left[\frac{d - b_{\mu_2}}{2(d-c)} f_{x_1}(c) + \frac{b_{\mu_2} - c}{2(d-c)} f_{x_1}(d)\right] d\mu_1(x_1) + \frac{k_1}{4}\int_a^b x_1^2\, d\mu_1(x_1) + \frac{k_2}{4}\int_c^d x_2^2\, d\mu_2(x_2) + \frac{k_1}{4}[(a+b)\, b_{\mu_1} - ab] + \frac{k_2}{4}[(c+d)\, b_{\mu_2} - cd] \tag{2-7}$$

$$\le \frac{(b - b_{\mu_1})(d - b_{\mu_2})}{(b-a)(d-c)} f(a, c) + \frac{(b - b_{\mu_1})(b_{\mu_2} - c)}{(b-a)(d-c)} f(a, d) + \frac{(b_{\mu_1} - a)(d - b_{\mu_2})}{(b-a)(d-c)} f(b, c) + \frac{(b_{\mu_1} - a)(b_{\mu_2} - c)}{(b-a)(d-c)} f(b, d) + \frac{k_1}{2}[(a+b)\, b_{\mu_1} - ab] + \frac{k_2}{2}[(c+d)\, b_{\mu_2} - cd]. \tag{2-8}$$

Proof. Set

$$b_{\mu_1} = \int_a^b x \, d\mu_1(x) \in [a, b], \quad b_{\mu_2} = \int_a^b x \, d\mu_2(x) \in [c, d].$$

It follows from Definition 2.3 that $f_{x_2}(x_1) := f(x_1, x_2)$ and $f_{x_1}(x_2) := f(x_1, x_2)$ are semiconvex of rate k_1 and k_2, respectively.

Applying Theorem 2.2 to $f_{x_2}(x_1)$ and $f_{x_1}(x_2)$ respectively, one has

$$f_{x_2}(b_{\mu_1}) \le \int_a^b f_{x_2}(x_1) \, d\mu_1(x_1) + \frac{k_1}{2} \int_a^b |x_1 - b_{\mu_1}|^2 \, d\mu_1(x_1)$$

$$= \int_a^b f_{x_2}(x_1) \, d\mu_1(x_1) + \frac{k_1}{2} \int_a^b x_1^2 \, d\mu_1(x_1) - \frac{k_1}{2} b_{\mu_1}^2 \tag{2-9}$$

$$\le \frac{b - b_{\mu_1}}{b - a} f_{x_2}(a) + \frac{b_{\mu_1} - a}{b - a} f_{x_2}(b) + \frac{k_1}{2}(b_{\mu_1} - a)(b - b_{\mu_1}), \tag{2-10}$$

and

$$f_{x_1}(b_{\mu_2}) \le \int_c^d f_{x_1}(x_2) \, d\mu_2(x_2) + \frac{k_2}{2} \int_c^d |x_2 - b_{\mu_2}|^2 \, d\mu_2(x_2)$$

$$= \int_c^d f_{x_1}(x_2) \, d\mu_2(x_2) + \frac{k_2}{2} \int_c^d x_2^2 \, d\mu_2(x_2) - \frac{k_2}{2} b_{\mu_2}^2 \tag{2-11}$$

$$\le \frac{d - b_{\mu_2}}{d - c} f_{x_1}(c) + \frac{b_{\mu_2} - c}{d - c} f_{x_1}(d) + \frac{k_2}{2}(b_{\mu_2} - c)(d - b_{\mu_2}). \tag{2-12}$$

Taking $x_2 = b_{\mu_2}$ in (2-9), $x_1 = b_{\mu_1}$ in (2-11), and adding the two resulting inequalities, one has

$$f(b_{\mu_1}, b_{\mu_2}) \le \frac{1}{2}\left[\int_a^b f(x_1, b_{\mu_2}) \, d\mu_1(x_1) + \int_c^d f(b_{\mu_1}, x_2) \, d\mu_2(x_2)\right] + \frac{k_1}{4} \int_a^b x_1^2 \, d\mu_1(x_1) + \frac{k_2}{4} \int_c^d x_2^2 \, d\mu_2(x_2) - \frac{k_1}{4} b_{\mu_1}^2 - \frac{k_2}{4} b_{\mu_2}^2.$$

This proves (2-5).

