Synchronization of Lur'e Networks via Heterogeneous Unknown Interconnections

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***Abstract*—This paper deals with synchronization of Lur'e networks diffusively coupled through heterogeneous unknown relative-state interconnections over connected undirected graphs. While the node nonlinearities are identical, unknown and incrementally sector-bounded, the edge nonlinearities are different, unknown and sector-bounded. These Lur'e nonlinearities can be regarded as model uncertainties or interconnection constraints. The S-Lemma and LMI techniques are employed following the absolute stability theory of Lur'e systems. In contrast to the case of homogeneous linear relative-state interconnections, here the obtained synchronization criterion is given by two isotypic LMIs, which involve the smallest and respectively the largest nonzero Laplacian eigenvalues. Finally, numerical simulations are presented to illustrate the effectiveness of the theoretical results.**

***Index Terms*—Lur'e network, synchronization, heterogeneous unknown interconnection, (incremental) sector-boundedness, LMI.**

I. Introduction

Synchronization of Lur'e networks has been extensively studied in the related fields, see [1]–[8] for instance. It draws much attention due to the fact that Lur'e systems can represent chaotic circuits, biochemical oscillators, and flexible joint robotic manipulators etc [9], and synchronization of Lur'e networks can be applied in secure communication [10], image encryption [11], and power grids [12], [13].

In [1], synchronization of connected undirected Lur'e networks via static linear relative-state interconnections was studied. Through dynamic linear relative-output interconnections, synchronization of undirected and directed Lur'e networks was studied successively in [2], [5]. These works handle *unknown input/output coupled MIMO* Lur'e nonlinearities at nodes under incremental sector-boundedness and require *no leader*. In other results on synchronization of Lur'e networks, the Lur'e node nonlinearities are SISO or input/output decoupled MIMO under slope-restrictedness and usually a leader is required, see for example [6], [14]. It should be noted that, slope-restrictedness is just a special case of incremental sector-boundedness. To the best of our knowledge, the latter is the most general assumption on the node nonlinearities in the context of Lur'e networks. Hence, in the study of Lur'e networks, it is natural to move forward along the works [1], [2], [5].

Besides linear interconnections, nonlinear ones were also studied in Lur'e networks. There are two types of nonlinear interconnections. One uses linear relative information feedback to design nonlinear interconnections for synchronization, see e.g. [7], [14]. The other has to confront nonlinear relative information feedback and to design suitable nonlinear interconnections, see e.g. [15], [16]. For example, a second type nonlinear interconnection was considered in [4], where a leader was employed to synchronize the Lur'e network subject to hybrid impulses. Certainly, there are also works involving both types of nonlinear interconnections, see e.g. [17]–[19]. Note that, the nonlinear interconnections therein have the passivity-like property.

Since the first type nonlinear interconnection has been mostly studied in Lur'e networks, the second type will be of interest in this paper. In fact, nonlinear interconnections via saturated feedback [20] or quantized feedback [8] belong to the second type. In addition, the sector-boundedness approach is often exploited to handle saturation and quantization in control system design. This inspires us to consider sector-bounded interconnections for synchronization of Lur'e networks, which can lead to a protocol suitable for more types of nonlinear interconnections.

Synchronization of dynamic networks via sector-bounded nonlinear interconnections has been explored to some extent. Output synchronization of a network of SISO linear systems was studied via slope-restricted nonlinear relative-output interconnections in [21], [22], where the absolute stability theory in the frequency domain was utilized. The same author considered delayed slope-restricted interconnections for single-integrator node dynamics in [23]. Also for single-integrator node dynamics, a kind of odd and increasing nonlinear interconnection works, which is in fact slope-restricted [24]. In [25], [26], the fully distributed synchronization problem of linear networks was studied through slope-restricted interconnections. However, the nonlinear interconnection function is used in the synchronization protocol therein and must be

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known exactly, which is not required in this paper. A network of stable LTI node dynamics was considered in [27], where the nonlinear interconnection function is sector-bounded but input/output decoupled.

In this paper, the node dynamics is described by the Lur'e system with an *incrementally sector-bounded unknown* nonlinearity, and the interconnection nonlinearities are *heterogeneous but also unknown and only required to be sector-bounded*. All the nonlinearities are not required to be input/output decoupled either. Here, these nonlinearities in fact denote all those within the corresponding sectors and cannot be approximated by for example neural networks. Moreover, it turns out that the systems matrix of the node dynamics is not required to be stable. Third, no leader is needed. Different from the static linear relative-state interconnection case [1], here two isotypic LMI synchronization conditions are obtained, which involve the smallest and respectively the largest nonzero Laplacian eigenvalues. This is similar to the dynamic linear relative-output interconnection case [2].

The reminder of this paper is organized as follows. Together with some preliminaries, the problem of interest is formulated in Section II. Section III presents a sector-bounded synchronization protocol design for connected undirected Lur'e networks. Its Lyapunov analysis is postponed to the Appendix. A numerical simulation example is given in Section IV to illustrate our obtained theoretical results. Finally, Section V closes this paper along with some concluding remarks.

