

DIRAC COHOMOLOGY FOR DEGENERATE AFFINE HECKE–CLIFFORD ALGEBRAS

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Abstract. In this paper, we study the Dirac cohomology theory on a class of algebraic structures. The main examples of this algebraic structure are the degenerate affine Hecke–Clifford algebra of type A_{n-1} by Nazarov and of classical types by Khongsap–Wang. The algebraic structure contains a remarkable subalgebra, which usually refers to Sergeev algebra for type A_{n-1} .

We define an analogue of the Dirac operator for those algebraic structures. A main result is to relate the central characters of modules of those algebras with the central characters of modules of the Sergeev algebra via the Dirac cohomology. The action of the Dirac operator on certain modules is also computed. Results in this paper could be viewed as a projective version of the Dirac cohomology of the degenerate affine Hecke algebra.

1. Introduction

Throughout this paper, we work over the ground field \mathbb{C} . Let W be a Weyl group. It is well known that W admits a non-trivial central extension

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \widetilde{W} \rightarrow W \rightarrow 1,$$

where \widetilde{W} is a distinguished double cover of W . The projective representations of W are linear representations of \widetilde{W} which do not factor through W . Those representations over \mathbb{C} have been known for a long time from the work of Schur, Morris, Read, Stembridge, and others [Mo1], [Mo2], [Re], [Sc], [St].

The degenerate affine Hecke–Clifford algebra for type A_{n-1} (see Definition 4.2) was introduced by Nazarov [Na] to study Young’s symmetrizers of the projective representations of S_n . The degenerate affine Hecke–Clifford algebra for other classical types was later constructed by Khongsap–Wang [WK]. Those algebras could

be viewed as the projective counterpart of the degenerate affine Hecke algebra of Lusztig.

The purpose of this paper is to establish Dirac cohomology theory for those classes of algebras. We first single out the algebraic structure (see Section 3) that is necessary to prove several important results for the Dirac cohomology, and then we show that the degenerate affine Hecke–Clifford algebras considered in [Na] and [WK] satisfy that algebraic structure. Our approach is an analogue of the one recently developed for degenerate affine Hecke algebras by Barbasch–Ciubotaru–Trapa [BCT] (also see a recent extension by Ciubotaru [Ci2]).

In more detail, let \mathcal{H}_W be the associative algebra with certain important properties (see Definitions 3.1 and 3.3). The algebra \mathcal{H}_W contains a remarkable subalgebra, namely $\text{Seg}(W)$ (see again Definition 3.1), which is the same as the Sergeev algebra when W is of type A_{n-1} . The Dirac type element in \mathcal{H}_W is defined as an analogue of the one in [BCT] and has some nice properties. In specific examples of \mathcal{H}_W in Section 4, the Dirac type element can be viewed as the square root of a certain Casimir type element (Theorem 4.23).

For an \mathcal{H}_W -module (π, X) , the Dirac cohomology is defined as

$$H_D(X) = \ker \pi(D) / (\ker \pi(D) \cap \text{im } \pi(D)),$$

which is a $\text{Seg}(W)$ -module. One of our main results (Theorem 3.5) says that if X is irreducible and $H_D(X)$ is nonzero, then any irreducible $\text{Seg}(W)$ -module in $H_D(X)$ determines the central character of X . This is an analogue to a statement for Harish-Chandra modules called Vogan’s conjecture [HP]. A key step in the proof of Theorem 3.5 is to establish a canonical algebra homomorphism from the center of \mathcal{H}_W to the center of $\text{Seg}(W)$ (Theorem 3.4). In the case of the degenerate affine Hecke–Clifford algebra of type A_{n-1} , this homomorphism is shown to map onto the even elements of the center of $\text{Seg}(W)$ via the study of the Dirac cohomology on some modules (Corollary 7.21). The homomorphism indeed agrees with another natural map arising from the Jucys–Murphy type elements (see more detail in Remark 7.22), and hence the property of surjectivity has already been covered in the result of [Ru].

For a Dirac cohomology in other settings (see, for example, [HP]), one may apply the Dirac operator and Dirac cohomology developed in this paper to study the representation theory of \mathcal{H}_W . More precisely, the action of the Dirac operator provides information about the $\text{Seg}(W)$ -module structure and central characters of some \mathcal{H}_W -modules (see Corollary 4.24 and Theorem 4.25).

We provide evidences that the Dirac cohomology can be useful in the representation theory by computing the action of the Dirac operators in several cases. In Section 5, we consider some basic modules for all classical types and show that the Dirac operator acts identically to zero on those modules. Those modules for type A_{n-1} were constructed and studied by Hill–Kujawa–Sussan [HKS]. In Section 7, we go further for type A_{n-1} and compute the action of the Dirac type element D on more interesting modules. We show that the Dirac cohomology of those examples does not vanish, and this indeed coincides with the expectation from the case of the degenerate affine Hecke algebra in [BCT]. While some computations

can also be done for other classical types, the picture is more complete for type A_{n-1} to date.

This paper is organized as follows. In Section 2, we review some properties of superalgebras. In Section 3, we define a certain algebraic structure \mathcal{H}_W and develop the Dirac cohomology theory for \mathbb{H}_W . We provide examples of \mathcal{H}_W in Section 4 and compute the square of the Dirac operator. In Section 5 and Section 7, we consider the Dirac cohomology for some particular modules. In Section 6, we review properties of Sergeev algebra which is needed for the computation of Section 7.

Acknowledgment. The author would like to thank Dan Ciubotaru and Peter Trapa for the suggestion of this topic and many useful discussions. He also thanks Professor Weiqiang Wang for his interest in the work and pointing out the reference [Wa]. The author would also like to thank the referees for useful suggestions and comments, and also thank one of the referees for pointing out the reference [Ru].

2. Preliminaries

2.1. Notation for modules

In this paper, all the algebras are associative with a unit over \mathbb{C} . Let \mathcal{A} be an algebra. An \mathcal{A} -module is denoted (π, X) or simply X , where X is a vector space and π is the map defining the action of \mathcal{A} on X . For $a \in \mathcal{A}$ and $x \in X$, the action of a on x is written by $\pi(a)x$ or $a.x$.

Let \mathcal{B} be a subalgebra of \mathcal{A} . Define $\text{Ind}_{\mathcal{B}}^{\mathcal{A}}$ to be the induction functor, i.e.,

$$\text{Ind}_{\mathcal{B}}^{\mathcal{A}} Y = \mathcal{A} \otimes_{\mathcal{B}} Y,$$

where Y is a \mathcal{B} -module. The left adjoint functor of $\text{Ind}_{\mathcal{B}}^{\mathcal{A}}$ is the restriction functor denoted $\text{Res}_{\mathcal{B}}^{\mathcal{A}}$.

2.2. Superalgebras and supermodules

A super vector space V is a \mathbb{Z}_2 -graded vector space $V = V_0 \oplus V_1$. A super vector subspace W of V is a subspace of V such that $W = (W \cap V_0) \oplus (W \cap V_1)$. We say an element a in V_0 (resp. V_1) has even (resp. odd) degree, denoted $\text{deg}(v) = 0$ (resp. $\text{deg}(v) = 1$).

A superalgebra \mathcal{A} is an algebra with a super vector space structure $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for $i, j \in \mathbb{Z}_2$. A subalgebra \mathcal{C} of a superalgebra \mathcal{A} is said to be a supersubalgebra of \mathcal{A} if $\mathcal{C} = (\mathcal{A}_0 \cap \mathcal{C}) \oplus (\mathcal{A}_1 \cap \mathcal{C})$. A super ideal \mathcal{I} of a superalgebra \mathcal{A} is an ideal of \mathcal{A} such that $\mathcal{I} = (\mathcal{A}_0 \cap \mathcal{I}) \oplus (\mathcal{A}_1 \cap \mathcal{I})$.

For superalgebras \mathcal{A} and \mathcal{B} , a superalgebra homomorphism from \mathcal{A} to \mathcal{B} is an algebra homomorphism with $f(\mathcal{A}_i) \subset \mathcal{B}_i$ for $i \in \mathbb{Z}_2$.

For superalgebras \mathcal{A} and \mathcal{B} , the super tensor product of \mathcal{A} and \mathcal{B} , denoted $\mathcal{A} \widetilde{\otimes} \mathcal{B}$, is a superalgebra isomorphic to $\mathcal{A} \otimes \mathcal{B}$ as vector spaces with the multiplication determined by:

$$(a \otimes b)(a' \otimes b') = (-1)^{\text{deg}(b) \text{deg}(a')} (aa' \otimes bb'),$$

where $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ are homogeneous elements.

Let \mathcal{A} be a superalgebra. An \mathcal{A} -supermodule X is an \mathcal{A} -module with a super vector space structure $X = X_0 \oplus X_1$ and the property that $\mathcal{A}_i \cdot X_j \subseteq X_{i+j}$, where $i, j \in \mathbb{Z}_2$. A supersubmodule Y of an \mathcal{A} -supermodule X is a submodule of X such that $Y = (X_0 \cap Y) \oplus (X_1 \cap Y)$. An \mathcal{A} -supermodule X is irreducible if there is no proper non-zero supersubmodule of X .

For an \mathcal{A} -supermodule $X = X_0 \oplus X_1$, define a map $\delta : X \rightarrow X$ such that $\delta(v) = v$ if $v \in X_0$ and $\delta(v) = -v$ if $v \in X_1$.

Let $\text{Mod}_{\text{sup}}(\mathcal{A})$ be the category of \mathcal{A} -supermodules. The morphisms in the category $\text{Mod}_{\text{sup}}(\mathcal{A})$ are the even homomorphisms between \mathcal{A} -supermodules. Let $\Pi : \text{Mod}_{\text{sup}}(\mathcal{A}) \rightarrow \text{Mod}_{\text{sup}}(\mathcal{A})$ be a parity change functor. That means for an \mathcal{A} -supermodule, $\Pi(M)$ and M are isomorphic as \mathcal{A} -modules, but have opposite \mathbb{Z}_2 -grading.

2.3. Relations between irreducible supermodules and irreducible modules

Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a superalgebra. Given an irreducible \mathcal{A} -module (π, Y) , we construct a supermodule as follows. Let $(\bar{\pi}, \bar{Y})$ be an irreducible \mathcal{A} -module such that \bar{Y} is identified with Y as vector spaces and the \mathcal{A} -action on \bar{Y} is determined for any homogenous element $a \in \mathcal{A}$ and for $v \in Y$ by

$$\bar{\pi}(a)v = (-1)^{\text{deg}(a)}\pi(a)v.$$

Let (π_{X_Y}, X_Y) be an \mathcal{A} -supermodule such that $X_Y = Y \oplus \bar{Y}$ as vector spaces and the action of \mathcal{A} on $X_Y = Y \oplus \bar{Y}$ is as: $\pi_{X_Y}(a)(v, \bar{v}) = (\pi(a)v, \bar{\pi}(a)\bar{v})$. Let $(X_Y)_0 = \{(v, \bar{v}) \in X_Y : v = \bar{v}\}$ and let $(X_Y)_1 = \{(v, \bar{v}) \in X_Y : v = -\bar{v}\}$. It is elementary to check that $X_Y = (X_Y)_0 \oplus (X_Y)_1$ is an \mathcal{A} -supermodule.

Lemma 2.1. *Let Y be an irreducible \mathcal{A} -module. Let $X_Y = Y \oplus \bar{Y}$ be an \mathcal{A} -supermodule with the supermodule structure described above. Then*

- (1) X_Y is an irreducible \mathcal{A} -supermodule if and only if Y and \bar{Y} are non-isomorphic as \mathcal{A} -modules.
- (2) If Y and \bar{Y} are isomorphic as \mathcal{A} -modules, then there is a supermodule structure on Y .

Proof. For (1), we first prove that if X_Y is an irreducible \mathcal{A} -supermodule, then Y and \bar{Y} are not isomorphic as \mathcal{A} -modules. Suppose instead there exists an \mathcal{A} -module isomorphism $f : Y \rightarrow \bar{Y}$, and we will derive a contradiction. Recall that \bar{Y} is identified with Y as vector spaces and thus there exists a natural vector space isomorphism $\theta : \bar{Y} \rightarrow Y$ such that $(-1)^{\text{deg}(a)}\pi(a)\theta = \theta\bar{\pi}(a)$ for any homogenous $a \in \mathcal{A}$. Then $\theta \circ f$ satisfies the property that for any homogenous element $a \in \mathcal{A}$,

$$\pi(a)(\theta \circ f)(x) = (-1)^{\text{deg}(a)}(\theta \circ f)(\pi(a)x) \quad .$$

Then the map $(\theta \circ f)^2$ is an \mathcal{A} -module automorphism of Y . Thus, by Schur's lemma and a suitable normalization, we may assume $(\theta \circ f)^2$ is an identity map. Then as vector spaces

$$Y = \ker(\theta \circ f - \text{Id}) \oplus \ker(\theta \circ f + \text{Id}).$$

For $\epsilon = 0, 1$, let

$$\text{Ker}_\epsilon = \{(v, (-1)^\epsilon v) \in X_Y : v \in \ker(\theta \circ f - (-1)^\epsilon \text{Id})\}.$$

Then it is straightforward to verify that $\text{Ker}_0 \oplus \text{Ker}_1 \subset X_Y$ gives a proper supersubmodule of X_Y .

We now prove that if Y and \overline{Y} are not isomorphic as \mathcal{A} -modules, X_Y is an irreducible \mathcal{A} -supermodule. Suppose instead that there exists a proper supersubmodule M of X_Y and we will get a contradiction. Let

$$M^i = \{v \in Y : (v, (-1)^i v) \in M \cap (X_Y)_i\}$$

for $i \in \mathbb{Z}_2$, which are regarded as vector subspaces of Y . We first see that $M^0 \cap M^1 = 0$. Otherwise, there exists some nonzero $v \in Y$ such that $(v, v) \in M$ and $(v, -v) \in M$, and so $(v, 0), (0, v) \in M$. The irreducibility of Y and \overline{Y} implies $M = X_Y$, contradicting that M is proper. Furthermore, the irreducibility of Y implies $Y = M^0 \oplus M^1$ (as vector spaces). Define a map $f : (\pi, Y) \rightarrow (\overline{\pi}, \overline{Y})$ determined by $f(v) = (-1)^i v$ for $v \in M^i$ ($i \in \mathbb{Z}_2$). One can check that f is an \mathcal{A} -module isomorphism and so this gives a contradiction.

We now consider (2). By (1), X_Y is not an irreducible \mathcal{A} -supermodule. Let X' be an irreducible supersubmodule of X_Y . Then by the construction of X_Y , X' is isomorphic to $Y = \overline{Y}$ as \mathcal{A} -modules. Then this gives a supermodule structure on Y . \square

We can also start with an irreducible \mathcal{A} -supermodule and decompose it into irreducible \mathcal{A} -module(s).

Lemma 2.2. *Let X be an irreducible \mathcal{A} -supermodule. Let δ be a linear automorphism on X such that $\delta(v) = (-1)^i v$ for $v \in X_i$ ($i = 0, 1$). If X is not an irreducible \mathcal{A} -module, then there exists an irreducible \mathcal{A} -submodule Y of X such that*

- (1) $\delta(Y)$ is also an \mathcal{A} -submodule of X and $\delta(Y) = \overline{Y}$; and
- (2) Y and $\delta(Y)$ are non-isomorphic \mathcal{A} -modules; and
- (3) $X = Y \oplus \delta(Y)$ as \mathcal{A} -modules.

Proof. (1) follows from $a.\delta(v) = (-1)^{\deg(a)}\delta(a.v)$ for any homogenous element $a \in \mathcal{A}$ and $v \in Y$. (2) and (3) are (a reformulation of) [BK, Lem. 2.3]. \square

Lemma 2.3. *Let X and X' be irreducible \mathcal{A} -supermodules. If X and X' are isomorphic as \mathcal{A} -modules, then X and X' are isomorphic, up to applying the functor Π , as \mathcal{A} -supermodules.*

Proof. Suppose X and X' are also irreducible \mathcal{A} -modules. Then X_0, X_1, X'_0, X'_1 are irreducible \mathcal{A}_0 -modules. Then either $X_0 = X'_0$ or $X_0 = X'_1$ as \mathcal{A}_0 -modules. Then either $X \cong X'$ or $X \cong \Pi(X')$ as \mathcal{A} -supermodules.

Suppose X is not an irreducible \mathcal{A} -module. Let $X = Y \oplus \delta(Y)$ and $X' = Y' \oplus \delta(Y')$ be the decomposition of X into \mathcal{A} -modules as in Lemma 2.2. Without loss of generality, we may assume $Y = Y'$ as \mathcal{A} -modules. Let $f : Y \rightarrow Y'$ be an \mathcal{A} -module isomorphism. Then f also induces an \mathcal{A} -module isomorphism $\overline{f} : \delta(Y) \rightarrow \delta(Y')$ such that $\overline{f} = \delta \circ f \circ \delta$. Then one can show that the map $f \oplus \overline{f}$ is an \mathcal{A} -supermodule isomorphism by checking that the map preserves grading. In particular, we also have $\Pi(X) = X$ as \mathcal{A} -supermodules in this case. \square

Let $\text{Irr}(\mathcal{A})$ (resp. $\text{Irr}_{\text{sup}}(\mathcal{A})$) be the set of irreducible \mathcal{A} -modules (resp. irreducible \mathcal{A} -supermodules). Let \sim be the equivalence relation on $\text{Irr}(\mathcal{A})$: $Y \sim Y'$ if and only if $Y = Y'$ or $Y = \overline{Y'}$. Let \sim_{Π} be the equivalence relation on $\text{Irr}_{\text{sup}}(\mathcal{A})$: $X \sim_{\Pi} X'$ if and only if $X = X'$ or $X = \Pi(X')$.

Proposition 2.4. *There is a natural bijection*

$$\text{Irr}_{\text{sup}}(\mathcal{A})/\sim_{\Pi} \longleftrightarrow \text{Irr}(\mathcal{A})/\sim .$$

Proof. Lemmas 2.1 and 2.3 define a map from $\text{Irr}(\mathcal{A})/\sim$ to $\text{Irr}_{\text{sup}}(\mathcal{A})/\sim_{\Pi}$. Lemma 2.2 defines a map in the opposite direction. The two maps are inverse to each other by Lemma 2.3. \square

2.4. Central characters of supermodules

For a superalgebra \mathcal{A} , let $Z(\mathcal{A})$ be the center of \mathcal{A} . Note that $Z(\mathcal{A})$ is a supersubalgebra of \mathcal{A} . Recall that $Z(\mathcal{A})_0$ is the set of even elements in $Z(\mathcal{A})$.

