Interval Observer-based Fault-tolerant Control for A Class of Positive Markov Jump Systems

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Abstract

The problem of fault-tolerant controller design for positive Markov jump systems subject to interval uncertainties and time-varying actuator faults is investigated in this paper. First, focusing on the actuator faults and assuming that the stochastic stability of the system is a prerequisite, an interval observer is developed to achieve the upper and lower estimates of the system state and actuator faults simultaneously. First, the stochastic stability of the system is assumed to be a prerequisite. Focusing on the actuator faults, an interval observer is developed to achieve the upper and lower estimates of the system state and actuator faults simultaneously. Then, a joint design of the interval observer and a state-estimate feedback controller is considered. By utilizing the obtained interval estimates, the controller is constructed to achieve a satisfactory performance and fault tolerance. Owing to positivity, the entire observer and controller design is formulated in terms of linear programming. Effectiveness of the proposed approach is demonstrated through simulation results.

Keywords: fault-tolerant control, interval observer, linear programming, L_1-gain performance, Markov jump system, positive system

1. Introduction

Intrinsically nonnegative variables are extensively present in real life. Positive system is an appropriate model to describe a family of systems whose state always resides in the nonnegative orthant. Increasing attention has been paid to positive systems for their theoretical and practical importance. Typical applications range from population dynamics in biology, pricing in economics, to congestion control in communication [37, 38]. More recent studies have also tried to explain the transmission of the ongoing COVID-19 in terms of positive systems [34]. Theoretical advantages of the positive feature have also contributed to many remarkable results on positive systems, such as simplification on characterization of stability and less conservative conditions for system stabilization [14, 28, 29, 35]. There are two main reasons for the recently increasing attention on positive

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systems: theoretical advantages of positive feature, such as simplification on characterization of stability and performance [14, 28, 29, 35], and wide applications in various fields, including population dynamics in biology, pricing in economics, and congestion control in communication [37, 38].

For a positive system, random jumps may occur due to various external causes, such as abrupt environmental changes for biotic population, unpredictable events in product pricing, random equipment failures over information communication. Generally speaking, markov Markov jump systems are a kind of stochastic hybrid systems which are suited for representing system dynamics in the form of random jumps among subsystems [6, 33, 36]. There are various external causes of random jumps in the above mentioned positive system, such as abrupt environmental changes for biotic population, unpredictable events in product pricing, random equipment failures over information communication. Therefore, by capturing the nonnegative state variables and stochastic factors, positive Markov jump systems (PMJSs) can be regarded as a unified modeling for many practical systems. Among the existing research results on PMJSs, two important issues in control theory, namely, stability criteria and performance evaluation, have been mainly studied. Concerning the stochastic stability issue of PMJSs, the notion of exponential mean stability is introduced and its relationship with different stability notions has been clarified in [3]. In addition, for PMJSs in continuous and discrete time, necessary and sufficient conditions are derived to guarantee the stochastic stability in the mean sense [20]. the mean stability is guaranteed through the derived necessary and sufficient conditions in [20]. Apart from stochastic stability analysis, to deal with the disturbance input of PMJSs, some specific measures have been investigated. The l_1-gain performance has been analyzed and a positive l_1-gain filter has been developed by employing the linear programming (LP) approach [49]. Taking time-delay factor into consideration, the L_∞-gain and L_1-gain performance indices have been established with necessary and sufficient condition provided in [26] and [50], respectively. Some basic analytical results of PMJSs have also been reported in the above works. For PMJSs of which only the sign structures of the subsystem matrices and mode transitions are feasible, the concept of sign-stability is introduced and is proved to be equivalent to the standard notions of stochastic stability in the structural framework [7]. Stochastic stability analysis for PMJSs with partially known transition rates [10] and for positive 2-D Markov jump systems [11] have also been extensively investigated. However, there are still many open control issues to be studied for in study of PMJSs.

Failures and faults in different system components are often encountered in dynamic systems, and the occurrence of unexpected faults in actuators and sensors may lead to instability or severely degraded performance. Therefore, reliability is an important issue in systems engineering to which serious attention should be paid. In order to attenuate the effects of system component failures, fault estimation is a prerequisite to design a functional controller that is capable of tolerating faults [15, 18]. To estimate the fault effectively, there are various representative results. The multi-constrained fault estimate estimation observer approach has been developed for continuous-time and discrete-time systems in [47]. Combined with the descriptor approach, static observer Static observer [16], adaptive observer [46], sliding mode observer [45] and interval observer [19, 43] according to the descriptor approach have been constructed to estimate different component faults. Among these different observer structures, interval observer is most suitable for state estimation of positive systems, which could explicitly define an interval within which the state lies [5]. As pointed out in [41, 42], the use of interval observer techniques allows the estimator to obtain the set of admissible values of system state at all times. In case of bounded parameter uncertainties or disturbances, these techniques could considerably simplify the design of a robust observer by using available information without any additional assumptions. It has therefore been widely adopted for a variety of positive systems, including interval systems with input or time delay [17, 25], LPV systems [8], switched systems [13], and Markov jump systems [4]. As a practical observer design strategy, interval observers have been applied to a wide range of areas such as chemical/biochemical processes [2], circuit systems [21, 32], and vehicle dynamics [22].

Following the remarkable development in interval observers, interval observer-based fault estimation could be achieved by resorting to positive systems theory and different augmentation approaches [39, 40, 48]. Interval observer-based fault estimation have also been considered by resorting to positive systems theory and different augmentation approaches [39, 40, 48]. It has been shown that in fault detection, the interval observer approach Taking advantage of the estimated intervals, such a fault estimator could determine the minimum detectable fault, which contributes to the decision of a triggering limit for fault alarm detection [30]. Moreover, for the purpose of control, interval observers provide more possibilities in practical implementation or transient performance improvement. Note that by applying the techniques of interval observers, a wide spectrum of uncertain systems could be stabilized [12, 44]. It has also been shown in [24] that the interval observer-based control strategies enable an on-line calculation of accurate bounds on the given trajectories comparing with the Luenberger observer-based ones. Besides, it is demonstrated in [5] that by exploiting the positive nature of the error dynamics, such observers will contribute to a more tractable design of peak-to-peak controllers. Therefore, the interval observer-based control strategies are widely applicable to many disciplines of engineering, especially for systems with bounded uncertainties and systems with bounding requirements. Typical applications include interval observer-based tracking control of quadrotors (to guarantee more robustness under bounded model uncertainties) [1], and interval observer-based control of thermo-fluidic systems (to prevent overshooting safety-critical temperature bounds) [23].

Motivated by the above, the idea of interval observer-based control is generalized for PMJSs with uncertainties and actuator faults in this work. An interval observer is first designed for the dynamic estimation of bounds on the system state as well as actuator faults in a positive Markov jump system. Based on the obtained observer, a fault-tolerant control (FTC) scheme is proposed to guarantee the stochastic stability and L_1-gain performance of the closed-loop system. Finally, the observer and controller are developed using linear programming. An illustrative example is also presented. Compared with the interval observer designs for systems with bounded uncertainties [8, 13], the presence of unbounded and time-varying actuator faults makes it nontrivial to design an interval observer. Therefore, a system transformation is applied to decouple the unbounded variables, after which the system state and actuator faults could be estimated simultaneously. It should also be noted that, different from the previous work on interval observer-based controller design in [24] where the controller and the observer are separately designed, a joint design of them is adopted in this work to compensate for the time-varying actuator faults online. However, new challenge arises as both nonnegativity and stochastic stability are required when simultaneously searching for observer gains and controller gains. To cope with this, a new construction of the controller is proposed for positive system which, compared with the commonly used iteration approaches in controller design [9, 39], could simplify the computation with conditions in terms of LP and avoid any iteration process. To conclude, as a significant extension of studies on positive Markov jump system, the main contributions of this work are as follows.

1) Focusing on actuator faults, an interval observer is designed by utilizing the nonnegativity of positive systems. The considered faults may be unbounded, while the estimator is able to encapsulate the system state and actuator faults at all times.

2) By simultaneously providing the interval estimates of both the system state and actuator faults, a fault-tolerant controller utilizing both the state and actuator faults estimates is designed to improve the reliability of PMJSs.

3) A systematic construction of the controller is proposed with conditions derived in terms of LP, that leads to more convenient implementation for analysis and parameter computation.

The structure of the work is given as follows. In Section II, some preliminaries including system description are provided. The main results of designing the positive interval observer-based fault tolerant controller are shown in Section III. Simulation results are given in Section IV to verify the proposed techniques and Section V presents the conclusions.

Notation: Denote $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$ as the set of n-dimensional vectors with real, nonnegative, and strictly positive components, respectively. $\mathbb{R}^{n\times m}$, $\mathbb{R}^{n\times m}_+$, and $\mathbb{R}^{n\times m}_{++}$ is the set of all $n \times m$ real matrices, nonnegative matrices, and strictly positive matrices, respectively. For a matrix $A \in \mathbb{R}^{n\times m}$, $A[i,j]$ denotes its (i,j)th entry. $A \succ 0$, $A \succeq 0$, $A \prec 0$, $A \preceq 0$, means mean that all its entries are positive, nonnegative, negative and nonpositive, respectively. For matrices A, B, $C \in \mathbb{R}^{n\times m}$, $A \in [B, C]$ indicates that $B \preceq A \preceq C$. $A^\dagger$ denotes the Moore–Penrose inverse of A. A square matrix $M \in \mathbb{R}^{n\times n}$ is called Metzler if all its off-diagonal elements are nonnegative. The space of all the vector-valued Lebesgue functions defined on $[0, +\infty)$ with finite L_1-norm is denoted by $L_1[0, +\infty)$. I_n is the $n \times n$ identity matrix and $0_{m\times n}$ is the $m \times n$ zero matrix. $\mathbf{1}_n$ is the $n \times 1$ vector of all ones. The mathematical expectation is denoted by $\mathbb{E}\{\cdot\}$ and $\mathbf{P}\{\cdot\}$ denotes the probability of an event. For a vector $v \in \mathbb{R}^n$, $v[j]$ denotes its jth element and $\|v\|_\infty = \max_{j=1,2,\ldots,n}\{|v[j]|\}$. Besides, $v_{\min}$ denotes the minimal element in the vector v and $v^+_{\min}$ denotes its minimal nonzero element. Given a set of vectors $v_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$, $\operatorname*{vec}_{i=1}^{N}\{v_i\} = [v_1^{\mathrm{T}}\ v_2^{\mathrm{T}}\ \cdots\ v_N^{\mathrm{T}}]^{\mathrm{T}}$.

