# Steady-State Solution to the LWR Model on a Single Origin-Destination Parellel Road Network

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(*Received 00 Month 20XX; final version received 00 Month 20XX*)

This paper describes the systematical study of a steady-state solution to the Lighthill-Witham-Richards (LWR) model on a single origin-destination parallel road network, in which the user-equilibrium condition is satisfied and shocks on links are permitted. This study derives a novel static traffic assignment model considering the complete fundamental diagram, including the congested branch. For a composite links unit that includes a set of parallel links between two junctions, the user-equilibrium condition is discussed in detail and thus, the so-called user-equilibrium curves are defined. For a single origin-destination parallel road network, we note that shock structures must be introduced to guarantee the existence of the solution when a bottleneck exists, and thus we establish the correlation between the total number of vehicles and the steady-state solution. Moreover, the uniqueness of the solution is proved by introducing priority coefficients when shocks appear. We analytically give the complete solving procedure of the steady-state solution and thus, avoid the iterative algorithms used in other static traffic assignment models. A numerical scheme of the LWR network model is designed to converge the traffic flow into the discussed steady-state solution, by determining the percentages and priority coefficients at junctions. A numerical example is given to validate the convergence for the designed numerical scheme on a road network with two $2 \times 2$ junctions.

**Keywords:** shock discontinuities; user-equilibrium conditions; composite links unit; bottleneck effects; static traffic assignment

## 1. Introduction

Traffic assignment (TA) problems have been studied for several decades, from both static and dynamic views. Static TA models focus on the flow rates and travel times during a specified time period, in which the traffic flow on a road link is assumed to be steady or stationary. Dynamic TA models relax the steady assumption and thus, can describe time-variant traffic flows, travel times, and congestion dynamics. Although the static TA problem has been widely studied, the modeling theory needs further supplementation and improvement.

Traditional static TA models assume link performance functions, e.g., the Bureau of

Public Roads function (Bureau of Public Roads 1964), and impose no capacity (i.e., the allowed maximum flow) nor storage (i.e., the maximum number of stored vehicles) constraints on links. This implies that the link flow can exceed the actual link capacity and thus, queueing and spillback are ignored. To improve the traditional static TA models, several capacity constraint models have been proposed, which can be classified into two categories: models with steady queues and models with non-steady queues (Bliemer and Raadsen 2020). The models with steady queues essentially add the link flow capacity constraints into the optimization problem, which is a natural extension of the traditional models. These models maintain the assumption that the link inflow equals the outflow, which means the queues on links are steady. Thompson and Payne (1975) first formulated the TA problem as a side-constrained optimization problem and introduced a Lagrange multiplier, which accounts for the queueing delay. The travel time over a link is then composed of the flow-related travel time (which is determined by the link performance function) and the queueing delay. This type of model usually assumes that the number of queueing vehicles is unlimited, which means the storage of a link, and thus, the queue spillback, are not considered (Smith 1987; Larsson and Patriksson 1995; Bell 1995; Meng, Lam, and Yang 2008; Yang and Yagar 2008). Smith, Huang, and Viti (2013) further developed the work of Smith (1987), and considered the storage constraint by introducing a maximum queue length. Models with non-steady queues assume that the link inflow is greater than the link outflow, and thus, the queue on a link can grow dynamically with time. This type of model is not static in the strict sense, and thus, is also known as a quasi-dynamic TA model, even though the traffic status on the links is steady. These models do not require all vehicles to complete their trips, but rather store them in residual queues (Bliemer et al. 2014; Lam and Zhang 2000). These improved models are still based on link performance functions, which assume that travel time increases with link flow. And then link flow increases with density, if we assume the velocity is a monotonically increasing function of density. Thus, this is only suitable for the un-congested situation corresponding to the left branch of the fundamental diagram.

For the dynamic TA problem, the models with side constraints also have been proposed, in which the side constraints can represent the restrictions on the traffic volume. Zhong et al. (2011) formulated the side-constraint dynamic TA problem as an infinite-dimensional variational inequality, and proposed an algorithm based on Euler's discretization scheme and nonlinear programming to solve the model. Then, Graf and Harks (2023) gave a conter-example to the result from (Zhong et al. 2011), and proposed a new framework for side-constrained dynamic TA problem. Hoang et al. (2019) developed a linear programming framework to solve the dynamic TA problem with general capacitated constraints. It is worth noting that the side-constraint TA model does not directly consider the right-hand side of the fundamental diagram. There are several dynamic traffic flow network models based on the fundamental diagram theory, including first-order models, e.g., Lighthill-Witham-Richards (LWR) model (Lighthill and Whitham 1955; Richards 1956) and the cell transmission model (Daganzo 1995), and higher-order models (Lebacque 1995; Coclite, Garavello, and Piccoli 2005; Garavello et al. 2007; Lin et al. 2015, 2022). These models implicit the side constraints, because the physical link capacity and storage are restricted in the fundamental diagram. Moreover, these models have been introduced in dynamic TA models (Lo and Szeto 2002; Friesz et al. 2013; Zhang, Wolshon, and Dixit 2015; Cheng, Liu, and Szeto 2019; Li et al. 2020). Therefore, it remains to establish a static TA model based on the fundamental diagram theory, in which physical queueing is described by the shock, and the link capacity and storage are naturally considered.

Bliemer and Raadsen (2020) presented a static model with non-steady queues, in which the network loading component is derived from a dynamic link transmission model, considering the fundamental diagram. The model was formulated as a variational inequality and the existence of the solution was proved. The method of successive averages was used to solve the equivalent fixed point problem. This model is a type of quasi-dynamic TA model, in which the queues are not stationary or time-invariant. However, Jin (2015, 2017) reported that empirical observations suggest a steady state on a network, in which the locations and sizes of the queues (shocks) are nearly time-independent. The observations can be described by steady-state solutions of dynamic traffic flow models, such as LWR model. In this case, the traffic states remain unchanged with time, the flow on the link is constant, but the density can be constant or a stationary shock. Jin (2015, 2017) studied the existence and stability of steady-state solutions based on the multi-commodity LWR network model (Jin 2012). And, a brute-force method was presented to solve the steady-state solutions for a simple diverge-merge network. However, the detailed solving method and procedures were not discussed for a general road network. Moreover, the routes of vehicles were predefined, which means the user-equilibrium principle was not satisfied. In fact, it is challenging to establish a static model with steady shock structures, that can be analytically solved, for a general network.

In this work, we consider a single origin-destination (OD) parallel road network composed of sequential junctions, between which are multiple parallel links. We systematically study the steady-state solution to the LWR model and then provide a potentially key idea for extension to a general road network. In the steady-state solution, the user-equilibrium route choice conditions are considered and the flow at the junction is maximized. For travel demand or the total number of users being in a specific range, we find that if the link density is assumed to be constant, similar to that in traditional TA models, there is no steady-state solution. We should note that the present paper is the extension of the two previous works (Zhang et al. 2021; Lyu et al. 2021). However, Zhang et al. (2021) and Lyu et al. (2021) dealed with the simplest network that includes just a 1×2 and a 2×1 junctions. In addition to the network structure, the main extensions can be summarized as follows. i) The composite links unit, which consists of multiple links connecting two junctions is introduced. Then, the user-equilibrium condition of the composite links unit is derived, and the complete solving procedure of the solution under arbitrary travel demand or user number from 0 to the maximum number is discussed. ii) The theoretical properties of the steady-state equilibrium solution, including its existence and uniqueness, are discussed and proved in detail. The situations (along with all discussions and results) in Zhang et al. (2021) and Lyu et al. (2021) can be taken as special cases of the present paper.

The remainder of this paper is organized as follows. In Section 2, the steady state on a single OD parallel road network is systematically discussed. The analytical solving procedure for the steady-state solution is presented in Section 3. In Section 4, we give an example to validate the presented steady-state solution and numerical scheme. Section 5 concludes the paper. Appendix A describes the numerical convergence scheme based on the LWR network model.

## 2. The Steady State on a Single OD Parallel Road Network

We study a single OD parallel road network composed of sequential junctions, between which are multiple parallel links. Figure 1 shows the case with three junctions including $n$ links between junction $J_1$ and junction $J$, and $m$ links between junction $J$ and junction $J_2$. We explored the steady state on this road network by considering the whole fundamental diagram including the right branch in which the flow decreases with increasing density. Our study of the network's steady state led us to develop a new type of static TA model that is totally consistent with the LWR model or fundamental diagram theory,

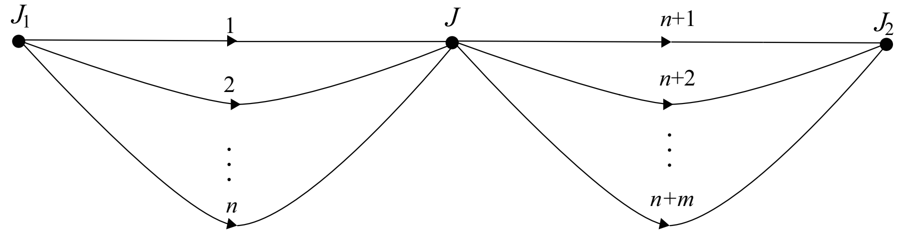



Figure 1. A simple single OD road network.

and thus, can describe both queues and spillback effects. Note that if we only consider the uncongested links, i.e., the left branch of the fundamental diagram, the model presented in this work is equivalent to the traditional static TA model. In this case, the theoretical properties, such as the existence and uniqueness of solutions, can be directly obtained. However, it is important to highlight that these theoretical properties cannot be directly extended to the model considering the complete fundamental diagram, due to the introduction of shock waves.

Note that if we only consider the uncongested links, i.e., the left branch of the fundamental diagram, the model presented in this work is equivalent to the traditional static TA model. In this case, the theoretical properties, such as the existence and uniqueness of solutions, can be directly obtained. It is obvious to similarly consider the congested links, i.e., the right branch of the fundamental diagram, in which case the solution can be uniquely derived. This appears to suggest a solution regarding to the complete fundamental diagram, in which case the density (rather than the flow) is taken as the solution variable. However, we find that the solution does not exist for the average density over (or the total number of users on) the road network, in which case shock waves must be introduced to complete the solution according to the LWR model (Section 2.2).