Proposition 2.5. *Let X be an irreducible \mathcal{A} -supermodule. For $z \in Z(\mathcal{A})_0$, z acts on X by the multiplication of a scalar.*

Proof. If X is an irreducible \mathcal{A} -module, then the statement follows from (ordinary) Schur’s lemma (for this case). If X is not an irreducible \mathcal{A} -module, then we can decompose $X = Y \oplus \delta(Y)$ as \mathcal{A} -modules as in Lemma 2.2. Then z acts on the two modules Y and $\delta(Y)$ by scalars, denoted λ and λ' respectively. Then for $v \in Y$,

$$z.(v + \delta(v)) = \frac{\lambda + \lambda'}{2}(v + \delta(v)) + \frac{\lambda - \lambda'}{2}(v - \delta(v)).$$

Note that $\delta(v + \delta(v)) = v + \delta(v)$ and so $v + \delta(v) \in X_0$, and similarly $v - \delta(v) \in X_1$. Then since z is of even degree, $\lambda = \lambda'$. \square

By Proposition 2.5, we can define the following:

Definition 2.6. Let \mathcal{A} be a superalgebra. Let (π, X) be an irreducible \mathcal{A} -supermodule. Define the central character χ_{π} to be the map from $Z(\mathcal{A})_0$ to \mathbb{C} such that $\chi_{\pi}(z)$ is the scalar of z acting on X .

The central character defined above is only for even elements in the center of a superalgebra. However, the central character indeed determines the action of odd elements in the center in the following sense:

Proposition 2.7. *Let $z \in Z(\mathcal{A})_1$. Let X be an irreducible \mathcal{A} -supermodule. If X is also an irreducible \mathcal{A} -module, then z acts by zero on X . If X is not an irreducible \mathcal{A} -module, then z acts on the two irreducible \mathcal{A} -submodules of X by two distinct scalars $\sqrt{\lambda}$ and $-\sqrt{\lambda}$, where λ is the scalar that $z^2 \in Z(\mathcal{A})_0$ acts on X .*

Proof. For (1), suppose X is an irreducible \mathcal{A} -module. Then by Schur’s Lemma, z acts on X by a scalar denoted by λ . Meanwhile by Lemmas 2.1 and 2.3, $X = \overline{X}$ as \mathcal{A} -modules. This implies z also acts by $-\lambda$ on X as z is an odd element. Hence $\lambda = 0$.

Now suppose X is not an irreducible \mathcal{A} -module. Then z^2 is an even element in the center and hence acts by a scalar, denoted λ . Then z acts on the irreducible \mathcal{A} -submodules of X by scalars $\sqrt{\lambda}$ and $-\sqrt{\lambda}$. \square

3. Dirac cohomology for \mathcal{H}_W

3.1. \mathcal{H}_W and a Dirac type element in \mathcal{H}_W

Fix a real reflection group W . Let V be a representation of W . Fix a W -invariant inner product on V . Let $\{a_1, \dots, a_n\}$ be an orthogonal basis for V .

Definition 3.1. An associative algebra $\mathcal{H}_W = \mathcal{H}_W(V)$ is said to have property $(*)$ if it satisfies the following properties. First \mathcal{H}_W is an algebra generated by symbols f_w ($w \in W$), c_i ($i = 1, \dots, n$) and a_i ($i = 1, \dots, n$) such that the map from $\mathbb{C}[W]$ to \mathcal{H}_W sending w to f_w is an injection and the algebra has a natural basis of elements having the form $a_1^{k_1} \cdots a_n^{k_n} c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} f_w$ (k_1, \dots, k_n non-negative integers, $w \in W$, $\epsilon_i = 0$ or 1). Again we shall write w for f_w for simplicity. Let $\text{Seg}(W)$ be the subalgebra of \mathcal{H}_W generated by all $w \in W$ and c_i ($i = 1, \dots, n$). Furthermore, the generators of \mathcal{H}_W satisfy the following relations:

$$wa_iw^{-1} = w(a_i), \tag{3.1}$$

$$[a_i, a_j] c_i c_j \in \text{Seg}(W) \quad \text{for } i \neq j,$$

$$c_j a_i = a_i c_j \quad \text{for } i \neq j, \tag{3.2}$$

$$c_i a_i = -c_i a_i, \tag{3.3}$$

$$c_i c_j = -c_j c_i \quad \text{for } i \neq j \quad \text{and} \quad c_i^2 = -1, \tag{3.4}$$

$$wc_i = w(c_i)w. \tag{3.5}$$

Here $w(a_i)$ is the action of w on V . Furthermore, we identify the linear space spanned by c_i with V via the map $a_i \mapsto c_i$ and hence there is a natural action of W on c_i , and $w(c_i)$ represents such action of w on c_i . Indeed, the algebra generated by the those c_i is isomorphic to the Clifford algebra on the vector space V , and the subalgebra $\text{Seg}(W)$ is the smash product of the Clifford algebra and the group algebra of W .

\mathcal{H}_W has a superalgebra structure with $\deg(c_i) = 1$, $\deg(a_i) = \deg(w) = 0$ ($i = 1, \dots, n$ and $w \in W$).

In the rest of this section, \mathcal{H}_W denotes an algebra satisfying the property $(*)$. Define a Dirac type element D in \mathcal{H}_W :

$$D = \sum_{i=1}^n a_i c_i. \tag{3.6}$$

The following two properties will be used several times:

Lemma 3.2.

- (1) $wD = Dw$ for any $w \in W$;
- (2) $c_i D = -Dc_i$ for any i .

Proof. (1) follows from the fact that $\{a_i\}$ forms an orthogonal basis and property (3.1). (2) follows from the properties (3.2), (3.3), and (3.4). \square

Two homogenous elements $h_1, h_2 \in \mathcal{H}_W$ are said to supercommute if $h_1 h_2 = (-1)^{\deg(h_1) \deg(h_2)} h_2 h_1$ for any homogenous $w \in \text{Seg}(W)$.

Definition 3.3. The algebra \mathcal{H}_W with the property $(*)$ is said to satisfy the property $(**)$ if for any $h \in \mathcal{H}_W$ such that h supercommutes with elements in $\text{Seg}(W)$, $D^2h - hD^2 = 0$.

In the next section, we shall give examples which satisfy the algebraic structure in Definitions 3.1 and 3.3. From now on, assume that \mathcal{H}_W satisfies the properties $(*)$ and $(**)$.

3.2. Relation between central characters for \mathcal{H}_W and $\text{Seg}(W)$

Let $d : \mathcal{H}_W \rightarrow \mathcal{H}_W$,

$$d(h) = Dh - (-1)^{\deg(h)}hD.$$

A relation between $Z(\mathcal{H}_W)_0$ and $Z(\text{Seg}(W))_0$ is the following:

Theorem 3.4. *For any $z \in Z(\mathcal{H}_W)_0$, there exists a unique element $\tilde{z} \in Z(\text{Seg}(W))_0$ such that*

$$z - \tilde{z} \in \text{im } d.$$

Let $\zeta : Z(\mathcal{H}_W)_0 \rightarrow Z(\text{Seg}(W))_0$ be the map that $\zeta(z)$ is such unique element \tilde{z} in $Z(\text{Seg}(W))_0$. Then ζ is an algebra homomorphism.

Our main result in this paper is the following, which says the central character of an \mathcal{H}_W -supermodule X is determined by the central characters of irreducible $\text{Seg}(W)$ -supermodules in the Dirac cohomology $H_D(X)$. Here $H_D(X)$ is defined in the theorem.

Theorem 3.5. *Let \mathcal{H}_W be an algebra satisfying property $(*)$ (Definition 3.1) and property $(**)$ (Definition 3.3). Let (π, X) be an irreducible \mathcal{H}_W -supermodule with the central character χ_π (Definition 2.6). Let the Dirac cohomology $H_D(X)$ of X be*

$$H_D(X) = \ker \pi(D) / (\ker \pi(D) \cap \text{im } \pi(D)).$$

Then $H_D(X)$ has a natural $\text{Seg}(W)$ -module structure. Let (σ, U) be an irreducible $\text{Seg}(W)$ -module with the central character χ_σ (Definition 2.6) such that

$$\text{Hom}_{\text{Seg}(W)}(U, H_D(X)) \neq 0.$$

Let $\zeta : Z(\mathcal{H}_W)_0 \rightarrow Z(\text{Seg}(W))_0$ be the map in Theorem 3.4. Let $\chi^\sigma : Z(\mathcal{H}_W)_0 \rightarrow \mathbb{C}$,

$$\chi^\sigma(z) = \chi_\sigma(\zeta(z)). \tag{3.7}$$

Then $\chi_\pi = \chi^\sigma$.

Since $wD = Dw$ and $c_iD = -Dc_i$ by Lemma 3.2, $\ker \pi(D)$ and $\ker \pi(D) \cap \text{im } \pi(D)$ are invariant under the action of $\text{Seg}(W)$. Thus $H_D(X)$ has a natural $\text{Seg}(W)$ -module structure from the \mathcal{H}_W -module structure. The proofs of Theorems 3.4 and 3.5 are given at the end of the next subsection. Theorem 3.5 directly follows from Theorem 3.4. Readers who only want to know how Theorem 3.4 implies Theorem 3.5 may jump to the end of the next subsection.

3.3. Proof of Theorems 3.4 and 3.5

The proofs of the theorems basically follow from the ideas of proofs in [HP, Chap. 3] and [BCT, Sect. 4]. We provide some technical details for this specific case.

Let $S^{\leq j}(V)$ be the vector space of polynomials of x_1, \dots, x_n with degree less than or equal to j . Let \mathcal{H}_W^j be the vector space spanned by elements of the form

$$\{pw : w \in \text{Seg}(W), p \in S^{\leq j}(V)\}.$$

Note that $\mathcal{H}_W^0 \subseteq \mathcal{H}_W^1 \subseteq \dots$ gives a filtration for \mathcal{H}_W . Define

$$\overline{\mathcal{H}}_W^r = \mathcal{H}_W^r / \mathcal{H}_W^{r-1},$$

for $r = 0, 1, \dots$ and $\mathcal{H}_W^{-1} = 0$. Let $\overline{\mathcal{H}}_W = \bigoplus_{j=0}^{\infty} \overline{\mathcal{H}}_W^j$. Note that $\overline{\mathcal{H}}_W$ has a natural superalgebra structure from \mathcal{H}_W .

Let $d_j : \overline{\mathcal{H}}_W^j \rightarrow \overline{\mathcal{H}}_W^{j+1}$ be the map induced from d and let $\overline{d} = \bigoplus_{j=0}^{\infty} d_j$. For any element $h \in \mathcal{H}_W$, we still write h for its corresponding element in $\overline{\mathcal{H}}_W$. Let $b_i = a_i c_i$ ($i = 1, \dots, n$). Let \mathcal{B} be the supersubalgebra of $\overline{\mathcal{H}}_W$ generated by all b_i . Note that $\overline{d}(\mathcal{B}) \subset \mathcal{B}$. Let \overline{d}' be the restriction of \overline{d} to \mathcal{B} .

In the following lemmas, one can see that $\ker \overline{d}'$, $\text{im } \overline{d}'$, $\ker \overline{d}$, $(\ker d \cap \text{im } d)^{\text{Seg}(W)}$ and so on are supersubspaces by using the fact that D is an homogenous element.

Lemma 3.6. *As supersubspaces of \mathcal{B} ,*

$$\ker \overline{d}' = \text{im } \overline{d}' \oplus \mathbb{C}.$$

Here \mathbb{C} is regarded as the \mathbb{C} -subalgebra of \mathcal{B} generated by 1.

Proof. Note that any element in \mathcal{B} can be uniquely written as a linear combination of elements of the form $pb_{i_1}b_{i_2} \dots b_{i_r}$ for $0 < i_1 < \dots < i_r \leq n$ and $p \in \mathbb{C}[b_1^2, \dots, b_n^2]$. Note that $D = \sum_{i=1}^n b_i$. Using the relations $b_i b_j = -b_j b_i$ (in \mathcal{B}) for $i \neq j$ and $b_i^2 b_j = b_j b_i^2$ (in \mathcal{B}) for any i, j , one can see that the action of \overline{d}' is determined by

$$\overline{d}'(pb_{i_1}b_{i_2} \dots b_{i_r}) = 2 \sum_{k=1}^r (-1)^{k-1} b_{i_k}^2 pb_{i_1} \dots \widehat{b_{i_k}} \dots b_{i_r},$$

where $p \in \mathbb{C}[b_1^2, \dots, b_n^2]$.

In order to apply the known cohomology of the Koszul complex, we identify \mathcal{B} with $\mathbb{C}[x_1, \dots, x_n] \otimes \wedge^\bullet \mathbb{C}^n$ as vector spaces, where $\wedge^\bullet \mathbb{C}^n$ is the exterior algebra, via the linear isomorphism η from $\mathbb{C}[x_1, \dots, x_n] \otimes \wedge^\bullet \mathbb{C}^n$ to \mathcal{B} determined by

$$\eta : p(x_1, \dots, x_n) \otimes e_{i_1} \wedge \dots \wedge e_{i_k} \mapsto p(b_1^2, \dots, b_n^2) b_{i_1} \dots b_{i_k},$$

where $\{e_1, \dots, e_n\}$ is the standard basis of \mathbb{C}^n . Then, by the above description of the action of \overline{d}' , the map $\eta^{-1} \circ \overline{d}' \circ \eta$ is a multiple of the differential map in the standard Koszul resolution. Then the result follows from the well known cohomology of the Koszul resolution. \square

Proposition 3.7. *As supersubspaces of $\overline{\mathcal{H}}_W$,*

$$\ker \overline{d} = \text{im } \overline{d} \oplus \text{Seg}(W).$$

Proof. By the property (*) of \mathcal{H}_W , $a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} w$ ($m_i \in \mathbb{Z}_{\geq 0}$, $\epsilon_i \in \{0, 1\}$ and $w \in W$) form a basis for $\overline{\mathcal{H}}_W$. Then $b_1^{m_1} b_2^{m_2} \cdots b_n^{m_n} c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} w$ ($m_i \in \mathbb{Z}_{\geq 0}$, $\epsilon_i \in \{0, 1\}$ and $w \in W$) also form a basis for $\overline{\mathcal{H}}_W$. Then as linear vector spaces, we may identify $\overline{\mathcal{H}}_W$ with $\mathcal{B} \otimes \text{Seg}(W)$ via the following map:

$$b_1^{m_1} b_2^{m_2} \cdots b_n^{m_n} c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} w \mapsto b_1^{m_1} \cdots b_n^{m_n} \otimes c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} w.$$

For any $h \in \overline{\mathcal{H}}_W$, $\overline{d}(hw) = \overline{d}(h)w$ for $w \in W$ and $\overline{d}(hc_i) = \overline{d}(h)c_i$. Then the map \overline{d} in $\overline{\mathcal{H}}_W$ is the same as $\overline{d}' \otimes \text{Id}$ in $\mathcal{B} \otimes \text{Seg}(W)$ under the above identification. Then by Lemma 3.6, one has

$$\begin{aligned} \ker \overline{d} &= \ker(\overline{d}' \otimes \text{Id}) = (\ker \overline{d}') \otimes \text{Seg}(W) \\ &= (\text{im } \overline{d}' \oplus \mathbb{C}) \otimes \text{Seg}(W) = \text{im } \overline{d} \oplus \text{Seg}(W). \quad \square \end{aligned}$$

For any subspace H of \mathcal{H}_W , define $H^{\text{Seg}(W)}$ to be the set of all elements supercommuting with elements in $\text{Seg}(W)$. If we view $\text{Seg}(W)$ as a subalgebra of $\overline{\mathcal{H}}_W$, we could similarly define $\overline{H}^{\text{Seg}(W)}$ for any subspace \overline{H} of $\overline{\mathcal{H}}_W$. Proposition 3.7 implies the following:

Corollary 3.8. *As supersubspaces of $\overline{\mathcal{H}}_W$,*

$$(\ker \overline{d})^{\text{Seg}(W)} = (\text{im } \overline{d})^{\text{Seg}(W)} \oplus Z(\text{Seg}(W)).$$

Lemma 3.9. *As supersubspaces of \mathcal{H}_W ,*

$$(\ker d)^{\text{Seg}(W)} = (\ker d \cap \text{im } d)^{\text{Seg}(W)} \oplus Z(\text{Seg}(W)).$$

Proof. It is clear that $Z(\text{Seg}(W))$ and $(\ker d \cap \text{im } d)^{\text{Seg}(W)}$ are subspaces of the space $(\ker d)^{\text{Seg}(W)}$ and thus $(\ker d \cap \text{im } d)^{\text{Seg}(W)} \oplus Z(\text{Seg}(W)) \subset (\ker d)^{\text{Seg}(W)}$. We will prove another inclusion by induction on the degree of filtration of an element in $(\ker d)^{\text{Seg}(W)}$.

Let h be an element in $(\ker d)^{\text{Seg}(W)}$ such that $h \in \mathcal{H}_W^i$ and $h \notin \mathcal{H}_W^{i-1}$ for some i . When $i = 0$, $\mathcal{H}_W^0 = \text{Seg}(W)$ and so the statement is clearly true. Now assume $i > 0$. Let \overline{h} be the image of h in $\overline{\mathcal{H}}_W^i$. Then by Corollary 3.8, $\overline{h} = \overline{d}(\overline{h}_0)$ for some unique \overline{h}_0 in $\overline{\mathcal{H}}_W^{i-1}$ such that $\overline{d}(\overline{h}_0) \in (\overline{\mathcal{H}}_W^i)^{\text{Seg}(W)}$. For any representative $h'_0 \in \mathcal{H}_W^{i-1}$ of \overline{h}_0 , let

$$h_0 = \frac{1}{2^n |W|} \sum_{k=1}^n \sum_{i_1 < \dots < i_k} \sum_{w \in W} (-1)^k (c_{i_1} \cdots c_{i_k}) w h'_0 w^{-1} (c_{i_1} \cdots c_{i_k})^{-1}.$$

By the uniqueness of the element \overline{h}_0 , \overline{h}_0 supercommutes with any element in $\text{Seg}(W)$. This implies h_0 is also a representative of \overline{h}_0 . Furthermore, h_0 supercommutes with elements in $\text{Seg}(W)$ and $d(h_0) \in (\mathcal{H}_W^i)^{\text{Seg}(W)}$. By the property (**), $d^2(h_0) = 0$ and so $d(h - d(h_0)) = 0$. By the induction hypothesis, $h - d(h_0) \in (\text{im } d)^{\text{Seg}(W)} \oplus Z(\text{Seg}(W))$. Hence, we also have $h \in (\text{im } d)^{\text{Seg}(W)} \oplus Z(\text{Seg}(W))$ since $d(h_0) \in (\text{im } d)^{\text{Seg}(W)}$. This completes the proof. \square

Lemma 3.10. $(\ker d)^{\text{Seg}(W)}$ is a supersubalgebra of \mathcal{H}_W and $(\ker d \cap \text{im } d)^{\text{Seg}(W)}$ is a two-sided super ideal of $(\ker d)^{\text{Seg}(W)}$.

