Optimal Gevrey Regularity for Supercritical Quasi-Geostrophic Equations

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Abstract: We consider the two dimensional surface quasi-geostrophic equations with super-critical dissipation. For large initial data in critical Sobolev and Besov spaces, we prove optimal Gevrey regularity endowed with the same decay exponent as the linear part. This settles several open problems in Biswas (J Differ Equ 257(6):1753–1772, 2014), Biswas et al. (J Funct Anal 269(10):3083–3119, 2015).

1. Introduction

We consider the following two-dimensional dissipative surface quasi-geostrophic equation:

$$\begin{cases} \partial_t \theta + (u \cdot \nabla)\theta + \nu D^\gamma \theta = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^2; \\ u = R^\perp \theta = (-R_2\theta, R_1\theta), & R_j = D^{-1}\partial_j; \\ \theta(0, x) = \theta_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\nu \geq 0$, $0 < \gamma \leq 2$, $D = (-\Delta)^{\frac{1}{2}}$, $D^\gamma = (-\Delta)^{\frac{\gamma}{2}}$, and more generally the fractional operator $D^s = (-\Delta)^{\frac{s}{2}}$ corresponds to the Fourier multiplier $|\xi|^s$, i.e. $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$ whenever it is suitably defined under certain regularity assumptions on f. The scalar-valued unknown θ is the potential temperature, and $u = D^{-1}\nabla^\perp \theta$ corresponds to the velocity field of a fluid which is incompressible. One can write $u = (-R_2\theta, R_1\theta)$ where R_j is the jth Riesz transform in 2D. The dissipative quasi-geostrophic equation (1.1) can be derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency [24]. It models the evolution of the potential temperature θ of a geostrophic fluid with velocity u on the boundary of a rapidly rotating half space. As such it is often termed surface quasi-geostrophic equations in the literature. If θ is a smooth solution to (1.1), then it obeys

the L^p-maximum principle, namely

$$\|\theta(t, \cdot)\|_{L^p(\mathbb{R}^2)} \le \|\theta_0\|_{L^p(\mathbb{R}^2)}, \qquad t \ge 0, \ \forall \, 1 \le p \le \infty. \tag{1.2}$$

Similar results hold when the domain $\mathbb{R}^2$ is replaced by the periodic torus $\mathbb{T}^2$. Moreover, if θ_0 is smooth and in $\dot{H}^{-\frac{1}{2}}(\mathbb{R}^2)$, then one can show that

$$\|\theta(t, \cdot)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^2)} \le \|\theta_0\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^2)}, \quad t > 0. \tag{1.3}$$

More precisely, for the inviscid case $\nu = 0$ one has conservation and for the dissipative case $\nu > 0$ one has dissipation of the $\dot{H}^{-\frac{1}{2}}$-Hamiltonian. Indeed for $\nu = 0$ by using the identity (below $P_{<J}$ is a smooth frequency projection to $\{|\xi| \le \text{constant} \cdot 2^J\}$)

$$\frac{1}{2}\frac{d}{dt}\|D^{-\frac{1}{2}}P_{<J}\theta\|_2^2 = -\int P_{<J}(\theta\mathcal{R}^\perp\theta) \cdot P_{<J}\mathcal{R}\theta \, dx,$$

one can prove the conservation of $\|D^{-\frac{1}{2}}\theta\|_2^2$ under the assumption $\theta \in L^3_{t,x}$. The two fundamental conservation laws (1.2) and (1.3) play important roles in the wellposedness theory for both weak and strong solutions. In [25] Resnick proved the global existence of a weak solution for $0 < \gamma \le 2$ in $L^\infty_t L^2_x$ for any initial data $\theta_0 \in L^2_x$. In [23] Marchand proved the existence of a global weak solution in $L^\infty_t H_x^{-\frac{1}{2}}$ for $\theta_0 \in \dot{H}_x^{-\frac{1}{2}}(\mathbb{R}^2)$ or $L^\infty_t L^p_x$ for $\theta_0 \in L^p_x(\mathbb{R}^2)$, $p \ge \frac{4}{3}$, when $\nu > 0$ and $0 < \gamma \le 2$. It should be pointed out that in Marchand's result, the non-dissipative case $\nu = 0$ requires $p > 4/3$ since the embedding $L^{\frac{4}{3}} \hookrightarrow \dot{H}^{-\frac{1}{2}}$ is not compact. On the other hand for the diffusive case one has extra $L^2_t \dot{H}^{\frac{\gamma}{2}-\frac{1}{2}}$ conservation by construction. In recent [10], non-uniqueness of stationary weak solutions were proved for $\nu \ge 0$ and $\gamma < \frac{3}{2}$. In somewhat positive direction, uniqueness of surface quasi-geostrophic patches for the non-dissipative case $\nu = 0$ with moving boundary satisfying the arc-chord condition was obtained in [7].

The purpose of this work is to establish optimal Gevrey regularity in the whole supercritical regime $0 < \gamma < 1$. We begin by explaining the meaning of super-criticality. For $\nu > 0$, the equation (1.1) admits a certain scaling invariance, namely: if θ is a solution, then for $\lambda > 0$

$$\theta_\lambda(t, x) = \lambda^{1-\gamma}\theta(\lambda^\gamma t, \lambda x) \tag{1.4}$$

is also a solution. As such the critical space for (1.1) is $\dot{H}^{2-\gamma}(\mathbb{R}^2)$ for $0 \le \gamma \le 2$. In terms of the L^∞ conservation law, (1.1) is L^∞-subcritical for $\gamma > 1$, L^∞-critical for $\gamma = 1$ and L^∞-supercritical for $0 < \gamma < 1$. Whilst the wellposedness theory for (1.1) is relatively complete for the subcritical and critical regime $1 \le \gamma \le 2$ (cf. [1,11,12,15,18] and the references therein), there are very few results in the supercritical regime $0 < \gamma < 1$ ([13,17,21,27]). In this connection we mention three representative works: (1) The work of Miura [22] which establishes for the first time the large data local wellposedness in the critical space $H^{2-\gamma}$; (2) The work of Dong [12] which via a new set of commutator estimates establishes optimal polynomial in time smoothing estimates for critical and supercritical quasi-geostrophic equations; (3) The work of Biswas, Martinez and Silva [3] which establishes short-time Gevrey regularity with an exponent **strictly less than** γ, namely:

$$\sup_{0<t<T} \|e^{\lambda t^{\frac{\alpha}{\gamma}}D^\alpha}\theta(t, \cdot)\|_{\dot{B}_{p,q}^{1+\frac{2}{p}-\gamma}} \lesssim \|\theta_0\|_{\dot{B}_{p,q}^{1+\frac{2}{p}-\gamma}}, \tag{1.5}$$

where $2 \leq p < \infty$, $1 \leq q < \infty$ and $\alpha < \gamma$.

Inspired by these preceding works, we develop in this paper an optimal local regularity theory for the super-critical quasi-geostrophic equation. Set $\nu = 1$ in (1.1). If we completely drop the nonlinear term and keep only the linear dissipation term, then the linear solution is given by

$$\theta_{\text{linear}}(t, x) = (e^{-tD^{\gamma}} \theta_0)(t, x). \tag{1.6}$$

Formally speaking, one has the identity $e^{tD^{\gamma}}(\theta_{\text{linear}}(t, \cdot)) = \theta_0$ for any $t > 0$. This shows that the best smoothing estimate one can hope for is

$$\|e^{tD^{\gamma}}(\theta_{\text{linear}}(t, \cdot))\|_X \lesssim \|\theta_0\|_X, \tag{1.7}$$

where X is a working Banach space. The purpose of this work, rough speaking, is to show that for the nonlinear local solution to (1.1) (say taking $\nu = 1$ for simplicity of notation), we have

$$\|e^{(1-\epsilon_0)tD^{\gamma}}(\theta(t, \cdot))\|_X \lesssim_{\epsilon_0} \|\theta_0\|_X, \tag{1.8}$$

where $\epsilon_0 > 0$ can be taken any small number, and X can be a Sobolev or Besov space. In this sense this is the *best possible* regularity estimate for this and similar problems.

We now state in more detail the main results. To elucidate the main idea we first showcase the result on the prototypical L^2-type critical $H^{2-\gamma}$ space. The following offers a substantial improvement of Miura [22] and Dong [12]. To keep the paper self-contained, we give a bare-hand harmonic-analysis-free proof. The framework we develop here can probably be applied to many other problems.

Theorem 1.1. *Let $\nu = 1$, $0 < \gamma < 1$ and $\theta_0 \in H^{2-\gamma}$. For any $0 < \epsilon_0 < 1$, there exists $T = T(\gamma, \theta_0, \epsilon_0) > 0$ and a unique solution $\theta \in C_t^0 H^{2-\gamma} \cap C_t^1 H^{1-\gamma} \cap L_t^2 H^{2-\frac{\gamma}{2}}([0, T] \times \mathbb{R}^2)$ to (1.1) such that $f(t, \cdot) = e^{\epsilon_0 t D^{\gamma}} \theta(t, \cdot) \in C_t^0 H^{2-\gamma} \cap L_t^2 H^{2-\frac{\gamma}{2}}([0, T] \times \mathbb{R}^2)$ and*

$$\sup_{0 < t < T} \|f(t, \cdot)\|_{H^{2-\gamma}}^2 + \int_0^T \|f(t, \cdot)\|_{H^{2-\frac{\gamma}{2}}}^2 \, dt \leq C \|\theta_0\|_{H^{2-\gamma}}^2,$$

where $C > 0$ is a constant depending on (γ, ϵ_0).

Our next result is devoted to the Besov case. In particular, we resolve the problem left open in [3], namely one can push to the **optimal threshold** $\alpha = \gamma$. Moreover we cover the whole regime $1 \leq p < \infty$.

Theorem 1.2. *Let $\nu = 1$, $0 < \gamma < 1$, $1 \leq p < \infty$ and $1 \leq q < \infty$. Assume the initial data $\theta_0 \in B_{p,q}^{1+\frac{2}{p}-\gamma}(\mathbb{R}^2)$. There exists $T = T(\gamma, \theta_0, p, q) > 0$ and a unique solution $\theta \in C_t^0([0, T], B_{p,q}^{1+\frac{2}{p}-\gamma})$ to (1.1) such that $f(t, \cdot) = e^{\frac{1}{2}t D^{\gamma}} \theta(t, \cdot) \in C_t^0([0, T], B_{p,q}^{1+\frac{2}{p}-\gamma})$ and*

$$\sup_{0 < t < T} \|e^{\frac{1}{2}t D^{\gamma}} \theta(t, \cdot)\|_{B_{p,q}^{1+\frac{2}{p}-\gamma}} \leq C \|\theta_0\|_{B_{p,q}^{1+\frac{2}{p}-\gamma}},$$

where $C > 0$ is a constant depending on (γ, p, q).

The techniques introduced in this paper may apply to many other similar models such as Burgers equations, generalized SQG models, and Chemotaxi equations (cf. recent very interesting works [4,8,14,28]). Also there are some promising evidences that a set of nontrivial multiplier estimates can be generalized from our work. All these will be explored elsewhere. The rest of this paper is organized as follows. In Sect. 2 we collect some preliminary materials along with the needed proofs. In Sect. 3 we give the nonlinear estimates for the $H^{2-\gamma}$ case. In Sect. 4 we give the proof of Theorem 1.1. In Sect. 5 we give the proof of Theorem 1.2.

2. Notation and Preliminaries

In this section we introduce some basic notation used in this paper and collect several useful lemmas.

We define the sign function $\mathrm{sgn}(x)$ on $\mathbb{R}$ as:

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0, \\ 0, & x = 0. \end{cases}$$

For any two quantities X and Y, we denote $X \lesssim Y$ if $X \leq CY$ for some constant $C > 0$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence. We denote $X \lesssim_{Z_1,\ldots,Z_N} Y$ if the implied constant depends on the quantities $Z_1, \ldots, Z_N$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$.