## 2.1. *Steady state on a link*

A link is defined as a stretch of road along which the number of lanes and free-flow speed are constant. Let $\rho(x,t)$ (veh/m) and $Q(x,t)$ (veh/s) denote the density per length and flow at time $t$ at position $x$. Velocity is then defined as $v(x,t) = Q(x,t)/\rho(x,t)$ (m/s), and we say that traffic flow is in a state of equilibrium if the following velocity–density or flow–density relationship is satisfied

$$V = V(\rho), \qquad Q = Q(\rho) = \rho V(\rho). \tag{1}$$

The LWR model can then be written as

$$\rho_t + Q(\rho)_x = 0. \tag{2}$$

We assume that all vehicles have the same size. Thus, the maximum $\rho_{jam}$ is constant and the link is blocked when $\rho = \rho_{jam}$. Equivalently, the storage of the link is $L\rho_{jam}$, where $L$ is the length of the link. The fundamental diagram, $V = V(\rho)$, is assumed to be a strictly decreasing function with

$$V(0) = V^f,\ V(\rho_{jam}) = 0,\ Q(\rho_{jam}) = 0.$$

Moreover, the function $Q$ is assumed to be sufficiently smooth and have, at most, one stagnation point at $\rho = \rho^*$, namely, $Q(\rho)$ strictly increases for $\rho \in [0, \rho^*]$ and strictly decreases for $\rho \in [\rho^*, \rho_{jam}]$. Thus, the capacity of the link is $Q(\rho^*)$.

According to the LWR model, traffic flow is viewed as being steady on a link if and only if the flow, $Q$, is constant. This suggests equilibrium solution satisfying (1), which can be generally expressed as follows

$$\rho(x) = \begin{cases} \rho^u, & 0 \le x < L^u, \\ \rho^d, & L^u \le x \le L, \end{cases} \qquad V(x) = \begin{cases} V(\rho^u), & 0 \le x < L^u, \\ V(\rho^d), & L^u \le x \le L, \end{cases} \tag{3}$$

where $x = L^u$ is the position of the shock interface, which divides the traffic into upstream and downstream subsections, with a low density, $\rho^u$, and a high density, $\rho^d$, respectively. Note that Eq. (3) also represents a constant solution with $\rho(x) = \rho^u$, for $L^u = L$, or $\rho(x) = \rho^d$, for $L^u = 0$. In the latter case, the queue propagates to the upstream junction, and the spillback arises.

However, these two types of constant solutions cannot cover the whole interval of the average density over (or the total number of users on) the studied road network, provided that the downstream junction functions as a bottleneck under the user-equilibrium condition. To complete the solution, a shock wave arises with $L^u > 0$, which is discussed in Section 3. Because the flow is static on the network, the shock structure must be stationary with

$$Q(\rho^u) = Q(\rho^d) = Q, \tag{4}$$

which obviously satisfies the Rankine-Hugoniot condition in hyperbolic conservation laws (Toro 1999).

Based on the assumption of $Q(\rho)$, Eq. (4) uniquely determines (Fig. 2)

$$\rho^u = \rho^u(Q) \in [0, \rho^*]; \qquad \rho^d = \rho^d(Q) \in (\rho^*, \rho_{jam}]. \tag{5}$$

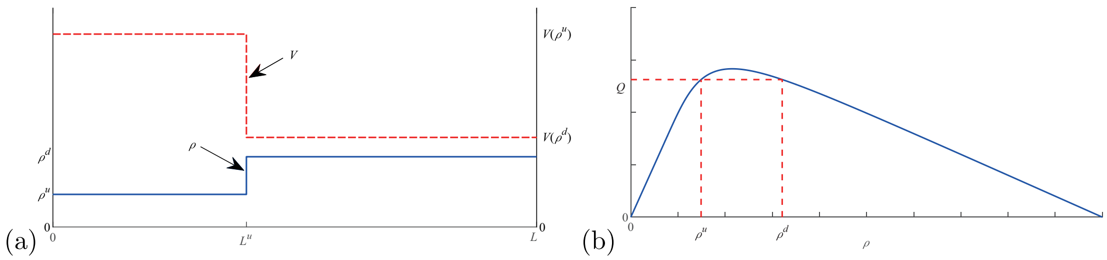



Figure 2. Steady state with shock structure on a link. (a) The density and velocity on a link, as described by Eq. (3); (b) One of $Q$, $\rho^u$, or $\rho^d$ can determine the other two variables, where $\rho^u$ and $\rho^d$ are strictly increasing and decreasing with flow, $Q$, respectively, and $\rho^d$ is strictly increasing with $\rho^u$.

The shock described in Eqs. (4)-(5) separates the upstream characteristics with a higher speed, $Q'(\rho^u)$, and the downstream characteristics with a lower speed, $Q'(\rho^d)$, which naturally satisfies the Rankine–Hugoniot condition for discontinuity (Whitham 1974; Toro 1999). The shock structure reduces to a trivial solution, $\rho(x) = \rho^u$ or $\rho(x) = \rho^d$ if and only if $L^u = L$ or $L^u = 0$.

## 2.2. *The user-equilibrium condition of a composite links unit*

A composite links unit is composed of $K$ links connected by an upstream junction, $J_u$, and a downstream junction, $J_d$, as shown in Fig. 3. Let $k$ index these links, such that $L_k$ and $l_k$ are the link length and number of lanes, respectively, and let $\rho_k$, $v_k$, and $q_k$

be the density, velocity and flow, respectively. This is similar to Eq. (1), and resulting in the following velocity–density and flow–density relationships on link $k$:

$$v_k = v_k(\rho_k), \quad q_k = \rho_k v_k(\rho_k) \equiv q_k(\rho_k), \;\; k = 1, \ldots, K,$$

together with similar properties.

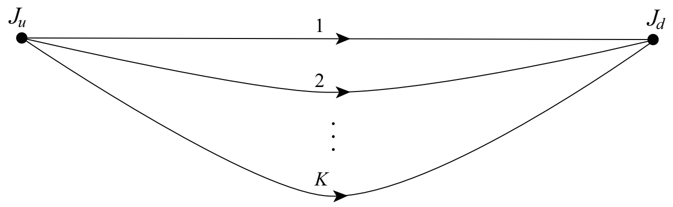



Figure 3. A composite links unit.

According to the discussion on a link, the steady state on a link is expressed as follows

$$\rho_k(x) = \begin{cases} \rho_k^u, & 0 \le x < L_k^u, \\ \rho_k^d, & L_k^u \le x \le L_k, \end{cases} \quad v_k(x) = \begin{cases} v_k(\rho_k^u), & 0 \le x < L_k^u, \\ v_k(\rho_k^d), & L_k^u \le x \le L_k, \end{cases} \quad k = 1, \ldots, K. \qquad (6)$$

The traveling time on each link can be computed by

$$T_k(\rho_k^u, L_k^u) = \frac{L_k^u}{v_k(\rho_k^u)} + \frac{L_k^d}{v_k(\rho_k^d)},$$

where $L_k^d = L_k - L_k^u$. Because $v_k(\rho_k^u) > v_k(\rho_k^d)$, $T_k$ is strictly increasing with $L_k^u$. When $L_k^u = L_k$, $T_k$ is strictly increasing with $\rho_k^u$. Therefore, we have

$$T_k(\rho_k^u, L_k^u) \ge T_k(\rho_k^u, L_k) \ge T_k(0, L_k) \equiv T_k^{min}.$$

$T_k = T_k^{min}$ if and only if $\rho_k^u = 0$ and $L_k^u = L_k$.

For convenience, we assume that any two of the minimal traveling times, $T_k^{min}$, are not equal, and any case that dose not satisfy this assumption can be taken as a degenerate or limit state of inequality. Without loss of generality, the $T_k^{min}$ values are sequenced from small to large, i.e.,

$$T_1^{min} < \cdots < T_K^{min}, \qquad (7)$$

which means that link 1 will be first utilized, followed by the others in sequence. Therefore, the user-equilibrium condition for traveling from $J_u$ to $J_d$ can be described as follows

$$T_k = \begin{cases} T_{k-1}, & T_{k-1} > T_k^{min}, \\ T_k^{min}, & T_{k-1} \le T_k^{min}, \end{cases} \qquad k = 2, \ldots, K. \qquad (8)$$

Equation (8) indicates that link $k$ is utilized with a traveling time equal to that on link $k-1$ if and only if the traveling time on link $k-1$ is greater than the minimal traveling time on link $k$. This condition is known as the user-equilibrium principle, which states that a link is utilized if and only if the traveling time is not greater than the traveling times on any of the other utilized links.

We then assume that there are no shock structures on any of the links, which means that the density, $\rho_k$ (as well as the velocity $v_k$), is constant,

$$\rho_k(x) = \rho_k,$$

where the constant $\rho_k = \rho_k^u$ or $\rho_k = \rho_k^d$, corresponding to $L_k^u = L_k$ or $L_k^d = L_k$, respectively, in Eq. (6). The user-equilibrium condition of (8) can then be expressed as follows

$$\frac{L_k}{v_k(\rho_k)} = \begin{cases} \dfrac{L_{k-1}}{v_{k-1}(\rho_{k-1})}, & \dfrac{L_{k-1}}{v_{k-1}(\rho_{k-1})} > \dfrac{L_k}{v_k^f}, \\ \dfrac{L_k}{v_k^f}, & \dfrac{L_{k-1}}{v_{k-1}(\rho_{k-1})} \leq \dfrac{L_k}{v_k^f}, \end{cases} \quad k = 2, \dots, K, \tag{9}$$

where $v_k^f$ is the free-flow velocity on link $k$.

Equation (9) suggests that the densities on the other links are implicit functions of the density on link 1, which can be computed using the following procedures.

Given $\rho_1$, implement the following for $k = 2, \dots, K$:

(1) Compute the critical density $\rho_{k-1}^c$ of $\rho_{k-1}$, by setting the equality of (9);
(2) If $\rho_{k-1} \leq \rho_{k-1}^c$, then set $\rho_k = \dots = \rho_K = 0$, and end the circulation;
(3) Otherwise, compute $\rho_k$ using the first equation of (9).

In this procedure, the existence and uniqueness of the solution are guaranteed by Eq. (7) and the strict monotonicity of $v_k(\cdot)$. Thus, $\rho_k = 0$ for $\rho_{k-1} \leq \rho_{k-1}^c$ by the second item of Eq. (9), and $\rho_k$ strictly increases with $\rho_{k-1}$ when $\rho_{k-1} \geq \rho_{k-1}^c$ by the first item of Eq. (9). Therefore, $\rho_k$ $(k > 1)$ is a function of $\rho_1$, and there exists some critical density, $\rho_1^k$ of $\rho_1$, such that $\rho_k = 0$ when $\rho_1 \leq \rho_1^k$, and $\rho_k$ strictly increases with $\rho_1$ when $\rho_1 \geq \rho_1^k$.