Proof. Let $z_1, z_2 \in (\ker d)^{\text{Seg}(W)}$. Then $d(z_i) = 0$ and so $Dz_i = \delta(z_i)D$. Now $d(z_1z_2) = D(z_1z_2) - \delta(z_1z_2)D = \delta(z_1z_2)D - \delta(z_1z_2)D = 0$. Hence $z_1z_2 \in (\ker d)^{\text{Seg}(W)}$. Hence $(\ker d)^{\text{Seg}(W)}$ is a subalgebra of \mathcal{H}_W .

We next show that $(\ker d \cap \text{im } d)^{\text{Seg}(W)}$ is a two-sided ideal of $(\ker d)^{\text{Seg}(W)}$. Let $z \in (\ker d)^{\text{Seg}(W)}$ and $z' \in (\ker d \cap \text{im } d)^{\text{Seg}(W)}$. We have to show $zz', z'z \in (\ker d \cap \text{im } d)^{\text{Seg}(W)}$. Write $z' = Dh - \delta(h)D$ for some $h \in \mathcal{H}_W$. Since $d(z) = Dz - \delta(z)D = 0$,

$$zz' = zDh - z\delta(h)D = D\delta(z)h - z\delta(h)D = D\delta(z)h - \delta(z\delta(h))D \in \text{im } d.$$

We also proved in the beginning that $zz' \in \ker d$ and thus $zz' \in (\ker d \cap \text{im } d)^{\text{Seg}(W)}$. The proof for $z'z \in (\ker d \cap \text{im } d)^{\text{Seg}(W)}$ is similar. \square

Proof of Theorem 3.4 Since $z \in Z(\mathcal{H}_W)_0 \subset \ker d^{\text{Seg}(W)}$, by Lemma 3.9, there exists a unique $\tilde{z} \in Z(\text{Seg}(W))$ such that $z - \tilde{z} \in (\ker d \cap \text{im } d)^{\text{Seg}(W)} \subset \text{im } d$. Note that \tilde{z} is in $Z(\text{Seg}(W))_0$ since the decomposition in Lemma 3.9 is between super vector spaces. Hence we have a map $\zeta : Z(\mathcal{H}_W)_0 \rightarrow Z(\text{Seg}(W))_0$.

It remains to prove that ζ is an algebra homomorphism. To see that ζ is an algebra map, let $z_i \in Z(\mathcal{H}_W) \subset \ker d^{\text{Seg}(W)}$ ($i = 1, 2$). Write $z_i = \zeta(z_i) + h_i$ for some $h_i \in (\ker d \cap \text{im } d)^{\text{Seg}(W)}$. Then $z_1z_2 = \zeta(z_1)\zeta(z_2) + \zeta(z_1)h_2 + \zeta(z_2)h_1 + h_1h_2$. By Lemma 3.10, $z_1z_2 - \zeta(z_1)\zeta(z_2) \in (\ker d \cap \text{im } d)^{\text{Seg}(W)}$. Thus $\zeta(z_1z_2) = \zeta(z_1)\zeta(z_2)$. This completes the proof. \square

Proof of Theorem 3.5. By our hypothesis, there exists a non-zero element $v \in H_D(X)$ such that v is in the isotypic component U of $H_D(X)$. Let \tilde{v} be a representative of v in $\ker \pi(D)$. Now by Theorem 3.4 for any $z \in Z(\mathcal{H}_W)_0$, $z - \zeta(z) = Da - \delta(a)D$ for some $a \in \mathcal{H}_W$. Then $\pi(z - \zeta(z))\tilde{v} = \pi(Da - \delta(a)D)\tilde{v} = \pi(Da)\tilde{v} \in \text{im } \pi(D)$. On the other hand, $\pi(z - \zeta(z))\tilde{v} = \chi_\pi(z)\tilde{v} - (\chi_\sigma(\zeta(z))\tilde{v} + \tilde{v}')$ for some $\tilde{v}' \in \ker \pi(D) \cap \text{im } \pi(D)$ and so $(\chi_\pi(z) - \chi_\sigma(\zeta(z)))\tilde{v} \in \text{im } \pi(D)$. We also have $(\chi_\pi(z) - \chi_\sigma(\zeta(z)))\tilde{v} \in \ker \pi(D)$ as $\tilde{v} \in \ker \pi(D)$. Thus $\chi_\pi(z)\tilde{v} - \chi_\sigma(\zeta(z))\tilde{v} \in \text{im } \pi(D) \cap \ker \pi(D)$. Since we choose $v \neq 0$, we can only have $\chi_\pi(z) = \chi_\sigma(\zeta(z)) = \chi^\sigma(z)$. This completes the proof. \square

4. Examples of \mathcal{H}_W and their Dirac cohomology theory

Let W be a classical Weyl group and let $R = R(W)$ be the root system associated to W . Let $\mathbf{k} : R \rightarrow \mathbb{C}$ be a function such that $\mathbf{k}(\alpha_1) = \mathbf{k}(\alpha_2)$ if $\alpha_1 = w(\alpha_2)$ for some $w \in W$. We shall write \mathbf{k}_α for $\mathbf{k}(\alpha)$. For any $\alpha \in R$, let s_α be the simple reflection associated to α .

Let e_1, \dots, e_n be the standard basis of \mathbb{R}^n . Let $\langle \cdot, \cdot \rangle$ be the inner product on \mathbb{R}^n such that $\langle e_i, e_j \rangle = \delta_{ij}$.

4.1. Type A_{n-1}

Notation 4.1. Set $W = W(A_{n-1})$ to be the Weyl group of type A_{n-1} . The root system $R(A_{n-1})$ of type A_{n-1} is the set

$$R(A_{n-1}) = \{e_i - e_j : 1 \leq i \neq j \leq n\}.$$

Fix a set R^+ of positive roots

$$R^+(\mathbf{A}_{n-1}) = \{e_i - e_j : 1 \leq i < j \leq n\}.$$

We usually write $\alpha > 0$ for $\alpha \in R^+(\mathbf{A}_{n-1})$ and write $\alpha < 0$ for $-\alpha \in R^+(\mathbf{A}_{n-1})$. The set of simple roots Δ is

$$\{e_i - e_{i+1} : i = 1, \dots, n - 1\}.$$

Since there is only one W -orbit for $R(\mathbf{A}_{n-1})$, we simply write \mathbf{k} for \mathbf{k}_α for any $\alpha \in R(\mathbf{A}_{n-1})$. For $i \neq j$, let

$$\alpha_{ij} = \begin{cases} e_i - e_j & \text{if } i < j, \\ e_j - e_i & \text{if } i > j. \end{cases} \tag{4.8}$$

Thus α_{ij} is always a positive root.

For a root $\alpha \in R(\mathbf{A}_{n-1})$, let s_α be the corresponding simple reflection in $W(\mathbf{A}_{n-1})$. For simplicity, set $s_{ij} = s_{\alpha_{ij}}$.

Definition 4.2 ([Na]). The degenerate affine Hecke–Clifford algebra for type \mathbf{A}_{n-1} , denoted $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$, is the associative algebra with a unit generated by the symbols $\{x_i\}_{i=1}^n$, $\{c_i\}_{i=1}^n$ and $\{f_w : w \in W(\mathbf{A}_{n-1})\}$ determined by the following properties:

- (1) the map from the group algebra $\mathbb{C}[W(\mathbf{A}_{n-1})] = \bigoplus_{w \in W(\mathbf{A}_{n-1})} \mathbb{C}w$ to $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$ given by $w \mapsto f_w$ is an algebra injection;
- (2) $x_i x_j = x_j x_i$ for all i, j ;
- (3) $x_i c_j = c_j x_i$ for $i \neq j$ and $x_i c_i = -c_i x_i$ for all i ;
- (4) $c_i c_j = -c_j c_i$ for $i \neq j$ and $c_i^2 = -1$ for all i ;
- (5) $f_w c_i = c_{w(i)} f_w$ for $w \in W(\mathbf{A}_{n-1})$ and for all i ;
- (6) $f_{s_{i,i+1}} x_i - x_{i+1} f_{s_{i,i+1}} = \mathbf{k}(-1 + c_i c_{i+1})$ for all $i = 1, \dots, n - 1$ and $f_{s_{i,i+1}} x_j = x_j f_{s_{i,i+1}}$ for all i, j with $|i - j| > 1$.

We later simply write w for f_w . The algebra has a superalgebra structure with $\deg(c_i) = 1$, $\deg(w) = 0$ for $w \in W(\mathbf{A}_{n-1})$, and $\deg(x_i) = 0$.

For $i \neq j$, define $c_{\alpha_{ij}}$ as

$$c_{\alpha_{ij}} = \begin{cases} \frac{\sqrt{2}}{2}(c_i - c_j) & \text{if } i < j, \\ \frac{\sqrt{2}}{2}(c_j - c_i) & \text{if } j < i. \end{cases} \tag{4.9}$$

Let $\tilde{s}_{\alpha_{ij}} = \tilde{s}_{ij} = s_{ij} c_{\alpha_{ij}}$.

The superalgebra $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$ admits a PBW type basis:

Proposition 4.3 ([Kl, Thm. 14.2.2]). *The set*

$$\{x_1^{m_1} \cdots x_n^{m_n} c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} w : m_1, \dots, m_n \in \mathbb{Z}_{\geq 0}, \epsilon_1, \dots, \epsilon_n \in \{0, 1\}, w \in W(\mathbf{A}_{n-1})\}$$

forms a basis for $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$.

The main statement of this subsection is Proposition 4.9, which says that $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$ satisfies property $(*)$ defined in Definition 3.1.

Let $\tilde{s}_\alpha = s_\alpha c_\alpha$. For later convenience, we also set $\tilde{s}_{ij} = \tilde{s}_{\alpha_{ij}} = s_{\alpha_{ij}} c_{\alpha_{ij}}$, $y_i = x_i c_i$, $y'_i = y_i + \frac{\sqrt{2}}{2} \sum_{i \neq j} \tilde{s}_{i,j}$ and $x'_i = -y'_i c_i$. Note that $\mathbb{C}[W(\mathbf{A}_{n-1})]^-$ embeds into $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$ via the map $\tilde{t}_\alpha \mapsto \tilde{s}_\alpha$.

The notations y'_i and x'_i will be used to define the Dirac type element in $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$ and are inspired by the setting in the degenerate affine Hecke algebra in [BCT].

Lemma 4.4.

- (1) $c_i y_j = -y_j c_i$ for any i, j ;
- (2) $\tilde{s}_{ij} c_k = -c_k \tilde{s}_{ij}$ for any i, j, k with $i \neq j$;
- (3) $c_i y'_j = -y'_j c_i$ for any i, j ;
- (4) For $\alpha \in R^+$ and $w \in S_n$, $w \tilde{s}_\alpha w^{-1} = \tilde{s}_{w(\alpha)}$ if $w(\alpha) > 0$, and $w \tilde{s}_\alpha w^{-1} = -\tilde{s}_{-w(\alpha)}$ if $w(\alpha) < 0$.

The above lemma is elementary. We skip the proof.

We shall use the natural permutation of $W(\mathbf{A}_{n-1})$ on the set $\{1, \dots, n\}$ below.

Lemma 4.5. *Let $w \in W(\mathbf{A}_{n-1})$. Then*

$$w y_i w^{-1} - y_{w(i)} = \sqrt{2\mathbf{k}} \sum_{\beta > 0, w^{-1}(\beta) < 0, \langle \beta, w(e_i) \rangle \neq 0} \tilde{s}_\beta.$$

In particular, for $\alpha > 0$,

$$\tilde{s}_\alpha y_i \tilde{s}_\alpha^{-1} + y_{s_\alpha(i)} = -\sqrt{2\mathbf{k}} \sum_{\beta > 0, s_\alpha^{-1}(\beta) < 0, \langle \beta, s_\alpha(e_i) \rangle \neq 0} \tilde{s}_\beta.$$

Proof. For $w \in W(\mathbf{A}_{n-1})$, define $l(w) = |\{e_i - e_j \in R^+(\mathbf{A}_{n-1}) : w(e_i - e_j) < 0\}|$. When $l(w) = 1$, $w = s_\alpha$ for some $\alpha \in \Delta$. We consider three cases. When $\langle e_i, \alpha \rangle = 0$, it is easy to see $s_\alpha y_i s_\alpha - y_i = 0$. Now consider the case $\langle e_i, \alpha \rangle = 1$. In this case, we have

$$\begin{aligned} s_\alpha y_i s_\alpha &= s_\alpha x_i c_i s_\alpha \\ &= x_{i+1} c_{i+1} + \mathbf{k}(-1 + c_i c_{i+1}) c_i s_\alpha \\ &= x_{i+1} c_{i+1} + \mathbf{k}(-c_i + c_{i+1}) s_\alpha \\ &= x_{i+1} c_{i+1} + \mathbf{k} s_\alpha (c_i - c_{i+1}) \\ &= y_{i+1} + \sqrt{2\mathbf{k}} \tilde{s}_\alpha. \end{aligned}$$

For $\langle e_i, \alpha \rangle = -1$, by using $s_\alpha \tilde{s}_\alpha s_\alpha = -\tilde{s}_\alpha$ and the computation in the case $\langle e_i, \alpha \rangle = 1$, we have

$$s_\alpha y_{i+1} s_\alpha = y_i + \sqrt{2\mathbf{k}} \tilde{s}_\alpha.$$

We now use an induction on $l(w)$. Assume $l(w) = k$ for some $k > 1$. Write $w = s_\alpha w'$ for some simple reflection s_α and $w' \in W(\mathbf{A}_{n-1})$ with $l(w') = k - 1$. Set

$\epsilon = 1$ if $\langle \alpha, w(e_i) \rangle \neq 0$ and $\epsilon = 0$ otherwise. Then

$$\begin{aligned}
wy_iw^{-1} &= s_\alpha w' y_i w'^{-1} s_\alpha \\
&= s_\alpha y_{w'(i)} s_\alpha + \sqrt{2}\mathbf{k} \sum_{\substack{\beta > 0, w'^{-1}(\beta) < 0, \\ \langle \beta, w'(e_i) \rangle \neq 0}} s_\alpha \tilde{s}_\beta s_\alpha \quad (\text{induction hypothesis}) \\
&= y_{s_\alpha w'(i)} + \epsilon \sqrt{2}\mathbf{k} \tilde{s}_\alpha + \sqrt{2}\mathbf{k} \sum_{\substack{\beta > 0, w'^{-1}(\beta) < 0, \\ \langle \beta, w'(e_i) \rangle \neq 0}} \tilde{s}_{s_\alpha(\beta)} \quad (\text{calculation for } l(w) = 1) \\
&= y_{s_\alpha w'(i)} + \epsilon \sqrt{2}\mathbf{k} \tilde{s}_\alpha + \sqrt{2}\mathbf{k} \sum_{\substack{\beta > 0, w'^{-1}(\beta) < 0, \\ \langle s_\alpha(\beta), s_\alpha w'(e_i) \rangle \neq 0}} \tilde{s}_{s_\alpha(\beta)} \\
&= y_{w(i)} + \sqrt{2}\mathbf{k} \sum_{\substack{\beta > 0, w^{-1}(\beta) < 0, \\ \langle \beta, w(e_i) \rangle \neq 0}} \tilde{s}_\beta.
\end{aligned}$$

This proves the first assertion. The second assertion follows from the first one with the equation that

$$\tilde{s}_\alpha y_i \tilde{s}_\alpha^{-1} = s_\alpha c_\alpha y_i (-c_\alpha s_\alpha) = s_\alpha (c_\alpha^2) y_i s_\alpha = -s_\alpha y_i s_\alpha. \quad \square$$

Lemma 4.6. For $i \neq j$, $[x'_i, x'_j] c_i c_j = y'_i y'_j + y'_j y'_i \in \text{Seg}_n$.

Proof.

$$\begin{aligned}
y'_i y'_j + y'_j y'_i &= \left(y_i + \frac{\sqrt{2}}{2} \mathbf{k} \sum_{k \neq i} \tilde{s}_{i,k} \right) \left(y_j + \mathbf{k} \frac{\sqrt{2}}{2} \sum_{l \neq j} \tilde{s}_{l,j} \right) \\
&\quad + \left(y_j + \frac{\sqrt{2}}{2} \mathbf{k} \sum_{l \neq j} \tilde{s}_{l,j} \right) \left(y_i + \frac{\sqrt{2}}{2} \mathbf{k} \sum_{k \neq i} \tilde{s}_{i,k} \right) \\
&= y_i y_j + y_j y_i + \frac{\sqrt{2}}{2} \mathbf{k} \left(\sum_{k \neq i} \tilde{s}_{i,k} y_j + y_j \sum_{k \neq i} \tilde{s}_{i,k} + y_i \sum_{l \neq j} \tilde{s}_{l,j} + \sum_{l \neq j} \tilde{s}_{l,j} y_i \right) \\
&\quad + \frac{1}{2} \mathbf{k}^2 \left(\sum_{k \neq i} \tilde{s}_{i,k} \sum_{l \neq j} \tilde{s}_{l,j} + \sum_{l \neq j} \tilde{s}_{l,j} \sum_{i \neq k} \tilde{s}_{i,k} \right) \\
&= \frac{\sqrt{2}}{2} \mathbf{k} \left(\sum_{k \neq i} \tilde{s}_{i,k} y_j + y_j \sum_{k \neq i} \tilde{s}_{i,k} + y_i \sum_{l \neq j} \tilde{s}_{l,j} + \sum_{l \neq j} \tilde{s}_{l,j} y_i \right) \\
&\quad + \frac{1}{2} \mathbf{k}^2 \left(\sum_{l \neq j} \sum_{k \neq i} \tilde{s}_{l,j} \tilde{s}_{i,k} + \sum_{l \neq j} \sum_{k \neq i} \tilde{s}_{i,k} \tilde{s}_{l,j} \right)
\end{aligned}$$

By Lemma 4.5, the term $\frac{\sqrt{2}}{2} \left(\sum_{k \neq i} \tilde{s}_{i,k} y_j + y_j \sum_{k \neq i} \tilde{s}_{i,k} + y_i \sum_{l \neq j} \tilde{s}_{l,j} + \sum_{l \neq j} \tilde{s}_{l,j} y_i \right)$ is in $\text{Seg}(W(A_{n-1}))$. This completes the proof. \square

Lemma 4.7.

- (1) $w x'_i w^{-1} = x'_{w(i)}$;
- (2) $c_i x'_i = -x'_i c_i$ and $c_j x'_i = x'_i c_j$ for $i \neq j$.