For any quantity X, we will denote by $X+$ the quantity $X + \epsilon$ for some sufficiently small $\epsilon > 0$. The smallness of such ϵ is usually clear from the context. The notation $X-$ is similarly defined. This notation is very convenient for various exponents in interpolation inequalities. For example instead of writing

$$\|fg\|_{L^1(\mathbb{R})} \leq \|f\|_{L^{\frac{2}{1+\epsilon}}(\mathbb{R})} \|g\|_{L^{\frac{2}{1-\epsilon}}(\mathbb{R})},$$

we shall write

$$\|fg\|_{L^1(\mathbb{R})} \leq \|f\|_{L^{2-}(\mathbb{R})} \|g\|_{L^{2+}(\mathbb{R})}. \tag{2.1}$$

For any two quantities X and Y, we shall denote $X \ll Y$ if $X \leq cY$ for some sufficiently small constant c. The smallness of the constant c (and its dependence on other parameters) is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of $\ll$ and $\gg$ here is *different* from the usual Vinogradov notation in number theory or asymptotic analysis.

We shall adopt the following notation for Fourier transform on $\mathbb{R}^n$:

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi} dx,$$

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} g(\xi)e^{ix\cdot\xi} d\xi.$$

Similar notation will be adopted for the Fourier transform of tempered distributions. For real-valued Schwartz functions $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, the usual Plancherel takes the form (note that $\overline{\hat{g}(\xi)} = \hat{g}(-\xi)$)

$$\int_{\mathbb{R}^n} f(x)g(x)dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(-\xi)d\xi.$$

We shall denote for $s > 0$ the fractional Laplacian $D^s = (-\Delta)^{s/2} = |\nabla|^s$ as the operator corresponding to the symbol $|\xi|^s$. For any $0 \le r \in \mathbb{R}$, the Sobolev norm $\|f\|_{\dot{H}^r}$ is defined as

$$\|f\|_{\dot{H}^r} = \|D^r f\|_2 = \|(-\Delta)^{r/2} f\|_2.$$

We will need to use the Littlewood–Paley (LP) frequency projection operators. To fix the notation, let $\phi_0 \in C_c^\infty(\mathbb{R}^n)$ and satisfy

$$0 \le \phi_0 \le 1, \quad \phi_0(\xi) = 1 \text{ for } |\xi| \le 1, \quad \phi_0(\xi) = 0 \text{ for } |\xi| \ge 7/6.$$

Let $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$ which is supported in $\frac{1}{2} \le |\xi| \le \frac{7}{6}$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, $j \in \mathbb{Z}$, define

$$\widehat{P_{\le j} f}(\xi) = \phi_0(2^{-j}\xi)\hat{f}(\xi), \quad P_{>j} f = f - P_{\le j} f,$$
$$\widehat{P_j f}(\xi) = \phi(2^{-j}\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}^n.$$

Sometimes for simplicity we write $f_j = P_j f$, $f_{\le j} = P_{\le j} f$, and $f_{[a,b]} = \sum_{a \le j \le b} f_j$. Note that by using the support property of ϕ, we have $P_j P_{j'} = 0$ whenever $|j - j'| > 1$. For $f \in \mathcal{S}'$ with $\lim_{j \to -\infty} P_{\le j} f = 0$, one has the identity

$$f = \sum_{j \in \mathbb{Z}} f_j, \quad (\text{in } \mathcal{S}')$$

and for general tempered distributions the convergence (for low frequencies) should be taken as modulo polynomials.

The Bony paraproduct for a pair of functions $f, g \in \mathcal{S}(\mathbb{R}^n)$ take the form

$$fg = \sum_{i \in \mathbb{Z}} f_i g_{[i-1,i+1]} + \sum_{i \in \mathbb{Z}} f_i g_{\le i-2} + \sum_{i \in \mathbb{Z}} g_i f_{\le i-2}.$$

For $s \in \mathbb{R}$, $1 \le p, q \le \infty$, the Besov norm $\|\cdot\|_{B_{p,q}^s}$ is given by

$$\|f\|_{B_{p,q}^s} = \begin{cases} \|P_{\le 0} f\|_p + (\sum_{k=1}^{\infty} 2^{sqk} \|P_k f\|_p^q)^{1/q}, & q < \infty; \\ \|P_{\le 0} f\|_p + \sup_{k \ge 1} 2^{sk} \|P_k f\|_p, & q = \infty. \end{cases}$$

The Besov space $B_{p,q}^s$ is then simply

$$B_{p,q}^s = \left\{ f : f \in \mathcal{S}', \|f\|_{B_{p,q}^s} < \infty \right\}.$$

Note that Schwartz functions are dense in $B_{p,q}^s$ when $1 \le p, q < \infty$.

In the following lemma we give refined heat flow estimate and frequency localized Bernstein inequalities for the fractional Laplacian $|\nabla|^\gamma$, $0 < \gamma < 2$. Note that for $\gamma > 2$ and $p \ne 2$ there are counterexamples to the frequency Bernstein inequalities (cf. Li and Sire [20]).

Lemma 2.1 *(Refined heat flow estimate and Bernstein inequality, case $0 < \gamma < 2$). Let the dimension $n \geq 1$. Let $0 < \gamma < 2$ and $1 \leq q \leq \infty$. Then for any $f \in L^q(\mathbb{R}^n)$, and any $j \in \mathbb{Z}$, we have*

$$\|e^{-t|\nabla|^\gamma} P_j f\|_q \leq e^{-c_1 t 2^{j\gamma}} \|P_j f\|_q, \quad \forall t \geq 0, \tag{2.2}$$

where $c_1 > 0$ is a constant depending only on (γ, n). For $0 < \gamma < 2$, $1 \leq q < \infty$, we have

$$\int_{\mathbb{R}^n} (|\nabla|^\gamma P_j f)|P_j f|^{q-2} P_j f dx \geq c_2 2^{j\gamma} \|P_j f\|_q^q, \quad \text{if } 1 < q < \infty; \tag{2.3}$$

$$\int_{\mathbb{R}^n} (|\nabla|^\gamma P_j f) \operatorname{sgn}(P_j f) dx \geq c_2 2^{j\gamma} \|P_j f\|_1, \quad \text{if } q = 1, \tag{2.4}$$

where $c_2 > 0$ depends only on (γ, n).
The $q = \infty$ formulation of (2.3) is as follows. Let $0 < \gamma < 2$. For any $f \in L^\infty(\mathbb{R}^n)$, if $j \in \mathbb{Z}$ and $|(P_j f)(x_0)| = \|P_j f\|_\infty$, then we have

$$\operatorname{sgn}(P_j f(x_0)) \cdot (|\nabla|^\gamma P_j f)(x_0) \geq c_3 2^{j\gamma} \|P_j f\|_\infty, \tag{2.5}$$

where $c_3 > 0$ depends only on (γ, n).

Remark. For $1 < q < \infty$ the first two inequalities also hold for $\gamma = 2$, one can see Propositions 2.5 and 2.7 below. On the other hand, the inequality (2.4) does not hold for $\gamma = 2$. One can construct a counterexample in dimension $n = 1$ as follows. Take $g(x) = \frac{1}{4}(3\sin x - \sin 3x) = (\sin x)^3$ which only has zeros of third order. Take $h(x)$ with $\hat{h}$ compactly supported in $|\xi| \ll 1$ and $h(x) > 0$ for all x. Set

$$f(x) = g(x)h(x)$$

which obviously has frequency localized to $|\xi| \sim 1$ and have same zeros as $g(x)$. Easy to check that $\|f\|_1 \sim 1$ but

$$\int_{\mathbb{R}} f''(x) \operatorname{sgn}(f(x)) dx = 0.$$

Remark 2.2. For $\gamma > 0$ sufficiently small, one can give a direct proof for $1 \leq q < \infty$ as follows. WLOG consider $g = P_1 f$ with $\|g\|_q = 1$, and let

$$I(\gamma) = \int_{\mathbb{R}^n} (|\nabla|^\gamma g)|g|^{q-2} g dx.$$

One can then obtain

$$I(\gamma) - I(0) = \int_{\mathbb{R}^n} \left(\int_0^\gamma T_s g ds\right)|g|^{q-2} g dx, \quad \widehat{T_s g}(\xi) = s(|\xi|^s \log|\xi|)\hat{g}(\xi).$$

Since g has Fourier support localized in $\{|\xi| \sim 1\}$, one can obtain uniformly in $0 < s \leq 1$,

$$\|T_s g\|_q \lesssim_n \|g\|_q = 1.$$

Note that $I(0) = 1$. Thus for $\gamma < \gamma_0(n)$ sufficiently small one must have $\frac{1}{2} \leq I(\gamma) \leq \frac{3}{2}$.

Remark. The inequality (2.5) was obtained by Wang-Zhang [26] by an elegant contradiction argument under the assumption that $f \in C_0(\mathbb{R}^n)$ (i.e. vanishing at infinity) and f is frequency localized to a dyadic annulus. Here we only assume $f \in L^\infty$ and is frequency localized. This will naturally include periodic functions and similar ones as special cases. Moreover we provide two different proofs. The second proof is self-contained and seems quite short.

Proof of Lemma 2.1. For the first inequality and (2.3), see [19] for a proof using an idea of perturbation of the Lévy semigroup. Since the constant $c_2 > 0$ depends only on (γ, n), the inequality (2.4) can be obtained from (2.3) by taking the limit $q \to 1$. (Note that since $f_j = P_j f \in L^1$ and has compact Fourier support, f_j can be extended to be an entire function on $\mathbb{C}^n$ and its zeros must be isolated.)

Finally for (2.5) we give two proofs. With no loss we can assume $j = 1$ and write $f = P_1 f$. By using translation we may also assume $x_0 = 0$. With no loss we assume $\|f\|_\infty = f(x_0) = 1$.

The first proof is to use (2.2) which yields

$$(e^{-t|\nabla|^\gamma} f)(0) \le e^{-ct},$$

where $c > 0$ depends only on (γ, n). Then since $f = P_1 f$ is smooth and

$$f - e^{-t|\nabla|^\gamma} f = \int_0^t e^{-s|\nabla|^\gamma} |\nabla|^\gamma f \, ds = t|\nabla|^\gamma f - \int_0^t \left(\int_0^s e^{-\tau|\nabla|^\gamma} |\nabla|^{2\gamma} f \, d\tau \right) ds.$$

One can then divide by $t \to 0$ and obtain

$$(|\nabla|^\gamma f)(0) \ge c.$$

The second proof is more direct. We note that $\int \psi(y) dy = 0$ where ψ corresponds to the projection operator P_1. Since $1 = (P_1 f)(0)$, we obtain

$$1 = \int \psi(y)(f(y) - 1) dy \le \sup_{y \neq 0}(|\psi(y)| \cdot |y|^{n+\gamma}) \cdot \int_{\mathbb{R}^n} \frac{1 - f(y)}{|y|^{n+\gamma}} dy$$

$$\lesssim_{\gamma,n} \int_{\mathbb{R}^n} \frac{1 - f(y)}{|y|^{n+\gamma}} dy.$$

Thus

$$(|\nabla|^\gamma f)(0) \gtrsim_{\gamma,n} 1.$$

□

In what follows we will give a different proof of (2.3) (and some stronger versions, see Propositions 2.5 and (2.7)) and some equivalent characterization. For the sake of understanding (and keeping track of constants) we provide some details.