Equation (9) can be converted to the following expression related to per-area density, $\bar{\rho}_k$:

$$\bar{\rho}_k = \begin{cases} 0, \ \bar{\rho}_1 < \tilde{\rho}_1^k, \\ \bar{v}_k^{-1}(\dfrac{L_k}{l_1}\bar{v}_1(\bar{\rho}_1)), \ \bar{\rho}_1 \geq \tilde{\rho}_1^k, \end{cases} \quad k = 2, \dots, K, \tag{10}$$

where $\bar{\rho}_k = \rho_k / l_k$, $\bar{v}_k = v_k(\bar{\rho}_k l_k)$, and $\bar{v}_k^{-1}(\cdot)$ is the inverse function of $\bar{v}_k(\cdot)$. The critical density, $\tilde{\rho}_1^k$, can be determined by $\rho_{k-1}^c$ in the computational procedure, namely, $\tilde{\rho}_1^k = \bar{v}_1^{-1}(\frac{L_1 v_{k-1}(\rho_{k-1}^c)}{L_{k-1}})$. After the conversion, the function remains monotonic. Figure 4(a) shows the functions, $\bar{\rho}_k = \bar{\rho}_k(\bar{\rho}_1)$, which are called user-equilibrium curves. Here, $K = 3$ and the function $v_k(\cdot)$ is

$$v_k(\rho_k) = V^f[1 - exp(1 - exp(0.2(\rho_k^{jam}/\rho - 1)))].$$

The curves in Fig. 4(a) represent the functions $\bar{\rho}_k = \bar{\rho}_k(\bar{\rho}_1), \ k = 2, 3$, which are defined by Eq. (10), indicating changes of the average densities $\bar{\rho}_2, \ \bar{\rho}_3$ on links 2 and 3 with respect to the average density $\bar{\rho}_1$ on link 1. Because we assume that $T_1^{min} < T_2^{min} < T_3^{min}$, link 1 is to be chosen, followed by links 2 and 3 in sequence. When $\bar{\rho}_1$ is relatively small, no road users choose links 2 and 3, i.e., $\bar{\rho}_2 = \bar{\rho}_3 = 0$. When $\bar{\rho}_1 > \tilde{\rho}_1^2$, which suggests that $T_2^{min} < T_1$, some vehicles begin to choose link 2 with $\bar{\rho}_2 > 0, \ \bar{\rho}_3 = 0$. When $\bar{\rho}_1 > \tilde{\rho}_1^3$, the vehicles are actually assigned to all three links.

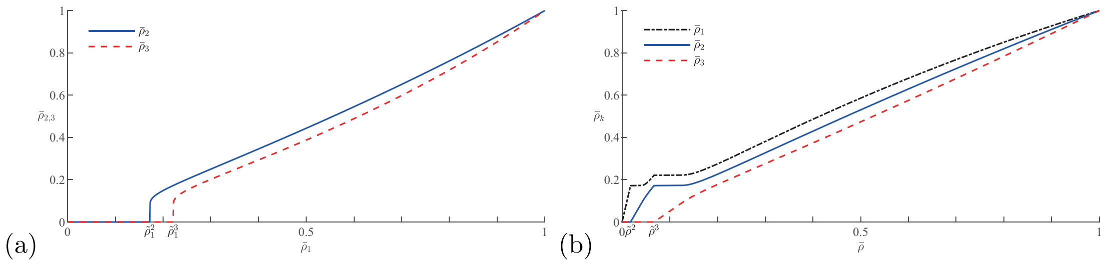



Figure 4. User-equilibrium curves. All densities are made dimensionless, by dividing through $\rho^{k,jam}$. $L_1/L_2 = L_2/L_3 = 0.8$, $l_1 = 1$, $l_2 = 2$, and $l_3 = 4$. (a) $\bar{\rho}_2$ and $\bar{\rho}_3$ vary with $\bar{\rho}_1$ (Eq. (10)); (b) $\bar{\rho}_1$, $\bar{\rho}_2$, and $\bar{\rho}_3$ vary with $\bar{\rho}$ (Eq. (11)).

We then define the average density over the area in the links unit as

$$\bar{\rho} = \frac{\sum_{k=1}^{K} N_k}{\sum_{k=1}^{K} L_k l_k} = \frac{\sum_{k=1}^{K} L_k \rho_k}{\sum_{k=1}^{K} L_k l_k} = \sum_{k=1}^{K} \frac{L_k l_k}{\sum_{k=1}^{K} L_k l_k} \bar{\rho}_k.$$

This equation indicates that $\bar{\rho}$ is the weighted mean of the average densities, $\bar{\rho}_k$. Note that $\bar{\rho}_k$ increases with $\bar{\rho}_1$, and $\bar{\rho}_1$ strictly increases with $\bar{\rho}_1$. Thus, $\bar{\rho}$ also increases with $\bar{\rho}_1$. This indicates that $\bar{\rho}_1$ can be replaced by $\bar{\rho}$, and then the following function holds

$$\bar{\rho}_k = \begin{cases} 0, \ \bar{\rho} < \tilde{\rho}^k, \\ \bar{v}_k^{-1}(\frac{L_k}{L_1}\bar{v}_1(\bar{\rho}_1(\bar{\rho}))), \ \bar{\rho} \geq \tilde{\rho}^k, \end{cases} \quad k = 1, \ldots, K, \tag{11}$$

where $\tilde{\rho}^1 = 0$, and other $\tilde{\rho}^k$ values are associated with the critical values of $\tilde{\rho}_1^k$. Setting the same parameters as those in Fig. 4(a), Fig. 4(b) shows the dependencies between $\bar{\rho}_k$ and $\bar{\rho}$, which are also called the user-equilibrium curves. The properties of these functions are similar to those in Fig. 4(a), in which case there always hold that $\bar{\rho}_1 \geq \bar{\rho}_2 \geq \bar{\rho}_3$.

Defining the total density of the links unit as $\rho = \sum_{k=1}^{K} l_k \bar{\rho}$ results in the following (total) flow–density relationship or fundamental diagram:

$$Q = \sum_{k=1}^{K} l_k \bar{q}_k(\bar{\rho}_k(\bar{\rho})) = \sum_{k=1}^{K} l_k \bar{q}_k(\bar{\rho}_k(\frac{\rho}{\bar{\rho}})) \equiv Q(\rho).$$

We note that the fundamental diagram is defined for a composite links unit rather than a single link. The definition indicates that, given the average density (or the total number of vehicles) on the composite links unit, we can obtain the total flow under the user-equilibrium condition of Eq. (8). This determines the fundamental diagram of the composite links unit, thus we can define the total demand and supply functions of the composite links unit similarly to that for a single link. Moreover, should introduce the shock structure to complete the steady-state solution.

### 2.3. *Ground rules for the steady-state solution*

In this section, we then discuss the ground rules for the steady-state solution for the single OD road network shown in Fig. 1. Based on Eq. (3), we denote the steady-state solutions on the upstream links unit and downstream links unit as

$$\rho_i(x) = \begin{cases} \rho_i^u, \ x \in [0, L_i^u], \\ \rho_i^d, \ x \in [L_i^u, L_i], \end{cases} \quad i = 1, \ldots, n, \tag{12}$$

and

$$\rho_j(x) = \begin{cases} \rho_j^u, \ x \in [0, L_j^u], \\ \rho_j^d, \ x \in [L_j^u, L_j], \end{cases} \quad j = n+1, \ldots, n+m,$$

where

$$\rho_k^u \in [0, \rho_k^*], \ \rho_k^d \in (\rho_k^*, \rho_k^{jam}], \ k = 1, \ldots, n+m.$$

Then, the steady state of the road network can be denoted as

$$q_k(\rho_k^u) = q_k(\rho_k^d) = q_k, \ k = 1, \ldots, n+m; \ \sum_{i=1}^{n} q_i = \sum_{j=n+1}^{n+m} q_j.$$

Considering the necessary conditions of the steady-state solution, we present the following ground rules.

(1) Supply–demand constraint condition. The flow of the upstream link, $q_i$, is not greater than its demand, $d_i$, while the flow of the downstream link, $q_j$, is not greater than its supply, $s_j$.
(2) User-equilibrium constraint condition. The upstream links unit and downstream links unit of junction $J$ satisfy Eq.(8).
(3) The principle of maximization of flow. The demand of upstream links should be satisfied and the supply of downstream links should be released, as possible, such that the total flow, $\sum_{i=1}^{n} q_i = \sum_{j=n+1}^{n+m} q_j$, at junction $J$ is maximized.
(4) Steady-keeping principle. If we set the initial steady-state solution, the numerical solution remains unchanged.

In Section 2.2, we discussed the user-equilibrium constraint condition without shocks, and presented the user-equilibrium curves. The case with shocks will be considered in the solving procedure in Section 3.2, and the steady-keeping principle of the solution will be reflected by numerical results without strict proving. Therefore, in this section, we discuss the other two ground rules.

First, we consider the supply–demand constraint condition. The demand function (Lebacque 1995) is defined as

$$d_i(\rho_i) = \begin{cases} q_i(\rho_i), \ \rho_i \in [0, \rho_i^*], \\ q_i(\rho_i^*), \ \rho_i \in [\rho_i^*, \rho_i^{jam}], \end{cases} \quad i = 1, \ldots, n,$$

where $\rho_i = \rho_i^u$ or $\rho_i^d$ is the density of the upstream link adjacent to junction $J$. The supply function (Lebacque 1995) is defined as

$$s_j(\rho_j) = \begin{cases} q_j(\rho_j^*), \ \rho_j \in [0, \rho_j^*], \\ q_j(\rho_j), \ \rho_j \in [\rho_j^*, \rho_j^{jam}], \end{cases} \quad j = n+1, \ldots, n+m,$$

where $\rho_j = \rho_j^u$ or $\rho_j^d$ is the density of the downstream link adjacent to junction $J$.

