On Quotients of Stochastic Networks over Finite Fields

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***Abstract*—This paper studies the set stability of stochastic finite-field networks (SFFNs) via the quotient-transition-system (QTS)-based method. The QTS is constructed to preserve complete probabilistic transition information of the original SFFN and has a comparatively smaller network scale. First, with respect to the initial partition of the state set, we obtain the smallest QTS by calculating the coarsest equivalence relation. Then, the stability relationship between SFFN and its corresponding QTS is explored. In particular, the smallest QTS corresponding to a synchronous n-node SFFN has no greater than $n+1$ nodes. This formal simplicity gives a solid foundation for the subsequent research. Moreover, we establish a visualization interface "Quotient Generator" to obtain the quotients for any SFFN. After that, we explore the necessary and sufficient conditions for the set stability in distribution and the finite-time set stability with probability one of SFFNs based on the QTS. Finally, an example concerning a 27-state SFFN is presented to demonstrate the theoretical results, indicating that its synchronization analysis can be completely characterized by the stability of a 4-node QTS. Furthermore, we analyze the relationships among the number of iterations to obtain the smallest QTS, the number of nodes in the obtained QTS, and the types of SFFNs.**

***Index Terms*—Finite-fields networks; Quotient; Stochastic systems; Set stability; Semi-tensor product of matrices.**

I. Introduction

Distributed collaboration is widely applied in multi-agent systems [1]–[3], sensor networks [4], power networks [5]–[7], robotics [8]–[10], estimation [11], and parallel computation [12], [13]. In a distributed network, each nodal state is decided by its neighbors' states, and all nodes cooperate with their neighbors to realize a desired global objective, such as consensus [14], stability [15]–[18], synchronization [19], [20], controllability [21], [22], observability [23], oscillatority [24], and optimal control [25]. Among these issues, set stability is a salient dynamic behavior that describes the stability of a network with respect to a preassigned subset of the state space, not just a single state or a cyclic trajectory. In particular, by adjusting the target set, the set stability issue can be converted into the synchronization problem or the stability with respect to a certain state.

The work was partially supported by General Research Fund (GRF) under Grant 17201820 and the National Natural Science Foundation of China (NSFC) under Grant 61973259.

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In recent years, finite-field networks (FFNs), as a kind of finite-valued networks, have attracted increasing attention, since they have the ability to model a system with finite communication bandwidths and limited storage capacities, such as wireless sensor networks [26]. When handling the issues of distribution estimation and quantized control, an FFN usually has a shorter convergence time compared with continuous dynamic networks and is resilient to communication noises [4]. For an FFN, its state is assigned from a finite integer-valued set with a prime number cardinality, and the operations $+_p$ and $\times_p$ are processed based on modular arithmetic [27]. It is worth mentioning that modular arithmetic is applicable in some practical applications, such as the relative measurements of pose estimation [4]. Since the directed information interaction between two nodes may encounter link failure or creation or intensity variation, an FFN may have time-varying network topologies. Such an FFN is generally called a stochastic FFN (SFFN), whose evolution switches among a set of candidate FFNs [28]–[30].

In the following, we briefly discuss some existing results on FFNs. Since the finite field is not algebraically closed, the analysis methods for real-valued networks are not directly applicable to FFNs. Over the years, some researchers tried to formulate the analysis and control strategies for FFNs. In 2012, Sundaram *et al.* discovered that the structural controllability and structural observability could be guaranteed if there exists a group of weight forests with self-loops on each vertex [31]. Subsequently, the necessary and sufficient conditions for the consensus and synchronization of FFNs were, respectively, obtained in [4] and [20], from the perspectives of state transition graphs and the characteristic polynomial of network matrices. These results were further extended to the cases of mode switching [28], [29] and time delay [28]. Some ten years ago, a new matrix product, named semi-tensor product (STP) of matrices, was proposed [32], which breaks the dimension-matching condition of conventional matrix product. By the STP of matrices, an FFN can be equivalently transformed into an algebraic state space representation (ASSR) form [33]–[35]. Based on ASSR, the leader-follower consensus and the finite-time consensus of FFNs with time delays were, respectively, investigated in [34] and [35].

Notably, for an n-node FFN over the finite field $\mathbb{F}_p := \{0, 1, \ldots, p-1\}$, both the transition graph and the transition matrix are p^n-dimensional, whose large scale makes any related algorithm have an exponential time complexity in the worst case. As obtained in [28], the consensus of an n-node SFFN with m modes is necessary and sufficient to achieve that all $(p^n - p)$-times product of A_s, $s \in \Omega$, denoted as

$M(k) \in \mathbb{F}_p^{n\times n}$, $k \in [1, m^{p^n - p}]$, holds $M(k)\mathbf{1}_n = \mathbf{1}_n$ (i.e., $M(k)$ is row-stochastic) and $f_{M(k)} = \lambda^{n-1}(\lambda - 1)$, where $f_{M(k)}$ is the characteristic polynomial of $M(k)$ (see Theorems 3 and 4 in [28]). Equivalently, its transition graph contains exact p cycles, all of which are unit cycles around the vertices $\mathbf{0}_n, \mathbf{1}_n, \ldots, (p-1)\mathbf{1}_n$ (see Theorem 2 in [28]). To determine the consensus of an SFFN, checking the former algebraic criterion needs $O(n^3 m^{2(p^n - p)})$ computational complexity; checking the latter graphical criterion is equivalent to finding all cycles in a graph with p^n vertices, which is an NP-hard problem. Likewise, checking the necessary and sufficient condition for the synchronization of SFFNs in [29] also needs relatively high computational complexity.

In order to handle large-scale finite-valued networks, some model reduction strategies have been proposed (see, e.g., [36]–[43]). In [36]–[38], Boolean networks are reduced by classifying the nodes based on their positions. Since such aggregation does not reveal the properties of the nodes, recently in [39] and [40], the invariant subspace approach was established to reduce Boolean networks by separating the logical functions. Besides, the quotient/bisimulation-based method, widely used in the control community [44], is also a great way to reduce the size of Boolean networks while preserving their properties relevant for analysis or synthesis [41]–[43]. In general cases, a quotient transition system (QTS) will be constructed from the original system by defining an equivalent relation on the state set. The obtained QTS can demonstrate almost all the dynamic behavior of the original system and is generally of a much smaller scale than the original system. However, to the best of our knowledge, there are no related works on the model reduction of FFNs as well as SFFNs.

In this paper, we shall study the dynamic behaviors of SFFN by transforming it into the equivalent discrete-time Markov chain (DTMC) and then constructing the corresponding QTS. The contributions of this work are concluded as follows:

- **Set stability criteria.** The concept of set stability is first defined for SFFNs, including the set stability in distribution (SSD) and the finite-time set stability with probability one (FTSSPO). Accordingly, set stability conditions are necessarily and sufficiently obtained based on the corresponding QTS. These criteria are applicable to the consensus and the synchronization of SFFNs (see Remarks 4.3 and 4.4) and are also applicable to the set stability of deterministic FFNs (simply let $m = 1$).
- **Algorithm QGA to obtain the smallest QTS.** For an SFFN, it is revealed that the coarsest equivalence relation contained in a given relation does indeed exist uniquely. This result facilitates the reduction of FFNs to the smallest scale. In particular, the smallest QTS corresponding to a synchronous SFFN has less than $n+1$ nodes (see Theorem 4.2). Furthermore, we build a visualization interface "Quotient Generator" to calculate a minimal number of quotients for an inputted SFFN or a randomly generated SFFN.
- **Remarkable scale-reduction effect.** Based on the QTS-based method, the consensus of a 27-state SFFN with 2 modes (studied in [28]) and the synchronization of a 27-state SFFN with 3 modes (studied in [29]) can be completely characterized by the stability of a 4-state QTS (see Example 4.1 and Subsection V-A). Compared with the QTS-based method applied in Boolean networks (see, e.g., [41]–[43]), here derived QTS for FFNs is generally on a smaller scale than the derived QTS for Boolean networks owing to the particular characteristics of FFNs (see Remarks 3.1 and 4.2).

The remainder of this paper is organized as follows. In Section II, some notations and the STP of matrices are introduced. The SFFN model and set stability definition are established in Section III. Section IV proposes the construction strategy for the coarsest equivalence relation and studies the set stability of SFFNs via the smallest QTS. Section V presents an example to illustrate the obtained results, followed by the discussion and conclusion in Section VI.

II. Preliminaries

In this section, we present some notations utilized in this paper and briefly retrospect the notion of the STP of matrices, as well as the ASSR approach.

Notations: $[a,b] := \{a, a+1, \ldots, b\}$. I_n is an $n\times n$ identity matrix. Given matrix A, $\mathrm{Col}_i(A)$ denotes the i-th column of A; $[A]_{i,j}$ denotes the (i,j)-th entry of A. $\delta_n^i := \mathrm{Col}_i(I_n)$ and $\Delta_n := \{\delta_n^i \mid i \in [1,n]\}$. $\mathbb{L}^{n\times m}$ is the set of $n\times m$ matrices whose columns belong to Δ_n. For an n-dimensional vector $\mathbf{a}$, it is called a stochastic vector if the sum of all entries equals 1. An n-dimensional stochastic vector $\mathbf{a}$ can be uniquely written as $\mathbf{a} := a_1\delta_n^1 + a_2\delta_n^2 + \cdots + a_n\delta_n^n$ with $0 \leq a_1, a_2, \ldots, a_n \leq 1$, and $\langle \mathbf{a} \rangle := \{\delta_n^i \mid a_i \neq 0, i \in [1,n]\}$. $\mathbf{b}_n := b\mathbf{1}_n$ with $\mathbf{1}_n := (1,1,\ldots,1)^\top$. $\delta_n\{i_1, i_2, \ldots, i_k\} := \{\delta_n^{i_1}, \delta_n^{i_2}, \ldots, \delta_n^{i_k}\}$. $|S|$ denotes the number of elements in set S. $\delta_n^i < \delta_n^j$ if and only if $i < j$.

Next, some matrix products are introduced as follows.

(i) The Kronecker product of $n\times u$ matrix $A := (a_{i,j})$ and $m\times w$ matrix B is defined as [45]:

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \cdots & a_{1,u}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \cdots & a_{n,u}B \end{bmatrix}.$$

(ii) The STP of $n\times u$ matrix A and $m\times w$ matrix B is defined as [32]:

$$A \ltimes B = (A \otimes I_{\frac{\mathrm{lcm}(u,m)}{u}})(B \otimes I_{\frac{\mathrm{lcm}(u,m)}{m}}),$$

where $\mathrm{lcm}(u,m)$ is the least common multiple of integers u and m.

Note: The STP of matrices retains all the properties of the conventional matrix product since $A \ltimes B = AB$ if $u = m$. In particular, it has some useful properties:

a) If $u = 1$, it has

$$A \ltimes B = (A \otimes I_m) \ltimes B;$$

b) If $u = w = 1$, it has

$$W_{[n,m]} \ltimes A \ltimes B = B \ltimes A,$$

where $W_{[n,m]} := [I_m \otimes \delta_n^1, I_m \otimes \delta_n^2, \ldots, I_m \otimes \delta_n^n]$.

The detailed properties and applications can be referred to [32] and the references therein.