Proof. For (1), it suffices to show when $w = s_\alpha$ for some $\alpha \in \Delta$. Fix an i . By the definition of x'_i , it suffices to show $s_\alpha y'_i s_\alpha = y'_{s_\alpha(i)}$. We consider two cases. In the case that $\langle e_i, \alpha \rangle = 0$, $s_\alpha(\alpha_{i,j}) > 0$ for any $j \neq i$. Then $s_\alpha \tilde{s}_{i,j} s_\alpha = \tilde{s}_{i,s_\alpha(j)}$ for any $j \neq i$. Thus, the last equality in Lemma 4.5 becomes

$$s_\alpha y'_i s_\alpha^{-1} = y_i + \frac{\sqrt{2}}{2} \mathbf{k} \sum_{j \neq i} \tilde{s}_{i,s_\alpha(j)} = y'_i$$

In the case that $\langle e_i, \alpha \rangle \neq 0$, let $k = i - 1$ or $i + 1$ such that $\alpha = \alpha_{i,k}$. Then, by Lemmas 4.4(4) and 4.5,

$$\begin{aligned} s_\alpha y'_i s_\alpha^{-1} &= y_{s_\alpha(i)} + \sqrt{2} \mathbf{k} \tilde{s}_\alpha - \frac{\sqrt{2}}{2} \mathbf{k} \tilde{s}_\alpha + \frac{\sqrt{2}}{2} \mathbf{k} \sum_{j \neq i,k} \tilde{s}_{k,j} \\ &= y_k + \frac{\sqrt{2}}{2} \mathbf{k} \sum_{j \neq k} \tilde{s}_{k,j} \\ &= y'_k. \end{aligned}$$

For (2), it is straightforward from Lemma 4.4 and $y'_i = x'_i c_i$. \square

Remark 4.8. The subalgebra of $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$ generated by the elements y_i and $\tilde{s}_{i,j}$ is the degenerate spin affine Hecke algebra of type \mathbf{A}_{n-1} defined in [Wan, Sect. 3.3]. (Other classical types for the degenerate spin affine Hecke algebra are established in [WK, Sect. 4].) The degenerate spin affine Hecke algebra can be regarded as a more elementary analogue of the degenerate affine Hecke algebra, and the notions of y'_i can be regarded as the Drinfeld presentation [Dr] under the analogue.

Proposition 4.9. *The degenerate affine Hecke–Clifford algebra $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$ satisfies the property (*) in Definition 3.1.*

Proof. We set W in Definition 3.1 equal to $W(\mathbf{A}_{n-1})$ and set a_i in Definition 3.1 to be x'_i . Using Lemmas 4.6 and 4.7, one can verify relations (3.1) to (3.5) in Definition 3.1. By Proposition 4.3 and expressions of $x'_i, (x'_1)^{m_1} \cdots (x'_n)^{m_n} c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} w$ ($m_1, \dots, m_n \in \mathbb{Z}, \epsilon_1, \dots, \epsilon_n \in \{0, 1\}, w \in W(\mathbf{A}_{n-1})$) form a basis for $\mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$. These verify the property (*). \square

4.2. Type \mathbf{B}_n

For type \mathbf{B}_n , we modify the original definition in [WK]. More precisely, the algebra we considered in Definition 4.11 is a deformation of the algebra in [WK]. It is not hard to do a similar modification for type \mathbf{A}_{n-1} . The main reason for this modification is to construct an explicit module in the next section, which cannot be done in the original definition of [WK] (by our approach). Considering the lack of existing literature for the representation theory of the degenerate affine Hecke–Clifford algebra for other classical types, such examples may be interesting and important.

Notation 4.10. Let $W = W(\mathbf{B}_n)$ be the Weyl group of type \mathbf{B}_n . Let the set $R(\mathbf{B}_n)$ of roots for type \mathbf{B}_n be

$$R(\mathbf{B}_n) = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm e_i : i = 1, \dots, n\}.$$

The roots $\pm e_i \pm e_j$ ($i \neq j$) are long, while the roots $\pm e_i$ are short. Fix a set $R^+(\mathbf{B}_n)$ of positive roots:

$$R^+(\mathbf{B}_n) = \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{e_i : i = 1, \dots, n\}.$$

The set Δ of simple roots is

$$\{e_i - e_{i+1} : i = 1, \dots, n - 1\} \cup \{e_n\}.$$

For $i \neq j > 0$, define α_{ij} as in (4.8), define $\alpha_{i,-j} = e_i + e_j$ and define $\alpha_i = e_i$. We also define $s_{ij} = s_{\alpha_{i,j}}$, $s_{i,-j} = s_{\alpha_{i,-j}}$ and $s_i = s_{\alpha_i}$.

We have a natural embedding $R(\mathbf{A}_{n-1}) \subset R(\mathbf{B}_n)$. and a natural embedding $W(\mathbf{A}_{n-1}) \subset W(\mathbf{B}_n)$ (i.e., the group $W(\mathbf{A}_{n-1})$ being the group generated by $s_{i,i+1}$ for $i = 1, \dots, n - 1$).

Definition 4.11. Let $N_{\mathbf{B}_n} \in \mathbb{C}$. Let $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}} = \mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}(\mathbf{k}, N_{\mathbf{B}_n})$ be the associative unital algebra generated by the symbols $\{x_i\}_{i=1}^n$, $\{c_i\}_{i=1}^n$ and $\{f_w : w \in W(\mathbf{B}_n)\}$ subject to the relations of (3), (4), (5), (6) in Definition 4.2 and additionally,

- (1) the map from the group algebra $\mathbb{C}[W(\mathbf{B}_n)] = \bigoplus_{w \in W(\mathbf{B}_n)} \mathbb{C}w$ to $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$ given by $w \mapsto f_w$ is an algebra injection;
- (2) $f_{s_n} c_n = -c_n f_{s_n}$ and $f_{s_n} c_i = c_i f_{s_n}$ for $i \neq n$;
- (3)

$$\begin{aligned} f_{s_n} x_n + x_n f_{s_n} &= -\sqrt{2} \mathbf{k}_{\alpha_n}, \\ f_{s_n} x_j - x_j f_{s_n} &= 0 \quad \text{for } j \neq n; \end{aligned}$$

- (4) $x_i x_j - x_j x_i = N_{\mathbf{B}_n} c_j c_i$ for $i \neq j$.

We shall again simply write w for f_w .

When $N_{\mathbf{B}_n} = 0$, $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}(\mathbf{k}, N_{\mathbf{B}_n})$ coincides with the degenerate affine Hecke–Clifford algebra of type \mathbf{B}_n in [WK, Def. 3.9].

For $N_{\mathbf{B}_n} \neq 0$, while x_i and x_j does not commute for $i \neq j$, we still have $x_i^2 x_j = x_j x_i^2$. The algebra $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$ hence still has some nice properties such as the commutation relations with intertwining operators (but we do not need this in this paper).

For $i \neq j > 0$, define $c_{\alpha_{ij}}$ as in (4.9) and define

$$c_{\alpha_{i,-j}} = \frac{\sqrt{2}}{2} (c_i + c_j).$$

Set $\tilde{s}_{i,-j} = s_{i,-j} c_{\alpha_{i,-j}}$. We also set $\tilde{s}_\alpha = \tilde{s}_i = s_i c_i$.

Since we have modified the original definition of the degenerate affine Hecke–Clifford algebra for type \mathbf{B}_n in [WK], we will give a proof for the existence of the PBW type basis.

Proposition 4.12. *The set*

$$\{x_1^{m_1} \dots x_n^{m_n} c_1^{\epsilon_1} \dots c_n^{\epsilon_n} w : m_1, \dots, m_n \in \mathbb{Z}_{\geq 0}, \epsilon_1, \dots, \epsilon_n \in \{0, 1\}, w \in W(\mathbf{B}_n)\}$$

forms a basis for $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$.

Proof. We follow the argument in [Kl, Thm. 3.2.2]. We consider the algebra $\widetilde{\mathbb{H}}$ generated by $\{\bar{x}_i\}$, $\{\bar{c}_i\}$ and $\{\bar{s}_{i,i+1}\}_{i=1}^{n-1} \cup \{\bar{s}_n\}$ subject to the relations (3), (4), (5), (6) in Definition 4.2 and the relation (2) (but not (1)) in Definition 4.18 (with a trivial replacement of notations). We resolve the minimal ambiguities according to the Bergman's diamond lemma [Be]. For example, we may consider an ordering $\bar{s} < \bar{c}_n < \cdots < \bar{c}_1 < \bar{x}_n < \cdots < \bar{x}_1$, where s is any simple reflection in $W(\mathbf{B}_n)$. This induces a semigroup ordering on $\langle \bar{x}_i, \bar{c}_i, \bar{s} \rangle$ ($i = 1, \dots, n$ and s runs for all simple reflections) from the length of words and the lexicographical ordering. Then one checks that

$$\begin{aligned} (\bar{s}_{i,i+1}\bar{x}_{i+1})\bar{x}_i &= (\bar{x}_i\bar{s}_{i,i+1} - \mathbf{k}_{\alpha_{i,i+1}}(-1 + \bar{c}_{i+1}\bar{c}_i))\bar{x}_i \\ &= \bar{x}_i\bar{s}_{i,i+1}\bar{x}_i - \mathbf{k}_{\alpha_{i,i+1}}(-1 + \bar{c}_{i+1}\bar{c}_i)\bar{x}_i \\ &= \bar{x}_i\bar{x}_{i+1}\bar{s}_{i,i+1} + \mathbf{k}_{\alpha_{i,i+1}}\bar{x}_i(-1 + \bar{c}_i\bar{c}_{i+1}) - \mathbf{k}_{\alpha_{i,i+1}}(-1 + \bar{c}_{i+1}\bar{c}_i)\bar{x}_i \\ &= \bar{x}_i\bar{x}_{i+1}\bar{s}_{i,i+1} \end{aligned}$$

and

$$\begin{aligned} \bar{s}_{i,i+1}(\bar{x}_{i+1}\bar{x}_i) &= \bar{s}_{i,i+1}(\bar{x}_i\bar{x}_{i+1} + N_{\mathbf{B}_n}\bar{c}_i\bar{c}_{i+1}) \\ &= \bar{s}_{i,i+1}\bar{x}_i\bar{x}_{i+1} + N_{\mathbf{B}_n}\bar{c}_{i+1}\bar{c}_i\bar{s}_{i,i+1} \\ &= \bar{x}_{i+1}\bar{s}_{i,i+1}\bar{x}_{i+1} + \mathbf{k}_{\alpha_{i,i+1}}(-1 + \bar{c}_i\bar{c}_{i+1})\bar{x}_{i+1} + N_{\mathbf{B}_n}\bar{c}_{i+1}\bar{c}_i\bar{s}_{i,i+1} \\ &= \bar{x}_{i+1}\bar{x}_i\bar{s}_{i,i+1} - \mathbf{k}_{\alpha_{i,i+1}}\bar{x}_{i+1}(-1 + \bar{c}_{i+1}\bar{c}_i) \\ &\quad + \mathbf{k}_{\alpha_{i,i+1}}(-1 + \bar{c}_i\bar{c}_{i+1})\bar{x}_{i+1} + N_{\mathbf{B}_n}\bar{c}_{i+1}\bar{c}_i\bar{s}_{i,i+1} \\ &= \bar{x}_i\bar{x}_{i+1}\bar{s}_{i,i+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} (\bar{s}_n\bar{x}_n)\bar{x}_j &= (-\bar{x}_n\bar{s}_n - \sqrt{2}\mathbf{k}_{\alpha_n})\bar{x}_j \\ &= -\bar{x}_n\bar{s}_n\bar{x}_j - \sqrt{2}\mathbf{k}_{\alpha_n}\bar{x}_j \\ &= -\bar{x}_n\bar{x}_j\bar{s}_n - \sqrt{2}\mathbf{k}_{\alpha_n}\bar{x}_j \\ &= -\bar{x}_j\bar{x}_n\bar{s}_n - N_{\mathbf{B}_n}c_jc_n\bar{s}_n - \sqrt{2}\mathbf{k}_{\alpha_n}\bar{x}_j \end{aligned}$$

and

$$\begin{aligned} \bar{s}_n(\bar{x}_n\bar{x}_j) &= \bar{s}_n\bar{x}_j\bar{x}_n + N_{\mathbf{B}_n}\bar{s}_n\bar{c}_j\bar{c}_n \\ &= \bar{x}_j\bar{s}_n\bar{x}_n + N_{\mathbf{B}_n}\bar{s}_n\bar{c}_j\bar{c}_n \\ &= -\bar{x}_j\bar{x}_n\bar{s}_n - \sqrt{2}\mathbf{k}_{\alpha_n}\bar{x}_j - N_{\mathbf{B}_n}\bar{c}_j\bar{c}_n\bar{s}_n. \end{aligned}$$

Similarly, for $i > j > k$,

$$\begin{aligned} (\bar{x}_i\bar{x}_j)\bar{x}_k &= \bar{x}_j\bar{x}_i\bar{x}_k + N_{\mathbf{B}_n}c_jc_i\bar{x}_k \\ &= \bar{x}_j\bar{x}_k\bar{x}_i + N_{\mathbf{B}_n}\bar{x}_j c_k c_i + N_{\mathbf{B}_n}c_j c_i \bar{x}_k \\ &= \bar{x}_k\bar{x}_j\bar{x}_i + N_{\mathbf{B}_n}c_k c_j \bar{x}_i + N_{\mathbf{B}_n}\bar{x}_j c_k c_i + N_{\mathbf{B}_n}c_j c_i \bar{x}_k \\ &= \bar{x}_k\bar{x}_j\bar{x}_i + N_{\mathbf{B}_n}c_k c_j \bar{x}_i + N_{\mathbf{B}_n}c_k c_i \bar{x}_j + N_{\mathbf{B}_n}c_j c_i \bar{x}_k. \end{aligned}$$

The calculation for $\bar{x}_i(\bar{x}_j\bar{x}_k)$ is similar. Other minimal ambiguities can be checked similarly.

Let \mathcal{I} be the two-sided ideal of $\tilde{\mathbb{H}}$ generated by the relations of $W(\mathbf{B}_n)$ (e.g., $\bar{s}^2 - 1$, $\bar{s}_{i,i+1}\bar{s}_{i+1,i+2}\bar{s}_{i,i+1} - \bar{s}_{i+1,i+2}\bar{s}_{i,i+1}\bar{s}_{i+1,i+2}$). Then $\tilde{H}/\mathcal{I} \cong \mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$. Let \mathcal{P} be the subalgebra of $\tilde{\mathbb{H}}$ generated by x_i and c_i . It is straightforward to check that $(\bar{s}^2 - 1)\mathcal{P} = \mathcal{P}(\bar{s}^2 - 1)$, $(\bar{s}_{i,i+1}\bar{s}_{i+1,i+2}\bar{s}_{i,i+1} - \bar{s}_{i+1,i+2}\bar{s}_{i,i+1}\bar{s}_{i+1,i+2})\mathcal{P} = \mathcal{P}(\bar{s}_{i,i+1}\bar{s}_{i+1,i+2}\bar{s}_{i,i+1} - \bar{s}_{i+1,i+2}\bar{s}_{i,i+1}\bar{s}_{i+1,i+2})$, and other similar equations. Those equations can also be deduced from Lemma 4.14 and its proof below. \square

Lemma 4.13. *For any root $\alpha > 0$, $c_i\tilde{s}_\alpha = -\tilde{s}_\alpha c_i$.*

Lemma 4.14.

$$wy_iw^{-1} = y_{w(i)} + \sqrt{2} \sum_{\alpha > 0, w^{-1}(\beta) < 0, \langle \beta, w(e_i) \rangle \neq 0} \mathbf{k}_\alpha \tilde{s}_\alpha.$$

In particular, for $\alpha > 0$

$$\tilde{s}_\alpha y_i \tilde{s}_\alpha^{-1} + y_{s_\alpha(i)} = -\sqrt{2} \sum_{\alpha > 0, s_\alpha^{-1}(\beta) < 0, \langle \beta, s_\alpha(e_i) \rangle \neq 0} \mathbf{k}_\beta \tilde{s}_\beta.$$

Proof. The relation $s_n x_n + x_n s_n = -\sqrt{2} \mathbf{k}_\alpha$ implies $s_n y_n - y_n s_n = -\sqrt{2} \mathbf{k}_\alpha c_n$. The latter equation is also equivalent to $s_n y_n s_n^{-1} = y_n + \sqrt{2} \mathbf{k}_\alpha \tilde{s}_n$. The remaining proof is just similar to the case of \mathbf{A}_{n-1} in the proof of Lemma 4.5. \square

For $i > 0$, define $y_i = x_i c_i$.

$$y'_i = y_i + \frac{\sqrt{2}}{2} \sum_{\alpha > 0, \langle \alpha, e_i \rangle \neq 0} \mathbf{k}_\alpha \tilde{s}_\alpha. \quad (4.10)$$

We also define $y_{-i} = y_i$ and $y'_{-i} = y'_i$.

There is a natural permutation of $W(\mathbf{B}_n)$ on the set $\{\pm 1, \dots, \pm n\}$.

Lemma 4.15.

- (1) For any $w \in W(\mathbf{B}_n)$, $wy'_i w^{-1} = y'_{w(i)}$.
- (2) For $i \neq j$, $y'_i y'_j + y'_j y'_i \in \text{Seg}(W(\mathbf{B}_n))$.