Considering the expression of the steady state in Eq. (12), we note that $\rho_i = \rho_i^d$ when $L_i^d \in (0, L_i]$ and $\rho_i = \rho_i^u$ when $L_i^d = 0$. Thus, the demand can be rewritten as

$$d_i(\rho_i^u) = sgn(L_i^d)d_i(\rho_i^d) + (1 - sgn(L_i^d))d_i(\rho_i^u),$$

where $sgn(\cdot)$ is the sign function, which is defined as

$$sgn(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Because $d_i(\rho_i^u) = q_i(\rho_i^u) = q_i,\ d_i(\rho_i^d) = q_i(\rho_i^*) \equiv q_i^*$, we have

$$d_i = d_i(q_i, L_i^d) = sgn(L_i^d)q_i^* + (1 - sgn(L_i^d))q_i.$$

Similarly, we note that $\rho_j = \rho_j^u$ when $L_j^u \in (0, L_j]$ and $\rho_j = \rho_j^d$ when $L_j^u = 0$. Thus, the supply can be rewritten as

$$s_j(\rho_j^u, L_j^u) = sgn(L_j^u)s_j(\rho_j^u) + (1 - sgn(L_j^u))s_j(\rho_j^d).$$

Because $s_j(\rho_j^u) = q_j(\rho_j^*) \equiv q_j^*,\ s_j(\rho_j^d) = q_j(\rho_j^d) = q_j$, we have

$$s_j(q_j, L_j^u) = sgn(L_j^u)q_j^* + (1 - sgn(L_j^u))q_j.$$

The supply-demand constraint condition requires $q_i \leq d_i(q_i, L_i^d)$ and $q_j \leq s_j(q_j, L_j^u)$. Thus, we have

$$sgn(L_i^d)q_i \leq sgn(L_i^d)q_i^*,\ i = 1, \dots, n, \tag{13}$$

$$sgn(L_j^u)q_j \leq sgn(L_j^u)q_j^*,\ j = n+1, \dots, n+m. \tag{14}$$

Note that Eqs. (13)–(14) always hold. Therefore, the steady-state solution always satisfies the supply–demand constraint condition.

Then, we discuss the principle of maximization of flow. For the steady-state solution with a shock structure, ($L_k^u \in (0, L_k)$), on link $k$, we have the following inequality:

$$q_k = (\frac{L_k^u}{L_k} + \frac{L_k^d}{L_k})q_k(\rho_k^d) < q_k(\frac{L_k^u}{L_k}\rho_k^u + \frac{L_k^d}{L_k}\rho_k^d).$$

Thus, if we reset the steady-state solution as $\check{\rho} = \frac{L_k^u}{L_k}\rho_k^u + \frac{L_k^d}{L_k}\rho_k^d$ without shock, the link has the same total number of users as the solution with shocks, and a greater flow.

We can also find the steady-state solution, $\hat{\rho}$ without shock, which has the same travel time as the solution with shocks, and greater flow. Specifically, we define function $f(\rho) = \frac{L_k^u}{v_k(\rho_k^u)} + \frac{L_k^d}{v_k(\rho_k^d)} - \frac{L_k}{v_k(\rho)}$, and can easily note that $f(\rho)$ strictly decreases with $\rho$ and $f(\rho_k^u) > 0,\ f(\rho_k^d) < 0$. Thus, there exists $\hat{\rho}_k \in (\rho_k^u, \rho_k^d)$, such that $q_k(\hat{\rho}_k) > q_k$ and $f(\hat{\rho}_k) = 0$, i.e.,

$$\frac{L_k}{v_k(\hat{\rho}_k)} = \frac{L_k^u}{v_k(\rho_k^u)} + \frac{L_k^d}{v_k(\rho_k^d)}.$$

Based on the above discussion, we conclude the following properties.

**Property 1.** For a steady-state solution with shocks on a link,

a. if the user number or average density is fixed, there is a steady-state solution without shock, such that the flow is greater than the solution with shocks.
b. If the travel time is fixed, there is a steady-state solution without shock, such that the flow is greater than the solution with shocks.

We can directly derive the following property through Property 1.

**Property 2.** For a links unit, the total flow is maximum if and only if all involved links have steady-state solutions without shocks.

These properties indicate that the solution with shocks is redundant and uneconomical. To satisfy the principle of maximization of flow, the steady-state solution with shocks should not be introduced if possible.

## 3. Analytical Solving Procedure for the Steady-State Solution

In this section, we discuss the steady-state solution for a single OD road network (Fig. 1), and analyze when and how to consider the shocks. The analytical solving procedure is then given for the steady-state solution.

### 3.1. *The steady-state equilibrium solution*

In Section 2.2, we discussed the steady-state solution of a links unit in detail and defined the user-equilibrium curves and the fundamental diagram. For the road network shown in Fig. 1, we denote the upstream and downstream links units and their variables using subscript $I$ and $II$. The fundamental diagram is written as

$$I : Q_I = Q_I(\rho_I),\ \rho_I = \sum_{i=1}^{n} l_i \bar{\rho}_I;\ II : Q_{II} = Q_{II}(\rho_{II}),\ \rho_{II} = \sum_{j=n+1}^{n+m} l_j \bar{\rho}_{II}.$$

As shown in Fig. 5, the maximum values of flow on links units $I$ and $II$ are denoted as $Q_I^* = Q_I(\rho_I^*)$ and $Q_{II} = Q_{II}(\rho_{II}^*)$, respectively. The maximum value of flow on the road network is then denoted as

$$Q_J^* = \min(Q_I^*, Q_{II}^*).$$

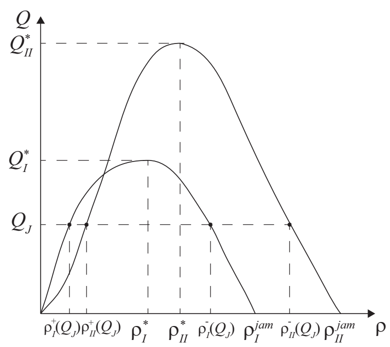



Figure 5. The fundamental diagrams of links units $I$ and $II$. There are, at most, two intersections between the horizontal line, $Q = Q_J$, and fundamental diagrams, $Q_I(\rho_I)$ or $Q_{II}(\rho_{II})$ (Eq. (15)).

We assume that $Q_I = Q_I(\rho_I)$ ($Q_{II} = Q_{II}(\rho_{II})$) has only one stationary point, $\rho_I^*$ ($\rho_{II}^*$), and thus, strictly increases when $\rho_I \le \rho_I^*$ ($\rho_{II} \le \rho_{II}^*$) and decreases when $\rho_I \ge \rho_I^*$ ($\rho_{II} \ge \rho_{II}^*$). Thus, there are two branches of inverse functions for $Q_I$ and $Q_{II}$:

$$I : \rho_I = \rho_I^{\pm}(Q_I);\ II : \rho_{II} = \rho_{II}^{\pm}(Q_{II}), \tag{15}$$

where the superscripts "+" and "−" denote the left and right branch inverse functions, respectively. Based on the fundamental diagrams, we define the total demand of the upstream links unit as

$$D_I(\rho_I) = \begin{cases} Q_I(\rho_I),\ \rho_I \in [0, \rho_I^*], \\ Q_I(\rho_I),\ \rho_I \in [\rho_I^*, \rho_I^{jam}], \end{cases}$$

and the total supply of the downstream links unit as

$$S_{II}(\rho_{II}) = \begin{cases} Q_{II}(\rho_{II}^*),\ \rho_{II} \in [0, \rho_{II}^*], \\ Q_{II}(\rho_{II}),\ \rho_{II} \in [\rho_{II}^*, \rho_{II}^{jam}], \end{cases}$$

where $\rho_I^{jam} = \sum_{i=1}^{n} l_i \bar{\rho}_{jam}$, $\rho_{II}^{jam} = \sum_{j=n+1}^{n+m} l_j \bar{\rho}_{jam}$.

We follow the user-equilibrium condition and the principle of maximization of flow, and present the following steady-state conditions without shocks:

(1) The flow at junction $Q_J$ is not greater than the demand of the upstream links unit, $D_I$, and the supply of the downstream links unit, $S_{II}$, and is equal to the maximum value under the constraints, namely,

$$Q_J = \min(D_I(\rho_I), S_{II}(\rho_{II})).$$

(2) The flow at junction, $Q_J$, is equal to the total flow of the upstream links unit or the total flow of the downstream links unit, namely,

$$Q_J = Q_I(\rho_I) = Q_{II}(\rho_{II}).$$

Thus, we have

$$\min(D_I(\rho_I), S_{II}(\rho_{II})) = Q_I(\rho_I) = Q_{II}(\rho_{II}).$$

If there is no bottleneck at junction $J_2$, then we give the possible steady-state solutions without shocks.

**Property 3.** For a given flow, $Q_J \le Q_J^*$, there are only the following two types of steady-state solutions without shocks:

a. Free-flow connection:

$$\rho_I = \rho_I^+(Q_J) \le \rho_I^*,\ \rho_{II} = \rho_{II}^+(Q_J) \le \rho_{II}^*; \tag{16}$$

b. Congestion connection:

$$\rho_I = \rho_I^-(Q_J) \ge \rho_I^*,\ \rho_{II} = \rho_{II}^-(Q_J) \ge \rho_{II}^*. \tag{17}$$

Note that given one of the three variables, $\rho_I$, $\rho_{II}$, or $Q_J$, the other two variables can be uniquely determined. We can then solve the densities for all links, $\rho_i$ and $\rho_j$, through user-equilibrium curves (Eq. (11)), based on $\rho_I$ and $\rho_{II}$.

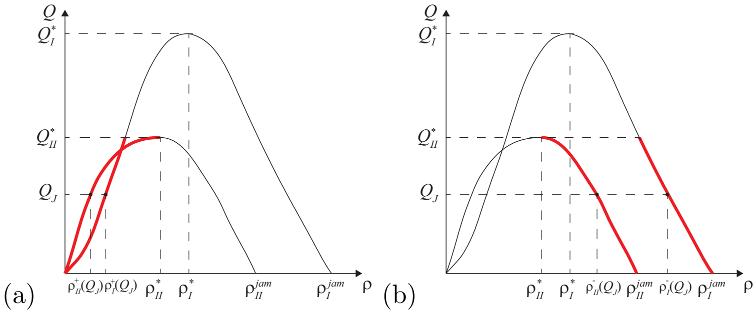



Figure 6. Steady states without shocks. (a) Free-flow connection; (b) Congestion connection. These two types of steady state cannot cover the whole range of $N$ from 0 to $N_{max}$, and thus, the shocks have to be introduced as a supplement.