2. Preliminaries and Problem Formulation

Consider a positive Markov jump linear system with actuator faults as follows:

$$\begin{cases} \dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))(u(t) + f(t)) + B_w(\sigma(t))w(t), \\ y(t) = C(\sigma(t))x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$ represent the system state, control input and measurement output, respectively, and $x(0) \succeq 0$; $f(t) \in \mathbb{R}^{n_u}$ is the actuator fault vector, whose components are not sign definite (that is, they may take positive or negative values); $w(t) \in \mathbb{R}_+^{n_w}$ is the external disturbance, which is an arbitrary signal in $L_1[0, +\infty)$. The Markov stochastic process $\{\sigma(t)\}$, $t \geq 0$, is time homogeneous with state space defined by $\mathscr{N} = \{1, 2, \dots, N\}$. The transition probabilities over the process are determined by

$$\mathbf{P}\{\sigma(t + \Delta t) = j \mid \sigma(t) = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t) & j \neq i, \\ 1 + \pi_{ii}\Delta t + o(\Delta t) & j = i, \end{cases} \tag{2}$$

where $\Delta t > 0$ is a small time increment, $o(\Delta t)$ is an infinitesimal of higher order with respect to Δt, $\lim_{\Delta t \to 0^+} \frac{o(\Delta t)}{\Delta t} = 0$; $\pi_{ij} \geq 0$ ($i, j \in \mathscr{N}$ and $i \neq j$) and $\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij}$. Define $\Pi = (\pi_{ij}) \in \mathbb{R}^{N\times N}$, $i, j \in \mathscr{N}$ as generator matrix of $\{\sigma(t)\}$, which governs the evolution of the Markov stochastic process. For simplicity, the matrices $A(\sigma(t))$, $B(\sigma(t))$, $B_w(\sigma(t))$ and $C(\sigma(t))$ are represented by A_i, B_i, B_{wi} and C_i, respectively, for each $\sigma(t) = i$, $i \in \mathscr{N}$. The positive Markov jump system is subject to interval uncertainties, in which A_i is an unknown constant matrix satisfying $A_i \in [\underline{A}_i, \overline{A}_i]$, $i \in \mathscr{N}$, where $\underline{A}_i \in \mathbb{R}^{n_x\times n_x}$ is Metzler; B_i, B_{wi} and C_i are known with $B_i \in \mathbb{R}_+^{n_x\times n_u}$, $B_{wi} \in \mathbb{R}_+^{n_x\times n_w}$, $C_i \in \mathbb{R}_+^{n_y\times n_x}$. Suppose that $f(t)$ is unknown, but it is differentiable up to the M-th ($M \geq 1$) order and its M-th derivative is assumed to satisfy $\underline{\delta}(t) \preceq f^{(M)}(t) \preceq \overline{\delta}(t)$ for some $\underline{\delta}(t)$, $\overline{\delta}(t) \in L_1[0, +\infty)$. One can see that the considered time-varying fault may be unbounded, which could cover a wide range of possible actuator faults.

Definition 1. *System (1) with $f(t) = 0$ (fault-free case) is a positive Markov jump system if for all $\sigma(0) \in \mathscr{N}$, $x(0) \succeq 0$, $u(t) \succeq 0$, and $w(t) \succeq 0$, the system state $x(t) \succeq 0$ and the output $y(t) \succeq 0$ for $t > 0$.*

Lemma 1. [14] *System (1) with $f(t) = 0$ (fault-free case) is positive if and only if A_i is Metzer, $B_i \succeq 0$, $B_{wi} \succeq 0$, and $C_i \succeq 0$.*

In this work, A_i is uncertain but it is constant and always belongs to the interval uncertain domain $[\underline{A}_i, \overline{A}_i]$. Since $\underline{A}_i$ is a Metzer matrix, B_i, B_{wi} and C_i are known nonnegative matrices, the PMJS (1) with $f(t) = 0$ is positive.

In the presence of unbounded actuator faults and interval parameter uncertainties, one may find it hard to design conventional Luenberger observers for system (1). Our first goal is to develop a group of interval observers to generate both the upper and lower boundary of the system state as well as actuator faults. Further considering that the plant may fail under unbounded actuator faults, our next phase is to develop an FTC scheme by utilizing the obtained interval estimates.

Results on stochastic stability for PMJS (1) without faults ($f(t) = 0$) taking into account are first given as follows.

Definition 2. [3, 50] *PMJS (1) with $w(t) = 0$, $u(t) = 0$ is mean stable if for any $\sigma(0) \in \mathscr{N}$ and $x(0) \succeq 0$,*

$$\lim_{t\to\infty} \mathbb{E}\{x(t)\} = 0. \tag{3}$$

Lemma 2. [3, 50] *PMJS (1) with $w(t) = 0$, $u(t) = 0$ is mean stable if and only if there exist strictly positive vectors p_i, $i \in \mathscr{N}$, such that*

$$p_i^{\mathrm{T}} A_i + \sum_{i=1}^{N} \pi_{ij} p_j^{\mathrm{T}} \prec 0. \tag{4}$$

3. Interval Observer-based Fault-tolerant Control

3.1. State and fault simultaneous observer design

We will present an interval observer to estimate the system state $x(t)$ and actuator fault $f(t)$ simultaneously with the known input $u(t)$ and measurement output $y(t)$. Note that here the mean stability of the system is a prerequisite for the construction of the interval observer; neither state-estimate feedback nor fault-compensated control is applied.

Denoting $\xi_m(t) = f^{(M-m)}(t)$, $m = 1, 2, \ldots, M$, and $x_\xi(t) = [x^{\mathrm{T}}(t), \xi_1^{\mathrm{T}}(t), \xi_2^{\mathrm{T}}(t), \ldots, \xi_M^{\mathrm{T}}(t)]^{\mathrm{T}}$, an augmented system can be constructed based on PMJS (1):

$$\begin{cases} \dot{x_\xi}(t) = \mathscr{A}_i x_\xi(t) + \mathscr{B}_i u(t) + \mathscr{B}_{wi} w(t) + \mathscr{W}_f f^{(M)}(t), \\ y(t) = \mathscr{C}_i x_\xi(t), \end{cases} \tag{5}$$

where

$$\mathscr{A}_i = \begin{bmatrix} A_i & 0 & \cdots & 0 & B_i \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \quad \mathscr{B}_i = \begin{bmatrix} B_i \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathscr{B}_{wi} = \begin{bmatrix} B_{wi} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathscr{W}_f = \begin{bmatrix} 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathscr{C}_i = \begin{bmatrix} C_i^{\mathrm{T}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^{\mathrm{T}}. \tag{6}$$

According to (5), both of the system state and derivatives of the fault signal are merged into a state vector of the augmented system. We then proceed to develop an interval observer for (5) in the form of

$$\begin{cases} \dot{\check{\nu}}(t) = \check{\mathscr{G}}_i \check{\nu}(t) + \check{T}_i \mathscr{B}_i u(t) + \check{H}_i y(t) + \mathscr{W}_f \underline{\delta}(t), \\ \check{x}_\xi(t) = \check{\nu}(t) + \check{L}_i y(t), \end{cases} \tag{7}$$

$$\begin{cases} \dot{\hat{\nu}}(t) = \hat{\mathscr{G}}_i\hat{\nu}(t) + \hat{T}_i\mathscr{B}_i u(t) + \hat{H}_i y(t) + \mathscr{W}_f\overline{\delta}(t), \\ \hat{x}_\xi(t) = \hat{\nu}(t) + \hat{L}_i y(t), \end{cases} \tag{8}$$

with $\check{\nu}(t), \hat{\nu}(t) \in \mathbb{R}^{n_e}$, $n_e = n_x + Mn_u$; and $\check{x}_\xi(t) = [\check{x}^{\mathrm{T}}(t), \check{\xi}_1^{\mathrm{T}}(t), \check{\xi}_2^{\mathrm{T}}(t), \dots, \check{\xi}_M^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_e}$, being the lower-bounding estimate of $x_\xi(t)$ and $\hat{x}_\xi(t) = [\hat{x}^{\mathrm{T}}(t), \hat{\xi}_1^{\mathrm{T}}(t), \hat{\xi}_2^{\mathrm{T}}(t), \dots, \hat{\xi}_M^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_e}$, being the upper-bounding estimate of $x_\xi(t)$; $\hat{\mathscr{G}}_i, \check{\mathscr{G}}_i, \check{T}_i, \hat{T}_i \in \mathbb{R}^{n_e \times n_e}$, $\check{H}_i, \hat{H}_i, \check{L}_i, \hat{L}_i \in \mathbb{R}^{n_e \times n_y}$, are observer matrices to be designed. Obviously, the lower- and upper-bounding estimates of the system state $x(t)$ and $f(t)$ could be obtained from $\check{x}_\xi(t)$ and $\hat{x}_\xi(t)$, respectively, for which $\check{f}(t) = \check{\xi}_M(t)$ and $\hat{f}(t) = \hat{\xi}_M(t)$. It is worth noting that by introducing $\check{v}(t)$ and $\hat{v}(t)$ into design of the interval observer, the derivative term of the output $y(t)$ is avoided, which eases control system synthesis.

Definition 3. *System (7) and system (8) form an interval observer for system (1) if for any $\check{x}_\xi(0) \preceq x_\xi(0) \preceq \hat{x}_\xi(0)$, and $x(0) \succeq 0$, $u(t) \succeq 0$, $w(t) \succeq 0$, the inequality $\check{x}_\xi(t) \preceq x_\xi(t) \preceq \hat{x}_\xi(t)$ holds for all $t > 0$.*

Remark 1. *According to Definition 3, the underlying idea is to guarantee that the estimation errors, namely, the lower-bounding estimation error $(x_\xi(t) - \check{x}_\xi(t))$ and upper-bounding estimation error $(\hat{x}_\xi(t) - x_\xi(t))$, are nonnegative for all $t > 0$, that is, the corresponding error dynamics should be positive systems. Therefore, the interval observer is constructed in such a way that both stochastic stability and positivity of the error dynamics can be guaranteed.*

With all the previous elements, the problem to be solved is formulated as follows.

Problem IOPMJS (Interval Observer of Positive Markov Jump System). For a positive Markov jump system (1), find an observer composed of (7) and (8) such that

1. when $\check{x}_\xi(0) \preceq x_\xi(0) \preceq \hat{x}_\xi(0)$, the estimation errors $(x_\xi(t) - \check{x}_\xi(t)) \succeq 0$, $(\hat{x}_\xi(t) - x_\xi(t)) \succeq 0$ for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $i \in \mathscr{N}$, $t > 0$;
2. $\lim\limits_{t\to\infty} \mathbb{E}\{x_\xi(t) - \check{x}_\xi(t)\} = 0$, and $\lim\limits_{t\to\infty} \mathbb{E}\{\hat{x}_\xi(t) - x_\xi(t)\} = 0$.