Proof. For (1), it suffices to check when $w = s_\alpha$ is a simple reflection. It is the direct consequence of the expression (4.10) for y'_i , Lemma 4.14, and the fact that $s_\alpha \tilde{s}_\alpha s_\alpha = -\tilde{s}_\alpha$. For (2), using the expression (4.10), we have

$$\begin{aligned} y'_i y'_j + y'_j y'_i &= y_i y_j + y_j y_i + \frac{\sqrt{2}}{2} \left[\sum_{\alpha > 0, \langle \alpha, e_j \rangle \neq 0} \mathbf{k}_\alpha (y_i \tilde{s}_\alpha + \tilde{s}_\alpha y_i) + \sum_{\alpha > 0, \langle \alpha, e_i \rangle \neq 0} \mathbf{k}_\alpha (y_j \tilde{s}_\alpha + \tilde{s}_\alpha y_j) \right] \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta > 0, \langle \alpha, e_i \rangle \neq 0, \langle \beta, e_i \rangle \neq 0} \mathbf{k}_\alpha \mathbf{k}_\beta (\tilde{s}_\alpha \tilde{s}_\beta + \tilde{s}_\beta \tilde{s}_\alpha). \end{aligned}$$

Since $y_i y_j + y_j y_i = N_{\mathbf{B}_n}$, we only need to consider and show that the middle term is in $\text{Seg}(W(\mathbf{B}_n))$:

$$\begin{aligned} & \sum_{\alpha > 0, \langle \alpha, e_j \rangle \neq 0} (y_i \tilde{s}_\alpha + \tilde{s}_\alpha y_i) + \sum_{\alpha > 0, \langle \alpha, e_i \rangle \neq 0} (y_j \tilde{s}_\alpha + \tilde{s}_\alpha y_j) \\ &= \left(\sum_{k \neq i} \mathbf{k}_{\alpha_i, k} \tilde{s}_{i, k} y_j + y_j \sum_{k \neq i} \mathbf{k}_{\alpha_i, k} \tilde{s}_{i, k} + y_i \sum_{l \neq j} \mathbf{k}_{\alpha_l, j} \tilde{s}_{l, j} + \sum_{l \neq j} \mathbf{k}_{\alpha_l, j} \tilde{s}_{l, j} y_i \right) \\ &+ \left(\sum_{k > 0, k \neq i} \tilde{s}_{i, -k} y_j + y_j \sum_{k > 0, k \neq i} \tilde{s}_{i, -k} + y_i \sum_{l > 0, l \neq j} \tilde{s}_{j, -l} + \sum_{l > 0, l \neq j} \tilde{s}_{j, -l} y_i \right) \\ &+ \mathbf{k}_{\alpha_i} (\tilde{s}_i y_j + y_j \tilde{s}_i) + \mathbf{k}_{\alpha_j} (\tilde{s}_j y_i + y_i \tilde{s}_j) \end{aligned}$$

which is in $\text{Seg}(W(\mathbf{B}_n))$ by Lemma 4.14. \square

Proposition 4.16. *The superalgebra $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$ satisfies the property $(*)$.*

Proof. Let $x'_i = -y'_i c_i$. We set W in Definition 3.1 to be $W(\mathbf{B}_n)$ and a_i to be x'_i . With Lemma 4.15, one can verify relations (3.1) to (3.5) in Definition 3.1 (also see more detail for type A_{n-1} in Section 4.1). By Proposition 4.12, $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$ has a PBW type basis. These show the proposition. \square

4.3. Type D_n

Notation 4.17. Let $W(D_n)$ be the Weyl group of type D_n . Let the set $R(D_n)$ of roots for type D_n be

$$R(D_n) = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \subset R(\mathbf{B}_n).$$

Let $R^+(D_n) = R(D_n) \cap R^+(\mathbf{B}_n)$ be a fixed set of positive roots. We shall again write $\alpha > 0$ for $\alpha \in R^+(D_n)$ and $\alpha < 0$ for $-\alpha \in R(D_n)$. The set of simple roots is given by

$$\Delta = \{e_i - e_{i+1} : i = 1, \dots, n-1\} \cup \{e_{n-1} + e_n\}.$$

Since there is only one W -orbit for $R(D_n)$, we simply write \mathbf{k} for \mathbf{k}_α for any $\alpha \in R(D_n)$.

We shall regard $W(D_n)$ as the subgroup of $W(\mathbf{B}_n)$ generated by elements $s_{i,j}$ and $s_{i,-j}$ for $i, j > 0$. We shall also keep using the notations in Notation 4.10.

Definition 4.18. Let $N_{D_n} \in \mathbb{C}$. Let $\mathbf{k}^B : R(\mathbf{B}_n) \rightarrow \mathbb{C}$ such that $\mathbf{k}^B|_{R(D_n)} = \mathbf{k}$ and $\mathbf{k}^B_\alpha = 0$ for any short root α in $R(\mathbf{B}_n)$. Let $\mathbb{H}_{W(D_n)}^{\text{Cl}} = \mathbb{H}_{W(D_n)}^{\text{Cl}}(\mathbf{k}, N_{D_n})$ be the supersubalgebra of $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}(\mathbf{k}^B, N_{D_n})$ generated by the elements $w \in W(D_n) \subset W(\mathbf{B}_n)$, $\{x_i\}_{i=1}^n$ and $\{c_i\}_{i=1}^n$.

Remark 4.19. We can explicitly write down the commutation formula from the algebra structure of $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$. For example,

$$\begin{aligned} s_{n-1, -n} x_{n-1} + x_n s_{n-1, -n} &= s_n s_{n-1, n} s_n x_{n-1} + x_n s_n s_{n-1, n} s_n \\ &= -s_n s_{n-1, n} x_n s_n + s_n x_{n-1} s_{n-1, n} s_n \\ &= s_n (-s_{n-1, n} x_n + x_{n-1} s_{n-1, n}) s_n \\ &= s_n (\mathbf{k}(-1 + c_n c_{n-1})) s_n \\ &= \mathbf{k}(-1 + c_{n-1} c_n). \end{aligned}$$

This agrees with a relation in [WK, Def. 3.6]. When $N_{D_n} = 0$, $\mathbb{H}_{W(D_n)}^{\text{Cl}}(\mathbf{k}, 0)$ is isomorphic to the degenerate affine Hecke–Clifford algebra of type D_n defined in [WK, Def. 3.6]. (We remark that in [WK], their convention for c_i satisfies $c_i^2 = 1$ rather than $c_i^2 = -1$.)

We again define

$$y'_i = y_i + \frac{\sqrt{2}}{2}\mathbf{k} \sum_{\alpha>0, \langle\alpha, e_i\rangle\neq 0} \tilde{s}_\alpha = y_i + \frac{\sqrt{2}}{2}\mathbf{k} \sum_{j\neq i} \tilde{s}_{ij} + \frac{\sqrt{2}}{2}\mathbf{k} \sum_{j\neq i} \tilde{s}_{i,-j}. \tag{4.11}$$

Again, for notational convenience, set $y'_{-i} = y'_i$.

Lemma 4.20.

- (1) $c_i y'_j = -y'_j c_i$ for any i, j ;
- (2) $s_\alpha y'_i s_\alpha^{-1} = y'_{s_\alpha(i)}$;
- (3) for $i \neq j$, $y'_i y'_j + y'_j y'_i \in \text{Seg}(W(D_n))$.

Proof. Note that y'_i is defined as the one in (4.10) for type B_n in Section 4.2 since we have $\mathbf{k}_\alpha^B = 0$ for any short root $\alpha \in R(B_n)$. Then the results can be established by Lemma 4.14 and by investigating the proof of Lemma 4.15. \square

Proposition 4.21. *The algebra $\mathbb{H}_{W(D_n)}^{\text{Cl}}$ satisfies the property (*) in Definition 3.1.*

Proof. This follows from $\mathbb{H}_{W(D_n)}^{\text{Cl}}$ forming a supersubalgebra of $\mathbb{H}_{W(B_n)}^{\text{Cl}}(\mathbf{k}^B, N_{D_n})$ and Remark 4.19. \square

4.4. Dirac element D

Let $\mathbb{H} = \mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$, $\mathbb{H}_{W(B_n)}^{\text{Cl}}$, or $\mathbb{H}_{W(D_n)}^{\text{Cl}}$. Using (3.6), the Dirac element D for \mathbb{H} is defined as

$$D = \sum_{i=1}^n x'_i c_i. \tag{4.12}$$

Using the expressions in Section 4.1, the explicit form of the Dirac element D is as:

- (1) Type A_{n-1} and D_n :

$$D = \sum_{i=1}^n x_i c_i + \sqrt{2} \sum_{\alpha>0} \mathbf{k}_\alpha s_\alpha c_\alpha = \sum_{i=1}^n y_i + \sqrt{2} \sum_{\alpha>0} \mathbf{k}_\alpha \tilde{s}_\alpha.$$

- (2) Type B_n :

$$D = \sum_{i=1}^n y_i + \sqrt{2} \sum_{\alpha>0, \alpha \text{ long}} \mathbf{k}_\alpha \tilde{s}_\alpha + \frac{\sqrt{2}}{2} \sum_{\alpha>0, \alpha \text{ short}} \mathbf{k}_\alpha \tilde{s}_\alpha.$$

In types A_{n-1} and D_n , we consider that all the roots are long.

Lemma 4.22.

$$\left(\sum_{\alpha>0, \alpha \text{ long}} \tilde{s}_\alpha \right)^2 = \sum_{\substack{\alpha>0, \beta>0, s_\alpha(\beta)<0 \\ \alpha, \beta \text{ long}}} \tilde{s}_\alpha \tilde{s}_\beta.$$

The above equality is also true if we replace all the long roots by short roots. Similarly, we also have

$$\left(\sum_{\alpha>0, \alpha \text{ long}} \tilde{s}_\alpha \right) \left(\sum_{\alpha>0, \alpha \text{ short}} \tilde{s}_\alpha \right) + \left(\sum_{\alpha>0, \alpha \text{ short}} \tilde{s}_\alpha \right) \left(\sum_{\alpha>0, \alpha \text{ long}} \tilde{s}_\alpha \right) = \sum_{\alpha>0, \beta>0, s_\alpha(\beta)<0} \tilde{s}_\alpha \tilde{s}_\beta,$$

where α and β run for all pairs of roots with distinct length.

Proof. We only prove for the first case, that is, the case of long roots only. It suffices to show that

$$\sum_{\substack{\alpha>0, \beta>0, s_\alpha(\beta)>0 \\ \alpha, \beta \text{ long}}} \tilde{s}_\alpha \tilde{s}_\beta = 0.$$

Set $\tilde{R} = \{(\alpha, \beta) \in R^+ \times R^+ : s_\alpha(\beta) > 0, \alpha \text{ and } \beta \text{ are long}\}$. Note that for any $(\alpha, \beta) \in \tilde{R}$, either $s_\beta(\alpha) > 0$ or $s_{s_\alpha(\beta)}(\alpha) > 0$. We define a map $\iota : \tilde{R} \rightarrow \tilde{R}$ such that

$$\iota(\alpha, \beta) = \begin{cases} (\beta, s_\beta(\alpha)) & \text{if } s_\beta(\alpha) > 0, \\ (s_\alpha(\beta), \alpha) & \text{if } s_{s_\alpha(\beta)}(\alpha) > 0. \end{cases}$$

It is not hard to verify that ι is well-defined and is an involution. For $\iota(\alpha, \beta) = (\alpha', \beta')$, one can also check that $\tilde{s}_\alpha \tilde{s}_\beta + \tilde{s}_{\alpha'} \tilde{s}_{\beta'} = 0$. Thus each term $\tilde{s}_\alpha \tilde{s}_\beta$ in the expression $\sum_{\alpha>0, \beta>0, s_\alpha(\beta)>0} \tilde{s}_\alpha \tilde{s}_\beta$ can be paired with another one and get canceled. This proves the expression is zero. \square

By Proposition 4.9, Proposition 4.16, and Proposition 4.21, \mathbb{H} satisfies the property $(*)$ and hence we can define $\text{Seg}(W)$ to be a subalgebra of \mathbb{H} according to Definition 3.1.

We compute the square of the Dirac element D . This is an analogue of [BCT, Thm. 2.11].

Theorem 4.23. *Let $\mathbb{H} = \mathbb{H}_{W(\mathbf{A}_{n-1})}^{\text{Cl}}$, $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$, or $\mathbb{H}_{W(\mathbf{D}_n)}^{\text{Cl}}$. Then*

$$D^2 = \Omega_{\mathbb{H}} - \Omega_{\text{Seg}(W)},$$

where

$$\begin{aligned} \Omega_{\mathbb{H}} &= \sum_{i=1}^n x_i^2, \\ \Omega_{\text{Seg}(W)} &= \frac{1}{2} \sum_{\alpha>0, \beta>0, s_\alpha(\beta)<0} |\langle \alpha, \alpha \rangle| |\langle \beta, \beta \rangle| \mathbf{k}_\alpha \mathbf{k}_\beta \tilde{s}_\alpha \tilde{s}_\beta. \end{aligned}$$

Moreover, \mathbb{H} satisfies the property $(**)$.

Proof. We only do for types A_{n-1} and B_n , and the case for type D_n follows from type B_n .

By Lemma 4.5 and Lemma 4.14, for any $\alpha > 0$,

$$\mathbf{k}_\alpha \left(\sum_{i=1}^n y_i \tilde{s}_\alpha + \tilde{s}_\alpha \sum_{i=1}^n y_i \right) = -\sqrt{2} \mathbf{k}_\alpha \sum_{\beta > 0, s_\alpha(\beta) < 0} \mathbf{k}_\beta |\langle \beta, \beta \rangle| \tilde{s}_\alpha \tilde{s}_\beta. \tag{4.13}$$

Now, by (4.13) and Lemma 4.22,

$$\begin{aligned} D^2 &= \left(\sum_{i=1}^n y_i + \frac{\sqrt{2}}{2} \sum_{\alpha > 0} \mathbf{k}_\alpha |\langle \alpha, \alpha \rangle| \tilde{s}_\alpha \right)^2 \\ &= \left(\sum_{i=1}^n y_i \right)^2 + \frac{\sqrt{2}}{2} \sum_{i=1}^n y_i \sum_{\alpha > 0} \mathbf{k}_\alpha |\langle \alpha, \alpha \rangle| \tilde{s}_\alpha + \frac{\sqrt{2}}{2} \sum_{\alpha > 0} \mathbf{k}_\alpha |\langle \alpha, \alpha \rangle| \tilde{s}_\alpha \sum_{i=1}^n y_i \\ &\quad + \frac{1}{2} \left(\sum_{\alpha > 0} \mathbf{k}_\alpha |\langle \alpha, \alpha \rangle| \tilde{s}_\alpha \right)^2 \\ &= \sum_{i=1}^n x_i^2 - \frac{1}{2} \sum_{\alpha > 0, \beta > 0, s_\alpha(\beta) < 0} \mathbf{k}_\alpha \mathbf{k}_\beta |\langle \alpha, \alpha \rangle| |\langle \beta, \beta \rangle| \tilde{s}_\alpha \tilde{s}_\beta. \end{aligned}$$

We can directly verify that $\Omega_{\mathbb{H}}$ is in the center of \mathbb{H} and $\Omega_{\text{Seg}(W)}$ is in the center of $\text{Seg}(W)$. Hence, \mathbb{H} has the property (**). \square

We obtain the following Parthasarathy-Dirac-type inequality. Examples satisfying the hypothesis of Corollary 4.24 below will be considered in Section 7 (see Proposition 7.12).

Corollary 4.24. *Suppose an irreducible \mathbb{H} -module (π, X) satisfies the property that X admits a non-degenerate positive-definite Hermitian form such that the adjoint operator of $\pi(D)$ is $-\pi(D)$. For any irreducible $\text{Seg}(W)$ -module (σ, U) ,*

$$\text{Hom}_{\text{Seg}(W)}(U, \text{Res}_{\text{Seg}(W)}^{\mathbb{H}} X) \neq 0$$

only if

$$\chi_\pi(\Omega_{\mathbb{H}}) \leq \chi_\sigma(\Omega_{\text{Seg}(W)}).$$

Proof. Let U_X be an U -isotypical component of X and let $u \in U$. The corollary follows from

$$0 \leq \langle D.u, D.u \rangle = \langle u, -D^2.u \rangle = -(\chi_\pi(\Omega_{\mathbb{H}}) - \chi_\sigma(\Omega_{\text{Seg}(W)})) \langle u, u \rangle. \quad \square$$

The conclusion of this section is a version of Theorem 3.5 in specific cases.

Theorem 4.25. *Let $\mathbb{H} = \mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$, $\mathbb{H}_{W(B_n)}^{\text{Cl}}$, or $\mathbb{H}_{W(D_n)}^{\text{Cl}}$. Let (π, X) be an irreducible supermodule of \mathbb{H} with the central character χ_π (Definition 2.6). Let D be the Dirac element in \mathbb{H} in (4.12). Define the Dirac cohomology $H_D(X)$ as in Theorem 3.5. Then $H_D(X)$ has a natural $\text{Seg}(W)$ -module structure. Suppose*

$$\text{Hom}_{\text{Seg}(W)}(U, H_D(X)) \neq 0,$$

for some $\text{Seg}(W)$ -module (σ, U) . Then $\chi_\pi = \chi^\sigma$, where χ^σ is defined as in (3.7).

Proof. This immediately follows from Theorem 3.5, Proposition 4.9, and Theorem 4.23. \square

5. Examples of non-vanishing Dirac cohomology

5.1. Construction of some modules

In this section, we construct some modules for the degenerate affine Hecke–Clifford algebra of classical types.

In type A_{n-1} , we follow the construction in [HKS, Sect. 4.1], which uses a Jucys–Murphy-type element. For type B_n , we use a slightly different approach. The underlying idea of the construction is to first consider a $\text{Seg}(W)$ -module and then try to extend the action to the entire degenerate affine Hecke–Clifford algebra. However, we may not expect that this process always works, and indeed, we can only do it for certain parameters.

Type A_{n-1} : Let Cl_n be the subalgebra of $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ generated by all c_i . Define $\tilde{\text{St}}_{W(A_{n-1})}$ to be an $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ -supermodule, which is identified with Cl_n as vector spaces and the action of \mathbb{H}^n on $\tilde{\text{St}}_{W(A_{n-1})}$ is determined by the following:

$$c_i \cdot 1 = c_i, \tag{5.14}$$

$$s_\alpha \cdot 1 = 1, \tag{5.15}$$

where 1 is the identity in Cl_n and

$$x_i \cdot v = \mathbf{k} \left(\sum_{1 \leq j < i \leq n} s_{i,j} (1 - c_i c_j) \right) \cdot v,$$

where v is any vector in Cl_n and the actions of $s_{i,j}$ and c_i, c_j are the ones defined in (5.14) and (5.15). The notation $\tilde{\text{St}}_{W(A_{n-1})}$ stands for a Steinberg-type module as it performs the role of Steinberg module in the degenerate affine Hecke algebra. It is straightforward to check that the above actions define an $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ -module by verifying the defining relations of $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$. Some details can be found in [HKS, Prop. 4.1.1].

Type B_n : Let α be a long root in $R(B_n)$ and let β be a short root in $R(B_n)$. Set $N_{B_n} = 2(n-1)\mathbf{k}_\alpha^2 + \sqrt{2}\mathbf{k}_\alpha\mathbf{k}_\beta$. Let Cl_n be the subalgebra of $\mathbb{H}_{W(B_n)}^{\text{Cl}}$ generated by the elements c_i , which is isomorphic to the Clifford algebra. Let $U(n)$ be an irreducible supermodule of Cl_n . The actions of $\mathbb{H}_{W(B_n)}^{\text{Cl}}$ on $U(n) \tilde{\otimes} U(n)$ are determined by the following:

$$x_i \cdot (u \otimes v) = -(-1)^{\text{deg}(u)} \sqrt{-1} \left(\left(\mathbf{k}_\alpha (c_1 + c_2 + \dots + c_{i-1} + (n-i)c_i) + \frac{\sqrt{2}}{2} \mathbf{k}_\beta c_i \right) \cdot u \right) \otimes (c_i \cdot v), \tag{5.16}$$

$$s_n \cdot (u \otimes v) = (-1)^{\text{deg}(u)} \sqrt{-1} (c_n \cdot u) \otimes (c_n \cdot v),$$

$$s_{i,j} \cdot (u \otimes v) = (-1)^{\text{deg}(u)} \sqrt{-1} \left(\frac{c_i - c_j}{\sqrt{2}} \cdot u \right) \otimes \left(\frac{c_i - c_j}{\sqrt{2}} \cdot v \right), \tag{5.17}$$

$$c_i \cdot (u \otimes v) = (-1)^{\text{deg}(u)} (u \otimes c_i \cdot v). \tag{5.18}$$

The above actions are indeed well defined:

Proposition 5.1. For $N_{\mathbf{B}_n} = 2(n-1)\mathbf{k}_\alpha^2 + \sqrt{2}\mathbf{k}_\alpha\mathbf{k}_\beta$, the actions (5.16)–(5.18) above on $U(n)\tilde{U}(n)$ define an $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}(\mathbf{k}, N_{\mathbf{B}_n})$ -module.