(iii) The Khatri-Rao product of $n \times m$ matrix A and $u \times m$ matrix B is defined as [46]:

$$A * B = [\mathrm{Col}_1(A) \ltimes \mathrm{Col}_1(B), \mathrm{Col}_2(A) \ltimes \mathrm{Col}_2(B), \cdots, \mathrm{Col}_m(A) \ltimes \mathrm{Col}_m(B)].$$

Define a bijective mapping $\sigma : \mathbb{F}_p \to \Delta_p$ as

$$\sigma(\varsigma) := \delta_p^{\varsigma+1}.$$

This bijection can be naturally extended to the one-to-one correspondence between $\mathbb{F}_p^n$ and Δ_{p^n} as

$$\bar{\sigma}(\varsigma_1, \varsigma_2, \ldots, \varsigma_n) := \ltimes_{i=1}^n \sigma(\varsigma_i) = \sigma(\varsigma_1) \ltimes \sigma(\varsigma_2) \ltimes \cdots \ltimes \sigma(\varsigma_n).$$

Lemma 2.1: ***(ASSR [32]).*** Given a p-valued logical function $f : \mathbb{F}_p^n \to \mathbb{F}_p$, its multi-linear form is expressed as

$$\sigma(f(\alpha_1, \alpha_2, \ldots, \alpha_n)) = F \ltimes_{i=1}^n \delta_p^{\alpha_i+1},$$

where $F \in \mathbb{L}^{p \times p^n}$ is called the structure matrix of f.

By resorting to Lemma 2.1, one can convert an arbitrary logical function to the corresponding algebraic form, formally called ASSR. Then, considering the functions $f_1(\alpha, \beta) = \alpha +_p \beta$ and $f_2(\alpha, \beta) = \alpha \times_p \beta$ with $\alpha, \beta \in \mathbb{F}_p$, by referring to [34], their structure matrices are, respectively, defined as

$$M_{+,p} = \delta_p[A_1, A_2, \ldots, A_p], \tag{1}$$

where $A_k = (k, k+1, \ldots, p, 1, 2, \ldots, k-1),\ k \in [1, p]$; and

$$M_{\times,p} = \delta_p[B_1, B_2, \ldots, B_p], \tag{2}$$

where $B_k = ([(0 \times k) \bmod p] + 1, [(1 \times k) \bmod p] + 1, \ldots, [((p-1) \times k) \bmod p] + 1),\ k \in [1, p]$.

III. Problem Formulation

An SFFN with n nodes has the following features:

- **(Finite field).** Each nodal state takes values from the field $\mathbb{F}_p$ with a prime characteristic p. The operation is composed of modular addition "$+_p$" and modular multiplication "$\times_p$" satisfying field axioms, which have been stated in many references (see, e.g., [4] and [27]) and thus is omitted here.
- **(Distributed connection with switching topology).** At each time instant t, the connection relationship between these n nodes can be characterized by a digraph $\mathscr{G}^{\theta(t)} := (\mathscr{V}, \mathscr{E}^{\theta(t)})$ with $\mathscr{V} := [1, n]$ and $\mathscr{E}^{\theta(t)} := \{(i, j) \mid$ there is a directed edge from node j to node i in mode $\theta(t)\} \cup \{(i, i) \mid i \in [1, n]\}$. Here, $\{\theta(t) \mid t \geq 0\}$ is the switching sequence, which is valued from a finite index set $\Omega := [1, m]$ and obeys an independently and identically distributed process. The probability distribution of these m modes is denoted as $\mathscr{D} = [d_1, d_2, \ldots, d_m]$.
- **(Evolution rule).** For each node $i \in \mathscr{V}$, its state at time $t+1$ is a weighted combination of the states of nodes in $\mathscr{N}_i^{\theta(t)} := \{j \mid (i, j) \in \mathscr{E}^{\theta(t)}\}$ at time t, that is,

$$x_i(t+1) = \sum_{j \in \mathscr{N}_i^{\theta(t)} \cup \{i\}} (a_{ij}^{\theta(t)} \times_p x_j(t)) \tag{3}$$

with $a_{ij}^{\theta(t)} \in \mathbb{F}_p$ being the weight of edge (i, j) at time t. Here, $\mathscr{N}_i^{\theta(t)}$ is called the in-node set of i at time t, and $\sum$ is the cumulative modular addition operator.

Therefore, such an SFFN can be described as

$$x(t+1) = A_{\theta(t)} \times_p x(t), \tag{4}$$

where $x := (x_1, x_2, \ldots, x_n)^\top \in \mathbb{F}_p^n$ is the network state, and $A_{\theta(t)} := (a_{ij}^{\theta(t)}) \in \mathbb{F}_p^{n \times n}$ is the weighted adjacency matrix of $\mathscr{G}^{\theta(t)}$. Furthermore, letting $\mathrm{x}_i := \bar{\sigma}(x_i) \in \Delta_p$, $\mathrm{a}_{ij}^{\theta(t)} := \sigma(a_{ij}^{\theta(t)}) \in \Delta_p$, and $\mathrm{x} := \sigma(x) \in \Delta_{p^n}$, one derives

$$\mathrm{x}_i(t+1) = D_{i,\theta(t)} \ltimes \mathrm{x}(t), \tag{5}$$

where $D_{i,\theta(t)} := (M_{+,p})^{n-1} \ltimes_{j=1}^n [I_{p^{n-j}} \otimes (M_{\times,p} \ltimes \mathrm{a}_{ij}^{\theta(t)})] \in \mathbb{L}^{p \times p^n}$. Then, SFFN (4) can be equivalently converted into the algebraic form (exactly the so-called ASSR):

$$\Sigma: \quad \mathrm{x}(t+1) = D_{\theta(t)} \ltimes \mathrm{x}(t),\ \mathrm{x}(t) \in \Delta_{p^n}, \tag{6}$$

where $D_{\theta(t)} := D_{1,\theta(t)} * D_{2,\theta(t)} * \cdots * D_{n,\theta(t)} \in \mathbb{L}^{p^n \times p^n}$. Subsequently, we introduce the notion of DTMC, which is uniquely determined by Σ (or equivalently, SFFN (4)) and is also known as the probability transition system.

Remark 3.1: As studied in [47]–[52], the ASSR of a probabilistic Boolean network is also in a similar form as Σ. However, there are many different properties between the ASSR of SFFNs (or FFNs) and probabilistic Boolean networks (or Boolean networks). For example, the transition graph of an FFN is composed of disjoint weakly-connected subgraphs, which contain exactly one cycle, possibly of unit length [53]. In contrast, the transition graph of probabilistic Boolean networks has no fixed pattern. Besides, an FFN achieves consensus or synchronization starting from any state in $\mathbb{F}_p^n$ if and only if it achieves consensus or synchronization starting from any state in Δ_n [20]. Owing to the particular characteristic of FFNs, it will result in a more remarkable scale-reduction for FFNs and SFFNs than Boolean networks and probabilistic Boolean networks. For example, in [41]–[43], there is no explicit discussion about how small a general Boolean network or probabilistic Boolean network can be reduced to. Hence, it is possible that the complexity of the reduced network will remain the same as that of the original network, or the reduced network still has a scale that is too large for efficient analysis. In contrast, we can obtain the upper bound of the reduction degree, that is, $\frac{n+1}{p^n}$, for SFFNs and FFNs, when studying the consensus and synchronization issues. A more detailed interpretation will be given in Subsection IV-B.

Definition 3.1: ***(DTMC [54]).*** A discrete-time Markov chain (DTMC) is a pair $(\mathscr{S}, \mathbf{P})$, where $\mathscr{S}$ is a set of states and $\mathbf{P}$ is the probability transition matrix (PTM) such that, for all $s \in \mathscr{S}$, it holds

$$\sum_{s' \in \mathscr{S}} \mathbf{P}(s, s') = 1.$$

Here, the DTMC in terms of Σ in (6) is denoted as

$$\Gamma(\Sigma) := (\Delta_{p^n}, \mathbf{P}) \tag{7}$$

with PTM $\mathbf{P}=\sum_{\upsilon=1}^{m}(d_\upsilon D_\upsilon)$. It can be demonstrated by a state transition graph with p^n nodes and the edges from node $j\in[1,p^n]$ to node $i\in[1,p^n]$ if and only if $[\mathbf{P}]_{i,j}>0$.

Noting that the consensus and synchronization issues of SFFN (4) are essentially the stability problem subject to a specific target set, we hence aim to explore the set stability of SFFN (4), including two forms: SSD and FTSSPO. By referring to [55], their definitions are given as follows.

Definition 3.2: ***(Set Stability).*** For a subset $\mathscr{O}\subseteq\mathbb{F}_p^n$, SFFN (4) achieves

- $\mathscr{O}$-stability in distribution if, $\lim_{t\to\infty}\mathbb{P}\{x(t;\theta_0,x_0)\in\mathscr{O}\}=1$ holds for any initial state $x_0\in\mathbb{F}_p^n$ and any initial switching $\theta_0\in\Omega$;
- finite-time $\mathscr{O}$-stability with probability one if, for each initial state $x_0\in\mathbb{F}_p^n$, there exists a corresponding integer $T(x_0)>0$ such that $\mathbb{P}\{x(t;\theta_0,x_0)\in\mathscr{O}\}=1$ holds for all $t\geq T(x_0)$ and any initial switching $\theta_0\in\Omega$.

Remark 3.2: The set stability of SFFN (4) can be transformed into the consensus or synchronization via assigning

$$\mathscr{O}=\mathscr{O}_c:=\{\mathbf{0}_n,\mathbf{1}_n,\ldots,(p-1)\mathbf{1}_n\},$$

whose equivalent algebraic form is

$$\mathscr{O}_c=\{\delta_{p^n}^1,\delta_{p^n}^{\frac{p^n-1}{p-1}+1},\delta_{p^n}^{2\cdot\frac{p^n-1}{p-1}+1},\ldots,\delta_{p^n}^{(p-1)\cdot\frac{p^n-1}{p-1}+1}\}$$

with a little abuse of the notation of $\mathscr{O}$ and $\mathscr{O}_c$. In particular, the consensus of SFFN (4) should additionally satisfy $A_\kappa\mathbf{1}_n=\mathbf{1}_n$ for all $\kappa\in\Omega$; the synchronization of SFFN (4) should additionally satisfy $A_\kappa\mathbf{1}_n=\beta(\kappa)\mathbf{1}_n$ with $\beta(\kappa)\in\mathbb{F}_p$ for each $\kappa\in\Omega$. ■

IV. Set Stability Analysis of SFFNs via QTS

This section devotes to establishing the set stability criteria for SFFN (4) via the QTS-based method, where the evolution of the constructed QTS can reflect the dynamic features of the original SFFN. In this sense, one can greatly reduce the computational complexity by analyzing the smallest QTS, instead of the probability transition of the whole state space.

A. Construction of the Smallest QTS

In this subsection, we construct the QTS for SFFN (4), before which we introduce the equivalence relation on Δ_{p^n}, termed as $\mathscr{R}$, and the quotient set of Δ_{p^n} by $\mathscr{R}$, denoted as $\Delta_{p^n}/\mathscr{R}$. For more details, please refer to literature [56].

Definition 4.1: ***(Equivalent Relation [56]).*** A binary relation $\mathscr{R}$ on Δ_{p^n} is said to be an equivalence relation, if and only if it is reflexive, symmetric, and transitive. That is, for any $\delta_{p^n}^i,\delta_{p^n}^j,\delta_{p^n}^k\in\Delta_{p^n}$, the following properties hold:

1) reflexivity: $(\delta_{p^n}^i,\delta_{p^n}^i)\in\mathscr{R}$;
2) symmetry: $(\delta_{p^n}^i,\delta_{p^n}^j)\in\mathscr{R}$ if and only if $(\delta_{p^n}^j,\delta_{p^n}^i)\in\mathscr{R}$;
3) transitivity: if $(\delta_{p^n}^i,\delta_{p^n}^j)\in\mathscr{R}$ and $(\delta_{p^n}^j,\delta_{p^n}^k)\in\mathscr{R}$, then $(\delta_{p^n}^i,\delta_{p^n}^k)\in\mathscr{R}$.