Before proceeding, we denote

$$\begin{aligned} &J_1 = [I_{n_x} \quad 0_{n_x \times Mn_u}]^{\mathrm{T}}, \quad J_2 = [0_{(M-1)n_u \times n_x} \quad I_{(M-1)n_u} \quad 0_{(M-1)n_u \times n_u}]^{\mathrm{T}}, \\ &J_3 = [0_{n_u \times (n_e - n_u)} \quad I_{n_u}]^{\mathrm{T}}, \quad S = [0_{(M-1)n_u \times (n_x + n_u)} \quad I_{(M-1)n_u}]^{\mathrm{T}}. \end{aligned} \tag{9}$$

Theorem 1. *For system (1) with $w(t) = 0$, an interval observer composed of (7) and (8) exists if there exist diagonal matrices $\check{P}_i$, $\hat{P}_i \in \mathbb{R}_+^{n_e \times n_e}$ with strictly positive diagonal elements, nonnegative matrices $\check{Z}_i^+$, $\check{Z}_i^-$, $\hat{Z}_i^+$, $\hat{Z}_i^- \in \mathbb{R}_+^{n_e \times n_y}$, matrices $\check{V}_i$, $\hat{V}_i \in \mathbb{R}^{n_e \times n_x}$, such that*

$$\begin{bmatrix} \Gamma_i^l & 0 \\ 0 & \Gamma_i^u \end{bmatrix} \quad \text{is Metzler,} \tag{10}$$

$$\begin{bmatrix}\underline{\Lambda}_i^l & -\overline{\Lambda}_i^u\end{bmatrix} \succeq 0, \tag{11}$$

$$\begin{bmatrix}\mathbf{1}_{n_e}^{\mathrm{T}}(\Gamma_i^l + \sum_{j=1}^{N} \pi_{ij}\check{P}_j^{\mathrm{T}}) \\ \mathbf{1}_{n_e}^{\mathrm{T}}(\Gamma_i^u + \sum_{j=1}^{N} \pi_{ij}\hat{P}_j^{\mathrm{T}})\end{bmatrix} \prec 0, \tag{12}$$

with $\check{Z}_i = \check{Z}_i^+ - \check{Z}_i^-$, $\hat{Z}_i = \hat{Z}_i^+ - \hat{Z}_i^-$, *and*

$$\begin{aligned}
\underline{\Lambda}_i^l &= \check{P}_i J_1 \underline{A}_i - \check{Z}_i^+ C_i \overline{A}_i + \check{Z}_i^- C_i \underline{A}_i - \check{V}_i, \\
\overline{\Lambda}_i^u &= \hat{P}_i J_1 \overline{A}_i - \hat{Z}_i^+ C_i \underline{A}_i + \hat{Z}_i^- C_i \overline{A}_i - \hat{V}_i, \\
\Gamma_i^l &= \check{V}_i J_1^{\mathrm{T}} + \check{P}_i (S J_2^{\mathrm{T}} + J_1 B_i J_3^{\mathrm{T}}) - \check{Z}_i C_i B_i J_3^{\mathrm{T}}, \\
\Gamma_i^u &= \hat{V}_i J_1^{\mathrm{T}} + \hat{P}_i (S J_2^{\mathrm{T}} + J_1 B_i J_3^{\mathrm{T}}) - \hat{Z}_i C_i B_i J_3^{\mathrm{T}}.
\end{aligned} \tag{13}$$

The interval observer matrix parameters are given by

$$\check{L}_i = \check{P}_i^{-1}\check{Z}_i, \quad \hat{L}_i = \hat{P}_i^{-1}\hat{Z}_i, \tag{14}$$

$$\check{\mathscr{G}}_i = \check{G}_i J_1^{\mathrm{T}} + S J_2^{\mathrm{T}} + (I - \check{L}_i \mathscr{C}_i) J_1 B_i J_3^{\mathrm{T}}, \tag{15}$$

$$\hat{\mathscr{G}}_i = \hat{G}_i J_1^{\mathrm{T}} + S J_2^{\mathrm{T}} + (I - \hat{L}_i \mathscr{C}_i) J_1 B_i J_3^{\mathrm{T}}, \tag{16}$$

with

$$\check{G}_i = \check{P}_i^{-1}\check{V}_i, \quad \hat{G}_i = \hat{P}_i^{-1}\hat{V}_i, \tag{17}$$

and

$$\check{H}_i = \check{\mathscr{G}}_i \check{L}_i, \quad \check{T}_i = I - \check{L}_i \mathscr{C}_i, \tag{18}$$

$$\hat{H}_i = \hat{\mathscr{G}}_i \hat{L}_i, \quad \hat{T}_i = I - \hat{L}_i \mathscr{C}_i. \tag{19}$$

Proof. Denote $\check{e}(t) = x_\xi(t) - \check{x}_\xi(t)$, and $\hat{e}(t) = \hat{x}_\xi(t) - x_\xi(t)$, one has

$$\check{e}(t) = x_\xi(t) - \check{x}_\xi(t) = x_\xi(t) - \check{\nu}(t) - \check{L}_i \mathscr{C}_i x_\xi(t) = (I - \check{L}_i \mathscr{C}_i) x_\xi(t) - \check{\nu}(t), \tag{20}$$

$$\hat{e}(t) = \hat{x}_\xi(t) - x_\xi(t) = \hat{\nu}(t) + \hat{L}_i \mathscr{C}_i x_\xi(t) - x_\xi(t) = \hat{\nu}(t) + (\hat{L}_i \mathscr{C}_i - I) x_\xi(t). \tag{21}$$

For the lower-bounding estimation error $\check{e}(t)$, we obtain from (5) and (7) that

$$\begin{aligned}
\dot{\check{e}}(t) &= (I - \check{L}_i \mathscr{C}_i)\mathscr{A}_i x_\xi(t) - \check{H}_i \mathscr{C}_i x_\xi(t) - \check{\mathscr{G}}_i \check{\nu}(t) + (I - \check{L}_i \mathscr{C}_i - \check{T}_i)\mathscr{B}_i u(t) \\
&\quad + (I - \check{L}_i \mathscr{C}_i)\mathscr{B}_{wi} w(t) + \mathscr{W}_f (f^{(M)}(t) - \underline{\delta}(t)) \\
&= [(I - \check{L}_i \mathscr{C}_i)\mathscr{A}_i - \check{\mathscr{G}}_i (I - \check{L}_i \mathscr{C}_i) - \check{H}_i \mathscr{C}_i] x_\xi(t) + \check{\mathscr{G}}_i \check{e}(t) \\
&\quad + (I - \check{L}_i \mathscr{C}_i - \check{T}_i)\mathscr{B}_i u(t) + (I - \check{L}_i \mathscr{C}_i)\mathscr{B}_{wi} w(t) + \mathscr{W}_f (f^{(M)}(t) - \underline{\delta}(t)).
\end{aligned} \tag{22}$$

When selecting $\check{H}_i$ and $\check{T}_i$ as in (18) and $\check{\mathscr{G}}_i$ in (15), the lower-bounding estimation error dynamic equation in (22) reduces to

$$\dot{\check{e}}(t) = (J_1 A_i - \check{L}_i C_i A_i - \check{G}_i) x(t) + \check{\mathscr{G}}_i \check{e}(t) + (I - \check{L}_i \mathscr{C}_i)\mathscr{B}_{wi} w(t)$$

$$+\mathscr{W}_f(f^{(M)}(t)-\underline{\delta}(t)). \tag{23}$$

Similarly, when selecting $\hat{H}_i$ and $\hat{T}_i$ as in (19) and $\hat{\mathscr{G}}_i$ in (16), the upper-bounding estimation error dynamic equation is

$$\begin{aligned}\dot{\hat{e}}(t) =&(\hat{L}_iC_iA_i - J_1A_i + \hat{G}_i)x(t) + \hat{\mathscr{G}}_i\hat{e}(t) + (\hat{L}_i\mathscr{C}_i - I)\mathscr{B}_{wi}w(t)\\ &+\mathscr{W}_f(\overline{\delta}(t) - f^{(M)}(t)).\end{aligned} \tag{24}$$

It follows from (10) that with diagonal matrices $\check{P}_i$, $\hat{P}_i \in \mathbb{R}_+^{n_e\times n_e}$, $\check{P}_i^{-1}\Gamma_i^l$ and $\hat{P}_i^{-1}\Gamma_i^u$ are Metzler matrices. Based on the definitions of Γ_i^l and Γ_i^u in (13), as well as the definitions in (14) and (17), one obtains that $\check{\mathscr{G}}_i$, $\hat{\mathscr{G}}_i$ are Metzler. Provided that $A_i \in [\underline{A}_i, \overline{A}_i]$, $C_i \succeq 0$, the inequalities

$$(\check{Z}_i^+ - \check{Z}_i^-)C_iA_i \preceq \check{Z}_i^+C_i\overline{A}_i - \check{Z}_i^-C_i\underline{A}_i, \tag{25}$$

$$\hat{Z}_i^+C_i\underline{A}_i - \hat{Z}_i^-C_i\overline{A}_i \preceq (\hat{Z}_i^+ - \hat{Z}_i^-)C_iA_i, \tag{26}$$

always hold with nonnegative $\check{Z}_i^+$, $\check{Z}_i^-$, $\hat{Z}_i^+$ and $\hat{Z}_i^-$, (11) implies that

$$\check{P}_iJ_1A_i - (\check{Z}_i^+ - \check{Z}_i^-)C_iA_i - \check{V}_i \succeq 0, \tag{27}$$

$$(\hat{Z}_i^+ - \hat{Z}_i^-)C_iA_i - \hat{P}_iJ_1A_i + \hat{V}_i \succeq 0. \tag{28}$$

Substitution of (14) and (17) into (27) and (28) yields

$$J_1A_i - \check{L}_iC_iA_i - \check{G}_i \succeq 0, \tag{29}$$

$$\hat{L}_iC_iA_i - J_1A_i + \hat{G}_i \succeq 0. \tag{30}$$

It is then straightforward to show that with $w(t) = 0$, if (10) and (11) are satisfied, then $\check{e}(t) \succeq 0$, $\hat{e}(t) \succeq 0$ when $\check{e}(0) \succeq 0$, $\hat{e}(0) \succeq 0$. With the stochastic stability of $x(t)$ being a prerequisite, the convergence to zero of the expectation of $\check{e}(t)$ and $\hat{e}(t)$ is guaranteed under condition (12) based on Lemma 2. □

Remark 2. *In Theorem 1, we deal with the case with $w(t) = 0$. When $w(t) \neq 0$, based on Lemma 1 as well as the estimation error dynamic equations (23) and (24), an interval observer composed of (7) and (8) exists for system (1) if the conditions in Theorem 1 are satisfied and the inequality*

$$\begin{bmatrix}\check{P}_iJ_1B_{wi} - \check{Z}_iC_iB_{wi} & \hat{Z}_iC_iB_{wi} - \hat{P}_iJ_1B_{wi}\end{bmatrix} \succeq 0, \tag{31}$$

holds, which guarantees that

$$(I - \check{L}_i\mathscr{C}_i)\mathscr{B}_{wi} \succeq 0, \tag{32}$$

$$(\hat{L}_i\mathscr{C}_i - I)\mathscr{B}_{wi} \succeq 0. \tag{33}$$

3.2. Fault-tolerant controller design

Actuator faults can cause operation failure of a control system. Moreover, different from previous studies on fault estimation for positive systems [27, 31], where actuator or sensor fault is assumed to be nonnegative, in this work a general class of actuator fault is considered, whose components are not sign definite. Thus the positivity of closed-loop system would not necessarily be preserved in the occurrence of the considered case of actuator fault. Our second goal is then to develop a state-estimate feedback control scheme in which an interval observer and a fault-tolerant controller are jointly designed in the L_1 sense. The positive Markov jump system is expected to be positive, mean stable, and satisfies satisfy a prescribed L_1-gain performance under the obtained controller.