II. Problem Formulation

Throughout this paper, we use standard notations. $\mathbb{R}$ denotes the field of real numbers. $\mathbb{R}^m$ denotes the set of $m\times 1$ real column vectors. $\mathbb{R}^{m\times n}$ denotes the space of $m\times n$ real matrices. $(\cdot)^T$ denotes the transpose of a real matrix. The Kronecker product of two matrices M_1 and M_2 is denoted by $M_1\otimes M_2$. $\mathrm{diag}(m_1,\cdots,m_d)$ denotes a block diagonal matrix with elements m_b, $b=1,\cdots,d$, on its diagonal. I_N denotes the $N\times N$ identity matrix. $\mathbf{0}$ denotes a zero matrix of compatible dimension. We define that $\mathbf{1}_N\triangleq[1,1,\cdots,1]^T_{1\times N}$.

Let $\mathcal{G}$ be a simple, connected undirected graph with N (≥ 2) nodes to represent the network topology. The eigenvalues of its Laplacian matrix $\mathcal{L}$ can be ordered as $\lambda_1=0<\lambda_2\leq\cdots\leq\lambda_N$. Moreover, there exists an orthogonal matrix $\mathcal{U}=\left[\frac{1}{\sqrt{N}}\mathbf{1}_N,\mathcal{U}_2\right]$, where $\mathcal{U}_2\in\mathbb{R}^{N\times(N-1)}$, such that $\mathcal{U}^T\mathcal{L}\mathcal{U}=\mathrm{diag}(0,\lambda_2,\cdots,\lambda_N)$. It is easily checked that $\mathcal{U}_2^T\mathcal{U}_2=I_{N-1}$ and $\mathcal{U}_2\mathcal{U}_2^T=I_N-\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$.

In this paper, we consider a network of N heterogeneous Lur'e systems described by

$$\dot{x}_i=Ax_i-Ef(Cx_i)+B\psi_i(u_i),\quad i=1,2,\cdots,N, \tag{1}$$

where $x_i(t)\in\mathbb{R}^n$ and $u_i(t)\in\mathbb{R}^m$ are the node state required to reach synchronization with neighbors and respectively the diffusive coupling control input to be designed. The system matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{s\times n}$, $E\in\mathbb{R}^{n\times s}$ are constant and given. The function $f(\cdot):\mathbb{R}^s\mapsto\mathbb{R}^s$ represents an *unknown* nonlinearity that is assumed to be incrementally sector-bounded within a *known* sector:

Definition 1: Let $\underline{S}_n,\bar{S}_n\in\mathbb{R}^{s\times s}$ be real symmetric matrices and $\bar{S}_n-\underline{S}_n$ be positive definite. Then, $f(\cdot)$ is called *incrementally sector-bounded* within the sector $[\underline{S}_n,\bar{S}_n]$ if it satisfies

$$\left[f(\rho)-f(\varrho)-\underline{S}_n(\rho-\varrho)\right]^T\left[f(\rho)-f(\varrho)-\bar{S}_n(\rho-\varrho)\right]\leq 0 \tag{2}$$

for all $\rho(t),\varrho(t)\in\mathbb{R}^s$.

The function $\psi_i(\cdot):\mathbb{R}^m\mapsto\mathbb{R}^m$ is also an *unknown* nonlinearity and assumed to be sector-bounded within a *known* sector:

Definition 2: Let $\underline{S}_e,\bar{S}_e\in\mathbb{R}^{m\times m}$ be real symmetric matrices and $\bar{S}_e-\underline{S}_e$ be positive definite. Then, $\psi_i(\cdot)$ is called *sector-bounded* within the sector $[\underline{S}_e,\bar{S}_e]$ if it satisfies

$$\left(\psi_i(\nu)-\underline{S}_e\nu\right)^T\left(\psi_i(\nu)-\bar{S}_e\nu\right)\leq 0 \tag{3}$$

for all $\nu(t)\in\mathbb{R}^m$.

Through local relative-state interconnections, the diffusive coupling control input is given by

$$u_i=K\sum_{j=1}^N a_{ij}(x_i-x_j), \tag{4}$$

where $[a_{ij}]_{N\times N}$ is the adjacency matrix associated with the graph $\mathcal{G}$, $K\in\mathbb{R}^{m\times n}$ is the coupling gain matrix to be designed such that $x_i(t)-x_j(t)\to\mathbf{0}$ as $t\to\infty$, $i=1,2,\cdots,N$, $j\neq i$, for any initial states, any $f(\cdot)$ within the sector $[\underline{S}_n,\bar{S}_n]$, and all $\psi_i(\cdot)$'s within the sector $[\underline{S}_e,\bar{S}_e]$, namely the Lur'e network (1) with (4) reaches synchronization.

III. Main Results

In this section, we present our main results.