Lemma 2.3. *Let* $0 < s < 1$. *Then for any* $g \in L^2(\mathbb{R}^n)$ *with* $\hat{g}$ *being compactly supported, we have*

$$\frac{1}{2} C_{2s,n} \cdot \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}} dx \, dy = \| |\nabla|^s g \|_2^2,$$

where $C_{2s,n}$ is a constant corresponding to the fractional Laplacian $|\nabla|^{2s}$ having the asymptotics $C_{2s,n} \sim_n s(1-s)$ for $0 < s < 1$. As a result, if $g \in L^2(\mathbb{R}^n)$ and $\| |\nabla|^s g\|_2 < \infty$, then

$$s(1-s) \cdot \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}} dxdy \sim_n \| |\nabla|^s g\|_2^2.$$

Similarly if $g \in L^2(\mathbb{R}^n)$ and $\int \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}} dxdy < \infty$, then

$$\| |\nabla|^s g\|_2^2 \sim_n s(1-s) \cdot \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}} dxdy.$$

Proof. Note that

$$\| |\nabla|^s g\|_2^2 = \int_{\mathbb{R}^n} (|\nabla|^{2s} g)(x) g(x) dx = \int_{\mathbb{R}^n} \left(\lim_{\epsilon \to 0} C_{2s,n} \int_{|y-x|>\epsilon} \frac{g(x) - g(y)}{|x-y|^{n+2s}} dy \right) g(x) dx,$$

where $C_{2s,n} \sim_n s(1-s)$. Now for each $0 < \epsilon < 1$, it is easy to check that (for the case $\frac{1}{2} \le s < 1$ one needs to make use of the regularised quantity $g(x) - g(y) + \nabla g(x) \cdot (y-x)$)

$$\left| \int_{|y-x|>\epsilon} \frac{g(x) - g(y)}{|x-y|^{n+2s}} dy \right| \lesssim |g(x)| + |\mathcal{M}\}(x)| + |\mathcal{M}(\partial g)(x)| + |\mathcal{M}(\partial^2 g)(x)|,$$

where $\mathcal{M}g$ is the usual maximal function. By Lebesgue Dominated Convergence, we then obtain

$$\| |\nabla|^s g\|_2^2 = C_{2s,n} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{|y-x|>\epsilon} \frac{g(x) - g(y)}{|x-y|^{n+2s}} dy g(x) dx.$$

Now note that for each $\epsilon > 0$, we have

$$\int_{\mathbb{R}^n} \int_{|y-x|>\epsilon} \frac{|g(x) - g(y)|}{|x-y|^{n+2s}} dy |g(x)| dx \lesssim_{\epsilon,n} \int_{\mathbb{R}^n} (|g(x)| + |\mathcal{M}g(x)|) |g(x)| dx < \infty.$$

Therefore by using Fubini, symmetrising in x and y and Lebesgue Monotone Convergence, we obtain

$$\| |\nabla|^s g\|_2^2 = \frac{1}{2} C_{2s,n} \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}} dxdy = \frac{1}{2} C_{2s,n} \int \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}} dxdy.$$

Now if $g \in L^2(\mathbb{R}^n)$ with $\| |\nabla|^s g\|_2 < \infty$, then by Fatou's Lemma, we get

$$s(1-s) \int \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}} dxdy$$

$$\le s(1-s) \cdot \liminf_{J \to \infty} \int \frac{|P_{\le J} g(x) - P_{\le J} g(y)|^2}{|x-y|^{n+2s}} dxdy \lesssim_n s(1-s) \| |\nabla|^s g\|_2^2.$$

On the other hand, note that

$$|P_{\leq J}g(x) - P_{\leq J}g(y)|$$

$$\leq \int |g(x-z) - g(y-z)| 2^{nJ} |\phi(2^J z)| dz$$

$$\lesssim_n \left(\int |g(x-z) - g(y-z)|^2 2^{nJ} |\phi(2^J z)| dz \right)^{\frac{1}{2}},$$

where $\phi \in L^1$ is a smooth function used in the kernel $P_{\leq J}$. The desired equivalence then easily follows. $\qquad\qquad\square$

Lemma 2.4. *Let* $1 < q < \infty$. *Then for any* $a, b \in \mathbb{R}$, *we have*

$$(a-b)(|a|^{q-2}a - |b|^{q-2}b) \sim_q (|a|^{\frac{q}{2}-1}a - |b|^{\frac{q}{2}-1}b)^2,$$

$$(a-b)(|a|^{q-2}a - |b|^{q-2}b) \geq \frac{4(q-1)}{q^2}(|a|^{\frac{q}{2}-1}a - |b|^{\frac{q}{2}-1}b)^2.$$

Proof. The first inequality is easy to check. To prove the second inequality, it suffices to show for any $0 < x < 1$,

$$\frac{1 + x^q - x - x^{q-1}}{(1 - x^{\frac{q}{2}})^2} \geq \frac{4(q-1)}{q^2} = \frac{q^2 - (q-2)^2}{q^2}.$$

Set $t = x^{\frac{q}{2}} \in (0, 1)$. The inequality is obvious for $q = 2$. If $2 < q < \infty$, then we need to show

$$\frac{t^{\frac{1}{q}} - t^{1-\frac{1}{q}}}{1 - t} < \frac{q-2}{q}.$$

If $1 < q < 2$, then we need

$$\frac{t^{1-\frac{1}{q}} - t^{\frac{1}{q}}}{1 - t} < \frac{2-q}{q}.$$

Set $\eta = \min\{\frac{1}{q}, 1 - \frac{1}{q}\} \in (0, \frac{1}{2})$. It then suffices for us to show the inequality

$$f(\eta) = 1 - 2\eta - \frac{t^\eta - t^{1-\eta}}{1 - t} \geq 0.$$

Note that $f(0) = f(1/2) = 0$ and $f''(\eta) = -(t^\eta - t^{1-\eta})(\log t)^2/(1 - t) < 0$. Thus the desired inequality follows. $\qquad\qquad\square$

Proposition 2.5. *Let* $1 < q < \infty$ *and* $0 < \gamma \leq 2$. *Then for any* $f \in L^q(\mathbb{R}^n)$ *and any* $j \in \mathbb{Z}$, *we have*

$$\int_{\mathbb{R}^n} (|\nabla|^\gamma P_j f)|P_j f|^{q-2} P_j f dx \sim_q \left\| |\nabla|^{\frac{\gamma}{2}} (|P_j f|^{\frac{q}{2}-1} P_j f) \right\|_2^2,$$

$$\int_{\mathbb{R}^n} (|\nabla|^\gamma P_j f)|P_j f|^{q-2} P_j f dx$$

$$\geq \frac{4(q-1)}{q^2} \left\| |\nabla|^{\frac{\gamma}{2}} (|P_j f|^{\frac{q}{2}-1} P_j f) \right\|_2^2 \geq \frac{4(q-1)}{q^2} \left\| |\nabla|^{\frac{\gamma}{2}} (|P_j f|^{\frac{q}{2}}) \right\|_2^2. \qquad (2.6)$$

Consequently if $\| P_j f \|_q = 1$, then for any $0 < s \leq 1$,

$$\| \, |\nabla|^s (|P_j f|^{\frac{q}{2}-1} P_j f) \|_2 \sim_{q,n} 2^{js}. \tag{2.7}$$

Also for any $0 < s \leq 1$,

$$\| \, |\nabla|^s (|P_j f|^{\frac{q}{2}}) \|_2 \sim_{q,n} 2^{js}. \tag{2.8}$$

Remark. In [16], by using a strong nonlocal pointwise inequality (see also Córdoba-Córdoba [6]), Ju Proved an inequality of the form: if $0 \leq \gamma \leq 2$, $2 \leq q < \infty$, θ, $|\nabla|^\gamma \theta \in L^q$, then

$$\int (|\nabla|^\gamma \theta) |\theta|^{q-2} \theta \, dx \geq \frac{2}{q} \| \, |\nabla|^{\frac{\gamma}{2}} (|\theta|^{\frac{q}{2}}) \|_2^2.$$

A close inspection of our proof below shows that the inequality (2.6) also works with $P_j f$ replaced by θ. Note that the present form works for any $1 < q < \infty$. Furthermore in the regime $q > 2$, we have $\frac{4(q-1)}{q^2} > \frac{2}{q}$ and hence the constant here is slightly sharper.

Remark. The inequality (2.7) was already obtained by Chamorro and P. Lemarié-Rieusset in [9]. Remarkably modulo a q-dependent constant it is equivalent to the corresponding inequality for the more localized quantity $\int (|\nabla|^\gamma P_j f) |P_j f|^{q-2} P_j f \, dx$. The inequality (2.8) for $q > 2$ was obtained by Chen, Miao and Zhang [5] by using Danchin's inequality $\| \nabla (|P_1 f|^{q/2}) \|_2^2 \sim_{q,n} \| P_1 f \|_q^q$ together with a fractional Chain rule in Besov spaces. The key idea in [5] is to show $\| \nabla P_{[N_0, N_1]} (|P_1 f|^{q/2}) \|_2 \gtrsim 1$ and in order to control the high frequency piece one needs the assumption $q > 2$ (so as to use $|\nabla|^{1+\epsilon_0}$-derivative for $\epsilon_0 > 0$ sufficiently small). Our approach here is different: namely we will not use Danchin's inequality and prove directly $\| \, |\nabla|^{s_0} (|P_1 f|^{q/2}) \|_2 \gtrsim 1$ for some s_0 sufficiently small (depending on (q, n)). Together with some further interpolation argument we are able to settle the full range $1 < q < \infty$. One should note that in terms of lower bound the inequality (2.8) is stronger than (2.7).

Proof. With no loss we can assume $j = 1$ and for simplicity write $P_1 f$ as f. Assume first $0 < \gamma < 2$. Then for some constant $C_{\gamma,n} \sim_n \gamma(2 - \gamma)$, we have (the rigorous justification of the computation below follows a similar argument as in the proof of Lemma 2.3)

$$\int (|\nabla|^\gamma f) |f|^{q-2} f \, dx = C_{\gamma,n} \int (\lim_{\epsilon \to 0} \int_{|y-x| > \epsilon} \frac{f(x) - f(y)}{|x - y|^{n+\gamma}} dy) |f|^{q-2} f(x) dx$$

$$= \frac{1}{2} C_{\gamma,n} \int \frac{(f(x) - f(y))(|f|^{q-2} f(x) - |f|^{q-2} f(y))}{|x - y|^{n+\gamma}} dx dy$$

$$\geq \frac{4(q-1)}{q^2} \cdot \frac{1}{2} C_{\gamma,n} \int \frac{(|f|^{\frac{q}{2}-1} f(x) - |f|^{\frac{q}{2}-1} f(y))^2}{|x - y|^{n+\gamma}} dx dy$$

$$= \frac{4(q-1)}{q^2} \| \, |\nabla|^{\frac{\gamma}{2}} (|f|^{\frac{q}{2}-1} f) \|_2^2,$$

where in the last two steps we have used Lemmas 2.4 and 2.3 respectively. One may then carefully take the limit $\gamma \to 2$ to get the result for $\gamma = 2$ (when estimating $\| \, |\nabla|^{\frac{\gamma}{2}} (|f|^{\frac{q}{2}-1} f) \|_2$, one needs to split into $|\xi| \leq 1$ and $|\xi| > 1$, and use Lebesgue Dominated Convergence and Lebesgue Monotone Convergence respectively). By the

simple inequality $||f|^{\frac{q}{2}-1}f(x)-|f|^{\frac{q}{2}-1}f(y)| \geq ||f|^{\frac{q}{2}}(x)-|f|^{\frac{q}{2}}(y)|$, we also obtain $|||\nabla|^{\frac{\gamma}{2}}(|f|^{\frac{q}{2}-1}f)||_2 \geq |||\nabla|^{\frac{\gamma}{2}}(|f|^{\frac{q}{2}})||_2$.

Next to show (2.7), we can use Remark 2.2 to obtain $|||\nabla|^s g||_2 \sim_{q,n} 1$ for any $0 < s \leq s_0(n)$ and $g = |f|^{\frac{q}{2}-1}f$. Since $\|g\|_2 = 1$ and $\|\nabla g\|_2 \lesssim_{q,n} 1$, a simple interpolation argument then yields $|||\nabla|^s g||_2 \sim_{q,n} 1$ uniformly for $0 < s \leq 1$.