Proof. The computation is straightforward for verifying the defining relations of $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$. For example,

$$\begin{aligned} & (s_{i,i+1}x_i - x_{i+1}s_{i,i+1}) \cdot (u \otimes v) \\ &= -\frac{1}{2}\mathbf{k}_\alpha((-n-i) + (n-i)c_i c_{i+1}) \cdot u \otimes ((-1 + c_i c_{i+1}) \cdot v) \\ &\quad + \frac{1}{2}\mathbf{k}_\alpha(((n-i-2) - (n-i)c_i c_{i+1}) \cdot u) \otimes (1 - c_i c_{i+1}) \cdot v \\ &= \mathbf{k}_\alpha u \otimes ((-1 + c_i c_{i+1}) \cdot v) \\ &= \mathbf{k}_\alpha(-1 + c_i c_{i+1}) \cdot (u \otimes v). \end{aligned}$$

Moreover, for $i < j$, note that

$$\begin{aligned} & \left(\mathbf{k}_\alpha(c_1 + c_2 + \cdots + c_{i-1} + (n-i)c_i) + \frac{\sqrt{2}}{2}\mathbf{k}_\beta c_i \right) \\ &\quad \cdot \left(\mathbf{k}_\alpha(c_1 + c_2 + \cdots + c_{j-1} + (n-j)c_j) + \frac{\sqrt{2}}{2}\mathbf{k}_\beta c_j \right) \\ &\quad + \left(\mathbf{k}_\alpha(c_1 + c_2 + \cdots + c_{j-1} + (n-j)c_j) + \frac{\sqrt{2}}{2}\mathbf{k}_\beta c_j \right) \\ &\quad \cdot \left(\mathbf{k}_\alpha(c_1 + c_2 + \cdots + c_{i-1} + (n-i)c_i) + \frac{\sqrt{2}}{2}\mathbf{k}_\beta c_i \right) \\ &= -2(i-1)\mathbf{k}_\alpha^2 + 2\mathbf{k}_\alpha \left(-(n-i)\mathbf{k}_\alpha - \frac{\sqrt{2}}{2}\mathbf{k}_\beta \right) \\ &= -2(n-1)\mathbf{k}_\alpha^2 - \sqrt{2}\mathbf{k}_\alpha\mathbf{k}_\beta \end{aligned}$$

and hence $x_i x_j - x_j x_i = (2(n-1)\mathbf{k}_\alpha^2 + \sqrt{2}\mathbf{k}_\alpha\mathbf{k}_\beta)c_j c_i$. Other relations can be verified similarly (and more easily). \square

Denote the above $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}$ -module by $\tilde{\text{St}}_{\mathbf{B}_n}$.

Type D_n : Set $N_{\mathbf{D}_n} = 2(n-1)\mathbf{k}_\alpha^2$. Recall that $\mathbb{H}_{W(\mathbf{D}_n)}^{\text{Cl}}$ is a subalgebra of the algebra $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}(\mathbf{k}^B, N_{\mathbf{D}_n})$ (see \mathbf{k}^B in Definition 4.18). By checking the parameter function, we have an $\mathbb{H}_{W(\mathbf{B}_n)}^{\text{Cl}}(\mathbf{k}^B, N_{\mathbf{D}_n})$ -module $\tilde{\text{St}}_{\mathbf{B}_n}$ defined above. Denote by $\tilde{\text{St}}_{\mathbf{D}_n}$ the restriction of $\tilde{\text{St}}_{\mathbf{B}_n}$ to an $\mathbb{H}_{W(\mathbf{D}_n)}^{\text{Cl}}$ -module.

5.2. Dirac cohomology

We keep using the notation in Section 5.

Proposition 5.2. Set $N_{\mathbf{B}_n} = 2(n-1)\mathbf{k}_\alpha^2 + \sqrt{2}\mathbf{k}_\alpha\mathbf{k}_\beta$ (with the notations in Section 5.1) and set $N_{\mathbf{D}_n} = 2(n-1)\mathbf{k}^2$. Let $\mathbb{H} = \mathbb{H}_{\mathbf{A}_{n-1}}^{\text{Cl}}, \mathbb{H}_{\mathbf{B}_n}^{\text{Cl}}(\mathbf{k}, N_{\mathbf{B}_n})$ or $\mathbb{H}_{\mathbf{D}_n}^{\text{Cl}}(\mathbf{k}, N_{\mathbf{D}_n})$. Let $X = \tilde{\text{St}}_{\mathbf{A}_{n-1}}, \tilde{\text{St}}_{\mathbf{B}_n}$ or $\tilde{\text{St}}_{\mathbf{D}_n}$ be an \mathbb{H} -module defined in Section 5.1. The Dirac operator D acts identically as zero on X . In particular, $H_D(X) \neq 0$.

Proof. Type A_{n-1} : For $v \in \tilde{S}t_{A_{n-1}}$,

$$\begin{aligned} \pi(D)v &= \sum_{1 \leq j < i \leq n} s_{ij}(1 - c_i c_j)c_i.v + \sqrt{2}\mathbf{k} \sum_{\alpha \in R^+} \tilde{s}_\alpha.v \\ &= \sum_{1 \leq j < i \leq n} \mathbf{k}s_{ij}(c_i - c_j).v + \sqrt{2}\mathbf{k} \sum_{\alpha \in R^+} \tilde{s}_\alpha.v \\ &= \left(-\sqrt{2} \sum_{1 < j < i < n} \mathbf{k}\tilde{s}_{ji} + \sqrt{2}\mathbf{k} \sum_{\alpha \in R^+} \tilde{s}_\alpha \right).v = 0. \end{aligned}$$

Type B_n : Recall that $\tilde{S}t_{B_n}$ is isomorphic to $U \tilde{\otimes} U$ as vector spaces in the notation of Section 5.1. For $u \otimes v \in U \tilde{\otimes} U$,

$$\begin{aligned} &(-1)^{\deg(u)} \sqrt{-1} \pi(D)(u \otimes v) \\ &= \sum_{i=1}^n \left(\left(\mathbf{k}_\alpha(c_1 + c_2 + \cdots + c_{i-1} + (n-i)c_i) + \frac{\sqrt{2}}{2} \mathbf{k}_\beta c_i \right).u \right) \otimes v \\ &\quad - \sqrt{2}\mathbf{k}_\alpha \sum_{1 \leq j < i \leq n} \left(\frac{c_i - c_j}{\sqrt{2}}.u \right) \otimes v - \sqrt{2}\mathbf{k}_\alpha \sum_{1 \leq j < i \leq n} \left(\frac{c_i + c_j}{\sqrt{2}}.u \right) \otimes v \\ &\quad - \frac{\sqrt{2}}{2} \mathbf{k}_\beta \sum_{i=1}^n (c_i.u) \otimes v \\ &= \sum_{i=1}^n \left(\left(2(n-i)c_i + \frac{\sqrt{2}}{2} \mathbf{k}_\beta c_i \right).u \right) \otimes v - \sqrt{2}\mathbf{k}_\alpha \sum_{i=1}^n (2(n-i)c_i.u) \otimes v \\ &\quad - \frac{\sqrt{2}}{2} \mathbf{k}_\beta \sum_{i=1}^n (c_i.u) \otimes v = 0. \end{aligned}$$

Type D_n : Recall that $\mathbb{H}_{W(D_n)}^{\text{Cl}}$ is a subalgebra of $\mathbb{H}_{W(B_n)}^{\text{Cl}}(\mathbf{k}^B, N_{D_n})$ (see the notation of \mathbf{k}^B in Definition 4.18). The Dirac operator for $\mathbb{H}_{W(D_n)}^{\text{Cl}}$ is the same as the Dirac operator for $\mathbb{H}_{W(B_n)}^{\text{Cl}}(\mathbf{k}^B, N_{D_n})$. Then the vanishing result follows from the result for type B_n , which has just been proven. \square

6. Sergeev algebra

The main purpose of this section is to review several results about Sergeev algebra, which will be useful for computing the Dirac cohomology of some modules for $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ in the next section. Some results can also be formulated to other types and one may refer to [WK, Sect. 2]. Starting from this section, we consider type A_{n-1} only and we shall usually use the notation S_n for $W(A_{n-1})$ (where S_n represents the symmetric group). Write R for $R(A_{n-1})$ and R^+ for $R^+(A_{n-1})$. Recall that Δ is the set of simple roots in R .

6.1. The superalgebra $\mathbb{C}[\tilde{S}_n]^-$

Let \tilde{S}_n be the group generated by the elements $\psi, \tilde{t}_{1,2}, \dots, \tilde{t}_{n-1,n}$ subject to the following relations:

$$\begin{aligned} (\tilde{t}_{i,i+1})^2 &= 1, \\ (\tilde{t}_{i,i+1}\tilde{t}_{i+1,i+2})^3 &= 1 \quad \text{for } i = 1, \dots, n-1, \\ \tilde{t}_{i,i+1}\tilde{t}_{j,j+1} &= \psi\tilde{t}_{j,j+1}\tilde{t}_{i,i+1} \quad \text{for } |i-j| > 1, \\ \psi\tilde{t}_{i,i+1} &= \tilde{t}_{i,i+1}\psi \quad \text{for } i = 1, \dots, n-1, \\ \psi^2 &= 1. \end{aligned}$$

Then \tilde{S}_n is a double cover of S_n via the map determined by sending \tilde{t}_{α_i} to the transposition between i and $i+1$, and $\psi \mapsto 1$. We also sometimes write $\tilde{t}_{\alpha_i, i+1}$ for $\tilde{t}_{i,i+1}$ if we want to refer to the simple root $\alpha_{i,i+1}$. Denote by $\mathbb{C}[\tilde{S}_n]$ the group algebra of \tilde{S}_n with a basis labeled as $\{e_{\tilde{w}} : \tilde{w} \in \tilde{S}_n\}$. Define $\mathbb{C}[\tilde{S}_n]^- := \mathbb{C}[\tilde{S}_n]/\langle e_\psi + 1 \rangle$. We shall simply write \tilde{w} for the image of $e_{\tilde{w}}$ in $\mathbb{C}[\tilde{S}_n]^-$. There is a superalgebra structure on $\mathbb{C}[\tilde{S}_n]^-$ with $\deg(\tilde{t}_\alpha) = 1$ for all $\alpha \in \Delta$.

Lemma 6.1. *Given an S_n -representation U and a $\mathbb{C}[\tilde{S}_n]^-$ -module U' , there exists a natural $\mathbb{C}[\tilde{S}_n]^-$ -module structure on $U \otimes U'$ characterized by*

$$\tilde{t}_\alpha.(u \otimes u') = (s_\alpha.u) \otimes (\tilde{t}_\alpha.u'),$$

where $\alpha \in \Delta$, $u \in U$, and $u' \in U'$.

Define an equivalence relation on $\text{Irr}(\mathbb{C}[\tilde{S}_n]^-)$: $U \sim_{\text{sgn}} U'$ if and only if $U = U'$ or $U = \text{sgn} \otimes U'$ as $\mathbb{C}[\tilde{S}_n]^-$ -modules, where sgn is the sign representation of S_n and the $\mathbb{C}[\tilde{S}_n]^-$ -module structure of $\text{sgn} \otimes U'$ is defined in Lemma 6.1.

Proposition 6.2. *There is a natural bijection*

$$\text{Irr}_{\text{sup}}(\mathbb{C}[\tilde{S}_n]^-)/\sim_\Pi \longleftrightarrow \text{Irr}(\mathbb{C}[\tilde{S}_n]^-)/\sim_{\text{sgn}}.$$

Proof. It suffices to see that the equivalence relation \sim in Proposition 2.4 is the same as \sim_{sgn} . This follows from $\deg(\tilde{t}_\alpha) = 1$ for all $\alpha \in \Delta$ and definitions. \square

6.2. Sergeev algebra

Definition 6.3. Recall that $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ is defined in Definition 4.2. The Sergeev algebra, denoted Seg_n , is the subalgebra of $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ generated by the elements $w \in W(A_{n-1}) = S_n$ and c_i ($i = 1, \dots, n$). In other words, since $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ satisfies the property (*), Seg_n is the same as $\text{Seg}(W_{A_{n-1}})$ in Definition 3.1. We shall use notations in Section 4.1 (e.g., $s_\alpha, c_\alpha, \tilde{s}_\alpha$).

Let Cl_n be the supersubalgebra of Seg_n generated by c_i ($i = 1, \dots, n$). There exists a unique, up to applying the functor Π , irreducible supermodule of Cl_n . Let $U(n)$ be a fixed choice of an irreducible supermodule of Cl_n . The dimension of $U(n)$ is $2^{n/2}$ for n even and $2^{(n+1)/2}$ for n odd.

The relation between subalgebras Seg_n and $\mathbb{C}[\tilde{S}_n]^-$ is the following.

Lemma 6.4 ([Kl, Lem. 13.2.3]). *Seg_n is isomorphic to $\mathbb{C}[\tilde{S}_n]^- \tilde{\otimes} \text{Cl}_n$ as superalgebras.*

Proof. Define a map:

$$s_\alpha \mapsto \tilde{t}_\alpha \otimes c_\alpha \quad (\alpha \in \Delta), \quad c_i \mapsto 1 \otimes c_i \quad (i = 1, \dots, n).$$

One can verify that the map is an isomorphism. \square

For any $\alpha \in R^+$, define $\tilde{t}_\alpha \in \mathbb{C}[S_n]^-$ such that s_α maps to $\tilde{t}_\alpha \otimes c_\alpha$ under the map in the proof of Lemma 6.4.

Here is an analogue of Lemma 6.1:

Lemma 6.5. *Given an S_n -representation U and a Seg_n -module U' , there exists a natural Seg_n -module structure on $U \otimes U'$ characterized by*

$$s_\alpha.(u \otimes u') = (s_\alpha.u) \otimes (s_\alpha.u'),$$

and

$$c_i.(u \otimes u') = u \otimes (c_i.u'),$$

where $\alpha \in \Delta$, $i = 1, \dots, n$, $u \in U$, and $u' \in U'$.

6.3. Relation between supermodules of $\mathbb{C}[\tilde{S}_n]^-$ and Seg_n

Recall from [BK] (our formulation here is a bit different) a natural functor F :

$$\begin{aligned} F : \text{Mod}_{\text{sup}}(\mathbb{C}[\tilde{S}_n]^-) &\rightarrow \text{Mod}_{\text{sup}}(\text{Seg}_n), \\ X &\mapsto X \otimes U(n). \end{aligned}$$

The Seg_n -supermodule structure of $X \otimes U(n)$ is characterized by

$$\begin{aligned} s_\alpha.(x \otimes u) &= -(-1)^{\text{deg}(x)}(\tilde{t}_\alpha.x) \otimes (c_\alpha.u) \quad (\alpha \in \Delta), \\ c_i.(x \otimes u) &= (-1)^{\text{deg}(x)}x \otimes (c_i.u) \quad (i = 1, \dots, n). \end{aligned}$$

It is straightforward to check that the above equations define a Seg_n -module. Next, define

$$\begin{aligned} G : \text{Mod}_{\text{sup}}(\text{Seg}_n) &\rightarrow \text{Mod}_{\text{sup}}(\mathbb{C}[\tilde{S}_n]^-), \\ Y &\mapsto \text{Hom}_{\text{Cl}_n}(U(n), Y). \end{aligned}$$

The $\mathbb{C}[\tilde{S}_n]^-$ -module structure is given by for $\theta \in \text{Hom}_{\text{Cl}_n}(U(n), Y)$,

$$(\tilde{t}_\alpha.\theta)(u) = (s_\alpha.c_\alpha).\theta(u) \quad (\alpha \in \Delta).$$

Proposition 6.6 ([BK, Thm. 3.4]). *The functors F and G form an adjoint pair, i.e., there is a natural isomorphism*

$$\text{Hom}_{\text{Seg}_n}(F(U), U') = \text{Hom}_{\mathbb{C}[\tilde{S}_n]^-}(U, G(U')).$$

Furthermore, if n is even, $G \circ F = \text{Id}$ and $F \circ G = \text{Id}$. If n is odd, $G \circ F = \text{Id} \oplus \Pi$ and $F \circ G = \text{Id} \oplus \Pi$, where Π is defined in Section 2.2.

Let U_{Cl_n} be a Seg_n -module defined by

$$U_{\text{Cl}_n} = \text{Ind}_{\mathbb{C}[S_n]}^{\text{Seg}_n} \text{triv} = \text{Seg}_n \otimes_{\mathbb{C}[S_n]} \text{triv},$$

where $\mathbb{C}[S_n]$ is regarded as the subalgebra of Seg_n generated by the elements f_{s_α} for all $\alpha \in \Delta$ and triv is the trivial representation of $\mathbb{C}[S_n]$. In particular, $\dim_{\mathbb{C}} U_{\text{Cl}_n} = 2^n$.

We define a corresponding $\mathbb{C}[\tilde{S}_n]^-$ -module U_{spin} as follows. If n is even, define $U_{\text{spin}} = G(U_{\text{Cl}_n})$. If n is odd, by [Kl, Prop. 13.2.2] and [Kl, Thm. 22.2.1], $G(U_{\text{Cl}_n}) = M \oplus \Pi(M)$ for some irreducible $\mathbb{C}[\tilde{S}_n]^-$ -module M . Then define $U_{\text{spin}} = M$.

An immediate consequence of Proposition 6.6 is given below.

Lemma 6.7. $F(U_{\text{spin}}) = U_{\text{Cl}_n}$.

7. Spectrum of the Dirac operator for type A_{n-1}

We have seen the action of the Dirac operator on certain modules. In this section, we will go further for type A_{n-1} and compute the action of D on some interesting $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ -modules. We shall see that Theorem 4.25 for $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ has interesting consequences. We shall write \mathbb{H}_n^{Cl} for $\mathbb{H}_{W(A_{n-1})}^{\text{Cl}}$ for simplicity. We keep using the notations in Section 4.1 and Section 6.

7.1. Further notation for the root system of type A_{n-1}

A partition of n is a sequence of positive integers (n_1, \dots, n_r) such that $n_1 \geq n_2 \geq \dots \geq n_r$ and $n_1 + \dots + n_r = n$. For a partition $\lambda = (n_1, \dots, n_r)$ of n , let $I_\lambda = \{1, \dots, n\} \setminus \{n_1, n_1 + n_2, \dots, n_1 + \dots + n_r\}$ and let

$$\Delta_\lambda = \{e_i - e_{i+1} : i \in I_\lambda\}.$$

Let V_λ be the real span of Δ_λ in \mathbb{R}^n and let $R_\lambda^+ = V_\lambda \cap R^+$.

7.2. Central characters for \mathbb{H}_n^{Cl}

The center of \mathbb{H}_n^{Cl} plays a role in the following computations.