Theorem 1: Consider a connected undirected graph $\mathcal{G}$. If there exists a positive definite matrix $Q\in\mathbb{R}^{n\times n}$, a matrix $H\in\mathbb{R}^{m\times n}$ and two positive real numbers α, β such that the following two LMIs

$$\left[\begin{array}{c|c|c} \begin{array}{c}\left[A-\frac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]Q+Q\left[A-\frac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]^T+\frac{\alpha}{2}EE^T\\ +\frac{1}{2}\lambda_i\left[B\left(\underline{S}_e+\bar{S}_e\right)H+H^T\left(\underline{S}_e+\bar{S}_e\right)B^T\right]+\frac{\beta}{2}BB^T\end{array} & QC^T & \lambda_i H^T \\ \hline CQ & -2\alpha\left(\bar{S}_n-\underline{S}_n\right)^{-2} & \mathbf{0} \\ \hline \lambda_i H & \mathbf{0} & -2\beta\left(\bar{S}_e-\underline{S}_e\right)^{-2} \end{array}\right]<\mathbf{0},\quad i=2,N \tag{5}$$

hold, then the Lur'e network (1) with (4) achieves synchronization, where $K:=HQ^{-1}$.

The proof is postponed to the Appendix.

Remark 1: Note that, the two LMIs (5) involve the smallest and respectively the largest nonzero Laplacian eigenvalues, similarly to some synchronization results for example [27]. In fact, the requirement of the smallest and the largest nonzero Laplacian eigenvalues is equivalent to that of only the smallest one. Both of them need to know the Laplacian matrix. In addition, the coupling gain matrix K is computed offline. Then the implementation is no more complex than the synchronization protocol for Lur'e networks via linear relative-state interconnections, see [1]. Certainly, the feasibility decreases.

Remark 2: By using the S-Lemma, (11) in the Appendix is equivalent to

$$\left[\begin{array}{c|c|c} \begin{array}{c}\left[A-\frac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C+\frac{\lambda_i}{2}B\left(\underline{S}_e+\bar{S}_e\right)K\right]Q+\frac{\beta}{2}BB^T+\\ Q\left[A-\frac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C+\frac{\lambda_i}{2}B\left(\underline{S}_e+\bar{S}_e\right)K\right]^T+\frac{\alpha}{2}EE^T\end{array} & \lambda_i QK^T & QC^T \\ \hline \lambda_i KQ & -2\beta\left(\bar{S}_e-\underline{S}_e\right)^{-2} & \mathbf{0} \\ \hline CQ & \mathbf{0} & -2\alpha\left(\bar{S}_n-\underline{S}_n\right)^{-2} \end{array}\right]<\mathbf{0}.$$

It follows that, the two LMIs (5) have feasible solutions *only if* the pair $(A-\frac{1}{2}E(\underline{S}_n+\bar{S}_n)C,\frac{1}{2}B(\underline{S}_e+\bar{S}_e))$ is stabilizable, rather than the pair (A,B). This is exactly the reason that A is not required to be stable.

Remark 3: Using the quadratic Lyapunov function and LMI techniques, Theorem 1 has conservatism, which is inevitable. In the absolute stability theory, the Lur'e-type Lyapunov function together with for example Zames-Falb multipliers is often used to reduce the conservatism. So it is possible to reduce the conservatism of Theorem 1 as well. This is an interesting problem for the future research.

If the node dynamics is described by the LTI system

$$\dot{x}_i=Ax_i+Bu_i,\quad i=1,2,\cdots,N, \tag{6}$$

then we have the following corollary:

Corollary 1: Consider a connected undirected graph $\mathcal{G}$. If there exists a positive definite matrix $Q\in\mathbb{R}^{n\times n}$, a matrix $H\in\mathbb{R}^{m\times n}$ and a positive real number β such that the following two LMIs

$$\begin{bmatrix} AQ+QA^T+\frac{\beta}{2}BB^T+\frac{1}{2}\lambda_i\left[B\left(\underline{S}_e+\bar{S}_e\right)H+H^T\left(\underline{S}_e+\bar{S}_e\right)B^T\right] & \lambda_i H^T \\ \lambda_i H & -2\beta\left(\bar{S}_e-\underline{S}_e\right)^{-2} \end{bmatrix}<\mathbf{0},\ i=2,N \tag{7}$$

hold, then the linear network (6) with (4) achieves synchronization, where $K:=HQ^{-1}$.

Proof. The proof can be performed similarly to that of Theorem 1 and omitted here. □

IV. Numerical Simulations

In this section, a given Lur'e network interconnected through specific heterogeneous sector-bounded nonlinearities will be studied over an unweighted undirected circle of 5 nodes. Its smallest and largest nonzero Laplacian eigenvalues are computed to be $\lambda_2=1.382$ and respectively $\lambda_5=3.618$.