We then assume that junction $J$ is a bottleneck, namely, $Q_I^* > Q_{II}^*$, as shown in Fig. 6. The black thin lines denote the fundamental diagrams of links unit $I$ and $II$, and the red thick lines in Fig. 6(a) and Fig. 6(b) indicate the free-flow connection of Eq. (16) and congestion connection of Eq. (17), respectively. We denote the total number of users on the road network as

$$N = N_I + N_{II}.$$

We then analyze the steady-state solution corresponding to $N$, from 0 to $N_{max}$. The analysis procedure shows that the two types of steady-state solutions without shocks cannot cover the whole range of $N$. Thus, the shocks must be introduced as a supplement, to achieve steady-state solutions for all user numbers, $N$.

State 1: Along with the user number, $N$, increasing from 0, the user numbers on links units $N_I$ and $N_{II}$ and the total flow on the road network, $Q_J$, increase until $Q_J = Q_{II}^*$. We then denote $N = N_1^*$. In this case, the steady-state solution on the road network is without shocks, as described by Eq. (16).

State 2: With $N$ gradually increasing, because of flow limitation, the flow on network, $Q_J$, cannot increase. Thus, there is no corresponding solution without shocks. In this case, the solution on links unit $II$ remains unchanged, and the extra users are added into the links unit $I$ by introducing shocks. The shock interface moves upstream, with $N$ increasing until it disappears, and now we denote $N = N_2^*$.

State 3: If $N$ continues to increase, the users are added to both links units $I$ and $II$, and the total flow, $Q_J$, decreases until $N = N_{max}$. In this case, the steady-state solution on the road network is without shocks and is described by Eq. (17).

### 3.2. *Solving procedure of the steady-state solution*

Based on the discussion in Section 3.1, we give the general solving procedure of the steady-state solution and discuss its existence and uniqueness.

(1) Calculate the critical values of the user numbers $N_1^*$ and $N_2^*$ as follows

$$N_1^* = N_I(\rho_I^+(Q_{II}^*)) + N_{II}(\rho_{II}^*),\ N_2^* = N_I(\rho_I^-(Q_{II}^*)) + N_{II}(\rho_{II}^*).$$

(2) If $N \in [0, N_1^*]$, the road network has a steady-state solution without shocks. We can then obtain $Q_J$ through

$$\frac{\sum_{i=1}^{n} L_i l_i}{\sum_{i=1}^{n} l_i}\rho_I^+(Q_J) + \frac{\sum_{j=n+1}^{n+m} L_j l_j}{\sum_{j=n+1}^{n+m} l_j}\rho_{II}^+(Q_J) = N.$$

Then, solve $\rho_I = \rho_I^+(Q_J)$ and $\rho_{II} = \rho_{II}^+(Q_J)$ and obtain the densities on all links through Eq. (15).

(3) If $N \in (N_1^*, N_2^*)$, the steady-state solution on links unit $II$ is the same as $N = N_1^*$, while the shocks on links unit $I$ are produced. Obtain the user number on links unit $I$ through $N_I = N - N_{II}(\rho_{II}^*)$, and determine the travel time, $T_I(N_I, Q_{II}^*) = N_I/Q_{II}^*$. Thus, obtain the ratios of user numbers and flow on all links as

$$\frac{N_{I,i}}{q_i} = \frac{N_I}{Q_{II}^*} = T_I(N_I, Q_{II}^*). \tag{18}$$

Thus, if the flow, $q_i$, is known, we can obtain $N_{I,i}$ and then, the other variables.

(4) If $N \in [N_2^*, N_{max}]$, the road network has a steady-state solution without shocks and we can obtain $Q_J$ through

$$\frac{\sum_{i=1}^{n} L_i l_i}{\sum_{i=1}^{n} l_i}\rho_I^-(Q_J) + \frac{\sum_{j=n+1}^{n+m} L_j l_j}{\sum_{j=n+1}^{n+m} l_j}\rho_{II}^-(Q_J) = N.$$

Then solve $\rho_I = \rho_I^-(Q_J)$ and $\rho_{II} = \rho_{II}^-(Q_J)$ and obtain the densities on all links through Eq. (17).

When $N \in (N_1^*, N_2^*)$, shocks emerge on links unit $I$. We now discuss the existence and uniqueness of the steady-state solution with shocks. For a given $N \in (N_1^*, N_2^*)$, we know that the user number on links unit $I$ is equal to $N_I = N - N_{II}(\rho_{II}^*)$, because the links unit $II$ has no shocks. There are two unknown variables on each link of links unit $I$, for example, $\rho_i^u$ and $L_i^u$, and thus, a total of $2n$ unknown variables. Note that Eq. (18) contains $n$ equations, and $Q_J = Q_{II}^*$ is the $(n+1)$th equation. Thus, we need $n-1$ equations to uniquely determine the steady-state solution with shocks on links unit $I$.

Note that if $N \in (N_1^*, N_2^*)$, the flow on links unit $I$ is $Q_I = Q_{II}^*$, and thus, we assume flow assignment ratios $\beta_i$, which are also known as priority coefficients, namely, $q_i = \beta_i Q_I$. Thus, $\beta_i$ should satisfy

$$\beta_i(N) \in [0, 1], \ \sum_{i=1}^{n} \beta_i(N) = 1. \tag{19}$$

and be continuously dependent on the total number, $N$ or $N_I$, which implies

$$\beta_i(N_1^*) = \frac{q_i(\rho_i(N_1^*))}{Q_{II}^*}, \ \beta_i(N_2^*) = \frac{q_i(\rho_i(N_2^*))}{Q_{II}^*}. \tag{20}$$

According to Eq. (18), we have

$$\begin{cases} N_{I,i} = \beta_i N_I, \ \dfrac{L_i}{v_i^f} \le \dfrac{N_I}{Q_{II}^*}, \\ \beta_i = 0, \ N_{I_i} = 0, \ \dfrac{L_i}{v_i^f} \ge \dfrac{N_I}{Q_{II}^*}. \end{cases} \tag{21}$$

Given $\beta_i$, we can solve the steady-state solution with shocks on links unit $I$ through the following procedure.

- For $i = 1, \ldots, n$, solve $\rho_i^u$ and $\rho_i^d$ through

$$\rho_i^u = \rho_i^+(\beta_i Q_{II}^*), \ \rho_i^d = \rho_i^-(\beta_i Q_{II}^*), \tag{22}$$

  where $\rho_i^{\pm}(\cdot)$ denotes the left or right branch inverse function.
- Calculate $N_{I,i} = \beta_i N_I$, and then solve $L_i^u$ through

$$L_i^u \rho_i^u + L_i^d \rho_i^d = N_{I,i}. \tag{23}$$

We then prove that $\beta_i$ always exists, such that all equations are solvable.

**Theorem 1.** The priority coefficient, $\beta_i$, can derive a steady-state solution on links unit $I$ if and only if

$$\beta_i \le \frac{L_i}{N_I} v_i^{-1}(\min(\frac{L_i Q_{II}^*}{N_I}, v_i^f)). \tag{24}$$

Moreover, the set of $\{\beta_i\}_{i=1}^n$ governed by inequality (24) is nonempty.

**proof** We first consider the first case in Eq. (21). Equation (23) indicates that the user number, $N_{I,i}$, should be limited, such that $L_i^u \in [0, L_i]$, which equals

$$L_i \rho_i^u \le N_{I,i} \le L_i \rho_i^d.$$

Considering Eq. (22) and Eq. (21), we have the necessary and sufficient conditions for solving Eq. (23):

$$\rho_i^+(\beta_i Q_{II}^*) \le \frac{\beta_i N_I}{L_i} \le \rho_i^-(\beta_i Q_{II}^*) \Leftrightarrow \beta_i Q_{II}^* \ge q_i(\frac{\beta_i N_I}{L_i}).$$

This is equivalent to

$$\frac{L_i Q_{II}^*}{N_I} \ge v_i(\frac{\beta_i N_I}{L_i}) \ or \ \beta_i \le \frac{L_i}{N_I} v_i^{-1}(\frac{L_i Q_{II}^*}{N_I}).$$

Note that the first equation of (21) indicates that the independent variable of $v_i^{-1}(\cdot)$ is not greater than $v_i^f$, while the second equation implies that the dependent variable of $v_i^{-1}$ is not smaller than $v_i^f$, and $\beta_i = 0$. Thus, we combine these two cases and obtain inequality (24).

Next, we need to prove that $\beta_i$, satisfying inequality (24), can be found. We rewrite inequality (24) as follows

$$\beta_i \leq \frac{q_i(\tilde{\rho}_i)}{\min(Q_{II}^*, \frac{N_I v_i^f}{L_i})}, \ \tilde{\rho}_i = v_i^{-1}(\tilde{v}_i), \ T_I = \frac{N_I}{Q_{II}^*}, \ \tilde{v}_i = \min(\frac{L_i}{T_I}, v_i^f), \tag{25}$$

where $T_I$ is the travel time on links unit $I$ and $\tilde{v}_i$ is the average velocity on link $i$. We denote $T_{I,i}$ as the travel time on link $i$, and have

$$T_{I,i} \geq T_I; \ T_{I,i} > T_I \Rightarrow \rho_i^d = \rho_i^u = 0. \tag{26}$$

$L_i/T_{I,i}$ can be represented as the weighted harmonic mean of $v_i(\rho_i^u)$ and $v_i(\rho_i^d)$, and we have

$$\tilde{v}_i = \min(\frac{T_{I,i}L_i}{T_I}(\frac{L_i^u}{v_i(\rho_i^u)} + \frac{L_i^d}{v_i(\rho_i^d)})^{-1}, v_i^f).$$

Thus, we can obtain

$$\min(\frac{T_{I,i}}{T_I}v_i(\rho_i^d), v_i^f) \leq \tilde{v}_i = v_i(\tilde{\rho}_i) \leq \min(\frac{T_{I,i}L_i}{T_I}v_i(\rho_i^u), v_i^f). \tag{27}$$

Considering Eq. (26), we have

$$\min(\frac{T_{I,i}}{T_I}v_i(\rho_i^d), v_i^f) = v_i(\rho_i^d), \ \min(\frac{T_{I,i}L_i}{T_I}v_i(\rho_i^u), v_i^f) = v_i(\rho_i^u). \tag{28}$$

Thus, Eq. (27) is equivalent to

$$\rho_i^d \geq \tilde{\rho}_i \geq \rho_i^u \Rightarrow q_i(\tilde{\rho}_i) \geq q_i(\rho_i), \tag{29}$$

where $\rho_i = \rho_i^u$ or $\rho_i = \rho_i^d$. Thus, we have

$$\sum_{i=1}^{n} q_i(\tilde{\rho}_i) \geq \sum_{i=1}^{n} q_i(\rho_i) = Q_{II}^*. \tag{30}$$

Based on the above derivations, the following statements can be verified.