Finally to show (2.8), we first use the simple fact that $\|\nabla(|g|)\|_2 \leq \|\nabla g\|_2$ to get

$$|||\nabla|(|f|^{\frac{q}{2}})||_2 \lesssim_q 1.$$

It then suffices for us to show $|||\nabla|^s(|f|^{\frac{q}{2}})||_2 \gtrsim_{q,n} 1$ for $0 < s \leq s_0(q,n)$ sufficiently small. To this end we consider the quantity

$$I(s) = \int_{\mathbb{R}^n} |\nabla|^s(|f|)|f|^{q-1}dx.$$

For $0 < s < 1$ this is certainly well defined since $|||\nabla|^s(|f|)||_q \lesssim \|f\|_q + \|\nabla(|f|)\|_q \lesssim 1$. To circumvent the problem of differentiating under the integral, one can further consider the regularized expression (later $N \to \infty$)

$$I_N(s) = \int_{\mathbb{R}^n} |\nabla|^s P_{\leq N}(|f|)|f|^{q-1}dx.$$

Then

$$I_N(s) - I_N(0) = \int_{\mathbb{R}^n}\int_0^s (T_{\tilde{s}}P_{\leq N}(|f|))d\tilde{s}|f|^{q-1}dx, \quad \widehat{T_{\tilde{s}}}(\xi) = \tilde{s}|\xi|^{\tilde{s}}\log|\xi|.$$

Define $\widehat{T_{\tilde{s}}^{(1)}}(\xi) = \tilde{s}|\xi|^{\tilde{s}}(\log|\xi|) \cdot \chi_{|\xi|<1/10}$ and $T_{\tilde{s}}^{(2)} = T_{\tilde{s}} - T_{\tilde{s}}^{(1)}$. It is not difficult to check that uniformly in $0 < \tilde{s} \leq \frac{1}{2}$,

$$\sup_{\xi \neq 0} \max_{|\alpha| \leq [n/2]} (|\xi|^{|\alpha|}|\partial_\xi^\alpha(\widehat{T_{\tilde{s}}^{(1)}}(\xi))|) \lesssim_n 1.$$

Thus by Hörmander we get $\|T_{\tilde{s}}^{(1)}P_{\leq N}(|f|)\|_q \lesssim_{n,q} \|f\|_q = 1$. For $T_{\tilde{s}}^{(2)}$ one can use $\|\nabla f\|_q \lesssim 1$ to get an upper bound which is uniform in $0 < \tilde{s} \leq \frac{1}{2}$. Therefore $\|T_{\tilde{s}}P_{\leq N}(|f|)\|_q \lesssim_{q,n} 1$ for $0 < \tilde{s} \leq \frac{1}{2}$. One can then obtain for $0 < s \leq s_0(q,n)$ sufficiently small that $\frac{1}{2} \leq I(s) \leq \frac{3}{2}$. Finally view $I(s)$ as

$$I(s) = \lim_{N \to \infty} \int_{\mathbb{R}^n} |\nabla|^s(Q_{\leq N}(|f|))(Q_{\leq N}(|f|))^{q-1}dx,$$

where $\widehat{Q_{\leq N}}(\xi) = \hat{q}(2^{-N}\xi)$, and $\hat{q} \in C_c^\infty$ satisfies $q(x) \geq 0$ for any $x \in \mathbb{R}^n$ (such q can be easily constructed by taking $q(x) = \phi(x)^2$ which corresponds to $\hat{q} = \hat{\phi} * \hat{\phi}$). By using the integral representation of the operator $|\nabla|^s$ and a symmetrization argument (similar to what was done before), we can obtain

$$I(s) \sim_{q,n} |||\nabla|^{\frac{s}{2}}(|f|^{\frac{q}{2}})||_2^2$$

and the desired result follows. $\qquad\square$

Lemma 2.6. *Let the dimension $n \geq 1$ and $0 < \gamma \leq 2$. Suppose $g \in L^2(\mathbb{R}^n)$ and for some $N_0 > 0$, $\epsilon_0 > 0$*

$$\|\hat{g}\|^2_{L^2(|\xi| \geq N_0)} \geq \epsilon_0 \|\hat{g}\|^2_{L^2_\xi(\mathbb{R}^n)}.$$

Then there exists $t_0 = t_0(\epsilon_0, N_0, \gamma) > 0$ such that for all $0 \leq t \leq t_0$, we have

$$\|e^{-t|\nabla|^\gamma} g\|_2 \leq e^{-\frac{1}{2}\epsilon_0 N_0^\gamma t} \|g\|_2.$$

Consequently if $\tilde{g} \in L^2(\mathbb{R}^n)$ satisfies $\|\tilde{g}\|_2 = C_1 > 0$, $\||\nabla|^s \tilde{g}\|_2 = C_2 > 0$ for some $s > 0$, then for any $0 < \gamma \leq 2$, there exists $t_0 = t_0(C_1, C_2, \gamma, n, s) > 0$, $c_0 = c_0(C_1, C_2, \gamma, n, s) > 0$, such that

$$\|e^{-t|\nabla|^\gamma} \tilde{g}\|_2 \leq e^{-c_0 t} \|\tilde{g}\|_2.$$

Proof. With no loss we assume $\|\hat{g}\|_{L^2_\xi} = 1$. Then

$$\int_{\mathbb{R}^n} e^{-2t|\xi|^\gamma} |\hat{g}(\xi)|^2 d\xi \leq 1 - \|\hat{g}\|^2_{L^2(|\xi| \leq N_0)} + e^{-2tN_0^\gamma} \|\hat{g}\|^2_{L^2(|\xi| > N_0)}$$

$$\leq 1 - \epsilon_0 + e^{-2tN_0^\gamma} \epsilon_0 \leq 1 - \epsilon_0 + (1 - \frac{3}{2}tN_0^\gamma)\epsilon_0 \leq e^{-\epsilon_0 N_0^\gamma t},$$

where in the last two steps we used the fact $e^{-x} \sim 1 - x + \frac{x^2}{2}$ for $x \to 0+$. The inequality for $\tilde{g}$ follows from the observation that $\||\xi|^s \hat{\tilde{g}}(\xi)\|_{L^2_\xi(|\xi| \leq N_0)} \ll 1$ for N_0 sufficiently small. $\qquad\square$

Proposition 2.7. *Let the dimension $n \geq 1$, $0 < \gamma \leq 2$ and $1 < q < \infty$. Then for any $f \in L^q(\mathbb{R}^n)$, any $j \in \mathbb{Z}$ and any $t > 0$, we have*

$$\|e^{-t|\nabla|^\gamma} P_j f\|_q \leq e^{-c 2^{j\gamma} t} \|P_j f\|_q,$$

where $c > 0$ is a constant depending on (γ, n, q).

Proof. With no loss we assume $j = 1$ and write $P_1 f$ simply as f. In view of the semigroup property of $e^{-t|\nabla|^\gamma}$ it suffices to prove the inequality for $0 < t \ll_{\gamma,q,n} 1$. Denote $e^{-t|\nabla|^\gamma} f = K * f$ and observe that K is a positive kernel with $\int K(z) dz = 1$. Consider first $2 \leq q < \infty$. Clearly

$$|\int K(x - y) f(y) dy|^{\frac{q}{2}} \leq \int K(x - y) |f(y)|^{\frac{q}{2}} dy.$$

By Lemma 2.6 and Proposition 2.5, we then get

$$\|e^{-t|\nabla|^\gamma} f\|_q^q \leq \|e^{-t|\nabla|^\gamma} (|f|^{\frac{q}{2}})\|_2^2 \leq e^{-ct} \|f\|_q^q.$$

For the case $1 < q < 2$, we observe

$$\|\int K(x - y) |f(y)| dy\|_{L_x^q}$$

$$\leq \|(\int K(x - y) |f(y)|^{\frac{q}{2}} dy)^{\frac{2(q-1)}{q}} \cdot (\int K(x - y) |f(y)|^q dy)^{\frac{2-q}{q}}\|_{L_x^q}$$

$$\leq \|(\int K(x - y) |f(y)|^{\frac{q}{2}} dy)^{\frac{2(q-1)}{q}}\|_{L_x^{\frac{q}{q-1}}} \|(\int K(x - y) |f(y)|^q dy)^{\frac{2-q}{q}}\|_{L_x^{\frac{q}{2-q}}}$$

$$\leq \|e^{-t|\nabla|^\gamma} (|f|^{\frac{q}{2}})\|_2^2 \cdot \|f\|_q^{2-q}.$$

Thus this case is also OK. $\qquad\square$

For the next lemma we need to introduce some terminology. Consider a function $F : (0, \infty) \to \mathbb{R}$. We shall say F is admissible if $F \in C^\infty$ and

$$|F^{(k)}(x)| \lesssim_{k,F} x^{-k}, \quad \forall k \geq 0, \ 0 < x < \infty.$$

It is easy to check that $\tilde{F}(x) = x F'(x)$ is admissible if F is admissible. A simple example of admissible function is $F(x) = e^{-x}$ which will show up in the bilinear estimates later.

Lemma 2.8. *Let $0 < \gamma < 1$ and $\sigma(\xi, \eta) = |\xi|^\gamma + |\eta|^\gamma - |\xi + \eta|^\gamma$ for $\xi, \eta \in \mathbb{R}^n$, $n \geq 1$. Then for $0 < |\xi| \ll 1$, $|\eta| \sim 1$, the following hold:*

(1) $|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \lesssim_{\alpha,\beta,\gamma,n} |\xi|^{\gamma - |\alpha|}$, *for any α, β.*
(2) $|\partial_\xi^\alpha (\sigma^{-m})| \lesssim_{\alpha,\gamma,m,n} |\xi|^{-m\gamma - |\alpha|}$ *for any $m \geq 1$ and any α.*
(3) $|\partial_\xi^\alpha (F(t\sigma))| \lesssim_{\alpha,\gamma,n,F} |\xi|^{-|\alpha|}$ *for any α, $t > 0$, and any admissible F.*
(4) $|\partial_\xi^\alpha \partial_\eta^\beta (F(t\sigma))| \lesssim_{\alpha,\gamma,\beta,n,F} |\xi|^{-|\alpha|}$, *for any α, β, and any $t > 0$, F admissible.*

Remark 2.9. The condition $|\xi| \ll 1$, $|\eta| \sim 1$ can be replaced by $0 < |\xi| \lesssim 1$, $|\eta| \sim 1$, $|\xi + \eta| \sim 1$.

Remark. This lemma also highlights the importance of the assumption $0 < \gamma < 1$. For $0 < \gamma < 1$, note that the function $g(x) = 1 + x^\gamma - (1 + x)^\gamma \sim \min\{x^\gamma, 1\}$. By the triangle inequality, this implies $\sigma(\xi, \eta) \geq |\xi|^\gamma + |\eta|^\gamma - (|\xi| + |\eta|)^\gamma \gtrsim \min\{|\xi|^\gamma, |\eta|^\gamma\}$ which does not vanish as long as $|\xi| > 0$ and $|\eta| > 0$. However for $\gamma = 1$, the phase $\sigma(\xi, \eta) = |\xi| + |\eta| - |\xi + \eta|$ no longer enjoys such a lower bound since $\sigma \equiv 0$ on the one-dimensional cone $\xi = \lambda\eta$, $\lambda \geq 0$.

Proof. With no loss we consider dimension $n = 1$. The case $n > 1$ is similar except some minor changes in numerology.

(1) Note that for $|\xi| \ll 1$, $|\eta| \sim 1$, we have

$$\sigma(\xi, \eta) = |\xi|^\gamma - \gamma \int_0^1 |\eta + \theta\xi|^{\gamma - 2}(\eta + \theta\xi)d\theta \cdot \xi.$$

Observe that

$$\partial_\eta \sigma(\xi, \eta) = (\text{OK}) \cdot \xi,$$

where we use the notation OK to denote any term which satisfy

$$|\partial_\xi^\alpha \partial_\eta^\beta (\text{OK})| \lesssim 1, \quad \forall \alpha, \beta.$$

This notation will be used throughout this proof. Thus for any $\beta \geq 1$, $\alpha \geq 0$, we have

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \lesssim |\xi|^{1 - |\alpha|}.$$

On the other hand, if $\beta = 0$ and $\alpha \geq 1$, then clearly

$$|\partial_\xi^\alpha \sigma(\xi, \eta)| = |\partial_\xi^\alpha (|\xi|^\gamma - |\xi + \eta|^\gamma)| \lesssim |\xi|^{\gamma - |\alpha|}.$$

(2) Observe that for $|\xi| \ll 1$ and $|\eta| \sim 1$, we always have $\sigma(\xi, \eta) \gtrsim |\xi|^\gamma$. One can then induct on α.