Definition 4.2: ***(Quotient Set).*** The equivalence class of $\delta_{p^n}^i\in\Delta_{p^n}$ under $\mathscr{R}$ is defined as

$$[\delta_{p^n}^i]:=\{\delta_{p^n}^j\in\Delta_{p^n}\mid(\delta_{p^n}^j,\delta_{p^n}^i)\in\mathscr{R}\}.$$

Then, the set of all equivalence classes in Δ_{p^n} under the equivalence relation $\mathscr{R}$ is called the quotient set of Δ_{p^n} by $\mathscr{R}$, denoted as $\Delta_{p^n}/\mathscr{R}$.

It indicates that every state $\delta_{p^n}^i\in\Delta_{p^n}$ is an element of the equivalence class $[\delta_{p^n}^i]$, which means $[\delta_{p^n}^i]\neq\emptyset$, and every two equivalence classes $[\delta_{p^n}^i]$ and $[\delta_{p^n}^j]$ are either equal or disjoint, that is, $[\delta_{p^n}^i]=[\delta_{p^n}^j]$ or $[\delta_{p^n}^i]\cap[\delta_{p^n}^j]=\emptyset$. Hence, quotient set $\Delta_{p^n}/\mathscr{R}$ forms a partition of Δ_{p^n} and is uniquely determined by $\mathscr{R}$. For convenience, we define $\Pi_{\mathscr{R}}:=\Delta_{p^n}/\mathscr{R}$ and $\omega_{\mathscr{R}}:=|\Pi_{\mathscr{R}}|$. With respect to equal relation $\mathscr{R}_e$ and total relation $\mathscr{R}_t$ [56], their induced quotient sets are $\Pi_{\mathscr{R}_e}:=\{\{\delta_{p^n}^i\}\mid i\in[1,p^n]\}$ and $\Pi_{\mathscr{R}_t}:=\Delta_{p^n}$, which depict the finest and the coarsest partitions of Δ_{p^n}, respectively. To reveal the relation between two quotient sets (or called partitions, we present the following definition for the "coarser" relation.

Definition 4.3: ***(Coarser Relation [56]).*** Given two equivalence relations $\mathscr{R}_1$ and $\mathscr{R}_2$ on Δ_{p^n}, quotient set $\Pi_{\mathscr{R}_2}$ is said to be coarser than $\Pi_{\mathscr{R}_1}$, expressed as $\Pi_{\mathscr{R}_1}\preceq\Pi_{\mathscr{R}_2}$, if each equivalence class under $\mathscr{R}_1$ is contained in a certain equivalence class under $\mathscr{R}_2$, that is, $\mathscr{R}_1\subseteq\mathscr{R}_2$.

For each $\delta_{p^n}^i\in\Delta_{p^n}$, denote the minimal state in the corresponding equivalent class by

$$\lfloor\delta_{p^n}^i\rfloor:=\min_{\delta_{p^n}^j\in[\delta_{p^n}^i]}\delta_{p^n}^j.$$

Then, let $\phi(\delta_{p^n}^i)$ be the order of $\lfloor\delta_{p^n}^i\rfloor$ in set $\{\lfloor\delta_{p^n}^j\rfloor\mid\delta_{p^n}^j\in\Delta_{p^n}\}$ whose elements are arranged from the smallest to the largest. In the following, the equivalence relation $\mathscr{R}$ on Δ_{p^n}, which can be characterized by a logical matrix $\mathscr{T}_{\mathscr{R}}\in\mathbb{L}^{\omega_{\mathscr{R}}\times p^n}$ with $\mathrm{Col}_i(\mathscr{T}_{\mathscr{R}})=\delta_{\omega_{\mathscr{R}}}^{\phi(\delta_{p^n}^i)}$, is required to satisfy

$$(\delta_{p^n}^i,\delta_{p^n}^j)\in\mathscr{R}\Leftrightarrow\mathscr{T}_{\mathscr{R}}\mathbf{P}\delta_{p^n}^i=\mathscr{T}_{\mathscr{R}}\mathbf{P}\delta_{p^n}^j.\tag{8}$$

It indicates that the states in the same equivalent class must have the same transitions to other equivalent classes. Hence, we can study SFFN (4), or equivalently, $\Gamma(\Sigma)$ in (7), by its corresponding QTS defined as follows.

Definition 4.4: ***(Corresponding QTS).*** Given a DTMC $\Gamma(\Sigma)$ and an equivalence relation $\mathscr{R}$ on Δ_{p^n}, the QTS corresponding to $\Gamma(\Sigma)$ is defined as

$$\Gamma(\Sigma)/\mathscr{R}:=(\Pi_{\mathscr{R}},\mathbf{Q}_{\mathscr{R}})\tag{9}$$

with $\mathbf{Q}_{\mathscr{R}}:=\mathscr{T}_{\mathscr{R}}\mathbf{P}\mathscr{T}_{\mathscr{R}}^\top$.

In this paper, we would like to find the coarsest equivalence relation contained by a given relation $\mathscr{R}^\circ$, which determines a preliminary partition of Δ_{p^n} and is preassigned in terms of the desired evolution behavior. For example, when considering the consensus or synchronization of SFFN (4), the preliminary relation $\mathscr{R}^\circ$ is defined as

$$\mathscr{R}^\circ=\{(\delta_{p^n}^i,\delta_{p^n}^j)\mid\delta_{p^n}^i,\delta_{p^n}^j\in\mathscr{O}_c\text{ or }\Delta_{p^n}\setminus\mathscr{O}_c\}.$$

In particular, the coarsest equivalence relation will induce a QTS of the smallest size, which is subsequently proved to uniquely exist in $\mathscr{R}^\circ$ and is termed as $\mathscr{R}^\star$.

*Proposition 4.1: **(Existence and Uniqueness of $\mathscr{R}^\star$).*** Given a binary relation $\mathscr{R}^\circ$ on Δ_{p^n}, there exists a unique coarsest equivalence relation $\mathscr{R}^\star \subseteq \mathscr{R}^\circ$ on Δ_{p^n}, satisfying condition (8).

Proof: First, we prove that $\mathscr{R}^\star \subseteq \mathscr{R}^\circ$ satisfying condition (8) is an equivalence relation on Δ_{p^n} based on Definition 4.1. Since $\mathscr{R}_e \subseteq \mathscr{R}^\star$, relation $\mathscr{R}^\star$ is reflexive. From (8), it is symmetric because of

$$(\delta^i_{p^n}, \delta^j_{p^n}) \in \mathscr{R}^\star \Leftrightarrow \mathscr{T}_{\mathscr{R}^\star}\mathbf{P}\delta^i_{p^n} = \mathscr{T}_{\mathscr{R}^\star}\mathbf{P}\delta^j_{p^n} \Leftrightarrow (\delta^j_{p^n}, \delta^i_{p^n}) \in \mathscr{R}^\star.$$

Furthermore, relation $\mathscr{R}^\star$ is transitive owing to

$$\begin{aligned} &(\delta^i_{p^n}, \delta^j_{p^n}) \in \mathscr{R}^\star \text{ and } (\delta^j_{p^n}, \delta^k_{p^n}) \in \mathscr{R}^\star \\ \Leftrightarrow\ & \mathscr{T}_{\mathscr{R}^\star}\mathbf{P}\delta^i_{p^n} = \mathscr{T}_{\mathscr{R}^\star}\mathbf{P}\delta^j_{p^n} = \mathscr{T}_{\mathscr{R}^\star}\mathbf{P}\delta^k_{p^n} \\ \Leftrightarrow\ & (\delta^i_{p^n}, \delta^k_{p^n}) \in \mathscr{R}^\star. \end{aligned}$$

Since equal relation $\mathscr{R}_e$ satisfies $\mathscr{R}_e \subseteq \mathscr{R}^\circ$ and condition (8), the existence of $\mathscr{R}^\star$ is guaranteed. If there exists another equivalence relation $\mathscr{R}' \subseteq \mathscr{R}^\circ$ satisfying $|\mathscr{R}'| = |\mathscr{R}^\star|$ and condition (8), the union $\mathscr{R}' \cup \mathscr{R}^\star$ not only meets condition (8) but is coarser than both $\mathscr{R}'$ and $\mathscr{R}^\star$. Then, equivalent relation $\mathscr{R}' \cup \mathscr{R}^\star$ becomes the coarsest equivalence relation, which ensures the uniqueness of $\mathscr{R}^\star$. ■

Regarding the DTMC $\Gamma(\Sigma)$ with preliminary $\mathscr{R}^\circ$, the coarsest equivalence relation $\mathscr{R}^\star$ on Δ_{p^n} can be obtained by the following iteration:

$$\begin{cases} \mathscr{R}[k+1] = \{(\delta^i_{p^n}, \delta^j_{p^n}) \in \mathscr{R}[k] \mid \mathscr{T}_{\mathscr{R}[k]}\mathbf{P}\delta^i_{p^n} = \mathscr{T}_{\mathscr{R}[k]}\mathbf{P}\delta^j_{p^n}\} \\ \mathscr{R}[0] = \mathscr{R}^\circ, \end{cases} \tag{10}$$

whose terminal condition is that there exists an integer $k^\star \geq 0$ such that $\mathscr{R}[k^\star+1] = \mathscr{R}[k^\star]$. Let $\omega_{\mathscr{R}^\star} := |\mathscr{R}[k^\star]|$. The obtained $\mathscr{R}^\star = \mathscr{R}[k^\star]$ can be characterized by matrix $\mathscr{T}_{\mathscr{R}^\star} \in \mathbb{L}^{\omega_{\mathscr{R}^\star} \times p^n}$ with

$$\mathrm{Col}_i(\mathscr{T}_{\mathscr{R}^\star}) = \delta^{\phi(\delta^i_{p^n})}_{\omega_{\mathscr{R}^\star}},\ i \in [1, p^n].$$

Accordingly, we design Algorithm **QGA** to find the minimal number of quotients for the DTMC $\Gamma(\Sigma)$ derived from an SFFN (4). Furthermore, we establish a "Quotient Generator " to calculate the minimal number of quotients for a preassigned SFFN or a randomly generated SFFN, whose visualization interface can be accessed via the link: http://zhihanjiang.com/quotient/main/index.html.

After that, one can derive the corresponding smallest QTS as follows:

$$\Sigma_{\mathscr{R}^\star} : \mathrm{x}_{\mathscr{R}^\star}(t+1) = \check{D}_{\theta(t)} \ltimes \mathrm{x}_{\mathscr{R}^\star}(t),\ \mathrm{x}_{\mathscr{R}^\star} \in \Delta_{\omega_{\mathscr{R}^\star}} \tag{11}$$

with $\check{D}_{\theta(t)} := \mathscr{T}_{\mathscr{R}^\star} D_{\theta(t)} \mathscr{T}^\top_{\mathscr{R}^\star}$. Here, $\mathrm{x}_{\mathscr{R}^\star}(t)$ is the state of the corresponding smallest QTS $\Sigma_{\mathscr{R}^\star}$. For $\Sigma_{\mathscr{R}^\star}$, its total number of states is $\omega_{\mathscr{R}^\star}$, and its PTM is $\mathbf{Q}_{\mathscr{R}^\star} := \mathscr{T}_{\mathscr{R}^\star}\mathbf{P}\mathscr{T}^\top_{\mathscr{R}^\star}$. In particular, $\Sigma_{\mathscr{R}^\star}$ can reflect the evolution of Σ because it holds

$$\mathbf{P}\mathrm{x} = p \Leftrightarrow \mathbf{Q}_{\mathscr{R}^\star}\mathrm{x}_{\mathscr{R}^\star} = p_{\mathscr{R}^\star}$$

with $\mathrm{x}_{\mathscr{R}^\star} := \mathscr{T}_{\mathscr{R}^\star}\mathrm{x}$ and $p_{\mathscr{R}^\star} := \mathscr{T}_{\mathscr{R}^\star} p$.