Utilizing the interval estimates of the system state as well as the actuator faults, a fault-tolerant controller is constructed as

$$u(t) = \check{K}_i\check{x}(t) + \hat{K}_i\hat{x}(t) + \check{K}_{fi}\check{f}(t) + \hat{K}_{fi}\hat{f}(t), \tag{34}$$

where $\check{K}_i, \hat{K}_i, \check{K}_{fi}, \hat{K}_{fi} \in \mathbb{R}^{n_u\times n_x}$, are controller gains to be designed. Applying the fault-tolerant controller (34) to system (1) yields

$$\begin{aligned}\dot{x}(t) &= A_ix(t) + B_i(\check{K}_i\check{x}(t) + \hat{K}_i\hat{x}(t) + \check{K}_{fi}\check{f}(t) + \hat{K}_{fi}\hat{f}(t)) + B_if(t) + B_{wi}w(t)\\ &= (A_i + B_i\check{K}_i + B_i\hat{K}_i)x(t) - B_i\check{K}_i(x(t) - \check{x}(t)) + B_i\hat{K}_i(\hat{x}(t) - x(t))\\ &\quad + (B_i + B_i\check{K}_{fi} + B_i\hat{K}_{fi})f(t) - B_i\check{K}_{fi}(f(t) - \check{f}(t)) + B_i\hat{K}_{fi}(\hat{f}(t) - f(t)) + B_{wi}w(t).\end{aligned} \tag{35}$$

Choose $\check{K}_{fi} = -B_i^{\dagger}B_iK_{fi}$ and $\hat{K}_{fi} = B_i^{\dagger}B_i(K_{fi} - I)$, where $K_{fi} \in \mathbb{R}^{n_u\times n_u}$ is the new controller gain to be determined later, we have

$$B_i(I + \check{K}_{fi} + \hat{K}_{fi}) = 0 \tag{36}$$

and

$$\begin{aligned}\dot{x}(t) = {}&(A_i + B_i(\check{K}_i + \hat{K}_i))x(t) - B_i\check{K}_iJ_1^{\mathrm{T}}\check{e}(t) + B_i\hat{K}_iJ_1^{\mathrm{T}}\hat{e}(t)\\ &+B_iK_{fi}J_3^{\mathrm{T}}\check{e}(t) + B_i(K_{fi} - I)J_3^{\mathrm{T}}\hat{e}(t) + B_{wi}w(t).\end{aligned} \tag{37}$$

If we further denote $\mathcal{X}(t) = [x^{\mathrm{T}}(t), \check{e}^{\mathrm{T}}(t), \hat{e}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_{\mathcal{X}}}$, where $n_{\mathcal{X}} = 3n_x+2Mn_u$, and $\boldsymbol{\omega}(t) = [w^{\mathrm{T}}(t), (f^{(M)}(t) - \underline{\delta}(t))^{\mathrm{T}}, (\overline{\delta}(t) - f^{(M)}(t))^{\mathrm{T}}]^{\mathrm{T}}$, it is obtained that

$$\begin{aligned}\dot{\mathcal{X}}(t) = {}&\begin{bmatrix} A_i + B_i(\check{K}_i + \hat{K}_i) & -B_i(\check{K}_iJ_1^{\mathrm{T}} - K_{fi}J_3^{\mathrm{T}}) & B_i(\hat{K}_iJ_1^{\mathrm{T}} + (K_{fi} - I)J_3^{\mathrm{T}}) \\ J_1A_i - \check{L}_iC_iA_i - \check{G}_i & \check{\mathscr{G}}_i & 0 \\ \hat{L}_iC_iA_i - J_1A_i + \hat{G}_i & 0 & \hat{\mathscr{G}}_i \end{bmatrix}\mathcal{X}(t)\\ &+ \begin{bmatrix} B_{wi} & 0 & 0 \\ (I - \check{L}_i\mathscr{C}_i)\mathscr{B}_{wi} & \mathscr{W}_f & 0 \\ (\hat{L}_i\mathscr{C}_i - I)\mathscr{B}_{wi} & 0 & \mathscr{W}_f \end{bmatrix}\begin{bmatrix} w(t) \\ f^{(M)}(t) - \underline{\delta}(t) \\ \overline{\delta}(t) - f^{(M)}(t) \end{bmatrix}\end{aligned}$$

$$\triangleq \mathcal{A}_i\mathcal{X}(t) + \mathcal{W}_i\boldsymbol{\omega}(t). \tag{38}$$

Remark 3. *According to (37) and (38), the actuator fault can be compensated through the introduction of $\check{K}_{fi}\check{f}(t)$ and $\hat{K}_{fi}\hat{f}(t)$ into the control law, whilst $\check{K}_i$, $\hat{K}_i$ and K_{fi} are to be designed together with the interval observer to guarantee the positivity as well as the mean stability of the dynamic system in (38).*

To measure the L_1-gain, we introduce a controlled output as follows:

$$z(t) = C_{zi}x(t) + \check{E}_i\check{e}(t) + \hat{E}_i\hat{e}(t) \triangleq \mathcal{C}_i\mathcal{X}(t), \tag{39}$$

with $\mathcal{C}_i = [C_{zi} \quad \check{E}_i \quad \hat{E}_i]$, where C_{zi}, $\check{E}_i$ and $\hat{E}_i$ are nonnegative matrices which are chosen *a priori*.

Noticed that in the augmented system (38), $\boldsymbol{\omega} \in L_1[0, +\infty)$, we can state the observer-based fault-tolerant control problem as follows.

Problem IOFTCPMJS (Interval Observer-based Fault-tolerant Control of Positive Markov Jump System). For a positive Markov jump system (1), find an interval observer of the form (7)–(8) and control strategy (34) such that

1. the augmented system (38) is positive;
2. the augmented system (38) is stochastically stable in the mean sense;
3. L_1-gain from the disturbance $\boldsymbol{\omega}(t)$ to the reference output $z(t)$ is no greater than a certain level γ.

Lemma 3. [50] *Given a PMJS in the form of (38) and $\gamma > 0$, the system is mean stable and satisfies $\|z\|_{L_1} \leq \gamma\|\boldsymbol{\omega}\|_{L_1}$ if and only if there exist vectors $p_i \succ 0$, $i = 1, 2, \ldots, N$, such that*

$$p_i^{\mathrm{T}}\mathcal{A}_i + \sum_{j=1}^{N}\pi_{ij}p_j^{\mathrm{T}} + \mathbf{1}^{\mathrm{T}}\mathcal{C}_i \prec 0, \tag{40}$$

$$p_i^{\mathrm{T}}\mathcal{W}_i - \gamma\mathbf{1}^{\mathrm{T}} \prec 0. \tag{41}$$

Before proceeding, it is worth noting that for a nonnegative matrix B_i, $i \in \mathscr{N}$, if it does not have a zero row, there always exists an invertible matrix $Q_i \in \mathbb{R}^{n_u\times n_u}$, such that $B_iQ_i \in \mathbb{R}_{++}^{n_x\times n_u}$. Let

$$\Psi_i \triangleq B_iQ_i \succ 0. \tag{42}$$

Denote $\psi_{i,j} \in \mathbb{R}_{++}^{n_x}$, $j = 1, 2, \ldots, n_u$, as the jth column of Ψ_i. Based on the prespecified matrices Q_i and Ψ_i, we have the following theorem with conditions guaranteeing positivity, mean stability, and L_1-gain performance of the augmented system (38).

Theorem 2. *For positive Markov jump system (1) such that (42) holds, by utilizing an interval observer composed of (7) and (8), a fault-tolerant controller given by (34) exists such that for any* $A_i \in [\underline{A}_i, \overline{A}_i]$*, the augmented system (38) is positive, mean stable and satisfies* $\|z\|_{L_1} \le \gamma \|\boldsymbol{\omega}\|_{L_1}$ *with a given* $\gamma > 0$*, if there exist diagonal matrices* $\check{P}_i$*,* $\hat{P}_i \in \mathbb{R}_+^{n_e \times n_e}$ *with strictly positive diagonal elements, nonnegative matrices* $\check{Z}_i^+$*,* $\check{Z}_i^-$*,* $\hat{Z}_i^+$*,* $\hat{Z}_i^- \in \mathbb{R}_+^{n_e \times n_y}$*, matrices* $\check{V}_i$*,* $\hat{V}_i \in \mathbb{R}^{n_e \times n_x}$*, strictly positive vectors* $p_{xi} \in \mathbb{R}_{++}^{nx}$*, nonnegative vectors* $\check{u}_{i,j}^+$*,* $\check{u}_{i,j}^-$*,* $\hat{u}_{i,j}^+$*,* $\hat{u}_{i,j}^- \in \mathbb{R}_+^{n_x}$*, and* $u_{fi,j}^+$*,* $u_{fi,j}^- \in \mathbb{R}_+^{n_u}$*, for* $i = 1, 2, \ldots, N$*,* $j = 1, 2, \ldots, n_u$*, such that*

$$\begin{bmatrix} \Gamma_i^l & 0 \\ 0 & \Gamma_i^u \end{bmatrix} \quad \text{is Metzler,} \tag{43}$$

$$\begin{bmatrix} \underline{\Lambda}_i^l & -\overline{\Lambda}_i^u \end{bmatrix} \succeq 0, \tag{44}$$

$$\begin{bmatrix} \check{P}_i J_1 B_{wi} - \check{Z}_i C_i B_{wi} & \hat{Z}_i C_i B_{wi} - \hat{P}_i J_1 B_{wi} \end{bmatrix} \succeq 0, \tag{45}$$