(3) One can induct on α. The statement clearly holds for $\alpha = 0$. Assume the statement holds for $\alpha \le m$ and any admissible F. Then for $\alpha = m + 1$, we have

$$\partial_\xi^{m+1}(F(t\sigma)) = \partial_\xi^m(\tilde{F}(t\sigma) \cdot \sigma^{-1} \cdot \partial_\xi \sigma),$$

where $\tilde{F}(x) = xF'(x)$ is again admissible. The result then follows from the inductive assumption, Leibniz and the estimates obtained in (1) and (2).

(4) Observe that $\partial_\eta(\frac{1}{\sigma}) = -\sigma^{-2}\partial_\eta \sigma = \sigma^{-2}\xi \cdot$ (OK), and in general for $\beta \ge 0$,

$$\partial_\eta^\beta(\frac{1}{\sigma}) = \sum_{0 \le m \le \beta} \sigma^{-m-1}\xi^m \cdot \text{(OK)}.$$

Note that for $\beta \ge 1$ the summand corresponding to $m = 0$ is actually absent (this is allowed in our notation since we can take the term (OK) to be zero). Similarly one can check for any admissible F and $t > 0$,

$$\partial_\eta^\beta(F(t\sigma)) = \sum_{0 \le m \le \beta} F_m(t\sigma) \cdot \left(\frac{\xi}{\sigma}\right)^m \cdot \text{(OK)},$$

where F_m are admissible functions. This then reduce matters to the estimate in (3). The result is obvious. $\qquad\square$

3. Nonlinear Estimates: $H^{2-\gamma}$ Case for $0 < \gamma < 1$

Lemma 3.1. Set $A = D^\gamma$, $s = 2 - \gamma$ and recall $R^\perp = (-D^{-1}\partial_2, D^{-1}\partial_1)$. For any real-valued f, $g \in L^2(\mathbb{R}^2)$ with $\hat{f}$ and $\hat{g}$ being compactly supported, it holds that

$$\left| \int_{\mathbb{R}^2} D^s(e^{-tA}R^\perp g \cdot \nabla e^{-tA} f)D^s e^{tA} f\,dx \right| \lesssim_\gamma \|g\|_{\dot{H}^s} \|f\|_{\dot{H}^{s+\frac{\gamma}{2}}}^2. \tag{3.1}$$

If in addition $\text{supp}(\hat{g}) \subset B(0, N_0)$ for some $N_0 > 0$, then

$$\left| \int_{\mathbb{R}^2} D^s(e^{-tA}R^\perp g \cdot \nabla e^{-tA} f)D^s e^{tA} f\,dx \right|$$
$$\lesssim_\gamma N_0^{2s+2}\|f\|_2^2\|g\|_2 + \|f\|_{\dot{H}^s} \cdot \|f\|_{\dot{H}^{s+\frac{\gamma}{2}}} \cdot \|g\|_2 \cdot N_0^{2-\frac{\gamma}{2}}, \tag{3.2}$$

$$\left| \int_{\mathbb{R}^2} D^s(e^{-tA}R^\perp f \cdot \nabla e^{-tA} g)D^s e^{tA} f\,dx \right| \lesssim_\gamma \|f\|_{\dot{H}^s}^2 \cdot N_0^2\|g\|_2 + N_0^{2s+2}\|f\|_2^2\|g\|_2, \tag{3.3}$$

$$\left| \int_{\mathbb{R}^2} D^s(e^{-tA}R^\perp g \cdot \nabla e^{-tA} g)D^s e^{tA} f\,dx \right| \lesssim_\gamma N_0^{2s+2}\|g\|_2^2\|\hat{f}\|_{L^2(|\xi| \le 2N_0)}. \tag{3.4}$$

Remark 3.2. Note that if $\hat{f}$ is localized to $|\xi| \gtrsim N_0$, then the low-frequency term $N_0^{2s+2}\|f\|_2^2\|g\|_2$ can be dropped in (3.2) and (3.3).

Proof. We first show (3.1). For simplicity of notation we shall write $R^\perp g$ as g. Note that in the final estimates the operator $R^\perp$ can be easily discarded since we are in the L^2 setting. On the Fourier side we express the LHS inside the absolute value as (up to a multiplicative constant)

$$\int |\xi|^{2s} e^{-t(|\eta|^\gamma + |\xi - \eta|^\gamma - |\xi|^\gamma)} \hat{g}(\eta) \cdot (\xi - \eta) \hat{f}(\xi - \eta) \hat{f}(-\xi) d\xi d\eta.$$

Observe that by a change of variable $\xi \to \eta - \tilde{\xi}$ (and dropping the tildes), we have

$$\int (\xi - \eta) |\xi|^{2s} e^{-t(|\xi - \eta|^\gamma - |\xi|^\gamma)} \hat{f}(\xi - \eta) \hat{f}(-\xi) d\xi$$

$$= \frac{1}{2} \int \left((\xi - \eta) |\xi|^{2s} e^{-t(|\xi - \eta|^\gamma - |\xi|^\gamma)} - \xi |\xi - \eta|^{2s} e^{-t(|\xi|^\gamma - |\xi - \eta|^\gamma)} \right) \hat{f}(\xi - \eta) \hat{f}(-\xi) d\xi.$$

Denote

$$\tilde{\sigma}(\xi, \eta) = e^{-t|\eta|^\gamma} \left((\xi - \eta) |\xi|^{2s} e^{-t(|\xi - \eta|^\gamma - |\xi|^\gamma)} - \xi |\xi - \eta|^{2s} e^{-t(|\xi|^\gamma - |\xi - \eta|^\gamma)} \right),$$

$$N(g^1, g^2, g^3) = \int \tilde{\sigma}(\xi, \eta) \widehat{g^1}(\eta) \widehat{g^2}(\xi - \eta) \widehat{g^3}(-\xi) d\xi d\eta.$$

We just need to bound $N(g, f, f)$. By frequency localization, we have

$$N(g, f, f) = \sum_j \left(N(g_{<j-9}, f_j, f) + N(g_{>j+9}, f_j, f) + N(g_{[j-9,j+9]}, f_j, f) \right).$$

Rewriting $\sum_j N(g_{>j+9}, f_j, f) = \sum_j N(g_j, f_{<j-9}, f)$, we obtain

$$N(g, f, f) = \sum_j \left(N(g_{<j-9}, f_j, f) + N(g_j, f_{<j-9}, f) + N(g_{[j-9,j+9]}, f_j, f) \right)$$

$$= \sum_j \left(N(g_{<j-9}, f_j, f_{[j-2,j+2]}) + N(g_j, f_{<j-9}, f_{[j-2,j+2]}) + N(g_{[j-9,j+9]}, f_j, f_{\leq j+11}) \right)$$

$$= \sum_j \left(N(g_{\ll j}, f_{\sim j}, f_{\sim j}) + N(g_{\sim j}, f_{\ll j}, f_{\sim j}) + N(g_{\sim j}, f_{\sim j}, f_{\lesssim j}) \right),$$

where $g_{\ll j}$ corresponds to $|\eta| \ll 2^j$, $g_{\sim j}$ means $|\eta| \sim 2^j$, and $g_{\lesssim j}$ means $|\eta| \lesssim 2^j$. These notations are quite handy since only the relative sizes of the frequency variables η, ξ and $\xi - \eta$ will play some role in the estimates. Note that we should have written $g_{\ll j}$ as $g_{\{l:\, 2^l \ll 2^j\}}$ according to our convention of the notation $\ll$ but we ignore this slight inconsistency here for the simplicity of notation.

1. Estimate of $N(g_{\ll j}, f_{\sim j}, f_{\sim j})$. Note that $|\eta| \ll 2^j$, $|\xi - \eta| \sim |\xi| \sim 2^j$. It is not difficult to check that in this regime

$$|\tilde{\sigma}(\xi, \eta)| \leq |(\xi - \eta) |\xi|^{2s} - \xi |\xi - \eta|^{2s}| \cdot e^{-t(|\eta|^\gamma + |\xi - \eta|^\gamma - |\xi|^\gamma)}$$

$$\quad + |\xi| |\xi - \eta|^{2s} \cdot |e^{-t(|\eta|^\gamma + |\xi - \eta|^\gamma - |\xi|^\gamma)} - e^{-t(|\xi|^\gamma + |\eta|^\gamma - |\xi - \eta|^\gamma)}|$$

$$\quad \lesssim 2^{2js} \cdot |\eta| + |\xi| |\xi - \eta|^{2s} |e^{-t(|\eta|^\gamma + |\xi - \eta|^\gamma - |\xi|^\gamma)} - e^{-t(|\xi|^\gamma + |\eta|^\gamma - |\xi - \eta|^\gamma)}|.$$

To bound the second term, we shall use Lemma 2.8. More precisely, denote $\tilde{\xi} = -2^{-j}\eta$, $\tilde{\eta} = 2^{-j}\xi$, $T = 2^{j\gamma}t$, $F(x) = e^{-x}$. Clearly (recall in Lemma 2.8, $\sigma(\xi, \eta) = |\xi|^\gamma + |\eta|^\gamma - |\xi + \eta|^\gamma$)

$$e^{-t(|\eta|^\gamma + |\xi-\eta|^\gamma - |\xi|^\gamma)} = e^{-T(|\tilde{\xi}|^\gamma + |\tilde{\xi}+\tilde{\eta}|^\gamma - |\tilde{\eta}|^\gamma)} = F(T\sigma(\tilde{\xi}, \tilde{\eta} - \tilde{\xi})),$$

$$e^{-t(|\xi|^\gamma + |\eta|^\gamma - |\xi-\eta|^\gamma)} = F(T\sigma(\tilde{\xi}, \tilde{\eta})). \tag{3.5}$$

Consider for $0 \le \theta \le 1$, the function $G(\theta) = F(T\sigma(\tilde{\xi}, \tilde{\eta} - \theta\tilde{\xi}))$. By Lemma 2.8, we have

$$|G(1) - G(0)| \le \int_0^1 |\partial_{\tilde{\eta}}(F(T\sigma(\tilde{\xi}, \tilde{\eta} - \theta\tilde{\xi})))|d\theta \cdot |\tilde{\xi}| \lesssim |\tilde{\xi}| = 2^{-j}|\eta|.$$

Thus

$$|\tilde{\sigma}(\xi, \eta)| \lesssim 2^{2js} \cdot |\eta|.$$

Then by taking $\tilde{\epsilon} = \gamma$ below (note that $0 < \gamma < 1$), we get (below "$*$" denotes the usual convolution)

$$\left| \sum_j N(g_{\ll j}, f_{\sim j}, f_{\sim j}) \right| \lesssim \sum_j 2^{2js} \cdot \|f_{\sim j}\|_2 \cdot (\| \, |\widehat{Dg_{\ll j}}| * |\widehat{f_{\sim j}}| \,)\|_2)$$

$$\lesssim \sum_j 2^{2js} \cdot \|f_{\sim j}\|_2 \cdot (\|Dg_{\ll j}\|_{\dot{H}^{1-\tilde{\epsilon}}} \cdot \|f_{\sim j}\|_{\dot{H}^{\tilde{\epsilon}}})$$

$$\lesssim \|g\|_{\dot{H}^s} \|f\|^2_{\dot{H}^{s+\frac{\gamma}{2}}}.$$

Here in the second inequality above, we have used the simple fact that

$$\| \, |\hat{A}| * |\hat{B}| \, \|_{L^2_\xi} \lesssim \|\hat{A}\|_{L^1_\xi} \|\hat{B}\|_{L^2_\xi} \lesssim \| |\xi|^\theta \hat{A}\|_{L^2_\xi} \| |\xi|^{1-\theta} \hat{B}\|_{L^2_\xi},$$

if $\text{supp}(\hat{A}) \subset \{|\xi| \lesssim 1\}$, $\text{supp}(\hat{B}) \subset \{|\xi| \sim 1\}$, and $\theta < 1$.