Here, the node system matrices are randomly generated in MATLAB and given by $A=[-0.8320,1.5686,1.2797;-2.0802,-0.6835,-4.8451;4.8406,-3.3283,-3.9378]$, $B=[-1.2759;-3.0188;-0.1031]$, $C=[-1.6051,4.5163,4.2033]$, $E=[-4.4732;2.3786;-2.3088]$. The node nonlinearity is chosen to be $f(y)=|0.5y+1|-|0.5y-1|$, which is a saturation function and lies in the sector $[0,1]$, see Fig. 1. The interconnection nonlinearities are chosen

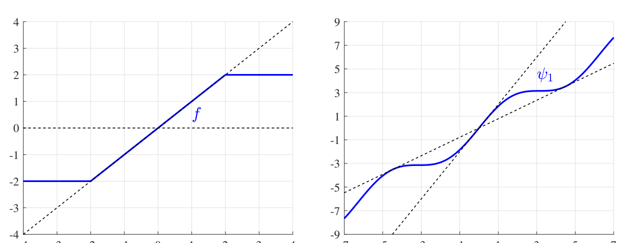

Fig. 1. Plots of $f(\cdot)$ and $\psi_1(\cdot)$

to be $\psi_i(u)=u+\frac{1}{i}\sin u$, $i=1,2,3,4,5$, which lie respectively in the sectors $[0.7828,2]$, $[0.8914,1.5]$, $[0.9276,1.3333]$, $[0.9457,1.25]$, $[0.9566,1.2]$, see the plot of ψ_1 in Fig. 1. We take the largest sector $[0.7828,2]$ to cover all these interconnection nonlinearities. So the assumption on the same sector makes sense. It is easily checked that the pair $(A-\frac{1}{2}E(\underline{S}_n+\bar{S}_n)C,\frac{1}{2}B(\underline{S}_e+\bar{S}_e))$ is stabilizable, where $\underline{S}_n=0$, $\bar{S}_n=1$, $\underline{S}_e=0.7828$, $\bar{S}_e=2$.

Then, by using the LMI Control Toolbox in MATLAB, we can compute feasible solutions to the two LMIs (5): $\alpha=0.0181$, $\beta=0.2874$, $H=[0.074,0.234,0.0673]$, $Q=[0.0828,0.1091,-0.1058;0.1091,0.2121,-0.1578;-0.1058,-0.1578,0.1436]$. Hence the coupling gain matrix can be computed to be $K:=HQ^{-1}=[51.7757,17.5743,57.9442]$.

Finally, we choose the initial states randomly and plot the corresponding state trajectories of the 5 nodes in the left figure in Fig. 2. In the right figure, we plot the synchronization

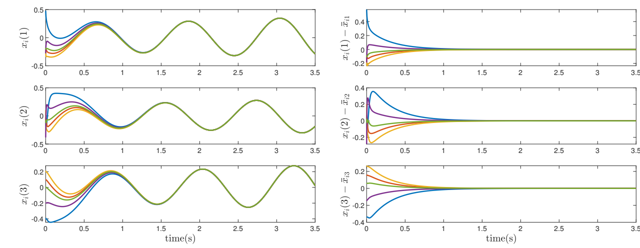

Fig. 2. State trajectories & synchronization errors

errors $x_i(k)-\bar{x}_{ik}$, where $\bar{x}_{ik}=\frac{1}{5}\sum_{i=1}^{5}x_i(k)$, $i=1,2,3,4,5$, $k=1,2,3$. Clearly, the network reaches synchronization.

V. Conclusions

We have achieved synchronization for connected undirected Lur'e networks via sector-bounded heterogeneous unknown interconnections in this paper. This work has generalized some results on synchronization in the presence of nonlinear interconnections and also leaves some interesting problems.

A possible future topic is to study synchronization of directed Lur'e networks via sector-bounded heterogeneous unknown interconnections. The relations between the asymmetric Laplacian matrix and the nonlinearities can not be decoupled easily. Another one is to study the case of sector-bounded heterogeneous unknown interconnections without full state information. Unfortunately, absolute stabilization by sector-bounded unknown partial state feedback is still an open problem and gives no clue to synchronization. Moreover, network synchronization via sector-bound heterogeneous unknown interconnections deserves more attention subject to various constraints for example time-delays and cyber-attacks.

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APPENDIX

Proof of Theorem 1. Before moving on, we define $\phi_i(u_i):=\bar{S}_e u_i-\psi_i(u_i)$, $i=1,2,\cdots,N$, which obviously satisfies the sector-boundedness condition

$$\phi_i^T(u_i)\big[\phi_i(u_i)-\big(\bar{S}_e-\underline{S}_e\big)u_i\big]\leq 0 \tag{8}$$

for all $u_i(t)\in\mathbb{R}^m$. That is, the unknown nonlinearity $\phi_i(\cdot):\mathbb{R}^m\mapsto\mathbb{R}^m$ lies in the sector $\big[\mathbf{0},\bar{S}_e-\underline{S}_e\big]$. Hence the ith Lur'e system becomes $\dot{x}_i=Ax_i-Ef(Cx_i)-B\phi_i(u_i)+B\bar{S}_e u_i$, which results in the following compact form:

$$\dot{x}=(I_N\otimes A)x-(I_N\otimes E)F(x)-(I_N\otimes B)\Phi(u)+\big(I_N\otimes B\bar{S}_e\big)u,$$

where $x=\big[x_1^T,x_2^T,\cdots,x_N^T\big]^T$, $F(x)=\big[f^T(Cx_1),f^T(Cx_2),\cdots,f^T(Cx_N)\big]$, $u=\big[u_1^T,u_2^T,\cdots,u_N^T\big]^T$, $\Phi(x)=\big[\phi_1^T(u_1),\phi_2^T(u_2),\cdots,\phi_N^T(u_N)\big]^T$. Moreover, we have $u=(\mathcal{L}\otimes K)x$ and thus get that

$$\dot{x}=\big(I_N\otimes A+\mathcal{L}\otimes B\bar{S}_e K\big)x-(I_N\otimes E)F(x)-(I_N\otimes B)\Phi(u). \tag{9}$$