Integrating (2-9) and (2-10) with respect to x_2 over $[c, d]$ we have

$$\begin{aligned} \int_c^d f_{x_2}(b_{\mu_1}) \, d\mu_2(x_2) &\le \int_c^d \int_a^b f(x_1, x_2) \, d\mu_1 \otimes \mu_2(x_1, x_2) + \frac{k_1}{2} \int_a^b x_1^2 \, d\mu_1(x_1) - \frac{k_1}{2} b_{\mu_1}^2 \\ &\le \int_c^d \left[\frac{b - b_{\mu_1}}{b - a} f_{x_2}(a) + \frac{b_{\mu_1} - a}{b - a} f_{x_2}(b)\right] d\mu_2(x_2) + \frac{k_1}{2}(b_{\mu_1} - a)(b - b_{\mu_1}). \end{aligned} \tag{2-13}$$

Integrating (2-11) and (2-12) with respect to x_1 over $[a, b]$, we get

$$\begin{aligned} \int_a^b f_{x_1}(b_{\mu_2}) \, d\mu_1(x_1) &\le \int_c^d \int_a^b f(x_1, x_2) \, d\mu_1 \otimes \mu_2(x_1, x_2) + \frac{k_2}{2} \int_c^d x_2^2 \, d\mu_2(x_2) - \frac{k_2}{2} b_{\mu_2}^2 \\ &\le \int_a^b \left[\frac{d - b_{\mu_2}}{d - c} f_{x_1}(c) + \frac{b_{\mu_2} - c}{d - c} f_{x_1}(d)\right] d\mu_1(x_1) + \frac{k_2}{2}(b_{\mu_2} - c)(d - b_{\mu_2}). \end{aligned} \tag{2-14}$$

Adding (2-13) and (2-14), we have

$$\frac{1}{2}\Big[\int_c^d f_{x_2}(b_{\mu_1})\,d\mu_2(x_2)+\int_a^b f_{x_1}(b_{\mu_2})\,d\mu_1(x_1)\Big]$$
$$\le \frac{1}{2}\Big[\int_c^d\!\int_a^b [f_{x_2}(x_1)+f_{x_1}(x_2)]\,d\mu_1\otimes\mu_2(x_1,x_2)\Big]+\frac{k_1}{4}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_2}{4}\int_c^d x_2^2\,d\mu_2(x_2)-\frac{k_1}{4}b_{\mu_1}^2-\frac{k_2}{4}b_{\mu_2}^2$$
$$\le \int_c^d\Big[\frac{b-b_{\mu_1}}{2(b-a)}f_{x_2}(a)+\frac{b_{\mu_1}-a}{2(b-a)}f_{x_2}(b)\Big]d\mu_2(x_2)+\frac{k_1}{4}(b_{\mu_1}-a)(b-b_{\mu_1})$$
$$+\int_a^b\Big[\frac{d-b_{\mu_2}}{2(d-c)}f_{x_1}(c)+\frac{b_{\mu_2}-c}{2(d-c)}f_{x_1}(d)\Big]d\mu_1(x_1)+\frac{k_2}{4}(b_{\mu_2}-c)(d-b_{\mu_2}),$$

and so

$$\begin{aligned}\text{(2-15)}\quad &\frac{1}{2}\Big[\int_c^d f_{x_2}(b_{\mu_1})\,d\mu_2(x_2)+\int_a^b f_{x_1}(b_{\mu_2})\,d\mu_1(x_1)\Big]\\
&\qquad+\frac{k_1}{4}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_2}{4}\int_c^d x_2^2\,d\mu_2(x_2)+\frac{k_1}{4}b_{\mu_1}^2+\frac{k_2}{4}b_{\mu_2}^2\\
&\le \int_c^d\!\int_a^b f(x_1,x_2)\,d\mu_1\otimes\mu_2(x_1,x_2)+\frac{k_1}{2}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_2}{2}\int_c^d x_2^2\,d\mu_2(x_2)\\
&\le \int_c^d\Big[\frac{b-b_{\mu_1}}{2(b-a)}f_{x_2}(a)+\frac{b_{\mu_1}-a}{2(b-a)}f_{x_2}(b)\Big]d\mu_2(x_2)+\frac{k_1}{4}\int_a^b x_1^2\,d\mu_1(x_1)\\
&\qquad+\int_a^b\Big[\frac{d-b_{\mu_2}}{2(d-c)}f_{x_1}(c)+\frac{b_{\mu_2}-c}{2(d-c)}f_{x_1}(d)\Big]d\mu_1(x_1)+\frac{k_2}{4}\int_c^d x_2^2\,d\mu_2(x_2)\\
&\qquad+\frac{k_1}{4}[(a+b)b_{\mu_1}-ab]+\frac{k_2}{4}[(c+d)b_{\mu_2}-cd],\end{aligned}$$

which shows (2-6) and (2-7).