2. Estimate of $N(g_{\sim j}, f_{\ll j}, f_{\sim j})$. In this case $|\eta| \sim |\xi| \sim 2^j$, $|\xi - \eta| \ll 2^j$. Since $s = 2 - \gamma \in (1, 2)$, in this regime we have

$$|\tilde{\sigma}(\xi, \eta)| \lesssim |\xi - \eta|2^{2js} + 2^j|\xi - \eta|^{2s} \lesssim 2^{2js}|\xi - \eta|.$$

Then

$$\left| \sum_j N(g_{\sim j}, f_{\ll j}, f_{\sim j}) \right| \lesssim \sum_j 2^{2js} \| \widehat{g_{\sim j}} * |\widehat{Df_{\ll j}}| \|_2 \cdot \|f_{\sim j}\|_2$$

$$\lesssim \sum_j 2^{2js} \|g_{\sim j}\|_{\dot{H}^{\frac{\gamma}{2}}} \cdot \|Df_{\ll j}\|_{\dot{H}^{1-\frac{\gamma}{2}}} \|f_{\sim j}\|_2 \lesssim \|g\|_{\dot{H}^s} \|f\|^2_{\dot{H}^{s+\frac{\gamma}{2}}}.$$

3. Estimate of $N(g_{\sim j}, f_{\sim j}, f_{\lesssim j})$. In this case $|\eta| \sim |\xi - \eta| \sim 2^j$, $|\xi| \lesssim 2^j$, and

$$\left| \sum_j N(g_{\sim j}, f_{\sim j}, f_{\lesssim j}) \right| \lesssim \sum_j 2^{2js} \| \widehat{g_{\sim j}} * |\widehat{f_{\sim j}}| \|_{L^\infty_\xi} \|\widehat{Df_{\lesssim j}}\|_{L^1_\xi}$$

$$\lesssim \sum_j 2^{2js} \|g_{\sim j}\|_2 \|f_{\sim j}\|_2 \cdot \| |\xi|^{1-\frac{\gamma}{2}} \widehat{Df_{\lesssim j}}\|_{L^2_\xi} \cdot 2^{j\frac{\gamma}{2}}$$

$$\lesssim \|g\|_{\dot{H}^s} \|f\|^2_{\dot{H}^{s+\frac{\gamma}{2}}}.$$

Now we turn to (3.2). Choose $J_0 \in \mathbb{Z}$ such that $2^{J_0-1} \leq N_0 < 2^{J_0}$. Clearly by frequency localization,

$$N(g, f, f) = N(g, f_{\lesssim J_0}, f_{\lesssim J_0}) + \sum_{j:2^j \gg N_0} N(g, f_{\sim j}, f_{\sim j}).$$

For the first term we have

$$|N(g, f_{\lesssim J_0}, f_{\lesssim J_0})| \lesssim N_0^{2s+1} \|\,|\widehat{g}| * |\widehat{f_{\lesssim J_0}}|\,\|_{L_\xi^\infty} \|\widehat{f_{\lesssim J_0}}\|_{L_\xi^1}$$

$$\lesssim N_0^{2s+2} \|g\|_2 \|f\|_2^2.$$

For the second term we can use the estimate of $N(g_{\ll j}, f_{\sim j}, f_{\sim j})$ and take $\tilde{\epsilon} = \frac{\gamma}{2}$ to get

$$\Big| \sum_{j:2^j \gg N_0} N(g, f_{\sim j}, f_{\sim j}) \Big| \lesssim \sum_{j:2^j \gg N_0} 2^{2js} \cdot \|f_{\sim j}\|_2 \cdot (\|Dg\|_{\dot{H}^{1-\tilde{\epsilon}}} \|f_{\sim j}\|_{\dot{H}^{\tilde{\epsilon}}})$$

$$\lesssim \|f\|_{\dot{H}^s} \cdot \|f\|_{\dot{H}^{s+\frac{\gamma}{2}}} \cdot \|g\|_2 \cdot N_0^{2-\frac{\gamma}{2}}.$$

The estimates of (3.3) and (3.4) are much simpler. We omit the details. □

4. Proof of Theorem 1.1

To simplify numerology we conduct the proof for the case $\epsilon_0 = 1/2$. Throughout this proof we shall denote $s = 2 - \gamma$.

Step 1. A priori estimate. Denote $A = \frac{1}{2}D^\gamma$ and $f = e^{tA}\theta$. It will be clear from Step 2 below that f is smooth and well-defined, and the following computations can be rigorously justified. Then f satisfies the equation

$$\partial_t f = -\frac{1}{2}D^\gamma f - e^{tA}(R^\perp e^{-tA} f \cdot \nabla e^{-tA} f).$$

Take $J_0 > 0$ which will be made sufficiently large later. Set $N_0 = 2^{J_0}$. Then

$$\frac{1}{2}\frac{d}{dt}(\|D^s P_{>J_0} f\|_2^2) + \frac{1}{2}\|D^{s+\frac{\gamma}{2}} P_{>J_0} f\|_2^2$$

$$= -\int D^s(R^\perp e^{-tA} f \cdot \nabla e^{-tA} f) D^s e^{tA} P_{>J_0}^2 f dx.$$

Now for convenience of notation we denote

$$N(g_1, g_2, g_3) = \int D^s(R^\perp e^{-tA} g_1 \cdot \nabla e^{-tA} g_2) D^s e^{tA} g_3 dx.$$

Denote $f_h = P_{>J_0}^2 f$ and $f_l = f - f_h$. Then clearly

$$N(f, f, f_h) = N(f_h, f_h, f_h) + N(f_l, f_h, f_h) + N(f_h, f_l, f_h) + N(f_l, f_l, f_h).$$

By Lemma 3.1 and noting that $\|f_l(t)\|_2 \lesssim e^{N_0^\gamma t}\|\theta_0\|_2$, we get (see Remark 3.2)

$$|N(f, f, f_h)|$$

$$\lesssim \|f_h\|_{\dot{H}^s}\|f_h\|_{\dot{H}^{s+\frac{\gamma}{2}}}^2 + \|f_h\|_{\dot{H}^s}\|f_h\|_{\dot{H}^{s+\frac{\gamma}{2}}} \cdot N_0^{2-\frac{\gamma}{2}} e^{N_0^\gamma t}\|\theta_0\|_2 + \|f_h\|_{\dot{H}^s}^2 \cdot N_0^2 e^{N_0^\gamma t}\|\theta_0\|_2$$

$$+ N_0^{2s+2} e^{4N_0^\gamma t}\|\theta_0\|_2^3.$$

This implies for $0 < t \le N_0^{-\gamma}$,

$$\frac{d}{dt}(\|D^s P_{>J_0} f\|_2^2) + \left(\frac{1}{2} - c_1\|D^s P_{>J_0} f\|_2\right) \cdot \|D^{s+\frac{\gamma}{2}} P_{>J_0} f\|_2^2$$

$$\le \underbrace{c_2 \cdot (N_0^2\|\theta_0\|_2 + N_0^{4-\gamma}\|\theta_0\|_2^2)}_{=:\beta} \cdot \|D^s P_{>J_0} f\|_2^2 + c_3 N_0^{2s+2}\|\theta_0\|_2^3,$$

where $c_1, c_2, c_3 > 0$ are constants depending on γ.

Thus as long as $\sup_{0\le s\le t} c_1\|D^s P_{>J_0} f(s)\|_2 < \frac{1}{10}$ and $t \le N_0^{-\gamma}$, we obtain

$$\sup_{0\le s\le t} \|D^s P_{>J_0} f(s)\|_2^2 \le e^{\beta t}\|D^s P_{>J_0}\theta_0\|_2^2 + te^{\beta t}c_3 N_0^{2s+2}\|\theta_0\|_2^3. \tag{4.1}$$

In particular, for any prescribed small constant $\epsilon_0 > 0$, we can first choose J_0 sufficiently large such that

$$\|D^s P_{>J_0}\theta_0\|_2 < \frac{1}{2}\epsilon_0. \tag{4.2}$$

Then by using (4.1) and choosing $T_0 = T_0(J_0, \theta_0, \epsilon_0)$ sufficiently small we can guarantee

$$\sup_{0\le s\le T_0} \|D^s P_{>J_0} f(s)\|_2 < \epsilon_0. \tag{4.3}$$

Step 2. Approximation system. For $n = 1, 2, 3, \cdots$, define $\theta^{(n)}$ as solutions to the system

$$\begin{cases} \partial_t\theta^{(n)} = -P_{<n}(R^\perp P_{<n}\theta^{(n)} \cdot \nabla P_{<n}\theta^{(n)}) - D^\gamma\theta^{(n)}, \\ \theta^{(n)}\big|_{t=0} = P_{<n}\theta_0. \end{cases}$$

The solvability of the above regularized system is not an issue thanks to frequency cut-offs. It is easy to check that $\theta^{(n)}$ has frequency supported in $|\xi| \lesssim 2^n$ and $\|\theta^{(n)}\|_2 \le \|\theta_0\|_2$. In particular for any $\tilde{s} \ge 0$ we have

$$\|D^{\tilde{s}} P_{\le J_0}\theta^{(n)}\|_2 \lesssim 2^{cJ_0\tilde{s}}\|\theta_0\|_2, \tag{4.4}$$

where $c > 0$ is a constant.

For any integer J_0 to be fixed momentarily, it is not difficult to check that

$$\frac{1}{2}\frac{d}{dt}(\|D^s P_{>J_0} e^{tA}\theta^{(n)}\|_2^2) + \frac{1}{2}\|D^{s+\frac{\gamma}{2}} P_{>J_0} e^{tA}\theta^{(n)}\|_2^2$$

$$= -\int D^s(R^\perp P_{<n}\theta^{(n)} \cdot \nabla P_{<n}\theta^{(n)})D^s e^{tA} P_{>J_0}^2 e^{tA} P_{<n}\theta^{(n)}dx.$$

Now fix J_0 sufficiently large such that

$$c_1 \| D^s P_{>J_0} \theta_0 \|_2 < 1/10.$$

By using the nonlinear estimate derived in Step 1 (easy to check that these estimates hold for $\theta^{(n)}$ with slight changes of the constants c_i if necessary), one can then find $T_0 = T_0(\gamma, \theta_0) > 0$ sufficiently small such that uniformly in n tending to infinity, we have

$$\sup_{0 \le t \le T_0} \| e^{tA} D^s \theta^{(n)}(t, \cdot) \|_2 + \int_0^{T_0} \| e^{tA} D^{s+\frac{\gamma}{2}} \theta^{(n)}(t, \cdot) \|_2^2 dt \lesssim 1.$$

By slightly shrinking T_0 further if necessary and repeating the argument for $\tilde{A} = \frac{4}{3} A = \frac{2}{3} D^\gamma$, we have uniformly in n tending to infinity,

$$\sup_{0 \le t \le T_0} \| e^{\frac{4}{3} tA} D^s \theta^{(n)}(t, \cdot) \|_2 \lesssim 1.$$

Furthermore for any prescribed small constant $\epsilon_0 > 0$, by using (4.3), we can choose J_0 and T_0 such that uniformly in n,

$$\sup_{0 \le t \le T_0} \| D^s P_{>J_0} e^{tA} \theta^{(n)} \|_2 < \epsilon_0.$$

Note that this implies

$$\sup_{0 \le t \le T_0} \| D^s P_{>J_0} \theta^{(n)} \|_2 < \epsilon_0. \tag{4.5}$$

The estimate (4.5) will be needed later.