$$p_{xi}^{\mathrm{T}} \mathbf{1}_{n_x} \underline{A}_i + \sum_{j=1}^{n_u} \psi_{i,j} \left[\frac{(\check{u}_{i,j}^+ + \hat{u}_{i,j}^+)^{\mathrm{T}}}{\psi_{i,j}^{\triangle}} - \frac{(\check{u}_{i,j}^- + \hat{u}_{i,j}^-)^{\mathrm{T}}}{\psi_{i,j}^{\triangledown}} \right] \quad \text{is Metzler,} \tag{46}$$

$$-\sum_{j=1}^{n_u} \psi_{i,j} \left(\frac{\check{u}_{i,j}^{+\mathrm{T}}}{\psi_{i,j}^{\triangledown}} - \frac{\check{u}_{i,j}^{-\mathrm{T}}}{\psi_{i,j}^{\triangle}} \right) J_1^{\mathrm{T}} + \sum_{j=1}^{n_u} \psi_{i,j} \left(\frac{u_{fi,j}^{+\mathrm{T}}}{\psi_{i,j}^{\triangle}} - \frac{u_{fi,j}^{-\mathrm{T}}}{\psi_{i,j}^{\triangledown}} \right) J_3^{\mathrm{T}} \succeq 0, \tag{47}$$

$$\sum_{j=1}^{n_u} \psi_{i,j} \left(\frac{\hat{u}_{i,j}^{+\mathrm{T}}}{\psi_{i,j}^{\triangle}} - \frac{\hat{u}_{i,j}^{-\mathrm{T}}}{\psi_{i,j}^{\triangledown}} \right) J_1^{\mathrm{T}} + \sum_{j=1}^{n_u} \psi_{i,j} \left(\frac{u_{fi,j}^{+\mathrm{T}}}{\psi_{i,j}^{\triangle}} - \frac{u_{fi,j}^{-\mathrm{T}}}{\psi_{i,j}^{\triangledown}} \right) J_3^{\mathrm{T}} - p_{xi}^{\mathrm{T}} \mathbf{1}_{n_x} B_i J_3^{\mathrm{T}} \succeq 0, \tag{48}$$

$$\begin{bmatrix} p_{xi}^{\mathrm{T}} \overline{A}_i + \sum_{j=1}^{n_u} (\check{u}_{i,j} + \hat{u}_{i,j})^{\mathrm{T}} + \mathbf{1}_{n_e}^{\mathrm{T}} (\overline{\Lambda}_i^l - \underline{\Lambda}_i^u) + \mathbf{1}_{n_x}^{\mathrm{T}} \sum_{j=1}^{N} \pi_{ij} P_{j1}^{\mathrm{T}} + \mathbf{1}_{n_y}^{\mathrm{T}} C_{zi} \\ -\sum_{j=1}^{n_u} (\check{u}_{i,j}^{\mathrm{T}} J_1^{\mathrm{T}} + u_{fi,j}^{\mathrm{T}} J_3^{\mathrm{T}}) + \mathbf{1}_{n_e}^{\mathrm{T}} (\Gamma_i^l + \sum_{j=1}^{N} \pi_{ij} P_{j2}^{\mathrm{T}}) + \mathbf{1}_{n_y}^{\mathrm{T}} \check{E}_i \\ \sum_{j=1}^{n_u} (\hat{u}_{i,j}^{\mathrm{T}} J_1^{\mathrm{T}} + u_{fi,j}^{\mathrm{T}} J_3^{\mathrm{T}}) - p_{xi}^{\mathrm{T}} B_i J_3^{\mathrm{T}} + \mathbf{1}_{n_e}^{\mathrm{T}} (\Gamma_i^u + \sum_{j=1}^{N} \pi_{ij} P_{j3}^{\mathrm{T}}) + \mathbf{1}_{n_y}^{\mathrm{T}} \hat{E}_i \end{bmatrix} \prec 0, \tag{49}$$

$$\begin{bmatrix} \Omega_{i1} - \gamma \mathbf{1}_{n_w}^{\mathrm{T}} & \Omega_{i2} - \gamma \mathbf{1}_{n_u}^{\mathrm{T}} & \Omega_{i3} - \gamma \mathbf{1}_{n_u}^{\mathrm{T}} \end{bmatrix} \prec 0, \tag{50}$$

where $\psi_{i,j}^{\triangle} = \|\psi_{i,j}\|_\infty$*,* $\psi_{i,j}^{\triangledown} = \psi_{i,j\,\min}$*;* $\check{u}_{i,j} = \check{u}_{i,j}^+ - \check{u}_{i,j}^-$*,* $\hat{u}_{i,j} = \hat{u}_{i,j}^+ - \hat{u}_{i,j}^-$*,* $u_{fi,j} = u_{fi,j}^+ - u_{fi,j}^-$*, and* $\check{Z}_i = \check{Z}_i^+ - \check{Z}_i^-$*,* $\hat{Z}_i = \hat{Z}_i^+ - \hat{Z}_i^-$*;*

$$\begin{aligned} \underline{\Lambda}_i^l &= \check{P}_i J_1 \underline{A}_i - \check{Z}_i^+ C_i \overline{A}_i + \check{Z}_i^- C_i \underline{A}_i - \check{V}_i, \\ \overline{\Lambda}_i^l &= \check{P}_i J_1 \overline{A}_i - \check{Z}_i^+ C_i \underline{A}_i + \check{Z}_i^- C_i \overline{A}_i - \check{V}_i, \\ \underline{\Lambda}_i^u &= \hat{P}_i J_1 \underline{A}_i - \hat{Z}_i^+ C_i \overline{A}_i + \hat{Z}_i^- C_i \underline{A}_i - \hat{V}_i, \\ \overline{\Lambda}_i^u &= \hat{P}_i J_1 \overline{A}_i - \hat{Z}_i^+ C_i \underline{A}_i + \hat{Z}_i^- C_i \overline{A}_i - \hat{V}_i, \\ \Gamma_i^l &= \check{V}_i J_1^{\mathrm{T}} + \check{P}_i (S J_2^{\mathrm{T}} + J_1 B_i J_3^{\mathrm{T}}) - \check{Z}_i C_i B_i J_3^{\mathrm{T}}, \\ \Gamma_i^u &= \hat{V}_i J_1^{\mathrm{T}} + \hat{P}_i (S J_2^{\mathrm{T}} + J_1 B_i J_3^{\mathrm{T}}) - \hat{Z}_i C_i B_i J_3^{\mathrm{T}}, \end{aligned} \tag{51}$$

and

$$\begin{aligned} \Omega_{i1} &= p_{xi}^{\mathrm{T}} B_{wi} + \mathbf{1}_{n_e}^{\mathrm{T}} [(\check{P}_i - \hat{P}_i) J_1 - (\check{Z}_i - \hat{Z}_i) C_i] B_{wi}, \\ \Omega_{i2} &= \mathbf{1}_{n_e}^{\mathrm{T}} \check{P}_i \mathscr{W}_f, \\ \Omega_{i3} &= \mathbf{1}_{n_e}^{\mathrm{T}} \hat{P}_i \mathscr{W}_f. \end{aligned} \tag{52}$$

The interval observer matrix parameters $\check{\mathscr{G}}_i$, $\check{L}_i$, $\check{H}_i$, $\check{T}_i$, $\hat{\mathscr{G}}_i$, $\hat{L}_i$, $\hat{H}_i$, $\hat{T}_i$ are given as in Theorem 1 while the controller matrix parameters are designed as

$$\check{K}_i = Q_i\check{R}_i, \quad \hat{K}_i = Q_i\hat{R}_i, \quad K_{fi} = Q_iR_{fi}, \tag{53}$$

with $\check{R}_i$, $\hat{R}_i$, and R_{fi} given by

$$\check{R}_i = \operatorname*{vec}_{j=1}^{n_u}\left\{\frac{\check{u}_{i,j}^{\mathrm{T}}}{p_{xi}^{\mathrm{T}}\psi_{i,j}}\right\},\ \hat{R}_i = \operatorname*{vec}_{j=1}^{n_u}\left\{\frac{\hat{u}_{i,j}^{\mathrm{T}}}{p_{xi}^{\mathrm{T}}\psi_{i,j}}\right\}, \tag{54}$$

$$R_{fi} = \operatorname*{vec}_{j=1}^{n_u}\left\{\frac{u_{fi,j}^{\mathrm{T}}}{p_{xi}^{\mathrm{T}}\psi_{i,j}}\right\}. \tag{55}$$

Proof. According to Theorem 1 and Remark 2, conditions (43)–(45) ensure that $\check{\mathscr{G}}_i$, $\hat{\mathscr{G}}_i$ are Metzler matrices, $J_1A_i - \check{L}_iC_iA_i - \check{G}_i \succeq 0$, $\hat{L}_iC_iA_i - J_1A_i + \hat{G}_i \succeq 0$, $(I - \check{L}_i\mathscr{C}_i)\mathscr{B}_{wi} \succeq 0$, and $(\hat{L}_i\mathscr{C}_i - I)\mathscr{B}_{wi} \succeq 0$. Based on the structures of $\check{K}_i$, $\hat{K}_i$ and K_{fi} in (53), and matrices $\check{R}_i$, $\hat{R}_i$ and R_{fi} given by (54)–(55), we have

$$B_i\check{K}_i = \sum_{j=1}^{n_u}\frac{\psi_{i,j}(\check{u}_{i,j}^{+} - \check{u}_{i,j}^{-})^{\mathrm{T}}}{p_{xi}^{\mathrm{T}}\psi_{i,j}}, \quad B_i\hat{K}_i = \sum_{j=1}^{n_u}\frac{\psi_{i,j}(\hat{u}_{i,j}^{+} - \hat{u}_{i,j}^{-})^{\mathrm{T}}}{p_{xi}^{\mathrm{T}}\psi_{i,j}}, \quad B_iK_{fi} = \sum_{j=1}^{n_u}\frac{\psi_{i,j}(u_{fi,j}^{+} - u_{fi,j}^{-})^{\mathrm{T}}}{p_{xi}^{\mathrm{T}}\psi_{i,j}}. \tag{56}$$

Provided that $\psi_{i,j}^{\triangle}$ return the maximal and $\psi_{i,j}^{\bigtriangledown}$ the minimal element of $\psi_{i,j}$, it always holds that

$$\frac{1}{\psi_{i,j}^{\triangle}} \le \frac{p_{xi}^{\mathrm{T}}\mathbf{1}_{n_x}}{p_{xi}^{\mathrm{T}}\psi_{i,j}} \le \frac{1}{\psi_{i,j}^{\bigtriangledown}}. \tag{57}$$