Taking $x_2=c,d$ in (2-10) and $x_1=a,b$ in (2-12), then (2-15) is

$$\le \frac{b-b_{\mu_1}}{2(b-a)}\Big[\frac{d-b_{\mu_2}}{d-c}f_a(c)+\frac{b_{\mu_2}-c}{d-c}f_a(d)+\frac{k_2}{2}(b_{\mu_2}-c)(d-b_{\mu_2})-\frac{k_2}{2}\int_c^d x_2^2\,d\mu_2(x_2)+\frac{k_2}{2}b_{\mu_2}^2\Big]$$
$$+\frac{b_{\mu_1}-a}{2(b-a)}\Big[\frac{d-b_{\mu_2}}{d-c}f_b(c)+\frac{b_{\mu_2}-c}{d-c}f_b(d)+\frac{k_2}{2}(b_{\mu_2}-c)(d-b_{\mu_2})-\frac{k_2}{2}\int_c^d x_2^2\,d\mu_2(x_2)+\frac{k_2}{2}b_{\mu_2}^2\Big]$$
$$+\frac{d-b_{\mu_2}}{2(d-c)}\Big[\frac{b-b_{\mu_1}}{b-a}f_c(a)+\frac{b_{\mu_1}-a}{b-a}f_c(b)+\frac{k_1}{2}(b_{\mu_1}-a)(b-b_{\mu_1})-\frac{k_1}{2}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_1}{2}b_{\mu_1}^2\Big]$$
$$+\frac{b_{\mu_2}-c}{2(d-c)}\Big[\frac{b-b_{\mu_1}}{b-a}f_d(a)+\frac{b_{\mu_1}-a}{b-a}f_d(b)+\frac{k_1}{2}(b_{\mu_1}-a)(b-b_{\mu_1})-\frac{k_1}{2}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_1}{2}b_{\mu_1}^2\Big]$$
$$+\frac{k_1}{4}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_2}{4}\int_c^d x_2^2\,d\mu_2(x_2)+\frac{k_1}{4}b_{\mu_1}^2+\frac{k_2}{4}b_{\mu_2}^2+\frac{k_1}{4}(b_{\mu_1}-a)(b-b_{\mu_1})+\frac{k_2}{4}(b_{\mu_2}-c)(d-b_{\mu_2})$$

$$= \frac{(b-b_{\mu_1})(d-b_{\mu_2})}{(b-a)(d-c)} f(a,c) + \frac{(b-b_{\mu_1})(b_{\mu_2}-c)}{(b-a)(d-c)} f(a,d)$$
$$+ \frac{(b_{\mu_1}-a)(d-b_{\mu_2})}{(b-a)(d-c)} f(b,c) + \frac{(b_{\mu_1}-a)(b_{\mu_2}-c)}{(b-a)(d-c)} f(b,d) + \frac{k_1}{2}[(a+b)b_{\mu_1}-ab] + \frac{k_2}{2}[(c+d)b_{\mu_2}-cd],$$

which proves (2-8). □

Definition 2.5 [1]. A function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said to be *convex on the coordinates* if for any $x_1 \in [a,b]$ and $x_2 \in [c,d]$, the partial maps

$$f_{x_1}(u) : [c,d] \to \mathbb{R}, \quad f_{x_1}(u) := f(x_1, u) \qquad \text{and} \qquad f_{x_2}(v) : [a,b] \to \mathbb{R}, \quad f_{x_2}(v) := f(v, x_2)$$

are convex.

It is obvious that if a function is convex on the coordinates then it is semiconvex of rate $(0,0)$ on the coordinates. As a direct consequence of Theorem 2.4, we have the following Hermite–Hadamard inequality for convex functions on the coordinates which include Dragomir's result (1-2) as a special case.

Corollary 2.6. *Let μ_1, μ_2 be two Borel probability measures on the intervals $[a,b]$ and $[c,d]$, respectively. If $f : [a,b] \times [c,d] \to \mathbb{R}$ is convex on the coordinates, then*