Step 3. Strong contraction of $\theta^{(n)}$ in $C_t^0 L_x^2$. Denote $\eta_{n+1} = \theta^{(n+1)} - \theta^{(n)}$. Then (below for simplicity of notation we write $-R^\perp$ as R)

$$\begin{aligned}
\partial_t \eta_{n+1} = {} & P_{<n+1}(R P_{<n+1} \eta_{n+1} \cdot \nabla P_{<n+1} \eta_{n+1}) - D^\gamma \eta_{n+1} \\
& + P_{<n+1}(R P_{<n+1} \theta^{(n)} \cdot \nabla P_{<n+1} \theta^{(n)}) - P_{<n}(R P_{<n} \theta^{(n)} \cdot \nabla P_{<n} \theta^{(n)}) \\
& + P_{<n+1}(R P_{<n+1} \theta^{(n)} \cdot \nabla P_{<n+1} \eta_{n+1}) \\
& + P_{<n+1}(R P_{<n+1} \eta_{n+1} \cdot \nabla P_{<n+1} \theta^{(n)}).
\end{aligned} \tag{4.6}$$

By using the divergence-free property, we have

$$\int (4.6) \cdot \eta_{n+1} dx$$
$$= - \int \left(P_{<n+1}(R P_{<n+1} \theta^{(n)} P_{<n+1} \theta^{(n)}) - P_{<n}(R P_{<n} \theta^{(n)} P_{<n} \theta^{(n)}) \right) \cdot \nabla \eta_{n+1} dx.$$

Clearly

$$\| P_{<n+1}(R P_{<n+1} \theta^{(n)} P_{<n+1} \theta^{(n)}) - P_{<n}(R P_{<n} \theta^{(n)} P_{<n} \theta^{(n)}) \|_2$$
$$\le \| P_{<n+1}(R(P_{<n+1} - P_{<n}) \theta^{(n)} P_{<n+1} \theta^{(n)}) \|_2 + \| P_{<n+1}(R P_{<n} \theta_n (P_{<n+1} - P_{<n}) \theta^{(n)}) \|_2$$
$$\quad + \| (P_{<n+1} - P_{<n})(R P_{<n} \theta^{(n)} P_{<n} \theta^{(n)}) \|_2$$
$$\lesssim 2^{-n(2-\gamma)} \| \theta^{(n)} \|_{H^{2-\gamma}}^2 + 2^{-n} \| \theta^{(n)} \|_{H^{2-\gamma}}^2 \lesssim 2^{-n},$$

where we have used the uniform Sobolev estimates in Step 2. Note that

$$\|\nabla \eta_{n+1}\|_2 \lesssim \|\theta^{(n)}\|_{H^{2-\gamma}} + \|\theta^{(n+1)}\|_{H^{2-\gamma}} \lesssim 1.$$

It follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\eta_{n+1}\|_2^2 + \|D^{\frac{\gamma}{2}}\eta_{n+1}\|_2^2 \\
&\quad \le \int (RP_{<n+1}\eta_{n+1} \cdot \nabla P_{<n+1}\theta^{(n)}) P_{<n+1}\eta_{n+1}\, dx \ + \mathrm{const}\cdot 2^{-n} \\
&\quad \le \mathrm{const}\cdot 2^{-n} + \int (RP_{<n+1}\eta_{n+1} \cdot \nabla P_{>J_0}P_{<n+1}\theta^{(n)}) P_{<n+1}\eta_{n+1}\, dx \\
&\qquad + \int (RP_{<n+1}\eta_{n+1} \cdot \nabla P_{\le J_0}P_{<n+1}\theta^{(n)}) P_{<n+1}\eta_{n+1}\, dx \\
&\quad \lesssim 2^{-n} + \|\eta_{n+1}\|_{(\frac{1}{2}-\frac{\gamma}{4})^{-1}}^2 \cdot \|\nabla P_{>J_0}P_{<n+1}\theta^{(n)}\|_{\frac{2}{\gamma}} \\
&\qquad + \|\eta_{n+1}\|_2^2 \cdot 2^{2J_0}\|\theta^{(n)}\|_2 \\
&\quad \lesssim 2^{-n} + \|D^{\frac{\gamma}{2}}\eta_{n+1}\|_2^2 \cdot \|D^s P_{>J_0}\theta^{(n)}\|_2 + 2^{2J_0}\|\eta_{n+1}\|_2^2.
\end{aligned}
$$

By using the nonlinear estimates in Step 2 and (4.5), one can choose J_0 sufficiently large (and slightly shrink T_0 further if necessary) such that the term $\|D^s P_{>J_0}\theta^{(n)}\|_2$ becomes sufficiently small (to kill the implied constant pre-factors in the above inequality). This implies

$$\frac{d}{dt}\|\eta_{n+1}\|_2^2 \lesssim 2^{-n} + 2^{2J_0}\|\eta_{n+1}\|_2^2. \tag{4.7}$$

Thus for some constants $\tilde{c}_1 > 0$, $\tilde{c}_2 > 0$, we have

$$\sup_{0\le t\le T_0}\|\eta_{n+1}\|_2^2 \le e^{\tilde{c}_1\cdot 2^{2J_0}T_0}\|\eta_{n+1}(0)\|_2^2 + e^{\tilde{c}_1\cdot 2^{2J_0}T_0}\cdot 2^{-n}\tilde{c}_2, \tag{4.8}$$

The desired strong contraction of $\theta^{(n)} \to \theta$ in $C_t^0 L_x^2$ follows easily.

    Step 4. Higher norms. By using the estimates in previous steps, we have for any $0 \le t \le T_0$,

$$
\begin{aligned}
\|D^s e^{\frac{4}{3}tA}\theta\|_2^2 &\le \limsup_{N\to\infty}\|D^s e^{\frac{4}{3}tA}P_{\le N}\theta\|_2^2 \\
&= \limsup_{N\to\infty}\lim_{n\to\infty}\|D^s e^{\frac{4}{3}tA}P_{\le N}\theta^{(n)}(t)\|_2^2 < B_1 < \infty,
\end{aligned}
$$

where the constant $B_1 > 0$ is independent of t.

    It follows easily that for any $0 \le s' < s$,

$$\|D^{s'}e^{tA}(\theta^{(n)}(t) - \theta(t))\|_{L_t^\infty L_x^2} \to 0, \quad \text{as } n \to \infty,$$

This implies $f(t) = e^{tA}\theta(t) \in C_t^0 H_x^{s'}$ for any $s' < s$. To show $f \in C_t^0 H_x^s$ it suffices to consider the continuity at $t = 0$ (for $t > 0$ one can use the fact $e^{\frac{1}{3}tA}f \in L_t^\infty H_x^s$ which controls frequencies $|\xi| \gg t^{-1/\gamma}$, and for the part $|\xi| \lesssim t^{-1/\gamma}$ one uses $C_t^0 L_x^2$). Since we are in the Hilbert space setting with weak continuity in time, the strong continuity then follows from norm continuity at $t = 0$ which is essentially done in Step 1.

5. Nonlinear Estimates for Besov Case: $0 < \gamma < 1$

For $\sigma = \sigma(\xi, \eta)$ we denote the bilinear operator

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^d_\eta} \int_{\mathbb{R}^d_\xi} \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta.$$

Lemma 5.1. *Suppose* $\mathrm{supp}(\sigma) \subset \{(\xi, \eta) : |\xi| < 1, \frac{1}{C_1} < |\eta| < C_1\}$ *for some constant* $C_1 > 0$. *Let* $n_0 = 2d + [d/2] + 1$ *and* $\Omega_0 = \{(\xi, \eta) : 0 < |\xi| < 1, \frac{1}{C_1} < |\eta| < C_1\}$. *Suppose* $\sigma \in C^{n_0}_{\mathrm{loc}}(\Omega_0)$ *and for some* $A_1 > 0$

$$\sup_{\substack{|\alpha| \leq [d/2]+1 \\ |\beta| \leq 2d}} \sup_{(\xi, \eta) \in \Omega_0} |\xi|^{|\alpha|} |\eta|^{|\beta|} |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq A_1.$$

Then for any $1 < p_1 < \infty$, $1 \leq p_2 \leq \infty$, $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\|T_\sigma(f, g)\|_r \lesssim_{d, C_1, A_1, p_1, p_2} \|f\|_{p_1} \|g\|_{p_2},$$

where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.

Similarly if $\mathrm{supp}(\sigma) \subset \{(\xi, \eta) : \frac{1}{\tilde{C}_1} < |\xi| < \tilde{C}_1, \frac{1}{\tilde{C}_2} < |\eta| < \tilde{C}_2\} = \Omega_1$ *for some constants* $\tilde{C}_1, \tilde{C}_2 > 0$. *Suppose* $\sigma \in C^{4d+1}_{\mathrm{loc}}(\Omega_1)$ *and for some* $\tilde{A}_1 > 0$

$$\sup_{|\alpha|+|\beta| \leq 4d+1} \sup_{(\xi, \eta) \in \Omega_1} |\xi|^{|\alpha|} |\eta|^{|\beta|} |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq \tilde{A}_1.$$

Then for any $1 \leq p_1 \leq \infty$, $1 \leq p_2 \leq \infty$, $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\|T_\sigma(f, g)\|_r \lesssim_{d, \tilde{C}_1, \tilde{C}_2, \tilde{A}_1, p_1, p_2} \|f\|_{p_1} \|g\|_{p_2},$$

where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.

Proof. For the first case see Theorem 3.7 in [3]. The idea is to make a Fourier expansion in the η-variable:

$$\sigma(\xi, \eta) = \sum_{k \in \mathbb{Z}^d} L^{-d} \int_{[-\frac{L}{2}, \frac{L}{2}]^d} \sigma(\xi, \tilde{\eta}) e^{-2\pi i \frac{k \cdot \tilde{\eta}}{L}} d\tilde{\eta} \, e^{2\pi i \frac{k \cdot \eta}{L}} \chi(\eta),$$

where $L = 8C_1$ and $\chi \in C^\infty_c((-\frac{L}{4}, \frac{L}{4})^d)$ is such that $\chi(\eta) \equiv 1$ for $1/C_1 < |\eta| < C_1$. A rough estimate on the number of derivatives required is $n_0 = 2d + [d/2] + 1$. Note that $r > 1/2$ and (by paying $2d$ derivatives) $2dr > d$ so that the resulting summation in k converges in l^r-norm. For the second case, one can make a Fourier expansion in (ξ, η). $\square$

Remark 5.2. For $t > 0, 0 < \gamma < 1, j \in \mathbb{Z}$, consider

$$\sigma_0(\xi, \eta) = e^{-t(|\xi|^\gamma + |\eta|^\gamma - |\xi + \eta|^\gamma)} \chi_{|\xi| \sim 2^j} \chi_{|\eta| \sim 2^j} \chi_{|\xi + \eta| \ll 2^j}.$$

By using the estimates $\|\mathcal{F}^{-1}(e^{t|\xi|^\gamma}\chi_{|\xi|\ll 1})\|_1 = \|\mathcal{F}^{-1}(e^{|\xi|^\gamma}\chi_{|\xi|\ll t^{\frac{1}{\gamma}}})\|_1 \lesssim e^{\tilde{c}t}$ $(\tilde{c} \ll 1)$, $\|\mathcal{F}^{-1}(e^{-t|\xi|^\gamma}\chi_{|\xi|\sim 1})\|_1 \lesssim e^{-Ct}$ $(C \sim 1)$, we have for any $1 \le r, p_1, p_2 \le \infty$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$,

$$\|T_{\sigma_0}(f,g)\|_r \lesssim_{\gamma,d} e^{-c2^{j\gamma}t}\|f\|_{p_1}\|g\|_{p_2} \lesssim \|f\|_{p_1}\|g\|_{p_2},$$

where $c > 0$ is a small constant. Denote

$$\sigma_1(\xi,\eta) = e^{-t(|\xi|^\gamma+|\eta|^\gamma-|\xi+\eta|^\gamma)}\chi_{|\xi|\ll 2^j}\chi_{|\eta|\sim 2^j},$$

$$\sigma_2(\xi,\eta) = e^{-t(|\xi|^\gamma+|\eta|^\gamma-|\xi+\eta|^\gamma)}\chi_{|\xi|\sim 2^j}\chi_{|\eta|\ll 2^j},$$

$$\sigma_3(\xi,\eta) = e^{-t(|\xi|^\gamma+|\eta|^\gamma-|\xi+\eta|^\gamma)}\chi_{|\xi|\sim 2^j}\chi_{|\eta|\sim 2^j}\chi_{|\xi+\eta|\sim 2^j}.$$

By using Lemmas 5.1, 2.8 and some elementary computations, it is not difficult to check that for any $\frac{1}{2} < r < \infty$, $1 < p_1, p_2 < \infty$, with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$,

$$\|T_{\sigma_l}(f,g)\|_r \lesssim_{\gamma,p_1,p_2,d} \|f\|_{p_1}\|g\|_{p_2}, \quad \forall l = 1, 2, 3.$$

We shall need to use these inequalities (sometimes without explicit mentioning) below.