Let $\hat{x}=\big(\mathcal{U}^T\otimes I_n\big)x$ and $\bar{x}=\big(\mathcal{U}_2^T\otimes I_n\big)x$. Then the synchronization error dynamics is obtained as

$$\begin{aligned}\dot{\bar{x}}=&\ \big(I_{N-1}\otimes A+\bar{\Lambda}\otimes B\bar{S}_e K\big)\bar{x}-\\ &(I_{N-1}\otimes E)\underbrace{\big(\mathcal{U}_2^T\otimes I_s\big)F(x)}_{\bar{F}(x)}-(I_{N-1}\otimes B)\underbrace{\big(\mathcal{U}_2^T\otimes I_m\big)\Phi(u)}_{\bar{\Phi}(u)},\end{aligned} \tag{10}$$

where $\bar{\Lambda}:=\mathrm{diag}(\lambda_2,\cdots,\lambda_N)$. It is well known that (9) is synchronized *if and only if* (10) is stabilized [1].

Consider the Lyapunov function candidate $V(\bar{x})=\bar{x}^T(I_{N-1}\otimes P)\bar{x}$, where $P:=Q^{-1}>\mathbf{0}$, which is obviously positive definite and radially unbounded. The time derivative of $V(\bar{x})$ along the trajectories of (10) is given by

$$\begin{aligned}&\dot{V}(\bar{x})\\ =&\ 2\bar{x}^T(I_{N-1}\otimes P)\big[\big(I_{N-1}\otimes A+\bar{\Lambda}\otimes B\bar{S}_e K\big)\bar{x}-(I_{N-1}\otimes E)\bar{F}-(I_{N-1}\otimes B)\bar{\Phi}\big]\\ &\big[\,\bar{x}^T\ \bar{F}^T\ \bar{\Phi}^T\,\big]\\ =&\begin{bmatrix} I_{N-1}\otimes\big(PA+A^TP\big)+ \bar{\Lambda}\otimes\big(PB\bar{S}_eK+K^T\bar{S}_eB^TP\big) & -I_{N-1}\otimes PE & -I_{N-1}\otimes PB\\ -I_{N-1}\otimes E^TP & \mathbf{0} & \mathbf{0}\\ -I_{N-1}\otimes B^TP & \mathbf{0} & \mathbf{0}\end{bmatrix}\\ &\big[\,\bar{x}^T\ \bar{F}^T\ \bar{\Phi}^T\,\big]^T.\end{aligned}$$

Meanwhile, it follows from Lemma 4 in [1] that

$$\begin{aligned}&\big[\,\bar{x}^T\ \bar{F}^T\,\big]\\ &\begin{bmatrix} I_{N-1}\otimes C^T\big(\underline{S}_n\bar{S}_n+\bar{S}_n\underline{S}_n\big)C & -I_{N-1}\otimes C^T\big(\underline{S}_n+\bar{S}_n\big)\\ -I_{N-1}\otimes\big(\underline{S}_n+\bar{S}_n\big)C & 2I_{N-1}\otimes I_s\end{bmatrix}\\ &\big[\,\bar{x}^T\ \bar{F}^T\,\big]^T\ \leq\ 0,\end{aligned}$$

$$\begin{aligned}&\big[\,\bar{x}^T\ \bar{F}^T\ \bar{\Phi}^T\,\big]\\ \Leftrightarrow&\begin{bmatrix} I_{N-1}\otimes C^T\big(\underline{S}_n\bar{S}_n+\bar{S}_n\underline{S}_n\big)C & -I_{N-1}\otimes C^T\big(\underline{S}_n+\bar{S}_n\big) & \mathbf{0}\\ -I_{N-1}\otimes\big(\underline{S}_n+\bar{S}_n\big)C & 2I_{N-1}\otimes I_s\ \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\\ &\big[\,\bar{x}^T\ \bar{F}^T\ \bar{\Phi}^T\,\big]^T\ \leq\ 0.\end{aligned}$$

On the other hand, by means of (8), we have that

$$\Phi^T\big\{\Phi-\big[I_N\otimes\big(\bar{S}_e-\underline{S}_e\big)\big]u\big\}=\textstyle\sum_{i=1}^N\phi_i^T\big[\phi_i-\big(\bar{S}_e-\underline{S}_e\big)u_i\big]\leq 0,$$

$$\Leftrightarrow\ \Phi^T\big(\mathcal{U}\mathcal{U}^T\otimes I_m\big)\big\{\Phi-\big[I_N\otimes\big(\bar{S}_e-\underline{S}_e\big)\big]u\big\}\leq 0,$$