(1) Inequality (25) is a necessary and sufficient condition of the set of $\beta_i$, governed by the condition that inequlity (24) is nonempty.
(2) If the equality holds in the above inequalities associated with link $i$, we have $\tilde{\rho}_i = \rho_i$ if and only if there is no shock on link $i$.
(3) The equality holds in inequality (30) if and only if there are no shocks on all involved links. In this case, we have $N = N_1^*$ or $N = N_2^*$.
(4) If $N = N_1^*$ or $N = N_2^*$, based on (26), inequality (24) degenerates to

$$\beta_i \leq \frac{q_i(\rho_i)}{Q_{II}^*} \Rightarrow \sum_{i=1}^{n} \beta_i \leq \frac{\sum_{i=1}^{n} q_i(\rho_i)}{Q_{II}^*} = 1.$$

This indicates that, only if the equality holds, the sum is equal to 1, and thus, we have (20).

There are infinite $\beta_i$ values that satisfy inequality (24), for example,

$$\beta_i = \frac{q_i(\tilde{\rho}_i)}{\sum_{i=1}^{n} q_i(\tilde{\rho}_i)}.$$

## 4. Numerical Simulation

The discussed stead-state flow is a solution to the dynamic LWR model. In this regard, any arbitrarily distributed traffic flow with fixed total number of vehicles on the network is expected to evolve into (or converge to) the discussed stead-state solution through numerical simulations of the LWR model. For the convergence, a "user-equilibrium oriented" numerical scheme of the LWR model is discussed in Appendix A. Then, a test example is given to verify the convergence to the corresponding analytical stead-state solution. This also demonstrates that the simulated steady-state solution satisfies the steady-keeping principle, and implies that the actual traffic flow can be controlled to approximately reach a steady state by adequate measures (e.g., by dynamic signal timing plan).

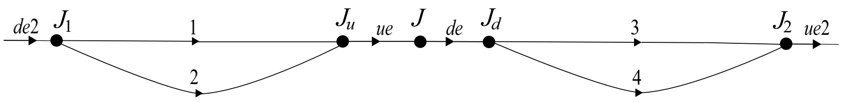



Figure 7. The inner extension for the road network with $n = m = 2$.

We consider the road network shown in Fig. 7, with a $2 \times 2$ junction after the inner extension, in the numerical simulations. The periodic boundary condition is used so that the user number, $N$, is unchanged in a single numerical simulation. This is equivalent to regarding it as a circular network. The analytical steady-state solutions are calculated based on the original network, which is obtained by shortening the extension links.

Links unit $I$ is composed of links 1 and 2, and links unit $II$ is composed of links 3 and 4. The free-flow speeds on all links are set as $V^f$, and the link parameters are

$$L_2/L_1 = 1.2,\ L_3/L_1 = L_4/L_1 = 1.0;\ l_1 = l_2,\ l_3 = l_4,\ l_3/l_1 = 0.5.$$

The lengths of the extension links are set as $L_{ue} = L_{de} = L_{ue2} = L_{de2} = L_e = 0.02\ L_1$, which is relatively short, such that the network after the inner extension is almost equivalent to the original network. The fundamental diagram on a link is adopted as

$$v_k(\bar{\rho}_k) = V^f(1 - \bar{\rho}_k)^{2.8}.$$

The densities, velocities, link lengths, flows, and user numbers are made dimensionless by dividing through $\rho_1^{jam}$, $V^f$, $L_1$, $\rho_1^{jam}V^f$, and $\rho_1^{jam}L_1$, respectively. We then have

$$\rho_1^* = \rho_2^* = 0.5263,\ \rho_3^* = \rho_4^* = 0.2632,\ \rho_I^* = 1.0526,\ \rho_{II}^* = 0.5263;$$
$$q_1^* = q_2^* = 0.2238,\ q_3^* = q_4^* = 0.1119,\ Q_I^* = 0.4451,\ Q_{II}^* = 0.2238;$$
$$N_1^* = 0.8285,\ N_2^* = 2.9894,\ N_{max} = 6.4.$$

The maximum flow on this network is

$$Q_J^* = \min(Q_I^*, Q_{II}^*) = Q_{II}^* = 0.2238.$$

This indicates that junction $J$ is a bottleneck, and shocks appear on links unit $I$. The fundamental diagrams of links units $I$ and $II$ are shown in Fig. 8.

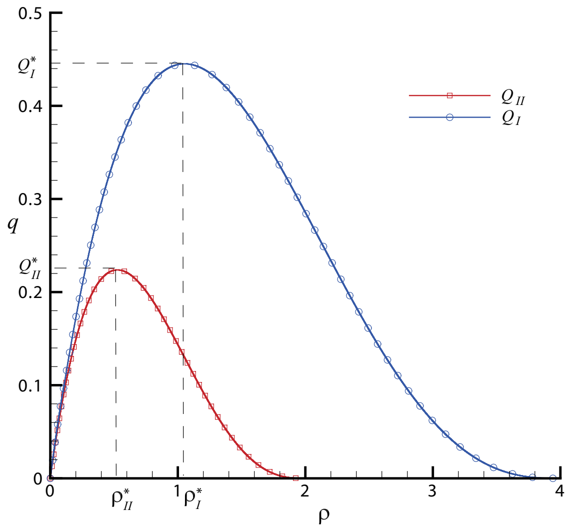



Figure 8. The fundamental diagrams of links units $I$ and $II$.

According to Eq. (A7), the priority coefficients at junctions $J_u$ and $J_2$ are set as

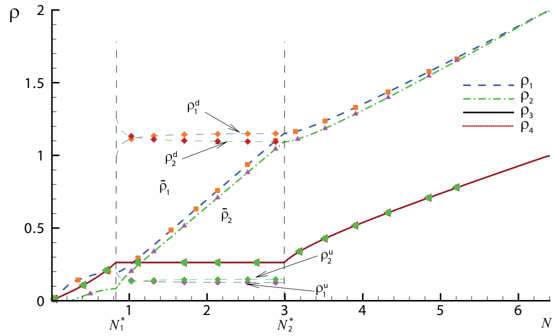
$$\beta = \frac{\eta_1}{\eta_1 + \eta_2}, \ \beta_{J_2} = \frac{\eta_3}{\eta_3 + \eta_4}.$$

For user numbers, $N$, from 0 to $N_{max}$, the densities and flows of the steady-state solutions are shown in Fig. 9 and Fig. 10. The upper and lower subfigures are analytical and numerical solutions, respectively. At the beginning of the numerical simulation, $N$ users are uniformly distributed on the road network. The convergence conditions Eq. (A6) are adopted, together with

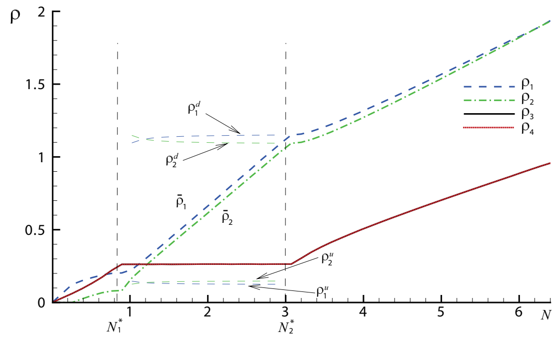
$$e_3^n = \max_{k,i} \frac{|q_{k,i}^n - q_{k,i+1}^n|}{q_{k,i}^n} < 10^{-5},$$

where $k = 1, \ldots, 4, i = 1, \ldots, N_x - 1$. First, we can observe that the analytical solutions and the numerical solutions have almost the same pattern. Then, because the inner extension links are introduced in the numerical simulation, all curves move right slightly, compared with those in the analytical solutions.

For $N \in [0, N_1^*]$, the traffic demand increases but is still so low that the network is uncongested with small densities on all links. For $N \in (N_1^*, N_2^*)$, the traffic demand reaches the highest so that the composite links unit $II$ cannot provide sufficient traffic supply, thus queuing arises on links 1 and 2 of composite links unit $I$. For $N \in [N_2^*, N_{max}]$, traffic demand remains the highest without queuing on the network, but with traffic flow being decreasingly in congested states. Specifically, when $N \in [0, N_1^*] \cup [N_2^*, N_{max}]$, the densities and flows on all links are constant and dependent only on the user number,

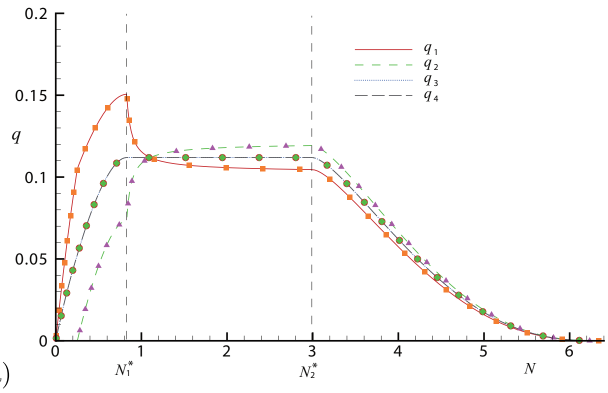



(a)

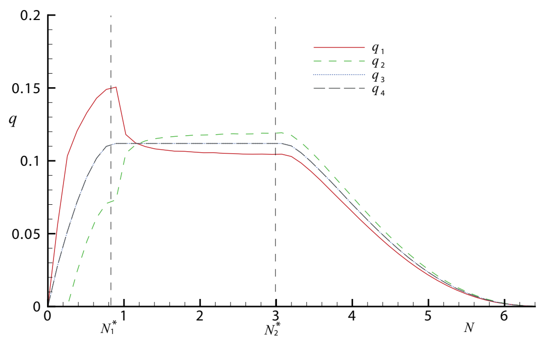



(b)

Figure 9. The densities on all links of the steady-state solutions for user number $N$, from 0 to $N_{max}$. Shocks appear on links 1 and 2 when $N \in (N_1^*, N_2^*)$. (a) Analytical solutions (continuous lines) and numerical solutions using non-periodic boundary conditions (discrete symbols); (b) Numerical solutions using periodic boundary conditions.




(a)




(b)

Figure 10. The flows on all links of the steady-state solutions for user numbers, $N$, from 0 to $N_{max}$. Shocks appear on links 1 and 2 when $N \in (N_1^*, N_2^*)$. (a) Analytical solutions (continuous lines) and numerical solutions using non-periodic boundary conditions (discrete symbols); (b) Numerical solutions using periodic boundary conditions.