It follows from the above that for any $A_i \in [\underline{A}_i, \overline{A}_i]$,

$$\begin{aligned}
&p_{xi}^{\mathrm{T}}\mathbf{1}_{n_x}(A_i + B_i(\check{K}_i + \hat{K}_i)) \succeq p_{xi}^{\mathrm{T}}\mathbf{1}_{n_x}\underline{A}_i + \sum_{j=1}^{n_u}\psi_{i,j}\left[\frac{(\check{u}_{i,j}^{+} + \hat{u}_{i,j}^{+})^{\mathrm{T}}}{\psi_{i,j}^{\triangle}} - \frac{(\check{u}_{i,j}^{-} + \hat{u}_{i,j}^{-})^{\mathrm{T}}}{\psi_{i,j}^{\bigtriangledown}}\right],\\
&-p_{xi}^{\mathrm{T}}\mathbf{1}_{n_x}B_i(\check{K}_iJ_1^{\mathrm{T}} - K_{fi}J_3^{\mathrm{T}}) \succeq -\sum_{j=1}^{n_u}\psi_{i,j}\left(\frac{\check{u}_{i,j}^{+\mathrm{T}}}{\psi_{i,j}^{\bigtriangledown}} - \frac{\check{u}_{i,j}^{-\mathrm{T}}}{\psi_{i,j}^{\triangle}}\right)J_1^{\mathrm{T}} + \sum_{j=1}^{n_u}\psi_{i,j}\left(\frac{u_{fi,j}^{+\mathrm{T}}}{\psi_{i,j}^{\triangle}} - \frac{u_{fi,j}^{-\mathrm{T}}}{\psi_{i,j}^{\bigtriangledown}}\right)J_3^{\mathrm{T}},\\
&p_{xi}^{\mathrm{T}}\mathbf{1}_{n_x}B_i(\hat{K}_iJ_1^{\mathrm{T}} + (K_{fi} - I)J_3^{\mathrm{T}}) \succeq \sum_{j=1}^{n_u}\psi_{i,j}\left(\frac{\hat{u}_{i,j}^{+\mathrm{T}}}{\psi_{i,j}^{\triangle}} - \frac{\hat{u}_{i,j}^{-\mathrm{T}}}{\psi_{i,j}^{\bigtriangledown}}\right)J_1^{\mathrm{T}} + \sum_{j=1}^{n_u}\psi_{i,j}\left(\frac{u_{fi,j}^{+\mathrm{T}}}{\psi_{i,j}^{\triangle}} - \frac{u_{fi,j}^{-\mathrm{T}}}{\psi_{i,j}^{\bigtriangledown}}\right)J_3^{\mathrm{T}} - p_{xi}^{\mathrm{T}}\mathbf{1}_{n_x}B_iJ_3^{\mathrm{T}}.
\end{aligned} \tag{58}$$

which implies that the matrix $(A_i + B_i(\check{K}_i + \hat{K}_i))$ is Metzler , $-B_i(\check{K}_iJ_1^{\mathrm{T}} - K_{fi}J_3^{\mathrm{T}}) \succeq 0$, and $B_i(\hat{K}_iJ_1^{\mathrm{T}} + (K_{fi} - I)J_3^{\mathrm{T}}) \succeq 0$ under conditions (46)–(48), respectively. This completes the proof for positivity of the augmented system (38).

For diagonal matrices $\check{P}_i$ and $\hat{P}_i$, the vectors $\check{p}_i = \check{P}_i\mathbf{1}$, and $\hat{p}_i = \hat{P}_i\mathbf{1}$ return strictly positive vectors of the main diagonal elements of these matrices respectively. Also, notice that

$$p_{xi}^{\mathrm{T}}B_i\check{K}_i = \sum_{j=1}^{n_u}(\check{u}_{i,j}^{+} - \check{u}_{i,j}^{-})^{\mathrm{T}}, \quad p_{xi}^{\mathrm{T}}B_i\hat{K}_i = \sum_{j=1}^{n_u}(\hat{u}_{i,j}^{+} - \hat{u}_{i,j}^{-})^{\mathrm{T}}. \tag{59}$$

Defining $p_i^{\mathrm{T}} = [p_{xi}^{\mathrm{T}} \quad \check{p}_i^{\mathrm{T}} \quad \hat{p}_i^{\mathrm{T}}]$, we obtain from (49) that

$$p_i^{\mathrm{T}}\begin{bmatrix}
\overline{A}_i + B_i(\check{K}_i + \hat{K}_i) & -B_i(\check{K}_iJ_1^{\mathrm{T}} - K_{fi}J_3^{\mathrm{T}}) & B_i(\hat{K}_iJ_1^{\mathrm{T}} + (K_{fi} - I)J_3^{\mathrm{T}})\\
J_1\overline{A}_i - \check{L}_i^{+}C_i\underline{A}_i + \check{L}_i^{-}C_i\overline{A}_i - \check{G}_i & \check{\mathscr{G}}_i & 0\\
\hat{L}_i^{+}C_i\overline{A}_i - \hat{L}_i^{-}C_i\underline{A}_i - J_1\underline{A}_i + \hat{G}_i & 0 & \hat{\mathscr{G}}_i
\end{bmatrix}$$

$$+\sum_{j=1}^{M}\pi_{ij}p_j^{\mathrm{T}}+\mathbf{1}^{\mathrm{T}}[C_{zi}\quad \check{E}_i\quad \hat{E}_i]\prec 0, \tag{60}$$

with $\check{L}_i^{+}=\check{P}_i^{-1}\check{Z}_i^{+}$, $\check{L}_i^{-}=\check{P}_i^{-1}\check{Z}_i^{-}$, $\hat{L}_i^{+}=\hat{P}_i^{-1}\hat{Z}_i^{+}$, $\hat{L}_i^{-}=\hat{P}_i^{-1}\hat{Z}_i^{-}$; which indicates that

$$p_i^{\mathrm{T}}\mathcal{A}_i+\sum_{j=1}^{N}\pi_{ij}p_j^{\mathrm{T}}+\mathbf{1}^{\mathrm{T}}\mathcal{C}_i\prec 0, \tag{61}$$

when $\underline{A}_i\preceq A_i\preceq \overline{A}_i$. Together with the condition in (50), we can conclude that system (38) is positive, mean stable and satisfies $\|z\|_{L_1}\leq\gamma\|\boldsymbol{\omega}\|_{L_1}$ based on Definition 1 and Lemma 3. □

Remark 4. *In order to guarantee positivity of the system (1) under the observer-based fault-tolerant controller, conditions (46)–(48) are presented, which are obtained based on the inequalities in (57) and (58). It is noted that the conditions (46)–(48) introduce conservatism in view of the difference between $\psi_{i,j}$ and $\psi_{i,j}^{\triangledown}$, as well as that between $\psi_{i,j}$ and $\psi_{i,j}^{\triangle}$. Therefore, one may reduce the conservatism by designing the invertible matrix Q_i such that the difference between $\psi_{i,j}^{\triangledown}$ and $\psi_{i,j}^{\triangle}$, $\forall i\in\mathscr{N}$, $j=1,2,\ldots,n_u$, is reduced as much as possible.*

For a positive Markov jump system (1) in which B_i has no zero row, Theorem 2 provides a computationally efficient solution to the interval observer-based fault-tolerant controller design problem. It is noted that if B_i has a zero row, it is impossible to find a matrix Q_i such that B_iQ_i is strictly positive, and the inequality in (57) will not hold. It thus fails to guarantee the specific structures of the matrices (that is, $A_i+B_i(\check{K}_i+\hat{K}_i)$ should be Metzler while $-B_i(\check{K}_iJ_1^{\mathrm{T}}+K_{fi}J_3^{\mathrm{T}})$ and $B_i(\hat{K}_iJ_1^{\mathrm{T}}+(K_{fi}-I)J_3^{\mathrm{T}})$ should be nonnegative) by exploiting conditions in (46)–(48).

In the case that $B_{\tilde{i}}\neq 0$ but $B_{\tilde{i}}[\tilde{r},1]=B_{\tilde{i}}[\tilde{r},2]=\ldots=B_{\tilde{i}}[\tilde{r},n_u]=0$, for some $\tilde{i}\in\mathscr{N}$, $\tilde{r}\in\{1,2,\ldots,n_x\}$, there always exists an invertible matrix $Q_{\tilde{i}}\in\mathbb{R}^{n_u\times n_u}$, such that

$$\varPhi_{\tilde{i}}\triangleq B_{\tilde{i}}Q_{\tilde{i}} \tag{62}$$

satisfying $\varPhi_{\tilde{i}}[r,c]>0$, $\forall r\neq\tilde{r}$. Denote $\phi_{\tilde{i},j}\in\mathbb{R}_{+}^{n_x}$, $j=1,2,\ldots,n_u$, as the jth column of $\varPhi_{\tilde{i}}$. Then the following proposition can be obtained based on Theorem 2.

Proposition 1. *For positive Markov jump system (1) such that (62) holds for $i=\tilde{i}$ and (42) holds for $i\in\mathscr{N}$, $i\neq\tilde{i}$, by utilizing an interval observer composed of (7) and (8), a fault-tolerant controller given by (34) exists such that for any $A_i\in[\underline{A}_i,\overline{A}_i]$, the augmented system (38) is positive, mean stable and satisfies $\|z\|_{L_1}\leq\gamma\|\boldsymbol{\omega}\|_{L_1}$ with a given $\gamma>0$, if there exist diagonal matrices $\check{P}_i$, $\hat{P}_i\in\mathbb{R}_{+}^{n_e\times n_e}$ with strictly positive diagonal elements, nonnegative matrices $\check{Z}_i^{+}$, $\check{Z}_i^{-}$, $\hat{Z}_i^{+}$, $\hat{Z}_i^{-}\in\mathbb{R}_{+}^{n_e\times n_y}$, matrices $\check{V}_i$, $\hat{V}_i\in\mathbb{R}^{n_e\times n_x}$, strictly positive vectors $p_{xi}\in\mathbb{R}_{++}^{nx}$, nonnegative vectors $\check{u}_{i,j}^{+}$, $\check{u}_{i,j}^{-}$, $\hat{u}_{i,j}^{+}$, $\hat{u}_{i,j}^{-}\in\mathbb{R}_{+}^{n_x}$, and $u_{fi,j}^{+}$, $u_{fi,j}^{-}\in\mathbb{R}_{+}^{n_u}$, for $i=1,2,\ldots,N$, $j=1,2,\ldots,n_u$, such that conditions in (43)–(50) are satisfied for $i\neq\tilde{i}$; for $i=\tilde{i}$, conditions in (43)–(45), (49)–(50) are satisfied, and the following conditions are satisfied:*