$$\Leftrightarrow\ \big[\big(\mathcal{U}^T\otimes I_m\big)\Phi\big]^T\big\{\big(\mathcal{U}^T\otimes I_m\big)\Phi-\big[\mathcal{U}^T\otimes\big(\bar{S}_e-\underline{S}_e\big)\big]u\big\}\leq 0,$$

$$\Leftrightarrow\ \Big[\Big(\tfrac{1}{\sqrt{N}}\mathbf{1}_N^T\otimes I_m\Big)\Phi\Big]^T\Big\{\Big(\tfrac{1}{\sqrt{N}}\mathbf{1}_N^T\otimes I_m\Big)\Phi-\Big[\tfrac{1}{\sqrt{N}}\mathbf{1}_N^T\otimes\big(\bar{S}_e-\underline{S}_e\big)\Big]u\Big\}$$

$$+\bar{\Phi}^T\left\{\bar{\Phi}-\left[\mathcal{U}_2^T\otimes\left(\bar{S}_e-\underline{S}_e\right)\right]u\right\}\leq 0,$$

$$\Leftrightarrow \left[\left(\tfrac{1}{\sqrt{N}}\mathbf{1}_N^T\otimes I_m\right)\Phi\right]^T\left(\tfrac{1}{\sqrt{N}}\mathbf{1}_N^T\otimes I_m\right)\Phi+\bar{\Phi}^T\left\{\bar{\Phi}-\left[\mathcal{U}_2^T\otimes\left(\bar{S}_e-\underline{S}_e\right)\right]u\right\}\leq 0$$

due to the fact that $\left[\frac{1}{\sqrt{N}}\mathbf{1}_N^T\otimes\left(\bar{S}_e-\underline{S}_e\right)\right]u=\left[\frac{1}{\sqrt{N}}\mathbf{1}_N^T\mathcal{L}\otimes\left(\bar{S}_e-\underline{S}_e\right)K\right]x=\mathbf{0}$,

$$\Rightarrow \bar{\Phi}^T\left\{\bar{\Phi}-\left[\mathcal{U}_2^T\otimes\left(\bar{S}_e-\underline{S}_e\right)\right]u\right\}\leq-\left[\left(\tfrac{1}{\sqrt{N}}\mathbf{1}_N^T\otimes I_m\right)\Phi\right]^T\left(\tfrac{1}{\sqrt{N}}\mathbf{1}_N^T\otimes I_m\right)\Phi\leq 0,$$

$$\Leftrightarrow \bar{\Phi}^T\left\{\bar{\Phi}-\left[\mathcal{U}_2^T\mathcal{L}\otimes\left(\bar{S}_e-\underline{S}_e\right)K\right]x\right\}\leq 0,$$

$$\Leftrightarrow \bar{\Phi}^T\left\{\bar{\Phi}-\left[\mathcal{U}_2^T\mathcal{L}\mathcal{U}\otimes\left(\bar{S}_e-\underline{S}_e\right)K\right]\hat{x}\right\}\leq 0,$$

$$\Leftrightarrow \bar{\Phi}^T\left\{\bar{\Phi}-\left[\left(\mathbf{0}_{(N-1)\times 1},\mathcal{U}_2^T\mathcal{L}\mathcal{U}_2\right)\otimes\left(\bar{S}_e-\underline{S}_e\right)K\right]\hat{x}\right\}\leq 0,$$

$$\Leftrightarrow \bar{\Phi}^T\left\{\bar{\Phi}-\left[\bar{\Lambda}\otimes\left(\bar{S}_e-\underline{S}_e\right)K\right]\bar{x}\right\}\leq 0,$$

$$\Leftrightarrow \begin{bmatrix}\bar{x}^T & \bar{\Phi}^T\end{bmatrix}\left[\begin{array}{c:c}\mathbf{0}_{(N-1)\times(N-1)}\otimes\mathbf{0}_{n\times n} & -\bar{\Lambda}\otimes K^T\left(\bar{S}_e-\underline{S}_e\right)\\ \hdashline -\bar{\Lambda}\otimes\left(\bar{S}_e-\underline{S}_e\right)K & 2I_{N-1}\otimes I_m\end{array}\right]\begin{bmatrix}\bar{x}^T & \bar{\Phi}^T\end{bmatrix}^T\leq 0,$$

$$\Leftrightarrow \begin{bmatrix}\bar{x}^T & \bar{F}^T & \bar{\Phi}^T\end{bmatrix}\left[\begin{array}{c:c:c}\mathbf{0}_{(N-1)n\times(N-1)n} & \mathbf{0} & -\bar{\Lambda}\otimes K^T\left(\bar{S}_e-\underline{S}_e\right)\\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \hdashline -\bar{\Lambda}\otimes\left(\bar{S}_e-\underline{S}_e\right)K & \mathbf{0} & 2I_{N-1}\otimes I_m\end{array}\right]\begin{bmatrix}\bar{x}^T & \bar{F}^T & \bar{\Phi}^T\end{bmatrix}^T\leq 0.$$