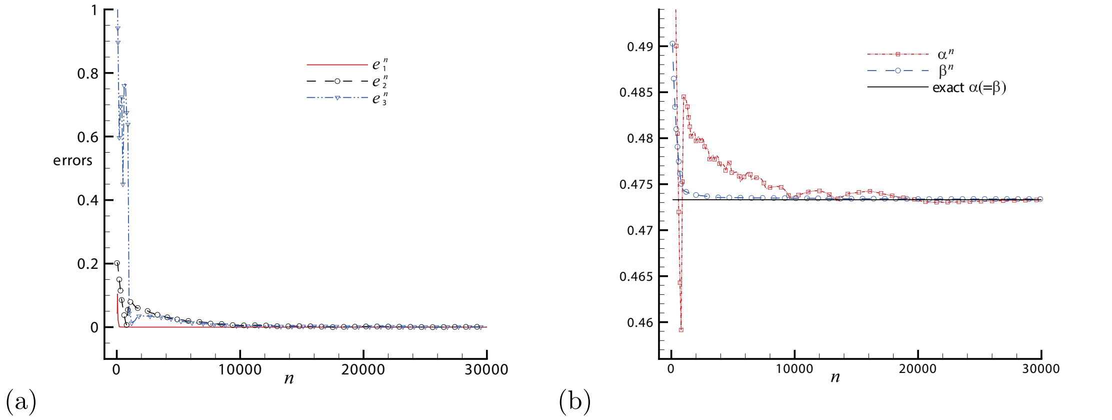



Figure 11. Convergence of the numerical scheme. (a) The errors with the time step of the numerical scheme; (b) The numerical flow distribution parameter and the priority coefficient.

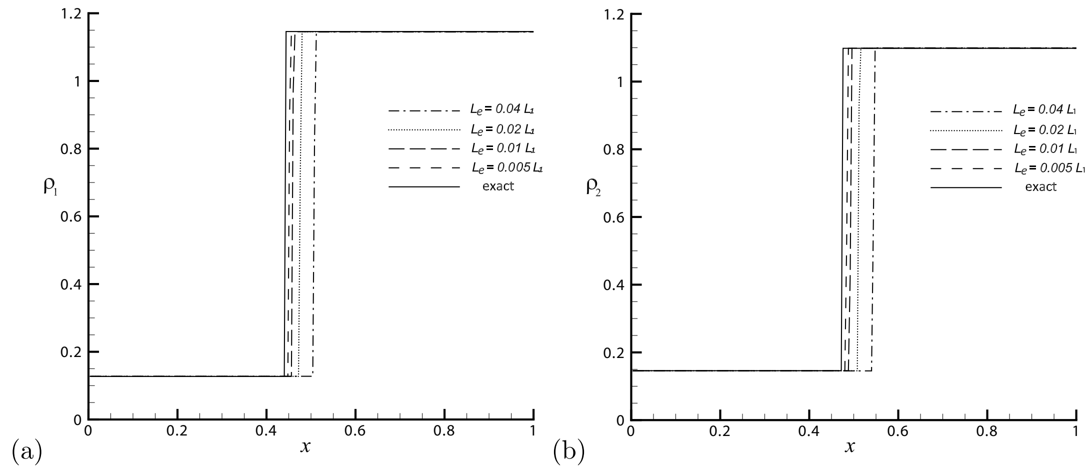



Figure 12. The numerical and analytical densities on links 1 and 2, with different lengths of the extension links.

$N$, and there are no shocks on the network. First, the densities on the links strictly increase with $N$. The flows on the links strictly increase and decrease with $N$ when $N \in [0, N_1^*]$ and $N \in [N_2^*, N_{max}]$, respectively. Then, there are more users choosing link 1, because of the user-equilibrium constraint. Finally, there are no users choosing link 2 until $N > 0.25$, and thus, the flow on link 1 equals the sum of the flows on links 3 and 4 when $N \leq 0.25$. When $N \in (N_1^*, N_2^*)$, the densities and flows on links 3 and 4 are constant and are independent of $N$, i.e., $\rho_3 = \rho_3^* = 0.2632$, $\rho_4 = \rho_4^* = 0.2632$ and $q_3 = q_3^* = 0.1119$, $q_4 = q_4^* = 0.1119$. Shocks appear on both link 1 and link 2. We can observe the following properties of the steady-state solution. First, the densities on the two sides of the shock interface of link 1 are $\rho_1^u$ and $\rho_1^d$, corresponding to $N = N_1^*$ and $N_2^*$, respectively. The average density, $\bar{\rho}_1$, continuously increase with $N$, from $\rho_1^u$ to $\rho_1^d$. This indicates that the queue length, $L_1^d$, continuously increases from 0 to $L_1$. The solution on link 2 is similar to that on link 1. The number of users choosing link 1 at junction $J_1$ is then more than the number of users choosing link 2 when $N < 1.1$. As $N$ increases, more users choose link 2 at junction $J_1$, namely, $\rho_2^u > \rho_1^u$ and $q_2 > q_1$ for $N > 1.1$. Finally, the number of users on link 1 is always more than the number on link 2, namely, $\bar{\rho}_1 > \bar{\rho}_2$ and $N_{I,1} > N_{I,2}$.

We then set $N = 3 \in (N_1^*, N_2^*)$ and $\lambda = 0.3$, and in the following section, discuss the numerical steady-state solutions in detail. Figure 11(a) illustrates the errors with the time step, $n$, of the numerical scheme and we can see the convergence of the numerical solution. Figure 11(b) shows the numerical flow distribution parameter, $\alpha$, and the priority coefficient, $\beta$, for link 1 at junctions $J_1$ and $J_u$, respectively. We can see that the parameters finally converge to the exact analytical values. Figure 12 compares the numerical and analytical densities on links 1 and 2, with different lengths of the extension links. As $L_e$ reduces, the numerical curves converge to the exact values, which indicates that the inner extension of the network is reasonable.

Note that in the above examples, we used periodic boundary conditions, so that the road network can be considered as a circular network. We then examine whether in an "open" road network (i.e., using non-periodic boundary conditions), the steady-state solution proposed in this work can also be obtained. In this case, it is only necessary to make the inner extension at junction $J$. The network is set to be empty initially. The inflow conditions at the origin ($J_1$) and the outflow conditions at the destination ($J_2$) are set adequately in the following.

When the traffic is in free flow (the demand is less than $Q^*_{II}$), the inflow condition at the origin ($J_1$) is set as $\rho_I(0,t) = \rho_0 < \rho^*_I$, and the corresponding OD demand is $D_I(\rho_I) = Q_I(\rho_I) < Q^*_{II}$. The outflow condition at the destination ($J_2$) is set as natural boundary conditions. These give

$$\begin{cases} \rho_I(0,t) = \rho_0 < \rho^*_I, \\ \dfrac{\partial \rho_3}{\partial x} = \dfrac{\partial \rho_4}{\partial x} = 0. \end{cases}$$

When the traffic is congested (the demand is larger than $Q^*_{II}$), the inflow condition at the origin ($J_1$) is set as $\rho_I(0,t)$, and the corresponding OD demand is $D_I(\rho_I) > Q^*_{II}$. The outflow condition at the destination ($J_2$) is limited by traffic congestion. These give

$$\begin{cases} \rho_I(0,t) = \rho_0 > \rho^*_I, \\ \dfrac{\partial \rho_3}{\partial x} = \dfrac{\partial \rho_4}{\partial x} = 0, \\ Q_{II}(L,t) = Q_I(\rho_I(0,t)), \end{cases}$$

where $L = L_3 = L_4$. When the traffic demand is comparable to the maximum flow of composite section II, $Q^*_{II}$, the boundary conditions are set as

$$\begin{cases} \rho_I(0,t) = \begin{cases} \rho^*_I, & t \le t_0, \\ \rho^-_I(Q^*_{II}), & t > t_0, \end{cases} \\ \dfrac{\partial \rho_3}{\partial x} = \dfrac{\partial \rho_4}{\partial x} = 0. \end{cases}$$

We select different values of $\rho_0$ and $t_0$ to obtain numerical solutions, as shown by the discrete symbols in Fig. 9(a) and Fig. 10(a). We can see that the numerical results are in good agreement with analytical solutions (by the same total number $N$ of vehicles on the network). However, we stress that these numerical examples are used mainly to demonstrate the mathematical property that the discussed static traffic states are truly steady-state solutions to the dynamic traffic flow (LWR) model, in that they can be involved from (any) traffic states through the dynamic model. In principle, the convergence could be guaranteed with balanced (equaled) in-flow and out-flow at each pair of origin and destination for $t$ sufficiently large, which is unnecessarily periodic.

## 5. Conclusions

In this study, a steady-state solution to the LWR model on a single OD parallel road network is studied. The density (or velocity) on a link permits the shock structure, and the user-equilibrium condition is considered. Thus, a novel static TA model is derived, considering queues and spillback effects. The solving procedure for any total number of vehicles is presented analytically, which indicated the existence of a solution. Moreover, the uniqueness of the solution is also proved by introducing priority coefficients when shocks appear. A numerical scheme of the LWR network model is designed to converge the traffic flow into the discussed steady-state solution, by adjusting the distribution percentages and priority coefficients at junctions. The numerical results show good convergence and validate the model and the numerical scheme.

It is significant to extend the model to a more general multi-OD road network in the future study. The following preliminary ideas should be instructive to the extension from the perspective of theoretical analysis and the optimization. First, we could define composite links units at different levels according to the set of OD-routes. Second, we should discuss the fundamental diagrams of different composite links units under the user-equilibrium conditions, which is similar to the discussion in the present paper. Third, the shock structures must be introduced to ensure the existence and uniqueness of the solution. For a multi-OD road network, the fundamental diagram for multi-class traffic flow (Wong and Wong 2002) should be introduced because there are vehicles of different OD-pairs on the same links. From the perspective of optimization, we could establish the optimization model similar to the classical static traffic assignment model. However, the variables on a link should be the upstream and downstream densities and the position of the shock. The principles proposed in the present paper ( e.g., the maximization of flow) should inevitably be applied to provide more constraints for the solution. Nevertheless, the problem would pose much more challenges than the problems in classical static traffic assignment modeling.

## Acknowledgements

The study was supported by grants from the National Natural Science Foundation of China (Grant Nos. 72101185, 72021002, 11972121), and the Research Grants Council of the Hong Kong Special Administrative Region, China (Grant No. 17204919). The sixth author was also supported by Francis S Y Bong Professorship in Engineering.