Next, we take the same example in [28] to illustrate the construction process of $\Sigma_{\mathscr{R}^\star}$ for the sake of checking the $\mathscr{O}_c$-stability of SFFN (4).

Algorithm 1 Quotient Generator Algorithm (**QGA**)

Input: P: The probability transition matrix $\mathbf{P}$. S: The initial partition for Δ_{p^n} obeying $\mathscr{R}^\circ$.

Output: $S^\star$: The quotient set $\Pi_{\mathscr{R}^\star} := \Delta_{p^n}/\mathscr{R}^\star$.

```
S* ← S;   ▷ Initialize the output set by the initial set.
repeat
    S ← S*;  ▷ Update the current set by latest output set.
    for each S_i ∈ S
        divide S_i into sets C_i that for each set C_{i,m} in C_i
            for each s_a, s_b ∈ C_{i,m}, a ≠ b, and each S_j ∈ S;
                p_{aj} ← Σ_{s_q ∈ S_j} P[s_a][s_q];
                p_{bj} ← Σ_{s_q ∈ S_j} P[s_b][s_q];
                ▷ Calculate the transition probability from state s_a
                  (state s_b) to all states in each set S_j.
                ensure that p_{aj} == p_{bj};
                ▷ Guarantee the transition probabilities of states in
                  C_{im} to each class to be the same.
            end for
        end for
    end for
    S* ← {C_1, C_2, ...}.   ▷ Update the output set.
until S* == S   ▷ End iteration when there is no update.
```

*Example 4.1: **(Smallest-$\Sigma_{\mathscr{R}^\star}$-construction).*** Consider SFFN (4) over $\mathbb{F}_3$, it has two modes as

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{12}$$

with the probability distribution $\mathscr{D} = [0.2, 0.8]$. Let $\mathscr{O}_c = \delta_{27}\{1, 14, 27\}$. The states therein are the vector form of synchronous states: $(0,0,0)^\top, (1,1,1)^\top, (2,2,2)^\top$. The preliminary partition of Δ_{27} is assigned as $\{\mathscr{O}_c, \Delta_{27}\backslash\mathscr{O}_c\}$. Because equality $\mathscr{T}_{\mathscr{R}^\circ}\mathbf{P}\delta^i_{p^n} = \mathscr{T}_{\mathscr{R}^\circ}\mathbf{P}\delta^j_{p^n}$ holds for any $\delta^i_{p^n}, \delta^j_{p^n}$ that belong to one of the sets: $C_{1,1} := \delta_{27}\{2, 3, 13, 15, 25, 26\}$, $C_{1,2} := \delta_{27}\{4, 7, 11, 17, 21, 24\}$, and $C_{1,3} := \Delta_{27}\backslash(\mathscr{O}_c \cup C_{1,1} \cup C_{1,2})$, one yields

$$\begin{aligned} \mathscr{R}[1] = \{&(\delta^i_{p^n}, \delta^j_{p^n}) \in \mathscr{R}^\circ \mid \\ &\delta^i_{p^n}, \delta^j_{p^n} \in C_{1,1} \text{ or } C_{1,2} \text{ or } C_{1,3} \text{ or } \mathscr{O}_c\}. \end{aligned}$$

Repeating this process to obtain the quotient set as

$$\Pi_{\mathscr{R}^\star} = \delta_{27}\{[1], [2], [4], [5]\},$$

where

$$[1] = \{1, 14, 27\},$$

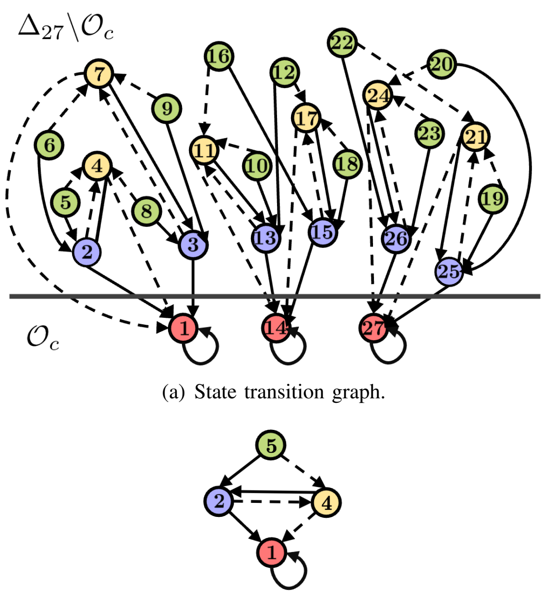
$$[2] = \{2, 3, 13, 15, 25, 26\},$$

$$[4] = \{4, 7, 11, 17, 21, 24\},$$

$$[5] = \{5, 6, 8, 9, 10, 12, 16, 18, 19, 20, 22, 23\}.$$

The state transition graph is demonstrated in Fig. 1(a), where node i represents state δ^i_{27}, and the nodes in the same color belong to the same quivalent class. The corresponding quotient transition graph is constructed in Fig. 1(b), where node i

represents the equivalent class $[\delta_{27}^i]$. Besides, the solid and dashed edges in Fig. 1(a) and Fig. 1(b) represent the state transition with probability 0.2 and 0.8, respectively.



(a) State transition graph.

(b) Quotient transition graph.

Fig. 1. Illustration of constructing QTS for Example 4.1.

The smallest QTS $\Sigma_{\mathscr{R}^\star}$ is obtained as (11) with $x_{\mathscr{R}^\star} \in \Delta_4$ and

$$\breve{D}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \breve{D}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be equivalently expressed as a DTMC $\Gamma(\Sigma)/\mathscr{R}^\star := (\Pi_{\mathscr{R}^\star}, \mathbf{Q}_{\mathscr{R}^\star})$ with

$$\mathbf{Q}_{\mathscr{R}^\star} = \begin{bmatrix} 1 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.2 & 0.2 \\ 0 & 0.8 & 0 & 0.8 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (13)$$

■

As indicated in Example 4.1, a 27-state SFFN can be characterized by a 4-state QTS. Although $\Sigma_{\mathscr{R}^\star}$ may not be an SFFN because $\omega_{\mathscr{R}^\star}$ may not be the power of p, we can still investigate the set stability of SFFN (4) via $\Sigma_{\mathscr{R}^\star}$, which has a smaller number of nodes.

B. Dynamic Equivalences between SFFNs and QTS

In this subsection, for a given subset $\mathscr{O} \subseteq \Delta_{p^n}$, we investigate the SSD and FTSSPO towards $\mathscr{O}$ of SFFN (4) via analyzing the stability of the obtained smallest $\Sigma_{\mathscr{R}^\star}$, which can be viewed as a DTMC $(\Delta_{\omega_{\mathscr{R}^\star}}, \mathbf{Q}_{\mathscr{R}^\star})$.

Proposition 4.2: ***(Necessary Condition of Set Stability).*** If SFFN (4) achieves set stability with respect to $\mathscr{O} \subseteq \Delta_{p^n}$, there must exist a state $\delta_{p^n}^{\nu} \in \mathscr{O}$ satisfying $[\delta_{p^n}^{\nu}] \subseteq \mathscr{O}$ and $\mathbf{Q}_{\mathscr{R}^\star}\delta_{\omega_{\mathscr{R}^\star}}^{\phi(\delta_{p^n}^{\nu})} = \delta_{\omega_{\mathscr{R}^\star}}^{\phi(\delta_{p^n}^{\nu})}$.

Proof: Assumed that SFFN (4) achieves $\mathscr{O}$-stability (that is, $\mathscr{O}$-SSD or $\mathscr{O}$-FTSSPO), there must exist a subset $\mathscr{O}^\star \subseteq \mathscr{O}$ such that $\langle \mathbf{P}x\rangle \subseteq \mathscr{O}^\star$ holds for all $x \in \mathscr{O}^\star$ (this statement obviously holds by referring to Theorem 1 and Proposition 1 in [57], where $\mathscr{O}^\star$ is generally called the largest invariant subset of $\mathbf{P}$ in $\mathscr{O}$). Therefore, $\mathscr{O}^\star$ is an equivalent class, termed as $[\delta_{p^n}^{\nu}]$ without loss of generality. If $\mathbf{Q}_{\mathscr{R}^\star}\delta_{\omega_{\mathscr{R}^\star}}^{\phi(\delta_{p^n}^{\nu})} \neq \delta_{\omega_{\mathscr{R}^\star}}^{\phi(\delta_{p^n}^{\nu})}$, that is,

$$\mathbb{P}\{x(1;\theta_0,x_0) \in [\delta_{p^n}^{\nu}] \mid x_0 \in [\delta_{p^n}^{\nu}]\} < 1,$$

it will conflict with the assumption $\mathbb{P}\{\langle \mathbf{P}x\rangle \subseteq [\delta_{p^n}^{\nu}] \mid x_0 \in [\delta_{p^n}^{\nu}]\} = 1$. Hence, $\mathbf{Q}_{\mathscr{R}^\star}\delta_{\omega_{\mathscr{R}^\star}}^{\phi(\delta_{p^n}^{\nu})} = \delta_{\omega_{\mathscr{R}^\star}}^{\phi(\delta_{p^n}^{\nu})}$. ■

Remark 4.1: In particular, if $\mathscr{O} = \mathscr{O}_c$, one has $[\delta_{p^n}^{\nu}] = \mathscr{O}_c$ in Proposition 4.2 because $A_\upsilon \mathbf{1}_n \in \mathscr{O}_c$ holds for all $\upsilon \in \Omega$ as proved in Theorems 4.2 and 4.4 in [29]. ■

Hereafter, suppose $\delta_{p^n}^{\nu}$ to be the state satisfying the conditions in Proposition 4.2, and denote $s^\star := \delta_{\omega_{\mathscr{R}^\star}}^{\phi(\delta_{p^n}^{\nu})}$. Next, we explore the equivalence relations between the $\mathscr{O}$-stability of SFFN (4) and $s^\star$-stability of QTS $\Sigma_{\mathscr{R}^\star}$.

Theorem 4.1: ***(Equivalence of Stability).*** SFFN (4) is globally stable towards $\mathscr{O}$ if and only if its corresponding QTS $\Sigma_{\mathscr{R}^\star}$ is globally stable towards $s^\star$.

Proof: Here, we only prove this theorem in the case of SSD, since this theorem for the finite-time case can be similarly proved.