By using the S-Lemma, $\dot{V}(\bar{x})$ is negative definite *for any* incrementally sector-bounded $f(\cdot)$ within $\left[\underline{S}_e,\bar{S}_e\right]$ and *for all* sector-bounded $\phi_i(\cdot)$'s within $\left[\mathbf{0},\bar{S}_n-\underline{S}_n\right]$ if there exist two positive real numbers $\bar{\alpha}$ and $\bar{\beta}$ such that

$$\left[\begin{array}{c:c:c}\begin{array}{c}I_{N-1}\otimes\left(PA+A^TP\right)+\\ \bar{\Lambda}\otimes\left(PB\bar{S}_eK+K^T\bar{S}_eB^TP\right)\end{array} & -I_{N-1}\otimes PE & -I_{N-1}\otimes PB\\ \hdashline -I_{N-1}\otimes E^TP & \mathbf{0} & \mathbf{0}\\ \hdashline -I_{N-1}\otimes B^TP & \mathbf{0} & \mathbf{0}\end{array}\right]$$
$$-\bar{\alpha}\left[\begin{array}{c:c:c}I_{N-1}\otimes C^T\left(\underline{S}_n\bar{S}_n+\bar{S}_n\underline{S}_n\right)C & -I_{N-1}\otimes C^T\left(\underline{S}_n+\bar{S}_n\right) & \mathbf{0}\\ \hdashline -I_{N-1}\otimes\left(\underline{S}_n+\bar{S}_n\right)C & 2I_{N-1}\otimes I_s & \mathbf{0}\\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]$$
$$-\bar{\beta}\left[\begin{array}{c:c:c}\mathbf{0}_{(N-1)n\times(N-1)n} & \mathbf{0} & -\bar{\Lambda}\otimes K^T\left(\bar{S}_e-\underline{S}_e\right)\\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \hdashline -\bar{\Lambda}\otimes\left(\bar{S}_e-\underline{S}_e\right)K & \mathbf{0} & 2I_{N-1}\otimes I_m\end{array}\right]<\mathbf{0}$$

holds, which is equivalent to

$$\left[\begin{array}{c:c:c}\begin{array}{c}I_{N-1}\otimes\left(PA+A^TP\right)+\bar{\Lambda}\otimes\left(PB\bar{S}_eK+K^T\bar{S}_eB^TP\right)\\ -\bar{\alpha}I_{N-1}\otimes C^T\left(\underline{S}_n\bar{S}_n+\bar{S}_n\underline{S}_n\right)C\end{array} & \begin{array}{c}\bar{\alpha}I_{N-1}\otimes C^T\left(\underline{S}_n+\bar{S}_n\right)\\ -I_{N-1}\otimes PE\end{array} & \begin{array}{c}\bar{\beta}\bar{\Lambda}\otimes K^T\left(\bar{S}_e-\underline{S}_e\right)\\ -I_{N-1}\otimes PB\end{array}\\ \hdashline \bar{\alpha}I_{N-1}\otimes\left(\underline{S}_n+\bar{S}_n\right)C-I_{N-1}\otimes E^TP & -2\bar{\alpha}I_{N-1}\otimes I_s & \mathbf{0}\\ \hdashline \bar{\beta}\bar{\Lambda}\otimes\left(\bar{S}_e-\underline{S}_e\right)K-I_{N-1}\otimes B^TP & \mathbf{0} & -2\bar{\beta}I_{N-1}\otimes I_m\end{array}\right]<\mathbf{0},$$

$$\Leftrightarrow\left[\begin{array}{c:c}\begin{array}{c}I_{N-1}\otimes\left(PA+A^TP\right)+\bar{\Lambda}\otimes\left(PB\bar{S}_eK+K^T\bar{S}_eB^TP\right)\\ -\bar{\alpha}I_{N-1}\otimes C^T\left(\underline{S}_n\bar{S}_n+\bar{S}_n\underline{S}_n\right)C\end{array} & \bar{\alpha}I_{N-1}\otimes C^T\left(\underline{S}_n+\bar{S}_n\right)-I_{N-1}\otimes PE\\ \hdashline \bar{\alpha}I_{N-1}\otimes\left(\underline{S}_n+\bar{S}_n\right)C-I_{N-1}\otimes E^TP & -2\bar{\alpha}I_{N-1}\otimes I_s\end{array}\right]$$
$$+\tfrac{1}{2\bar{\beta}}\left[\begin{array}{c}\bar{\beta}\bar{\Lambda}\otimes K^T\left(\bar{S}_e-\underline{S}_e\right)-I_{N-1}\otimes PB\\ \hdashline \mathbf{0}\end{array}\right]\left[\begin{array}{c:c}\bar{\beta}\bar{\Lambda}\otimes\left(\bar{S}_e-\underline{S}_e\right)K-I_{N-1}\otimes B^TP & \mathbf{0}\end{array}\right]<\mathbf{0},$$