$$f(b_{\mu_1}, b_{\mu_2}) \le \frac{1}{2}\left[\int_a^b f(x_1, b_{\mu_2})\, d\mu_1(x_1) + \int_c^d f(b_{\mu_1}, y_1)\, d\mu_2(y_1)\right] \le \int_a^b \int_c^d f(x_1,x_2)\, d\mu_1(x_1) \otimes \mu_2(x_2)$$
$$\le \frac{1}{2}\int_c^d \frac{b-b_{\mu_1}}{b-a} f_{x_2}(a) + \frac{b_{\mu_1}-a}{b-a} f_{x_2}(b)\, d\mu_2(x_2) + \frac{1}{2}\int_a^b \frac{d-b_{\mu_2}}{d-c} f_{x_1}(c) + \frac{b_{\mu_2}-c}{d-c} f_{x_1}(d)\, d\mu_1(x_1)$$
$$\le \frac{(b-b_{\mu_1})(d-b_{\mu_2})}{(b-a)(d-c)} f(a,c) + \frac{(b-b_{\mu_1})(b_{\mu_2}-c)}{(b-a)(d-c)} f(a,d)$$
$$+ \frac{(b_{\mu_1}-a)(d-b_{\mu_2})}{(b-a)(d-c)} f(b,c) + \frac{(b_{\mu_1}-a)(b_{\mu_2}-c)}{(b-a)(d-c)} f(b,d).$$

Remark 2.7. In case $\mu_1 := \frac{1}{b-a}\nu|_{[a,b]}$, $\mu_2 = \frac{1}{d-c}\nu|_{[c,d]}$, where ν is the one-dimensional Lebesgue measure, then $b_{\mu_1} = (b+a)/2$, $b_{\mu_2} = (c+d)/2$, and Corollary 2.6 reduces to Dragomir's result (1-2).

Another consequence of our Theorem 2.4 is a Hermite–Hadamard inequality for two-dimensional semiconvex functions of rate k. We start with the following.

Lemma 2.8. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be semiconvex of rate k, then f is semiconvex of rate (k,k) on the coordinates.*

Proof. It follows from the semiconvexity of f that the function

$$h(x_1,x_2) := f(x_1,x_2) + \frac{k}{2}(x_1^2 + x_2^2)$$

is convex in $[a,b] \times [c,d]$. As a result, $h(x_1,x_2)$ is convex on the coordinates (see Lemma 1 in [1]). That is, for any $x_1 \in [a,b]$ and $x_2 \in [c,d]$, both

$$h_{x_2}(x_1) := h(x_1,x_2) : [a,b] \to \mathbb{R} \quad \text{and} \quad h_{x_1}(x_2) := h(x_1,x_2) : [c,d] \to \mathbb{R}$$

are convex. As a result, for any $x_1 \in [a, b]$ and $x_2 \in [c, d]$,

$$g_{x_2}(v) := f(v, x_2) + \frac{k}{2}v^2 : [a, b] \to \mathbb{R} \quad \text{and} \quad g_{x_1}(u) := f(x_1, u) + \frac{k}{2}u^2 : [c, d] \to \mathbb{R}$$

are convex. Therefore, for any $x_1 \in [a, b]$ and $x_2 \in [c, d]$, $f_{x_2}(v)$ and $f_{x_1}(u)$ are semiconvex of rate k. □

Theorem 2.9. *If μ_1, μ_2 are Borel probability measures on an interval $[a, b]$ and $[c, d]$ respectively, then for every semiconvex function $f : [a, b] \times [c, d] \to \mathbb{R}$ of rate k, we have*

$$\begin{aligned}
&f(b_{\mu_1}, b_{\mu_2}) + \frac{k}{2}(b_{\mu_1}^2 + b_{\mu_2}^2)\\
&\leq \frac{1}{2}\left[\int_a^b f(x_1, b_{\mu_2})\, d\mu_1(x_1) + \int_c^d f(b_{\mu_1}, y_1)\, d\mu_2(y_1)\right] + \frac{k}{4}\left[\int_a^b x_1^2\, d\mu_1(x_1) + \int_c^d x_2^2\, d\mu_2(x_2)\right] + \frac{k}{4}(b_{\mu_1}^2 + b_{\mu_2}^2)\\
&\leq \int_a^b \int_c^d f(x_1, x_2)\, d\mu_1(x_1) \otimes \mu_2(x_2) + \frac{k}{2}\left[\int_a^b x_1^2\, d\mu_1(x_1) + \int_c^d x_2^2\, d\mu_2(x_2)\right]\\
&\leq \int_c^d \left[\frac{b - b_{\mu_1}}{2(b-a)} f_{x_2}(a) + \frac{b_{\mu_1} - a}{2(b-a)} f_{x_2}(b)\right] d\mu_2(x_2) + \frac{k}{4}\int_a^b x_1^2\, d\mu_1(x_1)\\
&\qquad + \int_a^b \left[\frac{d - b_{\mu_2}}{2(d-c)} f_{x_1}(c) + \frac{b_{\mu_2} - c}{2(d-c)} f_{x_1}(d)\right] d\mu_1(x_1)\\
&\qquad\qquad + \frac{k}{4}\int_c^d x_2^2\, d\mu_2(x_2) + \frac{k}{4}\left[(a+b)\, b_{\mu_1} + (c+d)\, b_{\mu_2} - ab - cd\right]\\
&\leq \frac{(b - b_{\mu_1})(d - b_{\mu_2})}{(b-a)(d-c)} f(a, c) + \frac{(b - b_{\mu_1})(b_{\mu_2} - c)}{(b-a)(d-c)} f(a, d)\\
&+ \frac{(b_{\mu_1} - a)(d - b_{\mu_2})}{(b-a)(d-c)} f(b, c) + \frac{(b_{\mu_1} - a)(b_{\mu_2} - c)}{(b-a)(d-c)} f(b, d) + \frac{k}{4}[(a+b)b_{\mu_1} + (c+d)b_{\mu_2} - ab - cd].
\end{aligned}$$