Assume that SFFN (4) achieves SSD towards $\mathscr{O}$. Proposition 4.2 indicates that there exists a state $\delta_{p^n}^{\nu} \in \mathscr{O}$ such that

$$\mathbb{P}\{x(t;\theta_0,x_0) \in [\delta_{p^n}^{\nu}] \mid x_0 \in [\delta_{p^n}^{\nu}]\} = 1, \forall\ \theta_0 \in \Omega, \forall\ t \geq 0.$$

Then, one has

$$\mathbb{P}\{x_{\mathscr{R}^\star}(t;\theta_0,s^\star) = s^\star\} = 1, \forall\ \theta_0 \in \Omega, \forall\ t \geq 0.$$

Hence, $s^\star$ is an equilibrium point of $\Sigma_{\mathscr{R}^\star}$. Next, from Definition 3.2, one has

$$\lim_{t\to\infty} \mathbb{P}\{x(t;\theta_0,x_0) \in [\delta_{p^n}^{\nu}]\} = 1, \forall\ x_0 \in \Delta_{p^n}, \forall\ \theta_0 \in \Omega.$$

According to the construction of $\mathscr{R}^\star$, one can derive

$$\lim_{t\to\infty} \mathbb{P}\{[x(t;\theta_0,[x_0])] = [\delta_{p^n}^{\nu}]\} = 1, \forall\ [x_0] \in \Pi_{\mathscr{R}^\star}, \forall\ \theta_0 \in \Omega, \quad (14)$$

where $[x(t;\theta_0,[x_0])]$ denotes the state of $\Gamma(\Sigma)/\mathscr{R}^\star$ at time t with initial state $[x_0]$ and initial switching θ_0. Let $s^0 = \delta_{\omega_{\mathscr{R}^\star}}^{\phi(x_0)}$. Since $\Sigma_{\mathscr{R}^\star}$ is derived from $\Gamma(\Sigma)/\mathscr{R}^\star$ by one-to-one mapping from $\Pi_{\mathscr{R}^\star}$ to $\Delta_{\omega_{\mathscr{R}^\star}}$, formula (14) is equal to

$$\lim_{t\to\infty} \mathbb{P}\{x_{\mathscr{R}^\star}(t;\theta_0,s^0) = s^\star\} = 1, \forall\ s^0 \in \Delta_{\omega_{\mathscr{R}^\star}}, \forall\ \theta_0 \in \Omega,$$

It indicates that $\Sigma_{\mathscr{R}^\star}$ achieves $s^\star$-stability in distribution. Subsequently, the proof of sufficiency can be derived backwards. ■

As indicated in Remark 3.2, if $\mathscr{O} = \mathscr{O}_c$, stable SFFN (4) achieves synchronization. In this case, an p^n-state SFFN (4) can be reduced into a QTS with the number of nodes no greater than $n+1$, which is revealed in the following theorem.

Theorem 4.2: ***(QTS for $\mathscr{O}_c$-stability has no greater than $n+1$ nodes).*** The number of nodes in the smallest QTS for solving the $\mathscr{O}_c$-stability of SFFN (4) is no greater than $n+1$.

Proof: In what follows, we prove the statement that SFFN (4) achieves global stability with respect to $\mathscr{O}_c$, if and only if $\Sigma_{\mathscr{R}^\star}$ is stable to $s^\star$ from

$$s^0 \in \Theta := \{\delta_{\omega_{\mathscr{R}^\star}}^{\phi(\sigma(\delta_n^\kappa))} | \kappa \in [1,n]\}.$$

If this statement is proved, we can analyze the stability of SFFN (4) just based on states $s^\star$ and $\delta_n^\kappa, \kappa \in [1,n]$. The smallest QTS induced by these $n+1$ states has no greater than $n+1$ nodes. Notably, SFFN (4) being $\mathscr{O}_c$-stable and QTS $\Sigma_{\mathscr{R}^\star}$ being $s^\star$-stable imply that $\mathscr{O}_c$ is an invariant subset of $A_{\theta(t)}$ in $\mathbb{F}_p^n$. It derives that $A_{\theta(t)}$ is row-stochastic, that is, $A_{\theta(t)}\mathbf{1}_n = \beta\mathbf{1}_n$ with $\beta \in \mathbb{F}_n$. Subsequently, the proof is divided into two parts: one for the $\mathscr{O}_c$-SSD, and another for the $\mathscr{O}_c$-FTSSPO.

SSD case: First, we prove that the SSD towards $\mathscr{O}_c$ of SFFN (4) is achieved starting from any $x_0 \in \mathbb{F}_p^n$ if and only if it is achieved starting from $\delta_n^\kappa, \kappa \in [1,n]$. The necessity obviously holds owing to $\{\delta_n^i \mid i \in [1,n]\} \subseteq \mathbb{F}_p^n$, and then we prove the sufficiency. On the one hand, denoting

$$\Xi(t;\sigma) := A_{\sigma(t-1)} \times_p A_{\sigma(t-2)} \times_p \cdots \times_p A_{\sigma(0)}$$

with $\sigma(\tau) \in \Omega$ for $\tau \in [0,t-1]$, it holds

$$\lim_{t\to\infty} \mathbb{P}\{\Xi(t;\sigma) \times_p \delta_n^\kappa \in \mathscr{O}_c\} = 1, \ \forall\ \kappa \in [1,n].$$

On the other hand, considering $\mathbf{b}_n \in \mathscr{O}_c$ with $b \in \mathbb{F}_p$, one obviously has $a \times_p \mathbf{b}_n \in \mathscr{O}_c$ and $a \times_p \mathbf{b}_n +_p c \times_p \mathbf{b}_n \in \mathscr{O}_c$ for any $a, c \in \mathbb{F}_p$. Since each state $x_0 \in \mathbb{F}_p^n$ of SFFN (4) can be expressed as $x_0 = \alpha_1\delta_n^1 +_p \alpha_2\delta_n^2 +_p \cdots +_p \alpha_n\delta_n^n$ with $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}_p$, one derives

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}\{x(t;\theta_0,x_0) \in \mathscr{O}_c\}\\
&= \lim_{t\to\infty} \mathbb{P}\{\Xi(t;\sigma) \times_p x_0 \in \mathscr{O}_c\}\\
&= \lim_{t\to\infty} \mathbb{P}\{\alpha_1\Xi(t;\sigma) \times_p \delta_n^1 +_p \alpha_2\Xi(t;\sigma) \times_p \delta_n^2 +_p \cdots\\
&\qquad +_p \alpha_n\Xi(t;\sigma) \times_p \delta_n^n \in \mathscr{O}_c\} = 1.
\end{aligned}$$

From Definition 3.2, SFFN (4) with $x_0 \in \mathbb{F}_p^n$ achieves SSD towards $\mathscr{O}_c$.

Invoking the construction of $\Sigma_{\mathscr{R}^\star}$ and Theorem 4.1, we can draw the conclusion that SFFN (4) achieves SSD towards $\mathscr{O}_c$ from $\delta_n^\kappa, \kappa \in [1,n]$, if and only if $\Sigma_{\mathscr{R}^\star}$ achieves stability in distribution towards $s^\star$ from any $s^0 \in \Theta$. Therefore, the proof for the SSD case is completed.

FTSSPO case: The necessity is evidently satisfied since any initial state $s^0 \in \Theta$ corresponds to $\delta_n^\kappa, \kappa \in [1,n]$, which belongs to $\mathbb{F}_p^n$. Next, we prove the sufficiency. Suppose that there exists a time step $T \in \mathbb{N}_{[1,p^n]}$ such that

$$\mathbb{P}\{\mathbf{Q}_{\mathscr{R}^\star}^t s^0 = s^\star\} = 1, \ \forall\ s^0 \in \Theta, \ \forall\ t \geq T,$$

which implies

$$\mathbb{P}\{\Xi(t;\sigma) \times_p \delta_n^\kappa \in \mathscr{O}_c\} = 1, \ \forall\ \kappa \in [1,n], \ \forall\ t \geq T.$$

Then, for each $x_0 \in \mathbb{F}_p^n$, one can derive that

$$\begin{aligned}
\Xi(T;\sigma) \times_p x_0 &= \alpha_1\Xi(T;\sigma) \times_p \delta_n^1 +_p \alpha_2\Xi(T;\sigma) \times_p \delta_n^2\\
&\quad +_p \cdots +_p \alpha_n\Xi(T;\sigma) \times_p \delta_n^n\\
&= \beta_1\mathbf{1}_n +_p \beta_2\mathbf{1}_n +_p \cdots +_p \beta_n\mathbf{1}_n\\
&= \beta^\star(x_0)\mathbf{1}_n \in \mathscr{O}_c,
\end{aligned}$$

where $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{F}_p$ and $\beta^\star(x_0) := \sum_{\kappa=1}^n \beta_\kappa$. In particular, $\beta^\star(x_0)\mathbf{1}_n$ is the state that SFFN (4) with initial state x_0 will evolve into at time instant T, and thus SFFN (4) with $x_0 \in \mathbb{F}_p^n$ achieves FTSSPO towards $\mathscr{O}_c$. Therefore, the proof for the FTSSPO case is completed. ■

Remark 4.2: ***(Network scale reduction from p^n to $n+1$).*** Theorem 4.2 provides a way to check whether SFFN (4) achieves SSD or FTSSPO towards $\mathscr{O}_c$ through n basis states instead of all p^n states. Accordingly, a smaller QTS with the number of nodes no greater than $n+1$ can be further constructed. This obtained QTS still contains enough transition information about the original SFFN, and the stability of the original SFFN can be inferred by analyzing the evolution of the QTS $\Sigma_{\mathscr{R}^\star}$ with a much smaller network scale. It is worth mentioning that this remarkable reduction does not arise in Boolean networks (see, e.g., [41]–[43]) owing to the fundamental differences between Boolean networks and FFNs as mentioned in Remark 3.1. Besides, it should be pointed out that such remarkable reduction must hold for consensus and synchronization issues, whose target set is $\mathscr{O}_c$, but may not hold for other issues. ■

In addition, as clarified in [4] and [28], if the states in $\mathscr{O}_c$ are all equilibrium points of SFFN (4), that is, the modes are all row stochastic, the synchronous $\Sigma_{\mathscr{R}^\star}$ achieves consensus. In this case, we can define preliminary relation $\tilde{\mathscr{R}}^\circ$ such that the preliminary partition of Δ_{p^n} is

$$\{\bar{\sigma}(\mathbf{0}_n)\}, \{\bar{\sigma}(\mathbf{1}_n)\}, \ldots, \{\bar{\sigma}((p-1)\mathbf{1}_n)\}, \ \Delta_{p^n}\backslash\{\bar{\sigma}(\mathbf{b}_n) \mid b \in \mathbb{F}_p\}.$$

Based on $\tilde{\mathscr{R}}^\circ$, one can derive the smallest QTS as $\Sigma_{\tilde{\mathscr{R}}^\star}$ with the size $\omega_{\tilde{\mathscr{R}}^\star}$. Then, we denote

$$\mathscr{S}^\star := \{\delta_{\omega_{\mathscr{R}^\star}}^{\phi(\bar{\sigma}(\mathbf{0}_n))}, \delta_{\omega_{\mathscr{R}^\star}}^{\phi(\bar{\sigma}(\mathbf{1}_n))}, \ldots, \delta_{\omega_{\mathscr{R}^\star}}^{\phi(\bar{\sigma}((p-1)\mathbf{1}_n))}\}$$

and deduce the following equivalent relation for the consensus of SFFN (4), which includes the consensus in distribution and finite-time consensus with probability one.

Corollary 4.1: ***(Equivalence of Consensus).*** SFFN (4) achieves global consensus, if and only if $\check{D}_{\theta(t)} \times_p a = a$ for all $a \in \mathscr{S}^\star$, and $\Sigma_{\tilde{\mathscr{R}}^\star}$ achieves $\mathscr{S}^\star$-stability from $s^0 \in \Theta$.