$$\Leftrightarrow\left[\begin{array}{c:c}\begin{array}{c}I_{N-1}\otimes\left(PA+A^TP\right)+\frac{1}{2\bar{\beta}}I_{N-1}\otimes PBB^TP+\\ \frac{1}{2}\bar{\Lambda}\otimes\left[PB\left(\underline{S}_e+\bar{S}_e\right)K+K^T\left(\underline{S}_e+\bar{S}_e\right)B^TP\right]-\\ \bar{\alpha}I_{N-1}\otimes C^T\left(\underline{S}_n\bar{S}_n+\bar{S}_n\underline{S}_n\right)C+\frac{\bar{\beta}}{2}\bar{\Lambda}^2\otimes K^T\left(\bar{S}_e-\underline{S}_e\right)^2K\end{array} & \bar{\alpha}I_{N-1}\otimes C^T\left(\underline{S}_n+\bar{S}_n\right)-I_{N-1}\otimes PE\\ \hdashline \bar{\alpha}I_{N-1}\otimes\left(\underline{S}_n+\bar{S}_n\right)C-I_{N-1}\otimes E^TP & -2\bar{\alpha}I_{N-1}\otimes I_s\end{array}\right]<\mathbf{0},$$

$$\begin{aligned}\Leftrightarrow I_{N-1}\otimes&\left\{P\left[A-\tfrac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]+\left[A-\tfrac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]^TP\right\}+\\ &\tfrac{1}{2}\bar{\Lambda}\otimes\left[PB\left(\underline{S}_e+\bar{S}_e\right)K+K^T\left(\underline{S}_e+\bar{S}_e\right)B^TP\right]+\tfrac{\bar{\alpha}}{2}I_{N-1}\otimes C^T\left(\bar{S}_n-\underline{S}_n\right)^2C\\ &+\tfrac{\bar{\beta}}{2}\bar{\Lambda}^2\otimes K^T\left(\bar{S}_e-\underline{S}_e\right)^2K+\tfrac{1}{2\bar{\beta}}I_{N-1}\otimes PBB^TP+\tfrac{1}{2\bar{\alpha}}I_{N-1}\otimes PEE^TP<\mathbf{0}.\end{aligned}\tag{11}$$

Let $H:=KQ$. Then (11) becomes

$$\begin{aligned}&I_{N-1}\otimes\left\{\left[A-\tfrac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]Q+Q\left[A-\tfrac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]^T\right\}+\\ &\tfrac{1}{2}\bar{\Lambda}\otimes\left[B\left(\underline{S}_e+\bar{S}_e\right)H+H^T\left(\underline{S}_e+\bar{S}_e\right)B^T\right]+\tfrac{\bar{\alpha}}{2}I_{N-1}\otimes QC^T\left(\bar{S}_n-\underline{S}_n\right)^2CQ\\ &+\tfrac{1}{2\bar{\beta}}I_{N-1}\otimes BB^T+\tfrac{\bar{\beta}}{2}\bar{\Lambda}^2\otimes H^T\left(\bar{S}_e-\underline{S}_e\right)^2H+\tfrac{1}{2\bar{\alpha}}I_{N-1}\otimes EE^T<\mathbf{0},\end{aligned}$$

$$\Leftrightarrow\left[\begin{array}{c:c:c}\begin{array}{c}I_{N-1}\otimes\left\{\left[A-\frac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]Q+Q\left[A-\frac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]^T\right\}+\\ \frac{1}{2}\bar{\Lambda}\otimes\left[B\left(\underline{S}_e+\bar{S}_e\right)H+H^T\left(\underline{S}_e+\bar{S}_e\right)B^T\right]+\frac{\beta}{2}I_{N-1}\otimes BB^T+\frac{\alpha}{2}I_{N-1}\otimes EE^T\end{array} & I_{N-1}\otimes QC^T & \bar{\Lambda}\otimes H^T\\ \hdashline I_{N-1}\otimes CQ & -2\alpha I_{N-1}\otimes\left(\bar{S}_n-\underline{S}_n\right)^{-2} & \mathbf{0}\\ \hdashline \bar{\Lambda}\otimes H & \mathbf{0} & -2\beta I_{N-1}\otimes\left(\bar{S}_e-\underline{S}_e\right)^{-2}\end{array}\right]<\mathbf{0},$$

where $\alpha:=1/\bar{\alpha}$ and $\beta:=1/\bar{\beta}$,

$$\Leftarrow\left[\begin{array}{c:c:c}\begin{array}{c}\left[A-\frac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]Q+Q\left[A-\frac{1}{2}E\left(\underline{S}_n+\bar{S}_n\right)C\right]^T+\\ \frac{1}{2}\lambda_i\left[B\left(\underline{S}_e+\bar{S}_e\right)H+H^T\left(\underline{S}_e+\bar{S}_e\right)B^T\right]+\frac{\beta}{2}BB^T+\frac{\alpha}{2}EE^T\end{array} & QC^T & \lambda_iH^T\\ \hdashline CQ & -2\alpha\left(\bar{S}_n-\underline{S}_n\right)^{-2} & \mathbf{0}\\ \hdashline \lambda_iH & \mathbf{0} & -2\beta\left(\bar{S}_e-\underline{S}_e\right)^{-2}\end{array}\right]<\mathbf{0},\quad i=2,\cdots,N,$$

which hold due to (5) and the convexity property of LMIs along with $\lambda_i=(1-\gamma)\lambda_2+\gamma\lambda_N$, $0\leq\gamma\leq 1$, $i=2,\cdots,N$. □