Before ending this section, we remark that Theorem 2.2 which is a one-dimensional Hermite–Hadamard inequality can also be seen as a special case of Theorem 2.4.

Remark 2.10. Theorem 2.2 which is a one-dimensional Hermite–Hadamard inequality can also be seen as a special case of Theorem 2.4. In fact, observe that when the interval $[c, d]$ degenerates to a point, the function $f(x, y)$ in Theorem 2.4 reduces to a semiconvex function $f(x) : [a, b] \to \mathbb{R}$. With suitable modifications, (2-5), (2-6), (2-7) and (2-8) in Theorem 2.4 reduce to (2-2) and (2-3) in Theorem 2.2.

3. Optimal mass transportation meaning of Theorem 2.4

We interpret the meaning of the new Hermite–Hadamard inequality obtained in the previous section from the point of view of optimal mass transportation problems.

A typical optimal mass transport problem is the Kantorovich problem, which is formulated as

$$\min_{\gamma \in \Pi(\nu_2, \nu_2)} \int_{\Omega \times \Omega} c(x, y)\, d\gamma(x, y), \tag{3-1}$$

where $\Omega \subset \mathbb{R}^n$, $\nu_1, \nu_2 \in \mathcal{P}(\Omega)$ = the space of Borel probability measures on Ω, $c(x, y) : \Omega \times \Omega \to [0, +\infty)$ is a *cost function*, and

$$\Pi(\nu_1, \nu_2) := \{\gamma \in \mathcal{P}(\Omega \times \Omega) : (\pi_1)_\sharp \gamma = \nu_1,\ (\pi_2)_\sharp \gamma = \nu_2\}$$

is the set of transport plans between ν_1 and ν_2. Here $\pi_1, \pi_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are the canonical projections on the first and second factors, respectively. We refer to [9; 10] for more information and references on optimal mass transportation theory.

For the sake of simplicity, in what follows we will only consider a Kantorovich problem on $\mathbb{R}^2$. For fixed $a, b, c, d \in \mathbb{R}$, let

$$\Omega := [\min\{a, b, 0\}, \max\{a, b, 0\}] \times [\min\{c, d, 0\}, \max\{c, d, 0\}] \subset \mathbb{R}^2,$$

and consider a cost function $c(x, y) : \Omega \times \Omega \to [0, +\infty)$ given by

$$c(x, y) := f(x - y) + \frac{k_1}{2}|x_1 - y_1|^2 + \frac{k_2}{2}|x_2 - y_2|^2, \tag{3-2}$$

where

$$f|_{[a,b]\times[c,d]} : [a, b] \times [c, d] \to [0, +\infty)$$

is semiconvex of rate (k_1, k_2) on the coordinates.

Before proceeding, we first recall some standard notations. Let μ_1, μ_2 be two Borel probability measures on the intervals $[a, b]$ and $[c, d]$, respectively, and using the notation in Theorem 2.2, b_{μ_i} will denote the barycenter of μ_i, for $i = 1, 2$. If δ_x denotes the Dirac measure at the point $x \in \mathbb{R}$, then the product measure $\delta_{b_{\mu_1}} \otimes \delta_{b_{\mu_2}}$ of $\delta_{b_{\mu_1}}$ and $\delta_{b_{\mu_2}}$ is given by

$$\delta_{b_{\mu_1}} \otimes \delta_{b_{\mu_2}}(A \times B) = \begin{cases} 1 & \text{if } b_{\mu_1} \in A,\ b_{\mu_2} \in B \\ 0 & \text{otherwise} \end{cases}$$

for any Borel measurable $A \subset [\min\{a, b, 0\}, \max\{a, b, 0\}]$ and $B \subset [\min\{c, d, 0\}, \max\{c, d, 0\}]$. Note that $\delta_{b_{\mu_1}} \otimes \delta_{b_{\mu_2}} \in \mathcal{P}(\Omega)$.