C. Set Stability Conditions

In what follows, we consecutively explore the necessary and sufficient conditions for the $\mathscr{O}$-stability of SFFN (4) based on the constructed QTS $\Sigma_{\mathscr{R}^\star}$. Before which, we introduce some notions in DTMC. A path π of DTMC $(\Delta_{\omega_{\mathscr{R}^\star}}, \mathbf{Q}_{\mathscr{R}^\star})$ with initial state $\mathrm{x}_{\mathscr{R}^\star}(0) := s^0$ is a (possibly infinite) sequence of states $\pi = s^0, s^1, \ldots$ satisfying $\mathbf{Q}_{\mathscr{R}^\star}(s^i, s^{i+1}) > 0$ for each integer $i \geq 0$, and its existence probability is calculated by

$$\mathbb{P}\{\pi\} = \prod_{i=0}^{\infty} \mathbf{Q}_{\mathscr{R}^\star}(s^i, s^{i+1}). \tag{15}$$

In particular, if $\pi = s^0, s^1, \ldots, s^n$, its probability is equal to $\mathbb{P}\{\pi\} = \prod_{i=0}^{n-1} \mathbf{P}(s^i, s^{i+1})$, which is also the probability of a path from s^0 to s^n. Denoting $R(s^0, s^\star)$ as the (possibly infinite) set of all paths from s^0 to $s^\star$, the probability of reaching $s^\star$ from s^0, defined as $\mathbb{PR}\{s^0, s^\star\}$, equals to the sum of the probabilities of all paths leading to $s^\star$ from s^0, that is,

$$\mathbb{PR}\{s^0, s^\star\} = \sum_{\pi \in R(s^0, s^\star)} \mathbb{P}\{\pi\}. \tag{16}$$

*Theorem 4.3: **(Criterion for SSD).*** SFFN (4) achieves SSD with respect to $\mathscr{O}$ if and only if $\mathbf{Q}_{\mathscr{R}^\star} s^\star = s^\star$ and $|R(\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star)| > 0$ for all $\delta^j_{\omega_{\mathscr{R}^\star}} \in \Delta_{\omega_{\mathscr{R}^\star}}$.

Proof: (Necessity). Assume that SFFN (4) achieves SSD towards $\mathscr{O}$. From Theorem 4.1 we acquire that $\Sigma_{\mathscr{R}^\star}$ achieves stability in distribution towards $s^\star$; by Proposition 4.2, we can deduce $\mathbf{Q}_{\mathscr{R}^\star} s^\star = s^\star$. If there exists a state $\delta^j_{\omega_{\mathscr{R}^\star}} \in \Delta_{\omega_{\mathscr{R}^\star}}$ such that $|R(\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star)| = 0$, it leads to $\mathbb{PR}\{\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star\} = 0$, implying $\mathrm{x}_{\mathscr{R}^\star}(t; \theta_0, \delta^j_{\omega_{\mathscr{R}^\star}}) \neq s^\star$ for all integers $t \geq 0$, which conflicts with the assumption.

(Sufficiency). $\mathbf{Q}_{\mathscr{R}^\star} s^\star = s^\star$ induces that $\mathbb{P}\{\mathrm{x}_{\mathscr{R}^\star}(1; \theta_0, s^\star) = s^\star\} = 1$ holds for any $\theta_0 \in \Omega$. Furthermore, $|R(\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star)| > 0$ for all $\delta^j_{\omega_{\mathscr{R}^\star}} \in \Delta_{\omega_{\mathscr{R}^\star}}$ guarantees that, for each $\delta^j_{\omega_{\mathscr{R}^\star}} \in \Delta_{\omega_{\mathscr{R}^\star}}$, there exists a path from $\delta^j_{\omega_{\mathscr{R}^\star}}$ to $s^\star$. By referring to Theorem 4.1 in [29], one can draw a conclusion that $\Sigma_{\mathscr{R}^\star}$ achieves $s^\star$-stability, then the sufficiency is proved by Theorem 4.1. ■

*Theorem 4.4: **(Criterion for FTSSPO).*** SFFN (4) achieves FTSSPO with respect to $\mathscr{O}$, if and only if $\mathbf{Q}_{\mathscr{R}^\star} s^\star = s^\star$ and $0 < |R(\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star)| < \infty$ for all $\delta^j_{\omega_{\mathscr{R}^\star}} \in \Delta_{\omega_{\mathscr{R}^\star}}$.

Proof: (Necessity). Noted that SFFN (4) achieving FTSSPO towards $\mathscr{O}$ must achieve SSD towards $\mathscr{O}$, the necessary conditions for SSD must hold here. Subsequently, we prove that $|R(\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star)| < \infty$ holds for all $\delta^j_{\omega_{\mathscr{R}^\star}} \in \Delta_{\omega_{\mathscr{R}^\star}}$. If there exists a state $\delta^j_{\omega_{\mathscr{R}^\star}} \in \Delta_{\omega_{\mathscr{R}^\star}}$ such that $|R(\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star)| = \infty$, there exists a state set $\mathscr{C} \subseteq \Delta_{\omega_{\mathscr{R}^\star}} \backslash \{s^\star\}$ satisfying

$$\mathbb{P}\{\mathrm{x}_{\mathscr{R}^\star}(t; \theta_0, \mathrm{x}_{\mathscr{R}^\star}(0)) \in \mathscr{C} \mid \mathrm{x}_{\mathscr{R}^\star}(0) \in \mathscr{C}\} =: \varepsilon(t) > 0$$

for any $\theta_0 \in \Omega$ and all integer $t \geq 0$. It leads to

$$\mathbb{P}\{\mathrm{x}_{\mathscr{R}^\star}(t; \theta_0, \mathrm{x}_{\mathscr{R}^\star}(0)) \in \mathscr{O} \mid \mathrm{x}_{\mathscr{R}^\star}(0) \in \mathscr{C}\} \leq 1 - \varepsilon(t) < 1$$

for any $\theta_0 \in \Omega$ and all integer $t \geq 0$. Hence, $\Sigma_{\mathscr{R}^\star}$ does not achieve finite-time $s^\star$-stability with probability one, and from Theorem 4.1, SFFN (4) does not achieve $\mathscr{O}$-FTSSPO, which conflicts with the assumption. Therefore, the necessity is proved.

(Sufficiency). $\mathbf{Q}_{\mathscr{R}^\star} s^\star = s^\star$ implies that $s^\star$ is an attractor of $\Sigma_{\mathscr{R}^\star}$; $|R(\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star)| > 0$ indicates $\mathbb{PR}\{\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star\} > 0$; $|R(\delta^j_{\omega_{\mathscr{R}^\star}}, s^\star)| < \infty$ for all $\delta^j_{\omega_{\mathscr{R}^\star}} \in \Delta_{\omega_{\mathscr{R}^\star}}$ implies that no attractor rather than $s^\star$ exists in $\Sigma_{\mathscr{R}^\star}$. Therefore, for each $\delta^j_{\omega_{\mathscr{R}^\star}} \in \Delta_{\omega_{\mathscr{R}^\star}}$, there exists a positive integer $T(\delta^j_{\omega_{\mathscr{R}^\star}})$ such that

$$\mathbb{P}\{\mathrm{x}_{\mathscr{R}^\star}(t; \theta_0, \delta^j_{\omega_{\mathscr{R}^\star}}) = s^\star\} = 1, \forall\ \theta_0 \in \Omega,\ \forall\ t \geq T(\delta^j_{\omega_{\mathscr{R}^\star}}).$$

Owing to the finiteness of state space $\Delta_{\omega_{\mathscr{R}^\star}}$, integer $T(\delta^j_{\omega_{\mathscr{R}^\star}})$ must be no greater than $\omega_{\mathscr{R}^\star}$. Letting $T := \max_{1 \leqslant j \leqslant \omega_{\mathscr{R}^\star}} T(\delta^j_{\omega_{\mathscr{R}^\star}})$, one has

$$\mathbb{P}\{\mathrm{x}_{\mathscr{R}^\star}(t; \theta_0, s^0) = s^\star\} = 1, \\ \forall\ \theta_0 \in \Omega,\ \forall\ s^0 \in \Delta_{\omega_{\mathscr{R}^\star}},\ \forall\ t \geq T.$$

Therefore, $\Sigma_{\mathscr{R}^\star}$ achieves finite-time $s^\star$-stability with probability one, and SFFN (4) achieves $\mathscr{O}$-FTSSPO by Theorem 4.1. ■

*Remark 4.3: **(Criterion for Synchronization).*** If $\mathscr{O} = \mathscr{O}_c$, set stability of SFFN (4) evolves into the synchronization of SFFN (4). By combining to Theorem 4.2, one can derive that SFFN (4) achieves SSD (resp., FTSSPO) with respect to $\mathscr{O}_c$ if and only if it holds $\mathscr{O}_c = [\mathbf{1}_n]$, $\mathbf{Q}_{\mathscr{R}^\star} s^\star = s^\star$ and $|R(s^0, s^\star)| > 0$ (resp., $0 < |R(s^0, s^\star)| < \infty$) for all $s^0 \in \Theta$. ■

*Remark 4.4: **(Criterion for Consensus).*** To check the consensus of SFFN (4), we can first determine whether its network matrices are row stochastic and next determine its $\mathscr{O}_c$-stability. This procedure can be mathematically expressed to meet the following conditions:

(i) $A_\upsilon \mathbf{1}_n = \mathbf{1}_n$ holds for all $\upsilon \in \Omega$;
(ii) $\mathbf{Q}_{\mathscr{R}^\star} s^\star = s^\star$;
(iii) (for consensus in distribution) $|R(s^0, s^\star)| > 0$ holds for all $s^0 \in \Theta$;
(iii') (for finite-time consensus with probability one) $0 < |R(s^0, s^\star)| < \infty$ holds for all $s^0 \in \Theta$.

■

V. Simulation and Analysis

In this section, we first illustrate the stability analysis for an SFFN via the QTS-based method, and then we explore the number of iterations to obtain the smallest QTS and the number of nodes in the obtained QTS for a series of randomly generated SFFNs.

A. An Illustrative Example

Here, we consider the Example 5.1 in [29], where SFFN (4) over finite field $\mathbb{F}_3$ has three modes as

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{17}$$

with the probability distribution $\mathscr{D} = [0.5, 0.3, 0.2]$. Next, we study the synchronization of this SFFN, and the desired set is preassigned as $\mathscr{O}_c = \{\mathbf{0}_3, \mathbf{1}_3, 2\mathbf{1}_3\}$. The state transition graph starting from the states $\delta^2_{27}, \delta^4_{27}, \delta^{10}_{27}$, corresponding to three basis vectors $\delta^1_3, \delta^2_3, \delta^3_3$, is depicted as Fig. 2. In Fig. 2, node i represents state δ^i_{27}, and each directed edge (i, j) refers to a possible transition from state δ^i_{27} to state δ^j_{27} with the transition probability marked above.