$$p_{x\tilde{i}}^{\mathrm{T}}\rho(\tilde{r})\underline{A}_{\tilde{i}}+\sum_{j=1}^{n_u}\phi_{\tilde{i},j}\left[\frac{(\check{u}_{\tilde{i},j}^{+}+\hat{u}_{\tilde{i},j}^{+})^{\mathrm{T}}}{\phi_{\tilde{i},j}^{\triangle}}-\frac{(\check{u}_{\tilde{i},j}^{-}+\hat{u}_{\tilde{i},j}^{-})^{\mathrm{T}}}{\phi_{\tilde{i},j,\tilde{r}}^{\triangledown}}\right]\quad\text{is Metzler,} \tag{63}$$

$$-\sum_{j=1}^{n_u}\phi_{\tilde{i},j}\left(\frac{\check{u}_{\tilde{i},j}^{+\mathrm{T}}}{\phi_{\tilde{i},j,\tilde{r}}^{\triangledown}}-\frac{\check{u}_{\tilde{i},j}^{-\mathrm{T}}}{\phi_{\tilde{i},j}^{\triangle}}\right)J_1^{\mathrm{T}}+\sum_{j=1}^{n_u}\phi_{\tilde{i},j}\left(\frac{u_{f\tilde{i},j}^{+\mathrm{T}}}{\phi_{\tilde{i},j}^{\triangle}}-\frac{u_{f\tilde{i},j}^{-\mathrm{T}}}{\phi_{\tilde{i},j,\tilde{r}}^{\triangledown}}\right)J_3^{\mathrm{T}}\succeq 0, \tag{64}$$

$$\sum_{j=1}^{n_u}\phi_{\tilde{i},j}\left(\frac{\hat{u}_{\tilde{i},j}^{+\mathrm{T}}}{\phi_{\tilde{i},j}^{\triangle}}-\frac{\hat{u}_{\tilde{i},j}^{-\mathrm{T}}}{\phi_{\tilde{i},j,\tilde{r}}^{\triangledown}}\right)J_1^{\mathrm{T}}+\sum_{j=1}^{n_u}\phi_{\tilde{i},j}\left(\frac{u_{f\tilde{i},j}^{+\mathrm{T}}}{\phi_{\tilde{i},j}^{\triangle}}-\frac{u_{f\tilde{i},j}^{-\mathrm{T}}}{\phi_{\tilde{i},j,\tilde{r}}^{\triangledown}}\right)J_3^{\mathrm{T}}-p_{x\tilde{i}}^{\mathrm{T}}\rho(\tilde{r})_{n_x}B_{\tilde{i}}J_3^{\mathrm{T}}\succeq 0, \tag{65}$$

where $\rho(\tilde{r})\in\mathbb{R}^{n_x}$ has its $\tilde{r}$-th element zero and the other elements one; $\phi_{\tilde{i},j}^{\triangle}=\|\phi_{\tilde{i},j}\|_\infty$, $\phi_{\tilde{i},j,\tilde{r}}^{\triangledown}=\phi_{\tilde{i},j\,\mathrm{min}}^{+}$. The observer matrices and controller matrix are given as in Theorem 2.

The proof of Proposition 1 is similar to that of Theorem 2 and is omitted here.

4. Illustrative Example

Consider a third-order positive Markov jump linear system of form (1) with $\sigma(t)\in\mathscr{N}=\{1,2\}$ and the following system matrices:

$$A_1=\begin{bmatrix}-2.5\pm0.2 & 0 & 1.5\\ 0 & -3.3\pm0.1 & 1.2\pm0.2\\ 0.7 & 1.1\pm0.05 & -4\pm0.15\end{bmatrix},\quad B_1=\begin{bmatrix}1.2 & 0.8\\ 0.5 & 0\\ 0 & 1\end{bmatrix},$$

$$A_2=\begin{bmatrix}-3.8\pm0.12 & 0.2 & 0.4\pm0.11\\ 0.5\pm0.2 & -3.95\pm0.2 & 0.3\pm0.02\\ 0.35 & 0.55 & -4\pm0.9\end{bmatrix},\quad B_2=\begin{bmatrix}1 & 2\\ 1.5 & 0.5\\ 0 & 0\end{bmatrix},$$

$$B_{w1}=B_{w2}=\begin{bmatrix}0.1 & 0.1 & 0.1\end{bmatrix}^{\mathrm{T}},\quad C_1=\begin{bmatrix}0.9 & 0.8 & 1.3\end{bmatrix},\quad C_2=\begin{bmatrix}0.6 & 1.2 & 1\end{bmatrix}.$$

The generator matrix is given as

$$\Pi=\begin{bmatrix}-3.9 & 3.9\\ 1.4 & -1.4\end{bmatrix}.$$

The reference output is given with $C_{z1}=C_1$, $C_{z2}=C_2$, and $\check{E}_1=\check{E}_2=\begin{bmatrix}1 & 0.8 & 0.9 & 0.8 & 1\end{bmatrix}$, $\hat{E}_1=\hat{E}_2=\begin{bmatrix}0.5 & 0.8 & 0.6 & 0.6 & 0.8\end{bmatrix}$. Initial conditions are set as $x(0)=\begin{bmatrix}1.3 & 1.5 & 2\end{bmatrix}$ and $w(t)=1.5e^{-\frac{1}{2}}|\sin(0.5\pi(t-0.5))|$.

The actuator fault is unknown in controller design but its first-order derivative $f^{(1)}$ is supposed to be bounded by $[\underline{\delta}(t),\overline{\delta}(t)]$, with $\underline{\delta}(t)=[-2e^{-\frac{t}{6}},0.01e^{-\frac{t}{2}}]^{\mathrm{T}}$, $\overline{\delta}(t)=[2e^{-\frac{t}{6}},0.1e^{-\frac{t}{6}}]^{\mathrm{T}}$. In other words, the designed interval observer-based FTC scheme could handle a large class of faults whose rate of change is within $[\underline{\delta}(t),\overline{\delta}(t)]$. Suppose that

$$f(t)=\begin{bmatrix}0.32-0.04e^{-\frac{t}{4}}(\sin 2t+8\cos 2t)\\ 0.2-0.2e^{-\frac{t}{5}}\end{bmatrix}.$$

Obviously,

$$\begin{bmatrix}-2e^{-\frac{t}{6}}\\ 0.01e^{-\frac{t}{2}}\end{bmatrix}\preceq f^{(1)}(t)=\begin{bmatrix}0.65e^{-\frac{t}{4}}\sin 2t\\ 0.04e^{-\frac{t}{5}}\end{bmatrix}\preceq\begin{bmatrix}2e^{-\frac{t}{6}}\\ 0.1e^{-\frac{t}{6}}\end{bmatrix}.$$

According to Remark 4, Q_1 is selected to reduce the difference between $\psi_{1,j}^{\triangle}$ and $\psi_{1,j}^{\triangledown}$, $j = 1, 2$, as much as possible. A possible solution is to parametrize Q_1 as

$$Q_1 = \begin{bmatrix} q_{1,1} & q_{1,2} \end{bmatrix} + \epsilon I,$$

where $q_{1,1}$, $q_{1,2} \in \mathbb{R}^2$, and $\epsilon > 0$. It is noted that for any matrix $[q_{1,1}, q_{1,2}]$, there always exists a sufficiently small constant $\epsilon > 0$ such that Q_1 is invertible. Based on the parametrization, Q_1 is designed by solving the following optimization problems:

$$\begin{aligned} &\min_{q_{1,1}} \quad (\|B_1 q_{1,1}\|_\infty - (B_1 q_{1,1})_{\min}) \quad \text{s.t.} \quad (B_1 q_{1,1})_{\min} \geq (b_{1,1})_{\min}^+, \\ &\min_{q_{1,2}} \quad (\|B_1 q_{1,2}\|_\infty - (B_1 q_{1,2})_{\min}) \quad \text{s.t.} \quad (B_1 q_{1,2})_{\min} \geq (b_{1,2})_{\min}^+, \end{aligned}$$

where $b_{1,1}, b_{1,2} \in \mathbb{R}^3$ are the first and second columns of B_1, respectively. One may consider the selection of Q_2 in a similar way. However, since there is a zero row in B_2, Q_2 is designed by solving the problems as follows:

$$\begin{aligned} &\min_{q_{2,1}} \quad \left(\|\tilde{B}_2 q_{2,1}\|_\infty - (\tilde{B}_2 q_{2,1})_{\min}\right) \quad \text{s.t.} \quad (\tilde{B}_2 q_{2,1})_{\min} \geq (b_{2,1})_{\min}^+, \\ &\min_{q_{2,2}} \quad \left(\|\tilde{B}_2 q_{2,2}\|_\infty - (\tilde{B}_2 q_{2,2})_{\min}\right) \quad \text{s.t.} \quad (\tilde{B}_2 q_{2,2})_{\min} \geq (b_{2,2})_{\min}^+, \end{aligned}$$

where $\tilde{B}_2 = \begin{bmatrix} 1 & 2 \\ 1.5 & 0.5 \end{bmatrix}$, $b_{2,1}, b_{2,2} \in \mathbb{R}^3$ are the first and second columns of $\tilde{B}_2$, respectively. $Q_2 = [q_{2,1} \quad q_{2,2}] + \epsilon I$. One then obtains

$$Q_1 = \begin{bmatrix} 1 & 1.6 \\ 0.5 & 0.8 \end{bmatrix} + 0.001I, \quad Q_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 0.1 \end{bmatrix} + 0.001I.$$

For a prescribed $\gamma = 0.82$, an interval observer composed of (7) and (8) is then constructed based on Theorem 2 and Proposition 1, with observer matrices given by

$$\check{G}_1 = \begin{bmatrix} -2.7 & 0 & 1.5 \\ 0 & -3.4 & 1 \\ 0.7 & 1.05 & -4.15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \check{L}_1 = \begin{bmatrix} 0 \\ -0.0593 \\ -0.0044 \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{G}_1 = \begin{bmatrix} -2.3 & 0 & 1.5 \\ 0 & -3.2 & 1.4 \\ 0.7 & 1.15 & -3.85 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \hat{L}_1 = \begin{bmatrix} 0 \\ -0.1351 \\ -0.4095 \\ 0 \\ 0 \end{bmatrix},$$

$$\check{G}_2 = \begin{bmatrix} -1.9008 & 0 & 0.9337 \\ 0 & -0.0004 & 0 \\ 0 & 0.3203 & -0.0005 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \check{L}_2 = \begin{bmatrix} -0.0464 \\ -0.0638 \\ -0.1258 \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{G}_2 = \begin{bmatrix} -3.7332 & 0 & 0.3842 \\ 0.5648 & -4.2585 & 0 \\ 0.2038 & 0 & -3.4461 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \hat{L}_2 = \begin{bmatrix} -0.0522 \\ -0.1328 \\ -0.1336 \\ 0 \\ 0 \end{bmatrix}.$$