In what follows, we will consider the Kantorovich problem with cost function $c(x, y)$ as defined in (3-2), for various mass distributions ν_1 and ν_2, and compare the transfer costs by applying Theorem 2.4.

Example 3.1. Take $\nu_1 = \delta_{b_{\mu_1}} \otimes \delta_{b_{\mu_2}}$, $\nu_2 = \delta_0 \otimes \delta_0 \in \mathcal{P}(\Omega)$, then $\Pi(\nu_1, \nu_2) = \{\nu_1 \otimes \nu_2\}$ is a singleton, and the optimal transportation cost from ν_1 to ν_2 is

$$\begin{aligned}
&\int_{\Omega\times\Omega} c(x, y)\, d\nu_1 \otimes \nu_2(x, y) \\
&= \int_{\Omega\times\Omega} \left[f(x_1 - y_1, x_2 - y_2) + \frac{k_1}{2}(x_1 - y_1)^2 + \frac{k_2}{2}(x_2 - y_2)^2 \right] \cdot d\delta_{b_{\mu_1}} \otimes \delta_{b_{\mu_2}} \otimes \delta_0 \otimes \delta_0(x_1, x_2, y_1, y_2) \\
&= f(b_{\mu_1}, b_{\mu_2}) + \left(\frac{k_1}{2} b_{\mu_1}^2 + \frac{k_2}{2} b_{\mu_2}^2 \right),
\end{aligned}$$

which is expression (2-4) in Theorem 2.4.

Example 3.2. Take $\nu_1=\frac{1}{2}\mu_1\otimes\delta_{b_{\mu_2}}+\frac{1}{2}\delta_{b_{\mu_1}}\otimes\mu_2,\ \nu_2=\delta_0\otimes\delta_0\in\mathcal{P}(\Omega)$, then $\Pi(\nu_1,\nu_2)=\{\nu_1\otimes\nu_2\}$ is a singleton, and the optimal transportation cost from ν_1 to ν_2 is

$$\begin{aligned}
&\int_{\Omega\times\Omega}c(x,y)\,d\nu_1\otimes\nu_2(x,y)\\
&=\frac{1}{2}\int_{\Omega\times\Omega}\Big[f(x_1-y_1,x_2-y_2)+\frac{k_1}{2}(x_1-y_1)^2+\frac{k_2}{2}(x_2-y_2)^2\Big]\cdot d\mu_1\otimes\delta_{b_{\mu_2}}\otimes\delta_0\otimes\delta_0(x_1,x_2,y_1,y_2)\\
&\quad+\frac{1}{2}\int_{\Omega\times\Omega}\Big[f(x_1-y_1,x_2-y_2)+\frac{k_1}{2}(x_1-y_1)^2+\frac{k_2}{2}(x_2-y_2)^2\Big]\cdot d\delta_{b_{\mu_1}}\otimes\mu_2\otimes\delta_0\otimes\delta_0(x_1,x_2,y_1,y_2)\\
&=\frac{1}{2}\Big[\int_a^b f(x_1,b_{\mu_2})\,d\mu_1(y_1)+\int_c^d f(b_{\mu_1},x_2)\,d\mu_2(x_2)\Big]\\
&\qquad\qquad+\Big[\frac{k_1}{4}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_2}{4}\int_c^d x_2^2\,d\mu_2(x_2)\Big]+\Big(\frac{k_1}{4}b_{\mu_1}^2+\frac{k_2}{4}b_{\mu_2}^2\Big),
\end{aligned}$$

which is expression (2-5) in Theorem 2.4.

Example 3.3. Take $\nu_1=\mu_1\otimes\mu_2,\ \nu_2=\delta_0\otimes\delta_0\in\mathcal{P}(\Omega)$, then $\Pi(\nu_1,\nu_2)=\{\nu_1\otimes\nu_2\}$ is a singleton, and the optimal transportation cost from ν_1 to ν_2 is

$$\begin{aligned}
&\int_{\Omega\times\Omega}c(x,y)\,d\nu_1\otimes\nu_2(x,y)\\
&=\int_{\Omega\times\Omega}\Big[f(x_1-y_1,x_2-y_2)+\Big(\frac{k_1}{2}(x_1-y_1)^2+\frac{k_2}{2}(x_2-y_2)^2\Big)\Big]\cdot d\mu_1\otimes\mu_2\otimes\delta_0\otimes\delta_0(x_1,x_2,y_1,y_2)\\
&=\int_c^d\int_a^b f(x_1,x_2)\,d\mu_1(x_1)\,d\mu_2(x_2)+\int_c^d\int_a^b\Big(\frac{k_1}{2}x_1^2+\frac{k_2}{2}x_2^2\Big)\,d\mu_1(x_1)\,d\mu_2(x_2)\\
&=\int_c^d\int_a^b f(x_1,x_2)\,d\mu_1(x_1)\,d\mu_2(x_2)+\Big[\frac{k_1}{2}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_2}{2}\int_c^d x_2^2\,d\mu_2(x_2)\Big]
\end{aligned}$$

which is expression (2-6) in Theorem 2.4.