Step 1: Construct the smallest QTS. By calling (10), one can derive the coarsest equivalence relation as

$$\mathscr{R}^\star = \{(\delta^i_{27}, \delta^j_{27}), (\delta^j_{27}, \delta^i_{27}) \mid i, j \in C_k, k = 1, 2, 3, 4\}$$

with

$$C_1 = \{1, 14, 27\}, \\ C_2 = \{2, 3, 13, 15, 20, 25\}, \\ C_3 = \{4, 7, 11, 17, 21, 24\}, \\ C_4 = \{5, 9, 10, 18, 19, 23\}.$$

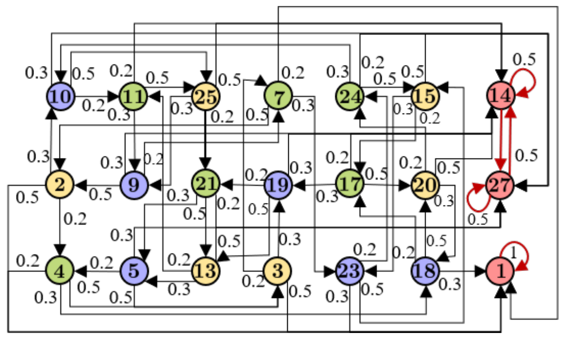


Fig. 2. State transition graph of SFFN (4) defined in Subsection V-A starting from the initial state $\delta_{27}^{2}, \delta_{27}^{4}, \delta_{27}^{10}$.

Besides, by Algorithm **QGA**, the smallest QTS is obtained as

$$\Gamma(\Sigma)/\mathscr{R}^{\star} = (\Pi_{\mathscr{R}^{\star}}, \mathbf{Q}_{\mathscr{R}^{\star}})$$

with $\Pi_{\mathscr{R}^{\star}} = \delta_{27}\{[1],[2],[4],[5]\}$ and

$$\mathbf{Q}_{\mathscr{R}^{\star}} = \begin{bmatrix} 1 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0.2 & 0 & 0.2 \\ 0 & 0.3 & 0.3 & 0 \end{bmatrix}.$$

The quotient transition graph is derived as in Fig. 3, where node i stands for quotient $[\delta_{27}^{i}]$, and each directed edge (i,j) refers to a possible transition from the state in quotient $[\delta_{27}^{i}]$ to the state in quotient $[\delta_{27}^{j}]$ with the transition probability marked above.

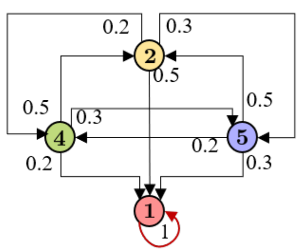


Fig. 3. Quotient transition graph corresponding to the state transition graph in Fig. 2.

Step 2: Verify the set stability criterion. With regards to the QTS $\Sigma_{\mathscr{R}^{\star}}$, it holds

$$\mathbf{Q}_{\mathscr{R}^{\star}}\delta_{4}^{1} = \delta_{4}^{1},$$

$$|R(\delta_{4}^{2},\delta_{4}^{1})| = \infty,\ |R(\delta_{4}^{3},\delta_{4}^{1})| = \infty,\ |R(\delta_{4}^{4},\delta_{4}^{1})| = \infty.$$

Therefore, by Theorem 4.3, one can draw the conclusion that this SFFN achieves SSD with respect to $\mathscr{O}_c$, more precisely, synchronization in distribution. However, by Theorem 4.4, such an SFFN does not achieve FTSSPO with respect to $\mathscr{O}_c$, more precisely, finite-time synchronization with probability one.

Remark 5.1: When checking whether SFFN (4) defined as (17) achieves synchronization, one can only check its corresponding QTS with only 4 nodes, which is far less than the original 27 states and less than 21 nodes indicated in Fig. 2 given in [29]. ■

B. Comparative Simulation

To explore the number of iterations required for obtaining the smallest QTS and the number of nodes in the induced QTS, we use our designed "Quotient Generator" to randomly generate three kinds of SFFNs and further obtain a minimal number of quotients by calculating the coarsest quotient set based on a preliminary partition of state set.

Type I: Deterministic FFNs. In this scenario, the PTM $\mathbf{P}$ is a logical matrix, rather than a stochastic matrix. We consider a 3-node FFN over $\mathbb{F}_3$ with the preliminary partition of state set

$$\{\{\delta_{27}^{1},\delta_{27}^{14},\delta_{27}^{27}\}, \Delta_{27}\backslash\{\delta_{27}^{1},\delta_{27}^{14},\delta_{27}^{27}\}\}. \tag{18}$$

We randomly generate 1600 FFNs or called samples. For each FFN, we count the number of iterations required for deriving the coarsest quotient set and calculate the number of nodes in the corresponding smallest QTS. Fig. 4 shows that the number of iterations is significantly less when the number of nodes in QTS is more than 14. Especially, there are 9.375% of Type I SFFNs whose number of iterations is equal to zero, which means the preliminary partition is a quotient set. As for the number of nodes in the smallest QTS, it is mainly distributed in the interval $[2,21]$, and its average is 7.944, which is significantly less than 27, that is, the total number of states. Most samples can be reduced to a 5-node QTS (190 in 1600 samples), and most samples only need 3 iterations to obtain the smallest QTS (240 in 1600 samples).

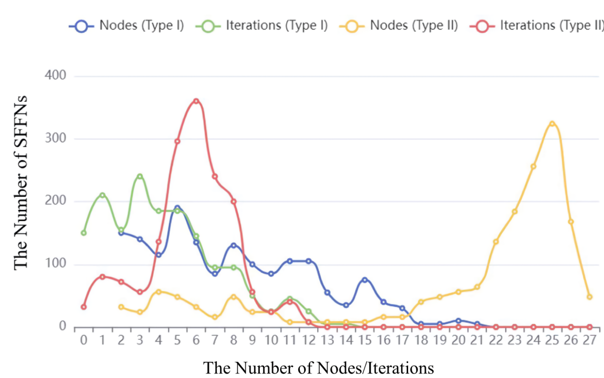


Fig. 4. Distributions of the number of nodes in the smallest QTS and iterations required for obtaining this smallest QTS among randomly generated 1600 Type I and Type II SFFNs over $\mathbb{F}_3$ with 27 states.

Type II: Equal-Probability-Switched FFNs. Next, we consider a kind of 3-node SFFNs with two modes, whose probability distribution is $\mathscr{D} = [0.5, 0.5]$. We generate 1600 SFFNs over the finite fields $\mathbb{F}_3$ and $\mathbb{F}_5$. For $\mathbb{F}_3$, the preliminary partition of Δ_{27} is (18); while for $\mathbb{F}_5$, the preliminary partition of Δ_{125} is

$$\{\{\delta_{125}^{125}\}, \Delta_{125}\backslash\{\delta_{125}^{125}\}\}. \tag{19}$$

Then, we count the number of iterations required for obtaining the coarsest quotient sets and the number of nodes in the corresponding smallest QTSs for $\mathbb{F}_3$ and $\mathbb{F}_5$, respectively. The results are presented in Fig. 4 and Fig. 5. The number of iterations mainly locates in the interval $[0,12]$ for the case of $\mathbb{F}_3$ as indicated in Fig. 4; while for the case of $\mathbb{F}_5$, it mainly locates in the interval $[0,20]$, as indicated in Fig. 5. In regard to the number of nodes in the obtained smallest QTS for the case of $\mathbb{F}_3$, it increases rapidly and even close to the total states when considering mode switching. In contrast, the distribution of the number of nodes in the smallest QTS derived from Type II SFFNs over $\mathbb{F}_5$ tends to be bipolar. More specifically, there are two different intervals ($[2,20]$ and $[96,120]$) with frequent samples. Besides, there are only 2% (that is, 32 in 1600 samples) of the generated Type II SFFNs over $\mathbb{F}_3$, whose number of nodes in the obtained QTS is 2; while there are 31% (that is, 496 in 1600 samples) of such SFFNs over $\mathbb{F}_5$. It indicates that the probability of an initial partition being the quotient set increases with the size of finite field $\mathbb{F}_p$, that is, the number of states in a Type II SFFN.

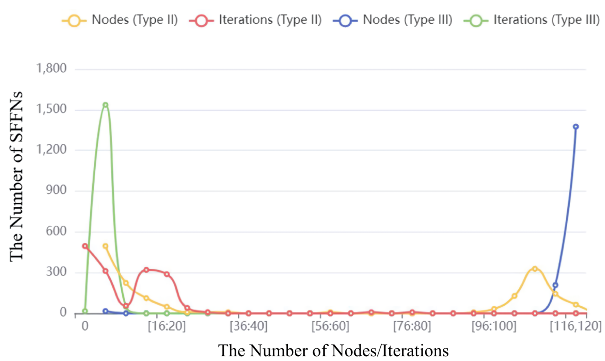


Fig. 5. Distributions of the number of nodes in the smallest QTS and iterations required for obtaining this smallest QTS among randomly generated 1600 Type II and Type III SFFNs over $\mathbb{F}_5$ with 125 states.

Type III: Random SFFNs. Furthermore, we randomly generate 1600 SFFNs over $\mathbb{F}_5$, which have three nodes and three modes obeying the probability distribution $\mathscr{D} = [0.1, 0.2, 0.7]$. Here, the state set is Δ_{125} with the preliminary partition as (19). As shown in Fig. 5, in this setup, the most frequent number of iterations required to obtain the smallest QTS locates in the interval $[0:20]$, which is similar to that for Type I and Type II SFFNs. Besides, the most of the numbers of nodes in the obtained smallest QTS are in the interval $[116,120]$, and only one Type III SFFN has been found to have two nodes in the smallest QTS. Therefore, it can be conjectured that, when the switching form becomes more complex (i.e., there are multiple switching modes with different probabilities of occurrence), the derived equivalent relation may be equal relation $\mathscr{R}_e$.

VI. Conclusion

In this paper, two kinds of set stability of SFFNs have been defined in terms of the stable time, that is, SSD and FTSSPO. The set stability analysis has been carried out by constructing quotients. First, by resorting to the STP of matrices, an SFFN has been transformed into a DTMC, based on which the preliminary relation has been preassigned according to the target set. Subsequently, the coarsest quotient set contained by this given preliminary relation has been calculated. In particular, the coarsest quotient set determines the coarsest partition of the state set, and will induce a QTS of the smallest size. Furthermore, the necessary and sufficient conditions have been established for the SSD and FTSSPO of SFFNs, respectively. They are, respectively, equivalent to the stability in distribution and finite-time stability with probability one of their corresponding QTS. Besides, the obtained criteria for the set stability of SFFNs have been applied to the cases of consensus and synchronization. In these cases, the scale in terms of the number of nodes in the QTS is much smaller than the scale of SFFN in the ASSR form.

It is worth mentioning that the formal simplicity of the obtained QTS makes it relatively convenient to handle classical control-theoretic problems for FFNs and SFFNs. In the future, we will try to utilize the QTS-based method to design a feasible control strategy (e.g., impulsive control, event-triggered control, and pinning control) such that an unstable FFN or SFFN can be stabilized to a target state set.

References

[1] G. Song, P. Shi, and C. P. Lim, "Distributed fault-tolerant cooperative output regulation for multiagent networks via fixed-time observer and adaptive control," *IEEE Transactions on Control of Network Systems*, vol. 9, no. 2, pp. 845–855, 2021.

[2] F. Rahimi and S. Ahmadpour, "Neighborhood-based distributed robust unknown input observer for fault estimation in nonlinear networked systems," *IET Control Theory & Applications*, vol. 16, no. 10, pp. 972–984, 2022.

[3] Y. Sun, J. Li, Z. Wang, X. He, Q. Fu, and Y. Zou, "Distributed formation-aggregation control algorithm for a cluster of quadrotors," *Journal of the Franklin Institute*, vol. 360, no. 3, pp. 1560–1581, 2023.

[4] F. Pasqualetti, D. Borra, and F. Bullo, "Consensus networks over finite fields," *Automatica*, vol. 50, no. 2, pp. 349–358, 2014.