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## Appendix A. Numerical scheme for LWR network model

Here, we introduce a "user-equilibrium oriented" numerical scheme of the LWR network model which converges the traffic flow into the proposed steady-state solution. Note that the numerical scheme is based on LWR network model, instead of using to solve the steady-state solution directly.

For the LWR model (Eq. (2)), on a link we adopt the following mature first-order finite volume scheme (Toro 1999) after spatial discretization (Fig. A1):

$$\rho_i^{n+1} = \rho_i^n - \Delta t^n \frac{\hat{q}(\rho_i^n, \rho_{i+1}^n) - \hat{q}(\rho_{i-1}^n, \rho_i^n)}{\Delta x_i}, \ \rho_i^0 = \rho_i(0),$$

where $\Delta t = t^{n+1} - t^n$, $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ and Godunov flux (Lebacque 1995):

$$\hat{q}(\rho_1, \rho_2) = \begin{cases} \min\limits_{\rho \in [\rho_1, \rho_2]} q(\rho), & \rho_1 \leq \rho_2, \\ \max\limits_{\rho \in [\rho_2, \rho_1]} q(\rho), & \rho_1 > \rho_2. \end{cases}$$

For each time step, $t^n$, the inflow and outflow as boundary conditions are obtained by solving the local Riemann problems at upstream and downstream junctions (Coclite, Garavello, and Piccoli 2005; Lin et al. 2015).

To avoid the complicated Riemann problem at an $n \times m$ junction, we make an inner extension at a junction. For junction $J$ of the road network shown in Fig. 1, the $n$ upstream links first connect a single link, $ue$ (upstream extended), which assumes the length, $L_u$; lane number, $\sum_{i=1}^{n} l_i$; and the same flow–density relationship as that in links

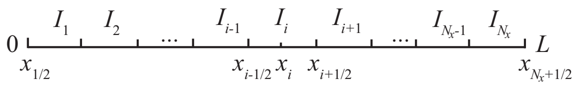


Figure A1. The spatial discretization of a link.

unit $I$. Thus, links unit $I$ and extension link $ue$ are connected naturally, and new traffic waves do not exist. The link $ue$ is then connected with a single link, $de$ (downstream extended), with a length, $L_d$; lane number, $\sum_{j=n+1}^{n+m} l_j$; and the same flow–density function as that in links unit $II$. This produces a $1 \times 1$ junction. Similarly, the link $de$ connects the links unit $II$ without new traffic waves.

After the inner extension, the road network is only composed of three types of junctions, namely, $n \times 1$, $1 \times 1$, and $1 \times m$ junctions. Figure 7 shows the inner extension of the road network when $n = m = 2$. In the following section, we discuss Riemann problems as boundary conditions at junctions, based on Fig. 7.

- **The $1 \times 1$ junction**

We take the inflow link, $ue$, and the outflow link, $de$, for example, and the junction is denoted as $J$. The flow–density relationships are $Q_I(\rho_u)$ and $Q_{II}(\rho_d)$ for $ue$ and $de$, which are the same as the links units $I$ and $II$, respectively.

For the time step, $t^n$, we denote the densities on links $ue$ and $de$, adjacent to junction $J$, as $\rho^n_{u,K_u}$ and $\rho^n_{d,1}$, where $K_u$ is the grid number of link $ue$. We can then obtain the flux at the interface of $ue$ and $de$:

$$\hat{q}^n_u = \hat{q}^n_d = \min(D_I(\rho^n_{u,K_u}), S_{II}(\rho^n_{d,1})).$$

- **The $1 \times 2$ junction**

We take the inflow link, $de$, and outflow links 3 and 4, for example, and the junction is denoted as $J_d$. The flow–density relationship of $de$ is the same as that of links unit $II$, namely, $Q_{II}(\rho_d)$. For the time step, $t^n$, we denote the densities on links adjacent to junctions $J_d$ as $\rho^n_{d,K_d}$, $\rho^n_{3,1}$, and $\rho^d_{4,1}$, and $K_d$, $K_3$, and $K_4$ are the grid numbers of the three links.

The fluxes for links at junction $J_d$ are determined by the following procedure.

(1) Calculate the user numbers and travel times over links 3 and 4 through

$$N^n_3 = \sum_{i=1}^{K_3} \Delta x \rho^n_{3,i},\ T^n_3 = \sum_{i=1}^{K_3} \frac{\Delta x}{v_3(\rho^n_{3,i})}, \tag{A1}$$

$$N^n_4 = \sum_{i=1}^{K_4} \Delta x \rho^n_{4,i},\ T^n_4 = \sum_{i=1}^{K_4} \frac{\Delta x}{v_4(\rho^n_{4,i})}. \tag{A2}$$

(2) Set the flow distribution coefficient, $\alpha^n$, of link 3, such that

$$\frac{\alpha^n}{1-\alpha^n} = \frac{N^n_3}{N^N_4}\frac{T^n_4 + \zeta^n \max(T^n_4 - T^n_3, 0)}{T^n_3 + \zeta^n \max(T^n_3 - T^n_4, 0)},$$

where

$$\eta^n = \frac{\theta \Delta t^n}{\max(T^n_3, T^n_4)}, \tag{A3}$$

and $\theta$ is a numerical constant parameter.

(3) Determine the fluxes of links at junction $J_d$ by

$$\hat{q}_d^n = \min(D_{II}(\rho_{d,K_d}^n), \frac{s_3(\rho_{3,1}^n)}{\alpha^n}, \frac{s_4(\rho_{4,1}^n)}{1-\alpha^n}),$$
$$\hat{q}_3^n = \alpha^n \hat{q}_d^n,\ \hat{q}_4^n = (1-\alpha^n)\hat{q}_d^n.$$

Note that $\alpha^n$ (or $1-\alpha^n$) is inversely proportional to the travel time $T_3^n$ (or $T_4^n$), such that more users choose the link with a shorter travel time (Eq. (A3)). From Eq. (A1) and Eq. (A3), we have

$$T_3^n = T_4^n \Leftrightarrow \frac{N_3^n}{\hat{q}_3^n} = \frac{N_4^n}{\hat{q}_4^n}, \tag{A4}$$

which is the necessary condition to reach a steady state on the downstream links unit. Thus, the numerical scheme can converge the traffic flow to satisfy the user-equilibrium condition and thus, the steady state by dynamically adjusting the inflow proportion $\alpha^n$. The necessary conditions of the steady state also include

$$\hat{q}_d^n = Q_{II}(\rho_{d,K_d}^n),\ \hat{q}_3^n = q_3(\rho_{3,1}^n),\ \hat{q}_4^n = q_4(\rho_{4,1}^n). \tag{A5}$$

Thus, the convergence conditions for the numerical solutions can be set as follows, considering Eq. (A4) and Eq. (A5):

$$e_1^n = \min(\frac{|\hat{q}_d^n - Q_{II}(\rho_{d,K_d}^n)|}{Q_{II}(\rho_{d,K_d}^n)+\epsilon'}, \frac{|\hat{q}_3^n - q_3(\rho_{3,1}^n)|}{q_3(\rho_{3,1}^n)+\epsilon'}, \frac{|\hat{q}_4^n - q_4(\rho_{4,1}^n)|}{q_4(\rho_{4,1}^n)+\epsilon'}) < \varepsilon_1,$$
$$e_2^n = \frac{|T_3^n - T_4^n|}{T_3^n} < \varepsilon_2. \tag{A6}$$

where $\epsilon' = 10^{-24}$, $\varepsilon_1 = 10^{-5}$, and $\varepsilon_2 = 0.001$ in the numerical experiments.

- **The $2 \times 1$ junction**

We take the inflow links, 1 and 2, and the outflow link, $ue$, for example, and the junction is denoted as $J_u$. The flow–density relationship of $ue$ is the same as that of links unit $I$, namely, $Q_I(\rho_u)$. Consider Eq. (24) and replace $Q_{II}^*$ with $N_I/T_I$, after which the priority coefficients for links 1 and 2 entering $ue$ should satisfy

$$\beta_i \le \frac{L_i}{N_I} v_i^{-1}(\min(\frac{L_i}{T_i}, v_i^f)) \equiv \eta_i;\ i = 1, 2,\ \beta_1 + \beta_2 = 1. \tag{A7}$$

We write $\beta_1 = \beta$ and $\beta_2 = 1-\beta$. For the time step, $t^n$, we denote the densities on links adjacent to junctions $J_u$ as $\rho_{1,K_1}^n$, $\rho_{2,K_2}^n$, and $\rho_{u,1}^n$, where $K_1$ and $K_2$ are the grid numbers of links 1 and 2, respectively. We then give the solving procedure of the fluxes for links at junction $J_u$ as follows.

(1) If the sum of demands on links 1 and 2 satisfy $d_1(\rho_{1,K_1}^n) + d_2(\rho_{2,K_2}^n) \le S_I(\rho_{u,1}^n)$, we set

$$\gamma^n = \frac{d_1}{d_1 + d_2}.$$

(2) If $d_1(\rho^n_{1,K_1}) + d_2(\rho^n_{2,K_2}) > S_I(\rho^n_{u,1})$, calculate the average travel time over links 1 and 2, and the user number on links unit $I$ through

$$T^n_I = \left( \sum_{i=1}^{K_1} \frac{\Delta x}{v_1(\rho^n_{1,i})} + \sum_{i=1}^{K_2} \frac{\Delta x}{v_2(\rho^n_{2,i})} \right) /2, \ N^n_I = \Delta x \left( \sum_{i=1}^{K_1} \rho^n_{1,i} + \sum_{i=1}^{K_2} \rho^n_{2,i} \right).$$

Then, determine $\eta^n_1$ and $\eta^n_2$ through Eq. (A7), and calculate $\gamma^n$ through

$$\gamma^n = \begin{cases} \beta^n, \ 1 - \dfrac{d_2}{S_I} < \beta^n < \dfrac{d_1}{S_I}, \\ \dfrac{S_I - d_2}{S_I}, \ \beta^n \leq 1 - \dfrac{d_2}{S_I}, \\ \dfrac{d_1}{S_I}, \ \beta^n \geq \dfrac{d_1}{S_I}. \end{cases}$$

(3) Determine the fluxes of links at junction $J_u$ by

$$\hat{q}^n_1 = \gamma^n \min(d_1 + d_2, S_I), \ \hat{q}^n_2 = (1 - \gamma^n) \min(d_1 + d_2, S_I), \ \hat{q}^n_u = \hat{q}^n_1 + \hat{q}^n_2.$$