Example 3.4. Take

$$\nu_1=\frac{d-b_{\mu_2}}{2(d-c)}\mu_1\otimes\delta_c+\frac{b_{\mu_2}-c}{2(d-c)}\mu_1\otimes\delta_d+\frac{b-b_{\mu_1}}{2(b-a)}\delta_a\otimes\mu_2+\frac{b_{\mu_1}-a}{2(b-a)}\delta_b\otimes\mu_2,\qquad \nu_2=\delta_0\otimes\delta_0$$

in $\mathcal{P}(\Omega)$, then $\Pi(\nu_1,\nu_2)=\{\nu_1\otimes\nu_2\}$ is a singleton, and the optimal transportation cost from ν_1 to ν_2 is

$$\begin{aligned}
&\int_{\Omega\times\Omega}c(x,y)\,d\nu_1\otimes\nu_2(x,y)\\
&=\frac{d-b_{\mu_2}}{2(d-c)}\Big[\int_{\Omega\times\Omega}f(x_1-y_1,x_2-y_2)\,d\mu_1\otimes\delta_c\otimes\delta_0\otimes\delta_0(x_1,x_2,y_1,y_2)+\frac{k_1}{2}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_2}{2}c^2\Big]\\
&\quad+\frac{b_{\mu_2}-c}{2(d-c)}\Big[\int_{\Omega\times\Omega}f(x_1-y_1,x_2-y_2)\,d\mu_1\otimes\delta_d\otimes\delta_0\otimes\delta_0(x_1,x_2,y_1,y_2)+\frac{k_1}{2}\int_a^b x_1^2\,d\mu_1(x_1)+\frac{k_2}{2}d^2\Big]
\end{aligned}$$

$$+\frac{b-b_{\mu_1}}{2(b-a)}\left[\int_{\Omega\times\Omega} f(x_1-y_1,x_2-y_2)\,d\delta_a\otimes\mu_2\otimes\delta_0\otimes\delta_0(x_1,x_2,y_1,y_2)+\frac{k_1}{2}a^2+\frac{k_2}{2}\int_c^d x_2^2\,d\mu_2(x_2)\right]$$

$$+\frac{b_{\mu_1}-a}{2(b-a)}\left[\int_{\Omega\times\Omega} f(x_1-y_1,x_2-y_2)\,d\delta_b\otimes\mu_2\otimes\delta_0\otimes\delta_0(x_1,x_2,y_1,y_2)+\frac{k_1}{2}b^2+\frac{k_2}{2}\int_c^d x_2^2\,d\mu_2(x_2)\right]$$

$$=\frac{1}{2}\int_c^d \frac{b-b_{\mu_1}}{b-a}f_{x_2}(a)+\frac{b_{\mu_1}-a}{b-a}f_{x_2}(b)\,d\mu_2(x_2)+\frac{k_1}{4}\int_a^b x_1^2\,d\mu_1(x_1)$$

$$+\frac{1}{2}\int_a^b \frac{d-b_{\mu_2}}{d-c}f_{x_1}(c)+\frac{b_{\mu_2}-c}{d-c}f_{x_1}(d)\,d\mu_1(x_1)+\frac{k_2}{4}\int_c^d x_2^2\,d\mu_2(x_2)$$

$$+\frac{k_1(b-b_{\mu_1})}{4(b-a)}a^2+\frac{k_1(b_{\mu_1}-a)}{4(b-a)}b^2+\frac{k_2(d-b_{\mu_2})}{4(d-c)}c^2+\frac{k_2(b_{\mu_2}-c)}{4(d-c)}d^2$$

$$=\int_c^d \frac{b-b_{\mu_1}}{2(b-a)}f_{x_2}(a)+\frac{b_{\mu_1}-a}{2(b-a)}f_{x_2}(b)\,d\mu_2(x_2)+\frac{k_1}{4}\int_a^b x_1^2\,d\mu_1(x_1)$$

$$+\int_a^b \frac{d-b_{\mu_2}}{2(d-c)}f_{x_1}(c)+\frac{b_{\mu_2}-c}{2(d-c)}f_{x_1}(d)\,d\mu_1(x_1)+\frac{k_2}{4}\int_c^d x_2^2\,d\mu_2(x_2)$$

$$+\frac{k_1}{4}\left[[(a+b)\,b_{\mu_1}-ab]+\frac{k_2}{4}[(c+d)\,b_{\mu_2}-cd]\right],$$

which is expression (2-7) in Theorem 2.4.