[5] D. K. Molzahn, F. Dörfler, H. Sandberg, S. H. Low, S. Chakrabarti, R. Baldick, and J. Lavaei, "A survey of distributed optimization and control algorithms for electric power systems," *IEEE Transactions on Smart Grid*, vol. 8, no. 6, pp. 2941–2962, 2017.

[6] Z. Li, M. Shahidehpour, and F. Aminifar, "Cybersecurity in distributed power systems," *Proceedings of the IEEE*, vol. 105, no. 7, pp. 1367–1388, 2017.

[7] Z. Zhong, Y. Zhu, C.-M. Lin, and T. Huang, "A fuzzy control framework for interconnected nonlinear power networks under tds attack: Estimation and compensation," *Journal of the Franklin Institute*, vol. 358, no. 1, pp. 74–88, 2021.

[8] R. Han, S. Chen, S. Wang, Z. Zhang, R. Gao, Q. Hao, and J. Pan, "Reinforcement learned distributed multi-robot navigation with reciprocal velocity obstacle shaped rewards," *IEEE Robotics and Automation Letters*, vol. 7, no. 3, pp. 5896–5903, 2022.

[9] L. Dou, S. Cai, X. Zhang, X. Su, and R. Zhang, "Event-triggered-based adaptive dynamic programming for distributed formation control of multi-uav," *Journal of the Franklin Institute*, vol. 359, no. 8, pp. 3671–3691, 2022.

[10] Y. Yu, Z. Miao, X. Wang, and L. Shen, "Cooperative circumnavigation control of multiple unicycle-type robots with non-identical input constraints," *IET Control Theory & Applications*, vol. 16, no. 9, pp. 889–901, 2022.

[11] Y. Chen, S. Kar, and J. M. Moura, "Resilient distributed estimation: Sensor attacks," *IEEE Transactions on Automatic Control*, vol. 64, no. 9, pp. 3772–3779, 2018.

[12] D. Bertsekas and J. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific, 2015.

[13] J. Zhu, Z. Ge, and Z. Song, "Distributed parallel pca for modeling and monitoring of large-scale plant-wide processes with big data," *IEEE Transactions on Industrial Informatics*, vol. 13, no. 4, pp. 1877–1885, 2017.

[14] Y. Xiao, N. Zhang, J. Li, W. Lou, and Y. T. Hou, "Distributed consensus protocols and algorithms," *Blockchain for Distributed Systems Security*, vol. 25, p. 40, 2019.

[15] S. Zhu, J. Cao, L. Lin, J. Lam, and S.-i. Azuma, "Towards stabilizable large-scale boolean networks by controlling the minimal set of nodes," *IEEE Transactions on Automatic Control*, to be published, doi: 10.1109/TAC.2023.3269321.

[16] A. J. Berger and L. Lapidus, "An introduction to the stability of distributed systems via a Liapunov functional," *AIChE Journal*, vol. 14, no. 4, pp. 558–568, 1968.

[17] M. Meng, J. Lam, J.-E. Feng, and K. C. Cheung, "Stability and stabilization of Boolean networks with stochastic delays," *IEEE Transactions on Automatic Control*, vol. 64, no. 2, pp. 790–796, 2018.

[18] L. Wang, Z. Wu, and J. Lam, "Necessary and sufficient conditions for security of hidden Markov Boolean control networks under shifting attacks," *IEEE Transactions on Network Science and Engineering*, to be published, doi: 10.1109/TNSE.2022.3208335.

[19] A. Manita, "Probabilistic issues in the node synchronization problem for large distributed systems," *Lobachevskii Journal of Mathematics*, vol. 38, no. 5, pp. 948–953, 2017.

[20] M. Meng, X. Li, and G. Xiao, "Synchronization of networks over finite fields," *Automatica*, vol. 115, Article ID: 108877, 2020.

[21] Y. Yu, J.-E. Feng, B. Wang, and P. Wang, "Sampled-data controllability and stabilizability of Boolean control networks: Nonuniform sampling," *Journal of the Franklin Institute*, vol. 355, no. 12, pp. 5324–5335, 2018.

[22] J. Liang, H. Chen, and J. Lam, "An improved criterion for controllability of Boolean control networks," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 6012–6018, 2017.

[23] S. Zhu, J. Cao, L. Lin, L. Rutkowski, J. Lu, and G. Lu, "Observability and detectability of stochastic labeled graphs," *IEEE Transactions on Automatic Control*, to be published, doi: 10.1109/TAC.2023.3278797.

[24] S.-i. Azuma, T. Yoshida, and T. Sugie, "Structural oscillatority analysis of Boolean networks," *IEEE Transactions on Control of Network Systems*, vol. 6, no. 2, pp. 464–473, 2019.

[25] L. Song, N. Wan, A. Gahlawat, C. Tao, N. Hovakimyan, and E. A. Theodorou, "Generalization of safe optimal control actions on networked multi-agent systems," *IEEE Transactions on Control of Network Systems*, to be published, doi: 10.1109/TCNS.2022.3203479.

[26] C.-X. Liu, Y. Liu, Z.-J. Zhang, and Z.-Y. Cheng, "The novel authentication scheme based on theory of quadratic residues for wireless sensor networks," *International Journal of Distributed Sensor Networks*, vol. 9, no. 3, p. 829048, 2013.

[27] R. Lidl and H. Niederreiter, *Finite Fields*. No. 20, Cambridge University Press, 1997.

[28] X. Li, M. Z. Chen, H. Su, and C. Li, "Consensus networks with switching topology and time-delays over finite fields," *Automatica*, vol. 68, pp. 39–43, 2016.

[29] L. Lin, J. Cao, S. Zhu, and P. Shi, "Synchronization analysis for stochastic networks through finite fields," *IEEE Transactions on Automatic Control*, vol. 67, no. 2, pp. 1016–1022, 2021.

[30] L. Lin, J. Cao, S. Zhu, and P. Shi, "Minimum-time and minimum-triggering impulsive stabilization for multi-agent systems over finite fields," *Systems & Control Letters*, vol. 155, p. 104991, 2021.

[31] S. Sundaram and C. N. Hadjicostis, "Structural controllability and observability of linear systems over finite fields with applications to multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 60–73, 2012.

[32] D. Cheng, H. Qi, and Z. Li, *Analysis and Control of Boolean Networks: A Semi-Tensor Product Approach*. London, U.K.: Springer-Verlag, 2011.

[33] L. Lin, J. Cao, X. Liu, G. Lu, and M. Abdel-Aty, "Cluster synchronization of finite-field networks," *IEEE Transactions on Cybernetics*, to be published, doi: 10.1109/TCYB.2023.3273571.

[34] Y. Li, H. Li, X. Ding, and G. Zhao, "Leader-follower consensus of multiagent systems with time delays over finite fields," *IEEE Transactions on Cybernetics*, vol. 49, no. 8, pp. 3203–3208, 2018.

[35] Y. Li, H. Li, and S. Wang, "Finite-time consensus of finite field networks with stochastic time delays," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 67, no. 12, pp. 3128–3132, 2020.

[36] J.-F. Pan and M. Meng, "Optimal one-bit perturbation in Boolean networks based on cascading aggregation," *Frontiers of Information Technology & Electronic Engineering*, vol. 21, no. 2, pp. 294–303, 2020.

[37] Y. Zhao, B. K. Ghosh, and D. Cheng, "Control of large-scale Boolean networks via network aggregation," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 27, no. 7, pp. 1527–1536, 2015.

[38] K. Zhang and K. H. Johansson, "Efficient verification of observability and reconstructibility for large boolean control networks with special structures," *IEEE Transactions on Automatic Control*, vol. 65, no. 12, pp. 5144–5158, 2020.

[39] D. Cheng, L. Zhang, and D. Bi, "Invariant subspace approach to Boolean (control) networks," *IEEE Transactions on Automatic Control*, to be published, doi: 10.1109/TAC.2022.3175248.

[40] D. Bi, L. Zhang, and K. Zhang, "Structural properties of invariant dual subspaces of boolean networks," *arXiv preprint arXiv:2301.10961*, 2023.

[41] R. Li, Q. Zhang, and T. Chu, "Bisimulations of probabilistic Boolean networks," *SIAM Journal on Control and Optimization*, vol. 60, no. 5, pp. 2631–2657, 2022.

[42] R. Li, Q. Zhang, and T. Chu, "Quotients of probabilistic Boolean networks," *IEEE Transactions on Automatic Control*, to be published, doi: 10.1109/TAC.2022.3144273.

[43] R. Li, Q. Zhang, and T. Chu, "On quotients of Boolean control networks," *Automatica*, vol. 125, Article ID: 109401,2021.

[44] A. Chutinan and B. H. Krogh, "Verification of infinite-state dynamic systems using approximate quotient transition systems," *IEEE Transactions on Automatic Control*, vol. 46, no. 9, pp. 1401–1410, 2001.

[45] A. Graham, *Kronecker products and matrix calculus with applications*. Courier Dover Publications, 2018.

[46] C. Khatri and C. R. Rao, "Solutions to some functional equations and their applications to characterization of probability distributions," *Sankhyā: The Indian Journal of Statistics, Series A*, pp. 167–180, 1968.

[47] R. Zhou, Y. Guo, and W. Gui, "Set reachability and observability of probabilistic Boolean networks," *Automatica*, vol. 106, pp. 230–241, 2019.

[48] S. Zhu, J. Lu, J. Zhong, Y. Liu, and J. Cao, "Sensors design for large-scale Boolean networks via pinning observability," *IEEE Transactions on Automatic Control*, vol. 67, no. 8, pp. 4162–4169, 2021.

[49] Q. Zhang, J.-E. Feng, and B. Wang, "Set reachability of Markovian jump Boolean networks and its applications," *IET Control Theory & Applications*, vol. 14, no. 18, pp. 2914–2923, 2020.

[50] S. Zhu, Y. Liu, Y. Lou, and J. Cao, "Stabilization of logical control networks: An event-triggered control approach," *Science China–Information Sciences*, vol. 63, Article ID: 112203, 2020.

[51] Y. Guo, P. Wang, W. Gui, and C. Yang, "Set stability and set stabilization of Boolean control networks based on invariant subsets," *Automatica*, vol. 61, pp. 106–112, 2015.

[52] S. Zhu, J. Lu, S.-i. Azuma, and W. X. Zheng, "Strong structural controllability of Boolean networks: Polynomial-time criteria, minimal node control, and distributed pinning strategies," *IEEE Transactions on Automatic Control*, to be published, doi: 10.1109/TAC.2022.3226701.

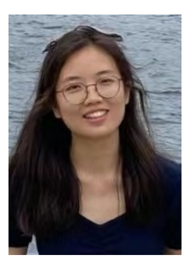
[53] R. A. Hernández Toledo, "Linear finite dynamical systems," *Communications in Algebra*, vol. 33, no. 9, pp. 2977–2989, 2005.

[54] A. Markov, *Dynamic Probabilistic Systems, volume 1: Markov Chains*. New York: John Wiley and Sons, 1971.

[55] Y. Guo, R. Zhou, Y. Wu, W. Gui, and C. Yang, "Stability and set stability in distribution of probabilistic Boolean networks," *IEEE Transactions on Automatic Control*, vol. 64, no. 2, pp. 736–742, 2018.

[56] G. Schmidt and T. Ströhlein, *Relations and Graphs: Discrete Mathematics for Computer Scientists*. Springer Science & Business Media, 2012.

[57] Y. Guo, Y. Ding, and D. Xie, "Invariant subset and set stability of Boolean networks under arbitrary switching signals," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 4209–4214, 2017.



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