Proposition 7.1 ([Kl, Thm. 14.3.1]). *The center $Z(\mathbb{H}_n^{\text{Cl}})$ of \mathbb{H}_n^{Cl} is the set of all symmetric polynomials in $\mathbb{C}[x_1^2, x_2^2, \dots, x_n^2]$. In particular, any element in $Z(\mathbb{H}_n^{\text{Cl}})$ is of even degree.*

Definition 7.2. Recall that the central character $\chi_\pi : Z(\mathbb{H}_n^{\text{Cl}})_0 \rightarrow \mathbb{C}$ of an irreducible supermodule (π, X) is defined in Definition 2.6. By Proposition 7.1, we can also write $\chi_\pi : Z(\mathbb{H}_n^{\text{Cl}}) \rightarrow \mathbb{C}$.

For an element $\gamma = (a_1, \dots, a_n) \in \mathbb{C}^n$, define $\chi'_\gamma : \mathbb{C}[x_1^2, \dots, x_n^2] \rightarrow \mathbb{C}$ such that $\chi'_\gamma(x_i^2) = a_i$. Define χ_γ to be the restriction of χ'_γ to $Z(\mathbb{H}_n^{\text{Cl}})$. For the central character χ_π of X , there exists a unique $\gamma \in \mathbb{C}^n$, up to permutations of coordinates, such that $\chi_\pi = \chi_\gamma$. We may also say that γ is the central character of X .

An \mathbb{H}_n^{Cl} -module (π, X) is said to be quasisimple if any element in $Z(\mathbb{H}_n^{\text{Cl}})$ acts by a scalar. In this case, γ defined as above is still called the central character of X .

7.3. Induced modules

Let us recall a construction of some \mathbb{H}_n^{Cl} -modules in [HKS, Sect. 4], which is indeed modified from the module of type A_{n-1} in Section 5.1. There are also some similar constructions of \mathbb{H}_n^{Cl} -modules in [Wa, Sect. 4]. Fix a partition $\lambda = (n_1, n_2, \dots, n_r)$ of n . Let S_λ be the subgroup of S_n generated by $s_{i,i+1}$ for $i = \{1, \dots, n\} \setminus \{n_1, n_1 + n_2, \dots, n_1 + \dots + n_r\}$. It is easy to see that S_λ is isomorphic to $S_{n_1} \times \dots \times S_{n_r}$. Let $\mathbb{H}_\lambda^{\text{Cl}}$ be the supersubalgebra of \mathbb{H}_n^{Cl} generated by all $w \in S_\lambda$, x_i ($i = 1, \dots, n$) and c_i ($i = 1, \dots, n$). Let Seg_λ be the supersubalgebra of $\mathbb{H}_\lambda^{\text{Cl}}$ generated by all $w \in S_\lambda$ and c_i ($i = 1, \dots, n$). Let $\tilde{\text{St}}_\lambda$ be an $\mathbb{H}_\lambda^{\text{Cl}}$ -module which is identified with Cl_n as vector spaces and the action of $\mathbb{H}_\lambda^{\text{Cl}}$ is characterized by:

$$c_i.1 = c_i \quad (i = 1, \dots, n), \quad s_\alpha.1 = 1 \quad (s_\alpha \in S_\lambda),$$

$$x_i.v = \left(\sum_{n_{k-1}+1 \leq j < i \leq n_k} s_{i,j}(1 - c_i c_j) \right).v \quad (i = n_k + 1, \dots, n_{k+1}),$$

where v is any vector in Cl_n and the actions of $s_{i,j}$ and c_i, c_j are the ones defined in (5.14) and (5.15). It is straightforward to check that the above actions define an $\mathbb{H}_\lambda^{\text{Cl}}$ -module by verifying the defining relations of $\mathbb{H}_\lambda^{\text{Cl}}$. Some details can be found in [HKS, Prop. 4.1.1].

Lemma 7.3. *The element x_i^2 acts on $\tilde{\text{St}}_\lambda$ by a scalar $(i - n_k - 1)(i - n_k)$ where $k = 0, \dots, r - 1$ and $i = n_k + 1, \dots, n_{k+1}$.*

Proof. Direct computation, or see [HKS, Prop. 4.1.1]. \square

Define the Dirac-type element D_λ in $\mathbb{H}_\lambda^{\text{Cl}}$ as:

$$D_\lambda = \sum_{i=1}^n y_i + \sqrt{2}\mathbf{k} \sum_{\alpha \in R_\lambda^+} \tilde{s}_\alpha.$$

Proposition 7.4. *The element D_λ acts as zero on the $\mathbb{H}_\lambda^{\text{Cl}}$ -module $\tilde{\text{St}}_\lambda$.*

Proof. It follows a similar computation of type A_{n-1} in the proof of Proposition 5.2. \square

Define

$$X_\lambda = \text{Ind}_{\mathbb{H}_\lambda^{\text{Cl}}}^{\mathbb{H}_n^{\text{Cl}}} \tilde{\text{St}}_\lambda = \mathbb{H}_n^{\text{Cl}} \otimes_{\mathbb{H}_\lambda^{\text{Cl}}} \tilde{\text{St}}_\lambda \tag{7.19}$$

with the map π_λ defining the action of \mathbb{H}_n^{Cl} on X_λ . Since $Z(\mathbb{H}_n^{\text{Cl}}) \subset Z(\mathbb{H}_\lambda^{\text{Cl}})$, any element of $Z(\mathbb{H}_n^{\text{Cl}})$ acts by a scalar on $\tilde{\text{St}}_\lambda$. With the definitions for X_λ and $Z(\mathbb{H}_n^{\text{Cl}})$, we have that X_λ is quasisimple (Definition 7.2). The central character of X_λ can be represented by

$$\underbrace{(1(1-1), \dots, n_1(n_1-1), \dots)}_{n_1 \text{ terms}}, \underbrace{1(1-1), \dots, n_r(n_r-1)}_{n_r \text{ terms}} \in \mathbb{R}^n.$$

To compute the Dirac cohomology of the above induced modules, we need some more information discussed in the next subsections.

7.4. S_n -structure and Seg_n -structure of (π_λ, X_λ)

We continue to fix a partition λ of n . Recall that in Definition 4.2(1), \mathbb{H}_n^{Cl} contains $\mathbb{C}[S_n]$ as a subalgebra. Let $(\pi_V, V = \mathbb{C}^n)$ be the S_n -representation such that elements in S_n permute the coordinates.

Lemma 7.5. *The restriction of X_λ to $\mathbb{C}[S_n]$ is isomorphic to*

$$\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{Res}_{\mathbb{C}[S_\lambda]}^{\mathbb{C}[S_n]} \left(\bigoplus_{i=0}^n \wedge^i V \right),$$

as $\mathbb{C}[S_n]$ -modules.

Proof. Note that the restriction of $\tilde{S}t_\lambda$ to $\mathbb{C}[S_\lambda]$ is isomorphic to $\text{Res}_{\mathbb{C}[S_\lambda]}^{\mathbb{C}[S_n]} (\bigoplus_{i=0}^n \wedge^i V)$. Then $\mathbb{H}_n^{\text{Cl}} \otimes_{\mathbb{H}_\lambda^{\text{Cl}}} \tilde{S}t_\lambda$ and $\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{Res}_{\mathbb{C}[S_\lambda]}^{\mathbb{C}[S_n]} (\bigoplus_{i=0}^n \wedge^i V)$ are isomorphic as $\mathbb{C}[S_n]$ -modules. \square

It is well known that we have the following $\mathbb{C}[S_n]$ -isomorphism:

$$\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{Res}_{S_\lambda}^{S_n} \left(\bigoplus_{i=0}^n \wedge^i V \right) \cong \left(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv} \right) \otimes \bigoplus_{i=0}^n \wedge^i V.$$

Here the module in the right-hand side is viewed as the tensor product of two S_n -representations. The isomorphism is given by

$$w \otimes (v_1 \wedge \cdots \wedge v_i) \mapsto (w \otimes 1) \otimes (\pi_V(w)v_1 \wedge \cdots \wedge \pi_V(w)v_i).$$

Note that the space $\bigoplus_{i=0}^n \wedge^i V$ can be identified with Cl_n via the map determined by

$$e_{i_1} \wedge \cdots \wedge e_{i_r} \mapsto c_{i_1} \cdots c_{i_r},$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $V = \mathbb{C}^n$. Thus $X_\lambda = \text{Ind}_{\mathbb{H}_\lambda^{\text{Cl}}}^{\mathbb{H}_n^{\text{Cl}}} \tilde{S}t_\lambda$ can be identified with, as vector spaces, $(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv}) \otimes U_{\text{Cl}_n}$ via the identification in Lemma 7.5 and the above identification between $\bigoplus_{i=1}^n \wedge^i V$ and Cl_n . Then if we translate the action of the subalgebra Seg_n under the above identifications, then we have:

$$\begin{aligned} \pi_\lambda(w)(w' \otimes 1 \otimes c_{i_1} \cdots c_{i_r}) &= ww' \otimes 1 \otimes c_{w(i_1)} \cdots c_{w(i_r)}, \\ \pi_\lambda(c_i)(w' \otimes 1 \otimes c_{i_1} \cdots c_{i_r}) &= w' \otimes 1 \otimes c_i c_{i_1} \cdots c_{i_r}. \end{aligned}$$

We have just proven that:

Lemma 7.6. *As Seg_n -supermodules,*

$$\text{Res}_{\text{Seg}_n}^{\mathbb{H}_n^{\text{Cl}}} X_\lambda = (\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv}) \otimes U_{\text{Cl}_n},$$

where the supermodule in the right hand side has the Seg_n -supermodule structure described in Lemma 6.5.

Recall that F is the functor defined in Section 6.3.

Proposition 7.7. *As Seg_n -supermodules,*

$$\text{Res}_{\text{Seg}_n}^{\mathbb{H}_n^{\text{Cl}}} X_\lambda = F((\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv}) \otimes U_{\text{spin}}),$$

where $(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv}) \otimes U_{\text{spin}}$ has $\mathbb{C}[\tilde{S}_n]^-$ -supermodule described in Lemma 6.1.

Proof. By Lemma 7.6, it suffices to show

$$(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv}) \otimes U_{\text{Cl}_n} = (\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv}) \otimes U_{\text{spin}} \otimes U(n).$$

By Lemma 6.7, there is a Seg_n -module isomorphism f from U_{Cl_n} to $F(U_{\text{spin}}) = U_{\text{spin}} \otimes U(n)$. Then define a vector space isomorphism of Seg_n -modules

$$(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv}) \otimes U_{\text{Cl}_n} \rightarrow (\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv}) \otimes U_{\text{spin}} \otimes U(n)$$

determined by

$$(w \otimes 1) \otimes (c_{i_1} \dots c_{i_r} \otimes 1) \mapsto (w \otimes 1) \otimes f(c_{i_1} \dots c_{i_r} \otimes 1).$$

Using the module structure described before Lemma 7.6, one can check the linear isomorphism is Seg_n -equivariant. \square

7.5. Hermitian form on (π_λ, X_λ)

We continue to fix a partition λ of n . In this subsection, we shall construct a Hermitian form on the \mathbb{H}_n^{Cl} -module (π_λ, X_λ) such that the adjoint operator of $\pi_\lambda(D)$ with respect to such form is $-\pi_\lambda(D)$. We will see this makes the computation for the Dirac cohomology $H_D(X)$ of those modules X much easier.

Recall that Seg_λ is a subalgebra of $\mathbb{H}_\lambda^{\text{Cl}}$.

Lemma 7.8. *There exists a Seg_λ -invariant positive definite Hermitian form on $\tilde{\text{St}}_\lambda$.*

Proof. Since $\text{Res}_{\text{Seg}_\lambda}^{\mathbb{H}_\lambda^{\text{Cl}}} \tilde{\text{St}}_\lambda = \text{Res}_{\text{Seg}_\lambda}^{\text{Seg}_n} U_{\text{Cl}_n}$ as Seg_λ -modules, it suffices to consider the case when $\lambda = (n)$. Recall that $U_{\text{Cl}_n} = \text{Seg}_n \otimes_{\mathbb{C}[S_n]} \text{triv}$ in Section 6.3. Define $\langle \cdot, \cdot \rangle : U_{\text{Cl}_n} \times U_{\text{Cl}_n} \rightarrow \mathbb{C}$ such that for $1 \leq i_1 < \dots < i_r \leq n$ and $1 \leq j_1 < \dots < j_s \leq n$,

$$\langle c_{i_1} c_{i_2} \dots c_{i_r} \otimes 1, c_{j_1} c_{j_2} \dots c_{j_s} \otimes 1 \rangle = \begin{cases} 1 & \text{if } \{i_1, \dots, i_r\} = \{j_1, \dots, j_s\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\langle \cdot, \cdot \rangle$ satisfies the desired properties. \square

We denote the Seg_λ -invariant Hermitian form on $\tilde{\text{St}}_\lambda$ in the above lemma by $\langle \cdot, \cdot \rangle_\lambda$. Recall that $X_\lambda = \mathbb{H}_n^{\text{Cl}} \otimes_{\mathbb{H}_\lambda^{\text{Cl}}} \tilde{\text{St}}_\lambda$. We define a bilinear form $\langle \cdot, \cdot \rangle$ on X_λ characterized by:

$$\langle w_1 \otimes v_1, w_2 \otimes v_2 \rangle = \delta_{w_1 S_\lambda, w_2 S_\lambda} \langle \pi_\lambda(w_2^{-1} w_1) v_1, v_2 \rangle_\lambda$$

where $w_1, w_2 \in S_n$ and $\delta_{w_1 S_\lambda, w_2 S_\lambda} = 1$ if $w_1 S_\lambda = w_2 S_\lambda$ and $\delta_{w_1 S_\lambda, w_2 S_\lambda} = 0$ otherwise.

Lemma 7.9. $\langle \cdot, \cdot \rangle$ defined above is a positive definite Hermitian form.

Proof. This follows from the property that $\langle \cdot, \cdot \rangle_\lambda$ is positive definite and Hermitian. \square

We next compute the adjoint operator of $\pi_\lambda(D)$ with respect to $\langle \cdot, \cdot \rangle$. We begin with some lemmas.

Lemma 7.10. For $v_1, v_2 \in \tilde{S}t_\lambda$, $\langle D \otimes v_1, 1 \otimes v_2 \rangle = \langle 1 \otimes v_1, D \otimes v_2 \rangle = 0$.

Proof. For $\alpha \in R^+ \setminus R_\lambda^+$, one has

$$\langle \tilde{s}_\alpha \otimes v_1, 1 \otimes v_2 \rangle = 0.$$

With the equality

$$D = D_\lambda + \sqrt{2}\mathbf{k} \sum_{\alpha > 0, \alpha \in R^+ \setminus R_\lambda^+} \tilde{s}_\alpha,$$

one has $\langle D \otimes v_1, 1 \otimes v_2 \rangle = \langle D_\lambda v_1, v_2 \rangle_\lambda$. Then we have $\langle D \otimes v_1, 1 \otimes v_2 \rangle = 0$ by Proposition 7.4. The proof for $\langle 1 \otimes v_1, D \otimes v_2 \rangle = 0$ is similar. \square

Lemma 7.11. Suppose $\beta_1 \neq \beta_2$ and $\beta_1, \beta_2 \in R^+ \setminus R_\lambda^+$. Then $s_{\beta_1} s_{\beta_2} \notin S_\lambda$.

Proof. In the following, we implicitly use several times the fact that any element in S_λ cannot send a positive root not in R_λ to a negative root. If $\langle \beta_1, \beta_2 \rangle = 0$, then $s_{\beta_1} s_{\beta_2}(\beta_2) = -\beta_2 < 0$. Since $\beta_2 \notin R_\lambda$, $s_{\beta_1} s_{\beta_2} \notin S_\lambda$. If $\langle \beta_1, \beta_2 \rangle = -1$, then $s_{\beta_2}(\beta_1) = \beta_1 + \beta_2 > 0$. Moreover, $s_{\beta_1} s_{\beta_2}(s_{\beta_2}(\beta_1)) = -\beta_1 < 0$. Since $\beta_1 + \beta_2 \notin R_\lambda$, $s_{\beta_1} s_{\beta_2} \notin S_\lambda$. If $\langle \beta_1, \beta_2 \rangle = 1$, then either $s_{\beta_1}(\beta_2) > 0$ or $s_{\beta_2}(\beta_1) > 0$. In the case that $s_{\beta_1}(\beta_2) > 0$, $s_{\beta_1} s_{\beta_2}(\beta_2) = -s_{\beta_1}(\beta_2) < 0$. Then since $\beta_2 \notin R_\lambda$, $s_{\beta_1} s_{\beta_2} \notin S_\lambda$. A similar argument by considering $(s_{\beta_1} s_{\beta_2})^{-1}$ can prove another case. \square

Proposition 7.12. The adjoint operator of $\pi_\lambda(D)$ with respect to $\langle \cdot, \cdot \rangle$ is $-\pi_\lambda(D)$.

Proof. It suffices to show that

$$\langle Dw_1 \otimes v_1, w_2 \otimes v_2 \rangle = \langle w_1 \otimes v_1, -Dw_2 \otimes v_2 \rangle$$

for any $w_1, w_2 \in S_n$ and $v_1, v_2 \in X_\lambda$. To this end, we consider two cases. Suppose $w_1 S_\lambda = w_2 S_\lambda$. Then,

$$\begin{aligned} \langle Dw_1 \otimes v_1, w_2 \otimes v_2 \rangle &= \langle w_2^{-1} Dw_1 \otimes v_1, 1 \otimes v_2 \rangle \\ &= \langle Dw_2^{-1} w_1 \otimes v_1, 1 \otimes v_2 \rangle \\ &= \langle D \otimes (w_2^{-1} w_1).v_1, 1 \otimes v_2 \rangle \\ &= 0 \quad (\text{by Lemma 7.10}). \end{aligned}$$

Similarly, we also have

$$\langle w_1 \otimes v_1, Dw_2 \otimes v_2 \rangle = 0.$$

and so $\langle Dw_1 \otimes v_1, w_2 \otimes v_2 \rangle = \langle w_1 \otimes v_1, -Dw_2 \otimes v_2 \rangle$.