Example 3.5. Take

$$\nu_1=\frac{(b-b_{\mu_1})(d-b_{\mu_2})}{(b-a)(d-c)}$$

$$\times\left(\delta_a\otimes\delta_c+\frac{(b_{\mu_1}-a)(d-b_{\mu_2})}{(b-a)(d-c)}\delta_b\otimes\delta_c+\frac{(b_{\mu_1}-a)(b_{\mu_2}-c)}{(b-a)(d-c)}\delta_b\otimes\delta_d+\frac{(b-b_{\mu_1})(b_{\mu_2}-c)}{(b-a)(d-c)}\delta_a\otimes\delta_d\right),$$

$$\nu_2=\delta_0\otimes\delta_0,$$

in $\mathcal{P}(\Omega)$, then $\Pi(\nu_1,\nu_2)=\{\nu_1\otimes\nu_2\}$ is a singleton, and the optimal transportation cost from ν_1 to ν_2 is

$$\int_{\Omega\times\Omega} c(x,y)\,d\nu_1\otimes\nu_2(x,y)$$

$$=\frac{(b-b_{\mu_1})(d-b_{\mu_2})}{(b-a)(d-c)}\left[f(a,c)+\frac{k_1}{2}a^2+\frac{k_2}{2}c^2\right]+\frac{(b-b_{\mu_1})(b_{\mu_2}-c)}{(b-a)(d-c)}\left[f(a,d)+\frac{k_1}{2}a^2+\frac{k_2}{2}d^2\right]$$

$$+\frac{(b_{\mu_1}-a)(d-b_{\mu_2})}{(b-a)(d-c)}\left[f(b,c)+\frac{k_1}{2}b^2+\frac{k_2}{2}c^2\right]+\frac{(b_{\mu_1}-a)(b_{\mu_2}-c)}{(b-a)(d-c)}\left[f(b,d)+\frac{k_1}{2}b^2+\frac{k_2}{2}d^2\right]$$

$$=\frac{(b-b_{\mu_1})(d-b_{\mu_2})}{(b-a)(d-c)}f(a,c)+\frac{(b-b_{\mu_1})(b_{\mu_2}-c)}{(b-a)(d-c)}f(a,d)+\frac{(b_{\mu_1}-a)(d-b_{\mu_2})}{(b-a)(d-c)}f(b,c)$$

$$+\frac{(b_{\mu_1}-a)(b_{\mu_2}-c)}{(b-a)(d-c)}f(b,d)+\frac{k_1}{2}[(a+b)\,b_{\mu_1}-ab]+\frac{k_2}{2}[(c+d)\,b_{\mu_2}-cd],$$

which is expression (2-8) in Theorem 2.4.

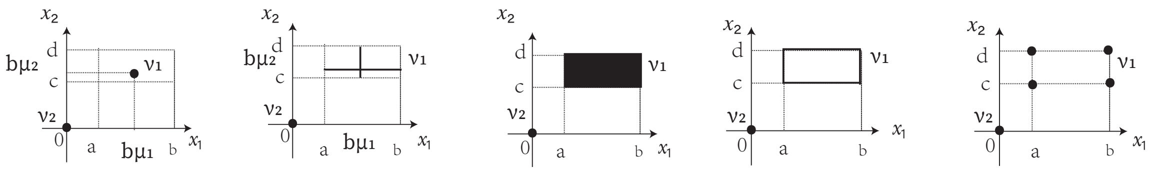

Figure 1. Mass transportation: Examples 3.1, 3.2, 3.3, 3.4, 3.5.

As the transport costs from ν_1 to ν_2 in Examples 3.1, 3.2, 3.3, 3.4 and 3.5 equal to each term in the Hermite–Hadamard inequality in Theorem 2.4, respectively, it follows that the transfer costs in these examples become more and more expensive (see Figure 1).

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PING CHEN: chenping200507@126.com
School of Mathematics and Information Technology, Jiangsu Second Normal University, Nanjing 210013, P. R. China.

WING-SUM CHEUNG: wscheung@hku.hk
Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong 999077, P. R. China.