The fault-tolerant controller in (34) has the following parameters:

$$\check{K}_1 = \begin{bmatrix} -0.0021 & -0.0074 & -0.4251 \\ 0 & -0.0109 & -0.2125 \end{bmatrix}, \hat{K}_1 = \begin{bmatrix} 0.7257 & 0.7219 & 0.1397 \\ 0.3629 & 0.3682 & 0.0699 \end{bmatrix} \times 10^{-3},$$

$$\check{K}_2 = \begin{bmatrix} -0.0071 & 0.0018 & -0.0718 \\ 0 & -0.0076 & -0.0023 \end{bmatrix}, \hat{K}_2 = \begin{bmatrix} -0.1794 & -0.12 & -0.1674 \\ -0.0604 & -0.04 & -0.0564 \end{bmatrix}.$$

As for $\check{K}_{f1}$, $\check{K}_{f2}$, $\hat{K}_{f1}$, and $\hat{K}_{f2}$, we have K_{f1} and K_{f2} as follows:

$$K_{f1} = \begin{bmatrix} 1.5330 & 2.8411 \\ 0.7665 & 1.4490 \end{bmatrix}, \quad K_{f2} = \begin{bmatrix} 0.0217 & 1.2914 \\ 0.0077 & 0.0018 \end{bmatrix}.$$

One can compute

$$\check{K}_{f1} = -B_1^{\dagger} B_1 K_{f1} = \begin{bmatrix} -1.5330 & -2.8411 \\ -0.7665 & -1.4490 \end{bmatrix}, \hat{K}_{f1} = B_1^{\dagger} B_1 (K_{f1} - I) = \begin{bmatrix} 0.5330 & 2.8411 \\ 0.7665 & 0.4490 \end{bmatrix},$$

$$\check{K}_{f2} = -B_2^{\dagger} B_2 K_{f2} = \begin{bmatrix} -0.0217 & -1.2914 \\ -0.0077 & -0.0018 \end{bmatrix}, \hat{K}_{f2} = B_2^{\dagger} B_2 (K_{f2} - I) = \begin{bmatrix} -0.9783 & 1.2914 \\ 0.0077 & -0.9982 \end{bmatrix}.$$

The simulation of the mode sequence is shown in Figure 1. Based on the designed controller, the system state components x_1, x_2, and x_3 are shown in Figures 2, 3, and 4, respectively. Corresponding interval estimates obtained from the interval observer are also provided in the figures. Figure 5 shows the time-varying actuator fault $f(t)$ and its estimates. It can be seen that As indicated by the figures, the system state and actuator fault can be encapsulated at all times and the convergence of estimation errors is achieved with the obtained interval observer. The designed fault-tolerant controller is also proved to be capable of stabilizing the system under actuator faults.

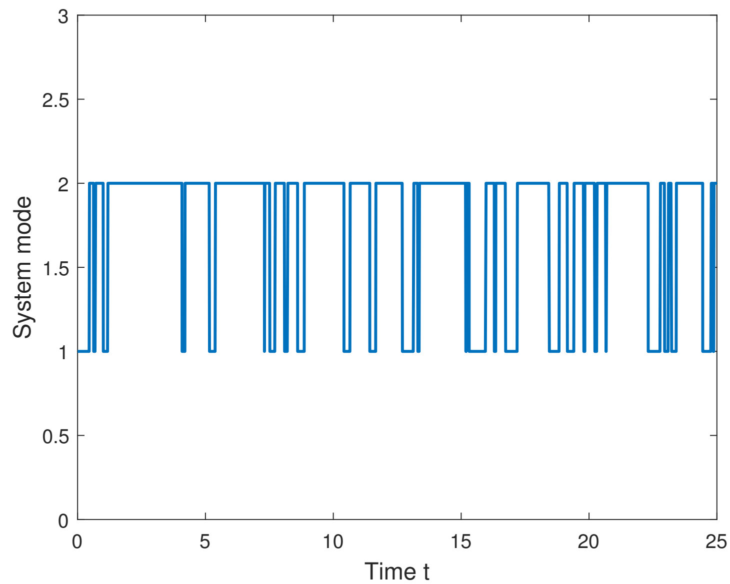

Figure 1: Markov jump process

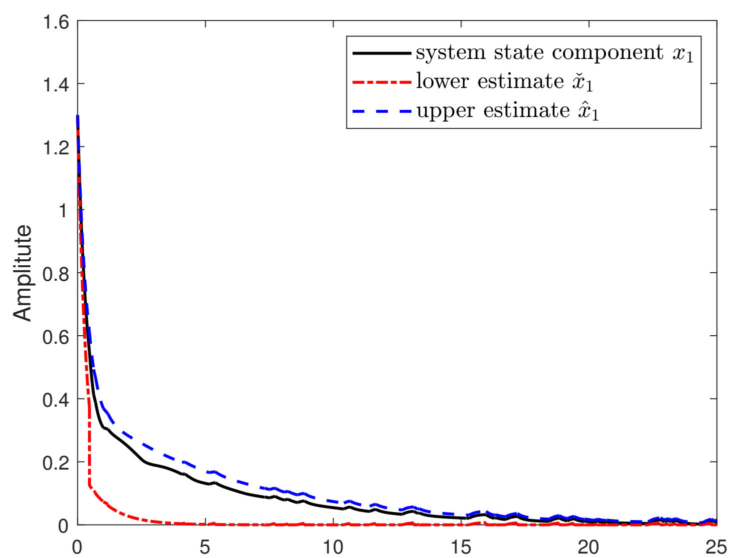

Figure 2: Trajectory of x_1 with interval estimates

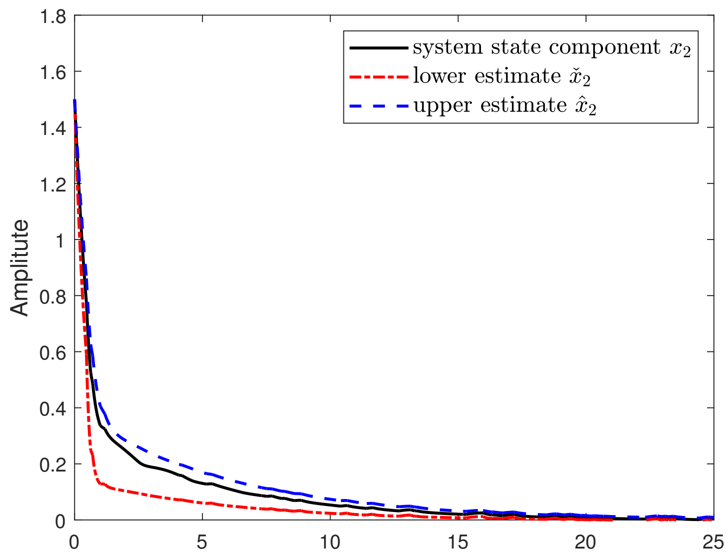

Figure 3: Trajectory of x_2 with interval estimates

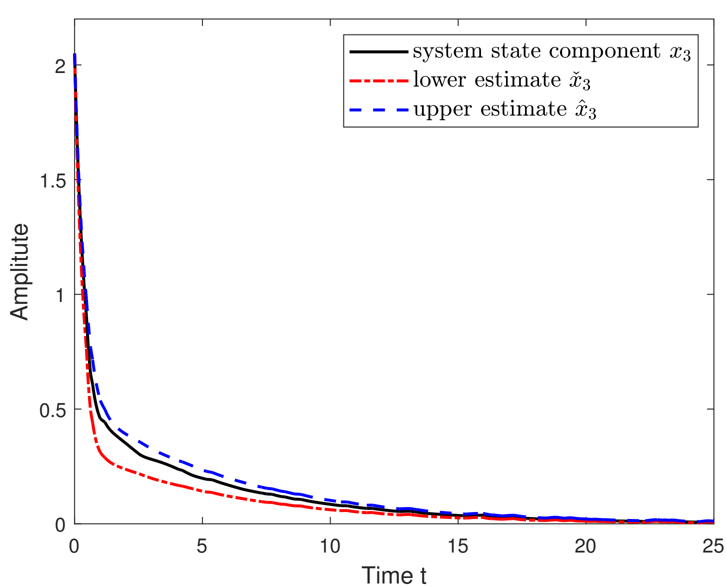

Figure 4: Trajectory of x_3 with interval estimates

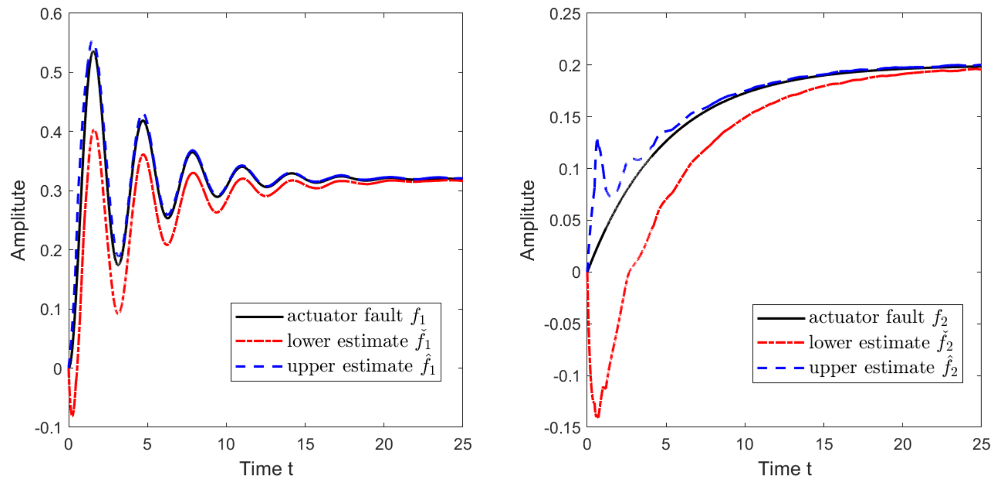

Figure 5: Trajectory of actuator fault with interval estimates

5. Conclusion

In this work, positive Markov jump systems subject to interval uncertainties and actuator faults have been considered. An interval observer design approach has been developed to provide guaranteed intervals for the system state and faults. Based on the interval observer framework, a linear programming approach has been proposed for the computation of the observer and controller to guarantee the stochastic stability and disturbance attenuation through L_1-gain performance. The effectiveness of the observer-based fault-tolerant scheme has been illustrated via a numerical example. For future developments, the influence of the observer gains and controller gains on the closed-loop performance will be further analyzed. New extensions will also be carried out on positive multi-agent systems subject to component faults by using the idea of interval observers.

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