Now we suppose that $w_1 S_\lambda \neq w_2 S_\lambda$. Without loss of generality, assume that $w_2^{-1} w_1$ is a minimal representative in $w_2^{-1} w_1 S_\lambda$.

$$\begin{aligned}
\langle w_2^{-1} w_1 D \otimes v_1, 1 \otimes v_2 \rangle &= \left\langle w_2^{-1} w_1 \sqrt{2} \sum_{\alpha > 0} \mathbf{k} \tilde{s}_\alpha \otimes v_1, 1 \otimes v_2 \right\rangle \\
&= \left\langle 1 \otimes v_1, \sqrt{2} \sum_{\alpha > 0} \mathbf{k} \tilde{s}_\alpha w_1^{-1} w_2 \otimes v_2 \right\rangle \\
&= -\langle 1 \otimes v_1, D w_1^{-1} w_2 \otimes v_2 \rangle + \langle 1 \otimes v_1, w_1^{-1} w_2 D \otimes v_2 \rangle \\
&\quad + \left\langle 1 \otimes v_1, \sqrt{2} \sum_{\alpha > 0} \mathbf{k} \tilde{s}_\alpha w_1^{-1} w_2 \otimes v_2 \right\rangle.
\end{aligned}$$

It remains to show

$$\langle 1 \otimes v_1, w_1^{-1} w_2 D \otimes v_2 \rangle + \sqrt{2} \left\langle 1 \otimes v_1, \sum_{\alpha > 0} \mathbf{k} \tilde{s}_\alpha w_1^{-1} w_2 \otimes v_2 \right\rangle = 0.$$

By Lemma 7.11, there exists at most one $\beta \in R^+ \setminus R_\lambda$ such that $w_1^{-1} w_2 s_\beta \in S_\lambda$. If such β does not exist, then the two terms in the left-hand side of the above equation are both zero and so the equation holds. If such unique β exists, let $\beta' = -w_1^{-1} w_2(\beta)$. Note that $\beta' > 0$, otherwise $w_1^{-1} w_2 s_\beta \notin S_\lambda$. Then

$$\begin{aligned}
&\langle 1 \otimes v_1, w_1^{-1} w_2 D \otimes v_2 \rangle + \sqrt{2} \left\langle 1 \otimes v_1, \sum_{\alpha > 0} \mathbf{k} \tilde{s}_\alpha w_1^{-1} w_2 \otimes v_2 \right\rangle \\
&= \sqrt{2} \mathbf{k} \langle 1 \otimes v_1, w_1^{-1} w_2 \tilde{s}_\beta \otimes v_2 \rangle \\
&\quad + \sqrt{2} \mathbf{k} \langle 1 \otimes v_1, \tilde{s}_{\beta'} w_1^{-1} w_2 \otimes v_2 \rangle \quad (\text{by definition of } D \text{ and } \langle \cdot, \cdot \rangle) \\
&= -\sqrt{2} \mathbf{k} \langle 1 \otimes v_1, \tilde{s}_{\beta'} w_1^{-1} w_2 \otimes v_2 \rangle \\
&\quad + \sqrt{2} \mathbf{k} \langle 1 \otimes v_1, \tilde{s}_{\beta'} w_1^{-1} w_2 \otimes v_2 \rangle \quad (\text{by Lemma 4.4(4)}) \\
&= 0.
\end{aligned}$$

This completes the proof. \square

Proposition 7.13. *Let (π_λ, X_λ) be the \mathbb{H}_n^{Cl} -module as in (7.19). Then*

$$\ker \pi_\lambda(D) = \ker \pi_\lambda(D^2)$$

and

$$\ker \pi_\lambda(D) \cap \text{im } \pi_\lambda(D) = 0.$$

In particular, $H_D(X_\lambda) = \ker \pi_\lambda(D^2)$.

Proof. It is clear that $\ker \pi_\lambda(D) \subset \ker \pi_\lambda(D^2)$. For $v \in \ker \pi_\lambda(D^2)$, we have $\langle \pi_\lambda(D)v, -\pi_\lambda(D)v \rangle = \langle \pi_\lambda(D^2)v, v \rangle = 0$ by Proposition 7.12. Since $\langle \cdot, \cdot \rangle$ is positive definite by Lemma 7.9, $\pi_\lambda(D)v = 0$. This proves the first equation, $\ker \pi_\lambda(D) = \ker \pi_\lambda(D^2)$. The equation $\ker \pi_\lambda(D) \cap \text{im } \pi_\lambda(D) = 0$ follows from the first one. \square

7.6. Dirac cohomology of X_λ

Let \mathcal{P}_n be the set of partitions of n . One can attach an element in \mathcal{P}_n to a point in \mathbb{R}^n via the Jacobson-Morozov triple. The map, denoted $\Phi_1 : \mathcal{P}_n \rightarrow \mathbb{R}^n$ can be explicitly described as:

$$(n_1, n_2, \dots, n_r) \mapsto (\underbrace{-n_1+1, -n_1+3, \dots, n_1-1}_{n_1 \text{ terms}}, \dots, \underbrace{-n_r+1, -n_r+3, \dots, n_r-1}_{n_r \text{ terms}}).$$

There is another way to attach an element in \mathcal{P}_n to a point in \mathbb{R}^n via the central characters of the modules X_λ . This map, denoted $\Phi_2 : \mathcal{P}_n \rightarrow \mathbb{R}^n$, is:

$$(n_1, n_2, \dots, n_r) \mapsto (\underbrace{\sqrt{(1-1)1}, \dots, \sqrt{(n_1-1)n_1}}_{n_1 \text{ terms}}, \dots, \underbrace{\sqrt{(1-1)1}, \dots, \sqrt{n_r(n_r-1)}}_{n_r \text{ terms}}).$$

The first interesting computational fact is the following:

Lemma 7.14. *For a partition λ of n , $|\Phi_1(\lambda)| = |\Phi_2(\lambda)|$, where $|\cdot|$ denotes the standard Euclidean norm in \mathbb{R}^n .*

Proof. This follows from the computation that

$$\sum_{k=1}^{n_i} (-n_i + 2k - 1)^2 = \sum_{k=1}^{n_i} k(k - 1) = \frac{1}{3}(n_i - 1)n_i(n_i + 1). \quad \square$$

For each $\lambda \in \mathcal{P}_n$, define a S_n -representation:

$$W_\lambda = \left(\text{Ind}_{\mathbb{C}[S_\lambda]}^{\mathbb{C}[S_n]} \text{triv} \right) \cap \left(\text{Ind}_{\mathbb{C}[S_{\lambda^t}]}^{\mathbb{C}[S_n]} \text{sgn} \right),$$

where sgn and triv are respectively the sign and trivial representations of S_λ , and λ^t is the conjugate of λ . It is well known that W_λ exhausts the list of irreducible representations of S_n .

Define

$$\Omega_{\mathbb{C}[\tilde{S}_n]^-} = 2\mathbf{k}^2 \sum_{\alpha > 0, \beta > 0, s_\alpha(\beta) < 0} \tilde{t}_\alpha \tilde{t}_\beta \in \mathbb{C}[\tilde{S}_n]^-.$$

Let $\mathcal{P}_n^{\text{dist}}$ be the set of partitions of n with distinct parts. Recall that we denote by $\text{Irr}_{\text{sup}} \mathbb{C}[\tilde{S}_n]^-$ (resp. $\text{Irr}_{\text{sup}} \text{Seg}_n$) the set of irreducible supermodules of $\mathbb{C}[\tilde{S}_n]^-$ (resp. Seg_n). Recall that the equivalence relation \sim_Π on $\text{Irr}_{\text{sup}} \mathbb{C}[\tilde{S}_n]^-$ or $\text{Irr}_{\text{sup}} \text{Seg}_n$ is defined in Section 2.3.

Proposition 7.15 ([Ci1, Part of Thm. 1.0.1] (also see [St])). *There exists a bijection $\Psi_1 : \mathcal{P}_n^{\text{dist}} \rightarrow \text{Irr}_{\text{sup}} \mathbb{C}[\tilde{S}_n]^- / \sim_\Pi$ such that for each partition λ of n , there exists a representative $(\sigma, U) \in \Psi_1(\lambda)$ with the properties that*

$$\mathbf{k}^2 |\Phi_1(\lambda)|^2 = \chi_\sigma(\Omega_{\mathbb{C}[\tilde{S}_n]^-})$$

and

$$\text{Hom}_{\mathbb{C}[\tilde{S}_n]^-}(U, W_\lambda \otimes U_{\text{spin}}) \neq 0.$$

Proof. In [Ci1, Thm. 1.0.1], the set $\text{Irr } \mathbb{C}[\tilde{S}_n]^- / \sim_{\text{sgn}}$ is considered instead of the set $\text{Irr}_{\text{sup}} \mathbb{C}[\tilde{S}_n]^- / \sim_{\Pi}$. By Proposition 6.2, there is a natural bijection between $\text{Irr } \mathbb{C}[\tilde{S}_n]^- / \sim_{\text{sgn}}$ and $\text{Irr}_{\text{sup}} \mathbb{C}[\tilde{S}_n]^- / \sim_{\Pi}$. Then one can now apply [Ci1, Thm. 1.0.1]. \square

Here is an analogue of Proposition 7.15. Recall that Ω_{Seg_n} (i.e., $\Omega_{\text{Seg}(W_{\Lambda_{n-1}})}$) is defined in Theorem 4.23.

Proposition 7.16. *There exists a bijection $\Psi_2 : \mathcal{P}_n^{\text{dist}} \rightarrow \text{Irr}_{\text{sup}} \text{Seg}_n / \sim_{\Pi}$ such that there exists a representative $(\sigma, U) \in \Psi_2(\lambda)$ with the properties that*

$$\mathbf{k}^2 |\Phi_2(\lambda)|^2 = \chi_{\sigma}(\Omega_{\text{Seg}_n})$$

and

$$\text{Hom}_{\text{Seg}_n}(U, F(W_{\lambda} \otimes U_{\text{spin}})) \neq 0.$$

Proof. Note that for an irreducible $\mathbb{C}[\tilde{S}_n]^-$ -supermodule U , $F(U)$ is either an irreducible supermodule or the direct sum of two irreducible supermodules of opposite grading. Thus we could define $\Psi_2(\lambda)$ to be the unique equivalence class in $\text{Irr}_{\text{sup}} \text{Seg}_n / \sim_{\Pi}$ containing the irreducible supermodule(s) in $F(U)$ for a representative $U \in \Phi_1(\lambda)$, where Φ_1 is defined in Proposition 7.15.

It remains to check those two properties. Recall that $F(U) = U \otimes U(n)$ and that the action of Seg_n on $F(U)$ is defined in Section 6.3. Then for $u \otimes u' \in U \otimes U(n)$,

$$\begin{aligned} \Omega_{\text{Seg}_n} \cdot (u \otimes u') &= 2\mathbf{k}^2 \sum_{\alpha, \beta > 0, s_{\alpha}(\beta) < 0} \tilde{s}_{\alpha} \tilde{s}_{\beta} \cdot (u \otimes u') \\ &= 2\mathbf{k}^2 \left(\sum_{\alpha, \beta > 0, s_{\alpha}(\beta) < 0} \tilde{t}_{\alpha} \tilde{t}_{\beta} \cdot u \right) \otimes u' \\ &= \chi_{\sigma}(\Omega_{\mathbb{C}[\tilde{S}_n]^-}) u \otimes u'. \end{aligned}$$

Thus for any irreducible supermodule (σ', U') in $F(U)$, $\chi_{\sigma'}(\Omega_{\text{Seg}_n}) = \chi_{\sigma}(\Omega_{\mathbb{C}[\tilde{S}_n]^-})$. Then combining this with Lemma 7.14 and Proposition 7.15, we have shown the first property.

The second property follows from

$$\text{Hom}_{\text{Seg}_n}(F(U'), F(W_{\lambda} \otimes U_{\text{spin}})) = \text{Hom}_{\mathbb{C}[\tilde{S}_n]^-}(U', G \circ F(W_{\lambda} \otimes U_{\text{spin}})) \neq 0,$$

where the last equality follows from Propositions 6.6 and 7.15. \square

Lemma 7.17. *For a partition λ of n with distinct parts, there exists a representative $U \in \Phi_2(\lambda)$ such that*

$$\text{Hom}_{\text{Seg}_n}(U, \text{Res}_{\text{Seg}_n}^{\mathbb{H}_n^{\text{Cl}}} X_{\lambda}) \neq 0.$$

Proof. This follows from

$$\begin{aligned} &\text{Hom}_{\text{Seg}_n}(U, \text{Res}_{\text{Seg}_n}^{\mathbb{H}_n^{\text{Cl}}} X_{\lambda}) \\ &= \text{Hom}_{\text{Seg}_n}(U, F((\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{\lambda}]} \text{triv}) \otimes U_{\text{spin}})) \quad (\text{by Proposition 7.7}) \\ &\supseteq \text{Hom}_{\text{Seg}_n}(U, F(W_{\lambda} \otimes U_{\text{spin}})) \quad (\text{by definition of } W_{\lambda}). \end{aligned}$$

The statement now follows from Proposition 7.16. \square

The following theorem states that the induced modules (π_λ, X_λ) with λ of distinct parts have non-zero Dirac cohomologies.

Theorem 7.18. *Let λ be a partition of n with distinct parts. Let (π_λ, X_λ) be the \mathbb{H}_n^{Cl} -module defined in (7.19). Let Ψ_2 be the map defined in Proposition 7.16. Then there exists a representative U in $\Psi_2(\lambda)$ such that*

$$\text{Hom}_{\text{Seg}_n}(U, H_D(X_\lambda)) \neq 0.$$

In particular, $H_D(X_\lambda)$ is non-zero.

Proof. For a fixed $\lambda \in \mathcal{P}_n^{\text{dist}}$, let U be a Seg_n -module with the property in Lemma 7.17. Then there exists a non-zero vector v in the isotypical component U of X_λ . By Theorem 4.23, Lemma 7.3, and Proposition 7.16, $\pi_\lambda(D^2)v = (\chi_{\pi_\lambda}(\Omega_{\mathbb{H}_n^{\text{Cl}}}) - \chi_{\Psi_2(\lambda)}(\Omega_{\text{Seg}_n}))v = (\mathbf{k}^2|\Phi_2(\lambda)|^2 - \chi_{\Psi_2(\lambda)}(\Omega_{\text{Seg}_n}))v = 0$. Hence, $v \in \ker(\pi_\lambda(D^2))$. By Proposition 7.13, $v \in H_D(X_\lambda) = \ker \pi_\lambda(D^2)$. This proves the theorem. \square

The Dirac cohomology $H_D(X_\lambda)$ also provides a way to realize irreducible Seg_n -supermodules.

Corollary 7.19. *For each of $\lambda \in \mathcal{P}_n^{\text{dist}}$, there exists a unique irreducible Seg_n -supermodule U , up to the equivalence of \sim_Π , such that $\text{Hom}_{\text{Seg}_n}(U, H_D(X_\lambda)) \neq 0$. Let $[H_D(X_\lambda)]$ be an irreducible submodule of $H_D(X_\lambda)$. Then*

$$\text{Irr}_{\text{sup}} \text{Seg}_n = \bigsqcup_{\lambda \in \mathcal{P}_n^{\text{dist}}} \{[H_D(X_\lambda)], \Pi([H_D(X_\lambda)])\},$$

where \bigsqcup means the disjoint union.

Proof. For the first assertion, the existence has been proved in Theorem 7.18 and we only have to prove the uniqueness. Let (σ', U') be an irreducible Seg_n -module such that

$$\text{Hom}_{\text{Seg}_n}(U', H_D(X_\lambda)) \neq 0.$$

Then $\chi_{\pi_\lambda} = \chi^{\sigma'}$ by Theorem 4.25 and Theorem 7.18. On the other hand, by Proposition 7.16, (σ', U') is in $\Phi_2(\lambda')$ for some $\lambda' \in \mathcal{P}_n^{\text{dist}}$. Then

$$\text{Hom}_{\text{Seg}_n}(U', H_D(X_{\lambda'})) \neq 0$$

and by Theorem 4.25 again, $\chi_{\pi_{\lambda'}} = \chi^{\sigma'}$. Thus $\chi_{\pi_{\lambda'}} = \chi_{\pi_\lambda}$ and so $\lambda = \lambda'$. This implies the uniqueness.

The second assertion follows from the first assertion and the bijectivity of Φ_2 in Proposition 7.16. \square

Let $K(\mathbb{H}_n^{\text{Cl}})$ (resp. $K(\text{Seg}_n)$) be the Grothendieck group of finite-dimensional \mathbb{H}_n^{Cl} -supermodules (resp. finite-dimensional Seg_n -supermodules). Then the Dirac cohomology H_D induces a map, still denoted H_D , from $K(\mathbb{H}_n^{\text{Cl}})$ to $K(\text{Seg}_n)$. Corollary 7.19 implies the following:

Corollary 7.20. *The image of $H_D : K(\mathbb{H}_n^{\text{Cl}}) \rightarrow K(\text{Seg}_n)$ has finite index in $K(\text{Seg}_n)$.*

Recall that the algebra homomorphism $\zeta : Z(\mathbb{H}_n^{\text{Cl}}) \rightarrow Z(\text{Seg}_n)_0$ is defined in Theorem 3.4. We also have:

Corollary 7.21. *The map $\zeta : Z(\mathbb{H}_n^{\text{Cl}}) \rightarrow Z(\text{Seg}_n)_0$ is surjective.*

Proof. It suffices to show that $\dim(\text{im } \zeta) \geq \dim Z(\text{Seg}_n)_0$. By Theorem 4.25 and Theorem 7.18, for any partition $\lambda \in \mathcal{P}_n^{\text{dist}}$, there exists $(\sigma_\lambda, U_\lambda) \in \text{Irr}_{\text{sup}} \text{Seg}_n$, such that $\chi_{\pi_\lambda} = \chi^{\sigma_\lambda}$. Since the central characters $\{\chi_{\pi_\lambda}\}_{\lambda \in \mathcal{P}_n^{\text{dist}}}$ are linearly independent over \mathbb{C} , $\{\chi^{\sigma_\lambda}\}_{\lambda \in \mathcal{P}_n^{\text{dist}}}$ are also linearly independent. Then we have that $\dim(\text{im } \zeta)$ is not less than the cardinality of $\mathcal{P}_n^{\text{dist}}$. Now the statement follows from the fact that $\dim Z(\text{Seg}_n)_0$ is equal to the cardinality of $\text{Irr}_{\text{sup}}(\text{Seg}_n)/\sim_\Pi$, which is the same as the cardinality of $\mathcal{P}_n^{\text{dist}}$. \square

Remark 7.22. The author would like to thank Professor Weiqiang Wang for pointing out that there is a canonical surjective superalgebra morphism from \mathbb{H}_n^{Cl} to Seg_n [Kl, Rem. 15.4.7]. Denote the map to be ζ' . According to [Kl, Rem. 15.4.7], the map ζ' sends x_i to the Jucys–Murphy type element

$$\zeta'(x_i) = \sum_{1 \leq j < i} s_{i,j}(1 - c_i c_j),$$

and ζ' is an identity on Seg_n . It is straightforward to check that $\zeta'(D) = 0$. By considering

$$z = \zeta(z) + Dh + hD$$

and applying ζ' on both sides, $\zeta'(z) = \zeta(z)$. Hence ζ' agrees with ζ on $Z(\mathbb{H}_n^{\text{Cl}})$. The author would like to thank one of the referees for pointing out that the map ζ' has already been proven to be surjective in [Ru] as a special case. This in turn gives another way to see that ζ is surjective.

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