Fix $t > 0$, $j \in \mathbb{Z}$, $0 < \gamma < 1$, and denote

$$B_j(f,g) = [P_j e^{tD^\gamma}, e^{-tD^\gamma}R^\perp f] \cdot \nabla e^{-tD^\gamma}g$$

$$= P_j e^{tD^\gamma}(e^{-tD^\gamma}R^\perp f \cdot \nabla e^{-tD^\gamma}g) - e^{-tD^\gamma}R^\perp f \cdot \nabla P_j g.$$

For integer $J_0 \ge 10$ which will be made sufficiently large later, we decompose

$$B_j(f,g)$$
$$= B_j(f_{\le J_0+2}, g_{\le J_0+4}) + B_j(f_{\le J_0+2}, g_{>J_0+4}) + B_j(f_{>J_0+2}, g_{\le J_0+2}) + B_j(f_{>J_0+2}, g_{>J_0+2})$$
$$= B_j(f_{\le J_0+2}, g_{\le J_0+4}) + B_j(f_{[J_0+2,J_0+4]}, g_{\le J_0+2})$$
$$\quad + B_j(f_{>J_0+4}, g_{\le J_0+2}) + B_j(f_{\le J_0+2}, g_{>J_0+4}) + B_j(f_{>J_0+2}, g_{>J_0+2}).$$

Lemma 5.3. $B_j(f_{\le J_0+2}, g_{\le J_0+4}) = 0$ and $B_j(f_{[J_0+2,J_0+4]}, g_{\le J_0+2}) = 0$ for $j > J_0 + 6$. For $j \le J_0 + 6$ and $1 \le p < \infty$,

$$\|B_j(f_{\le J_0+2}, g_{\le J_0+4})\|_p + \|B_j(f_{[J_0+2,J_0+4]}, g_{\le J_0+2})\|_p$$
$$\lesssim e^{c_1(1+t)}(\|P_{\le J_0+10}f\|_p^2 + \|P_{\le J_0+10}g\|_p^2),$$

where $c_1 > 0$ depends on (J_0, p, γ).

Proof. We only deal with $B_j(f_{\le J_0+2}, g_{\le J_0+4})$ as the estimate for $B_j(f_{[J_0+2,J_0+4]}, g_{\le J_0+2})$ is similar and therefore omitted. Clearly for $j \le J_0 + 6$ (below the notation $\infty-$, $p+$ is defined in the same way as in (2.1)),

$$\|e^{-tD^\gamma}R^\perp f_{\le J_0+2} \cdot \nabla P_j g_{\le J_0+4}\|_p \lesssim \|R^\perp f_{\le J_0+2}\|_{p+}2^{J_0}\|P_{\le J_0+10}g\|_{\infty-}$$
$$\lesssim 2^{J_0(1+\frac{2}{p})}(\|P_{\le J_0+10}f\|_p^2 + \|P_{\le J_0+10}g\|_p^2).$$

Here $p+$ is needed for $p = 1$ so that the Riesz transform can be discarded. On the other hand for $j \le J_0 + 6$,

$$\|P_j e^{tD^\gamma}(e^{-tD^\gamma}R^\perp f_{\le J_0+2} \cdot \nabla e^{-tD^\gamma}g_{\le J_0+4})\|_p \lesssim e^{c_1(1+t)}(\|P_{\le J_0+10}f\|_p^2 + \|P_{\le J_0+10}g\|_p^2).$$

$\square$

Lemma 5.4. *For $j \leq J_0 + 6$ and $0 < t \leq 1$,*

$$\|B_j(f_{\leq J_0+2}, g_{> J_0+4})\|_p \lesssim c_2 \|e^{-tD^\gamma} f_{\leq J_0+2}\|_p \|g_{[J_0+5, J_0+10]}\|_p.$$

For $j > J_0 + 6$, $t > 0$ and $1 \leq p < \infty$,

$$\|B_j(f_{\leq J_0+2}, g_{> J_0+4})\|_p$$
$$\lesssim c_2 \cdot 2^{j0+} \|P_{\leq J_0+2} f\|_p \|g_{[j-2, j+2]}\|_p + c_2 (t + t^2) \|P_{\leq J_0+2} f\|_p \cdot 2^{j\gamma} \|g_{[j-2, j+2]}\|_p,$$

where $c_2 > 0$ depends on (p, J_0), and the notation $0+$ is defined in the paragraph preceding (2.1).

Proof. The first inequality for $j \leq J_0 + 6$ is obvious. Consider now $j > J_0 + 6$. Observe that by frequency localization $B_j(f_{\leq J_0+2}, g_{> J_0+4}) = B_j(f_{\leq J_0+2}, P_{> J_0+4} g_{[j-2, j+2]})$. We just need to consider $T_\sigma(R^\perp f_{\leq J_0+2}, \nabla P_{> J_0+4} g_{[j-2, j+2]})$ with $|\xi| \ll 2^j$, $|\eta| \sim 2^j$, and

$$\sigma(\xi, \eta) = [\phi(2^{-j}(\xi + \eta)) e^{-t(|\xi|^\gamma + |\eta|^\gamma - |\xi+\eta|^\gamma)} - \phi(2^{-j}\eta) e^{-t|\xi|^\gamma}] \chi_{|\xi| \ll 2^j} \chi_{|\eta| \sim 2^j}$$

$$= (\phi(2^{-j}(\xi + \eta)) - \phi(2^{-j}\eta)) e^{-t(|\xi|^\gamma + |\eta|^\gamma - |\xi+\eta|^\gamma)} \chi_{|\xi| \ll 2^j} \chi_{|\eta| \sim 2^j}$$

$$- \phi(2^{-j}\eta)(e^{-t(|\xi|^\gamma + |\eta|^\gamma - |\xi+\eta|^\gamma)} - e^{-t|\xi|^\gamma}) \chi_{|\xi| \ll 2^j} \chi_{|\eta| \sim 2^j}$$

$$=: \sigma_1(\xi, \eta) + \sigma_2(\xi, \eta).$$

By Lemma 5.1, it is easy to check that for some $c_2 > 0$ depending on (J_0, p),

$$\|T_{\sigma_1}(R^\perp f_{\leq J_0+2}, \nabla P_{> J_0+4} g_{[j-2, j+2]})\|_p$$
$$\lesssim \|\partial R^\perp f_{\leq J_0+2}\|_{\infty-} \|g_{[j-2, j+2]}\|_{p+} \lesssim c_2 \cdot 2^{j0+} \|P_{\leq J_0+2} f\|_p \|g_{[j-2, j+2]}\|_p.$$

On the other hand for σ_2, we introduce for $0 \leq \tau \leq 1$

$$F(\tau) = e^{-t(|\xi|^\gamma + \tau(|\eta|^\gamma - |\xi+\eta|^\gamma))}, \quad F_1(\tau) = |\eta + \tau\xi|^\gamma.$$

Then observe

$$F(1) - F(0) = F'(0) + \int_0^1 F''(\tau)(1-\tau)d\tau$$

$$= e^{-t|\xi|^\gamma} t(|\xi + \eta|^\gamma - |\eta|^\gamma) + \int_0^1 F(\tau) t^2 (|\eta|^\gamma - |\xi+\eta|^\gamma)^2 (1-\tau)d\tau$$

$$= \gamma t e^{-t|\xi|^\gamma} |\eta|^{\gamma-2} (\eta \cdot \xi) + t e^{-t|\xi|^\gamma} \int_0^1 F_1''(\tau)(1-\tau)d\tau$$

$$+ \int_0^1 F(\tau) t^2 (|\eta|^\gamma - |\xi+\eta|^\gamma)^2 (1-\tau)d\tau.$$

To handle the last term above we make the following observation. Note that for $|\xi| \ll 2^j$, $|\eta| \sim 2^j$, one has $||\eta|^\gamma - |\xi+\eta|^\gamma| = \mathcal{O}(|\eta|^{\gamma-1} \cdot |\xi|) \leq \frac{\alpha_0}{2} |\xi|^\gamma$, for some constant $0 < \alpha_0 < 1$. In particular one may write

$$F(\tau) = e^{-t(1-\alpha_0)|\xi|^\gamma} e^{-t(\alpha_0|\xi|^\gamma + \tau(|\eta|^\gamma - |\xi+\eta|^\gamma))}. \tag{5.1}$$

The symbol corresponding to $e^{-t(\alpha_0|\xi|^\gamma + \tau(|\eta|^\gamma - |\xi+\eta|^\gamma))}$ is clearly good for us whilst the term $e^{-t(1-\alpha_0)|\xi|^\gamma}$ can be used to extract additional decay (see below).

It is then clear that

$$\|T_{\sigma_2}(R^\perp f_{\leq J_0+2}, \nabla P_{>J_0+4}g_{[j-2,j+2]})\|_p$$
$$\lesssim t\|e^{-tD^\gamma}\partial R^\perp f_{\leq J_0+2}\|_\infty \cdot 2^{j\gamma}\|g_{[j-2,j+2]}\|_p$$
$$+ 2^{j(\gamma-1)} \cdot t\|e^{-tD^\gamma}\partial^2 R^\perp f_{\leq J_0+2}\|_{\infty-}\|g_{[j-2,j+2]}\|_{p+}$$
$$+ 2^{j(2\gamma-1)} \cdot t^2\|\partial^2 e^{-t(1-\alpha_0)D^\gamma}R^\perp f_{\leq J_0+2}\|_{\infty-}\|g_{[j-2,j+2]}\|_{p+}$$
$$\lesssim c_2 \cdot (t+t^2)\|P_{\leq J_0+2}f\|_p \cdot 2^{j\gamma}\|g_{[j-2,j+2]}\|_p.$$

$\square$

Lemma 5.5. *Let* $1 \leq p < \infty$. *For* $j \leq J_0 + 6$, $0 < t \leq 1$, *we have*

$$\|B_j(f_{>J_0+4}, g_{\leq J_0+2})\|_p \lesssim c_2\|e^{-tD^\gamma}f_{>J_0+4}\|_p\|g_{\leq J_0+2}\|_p.$$

For $j > J_0 + 6$ *and any* $t > 0$, *we have*

$$\|B_j(f_{>J_0+4}, g_{\leq J_0+2})\|_p \lesssim c_2 \cdot 2^{j0+}\|P_{\leq J_0+2}g\|_p\|f_{[j-2,j+2]}\|_p,$$

In the above $c_2 > 0$ *depends on* (γ, p, J_0).

Proof. The estimate for $j \leq J_0 + 6$ is obvious. Observe that for $j > J_0 + 6$,

$$B_j(f_{>J_0+4}, g_{\leq J_0+2}) = P_j e^{tD^\gamma}(e^{-tD^\gamma}R^\perp P_{>J_0+4}f_{[j-2,j+2]} \cdot \nabla e^{-tD^\gamma}P_{\leq J_0+2}g).$$

Thus by Lemma 5.1,

$$\|B_j(f_{>J_0+4}, g_{\leq J_0+2})\|_p$$
$$\lesssim \|R^\perp f_{[j-2,j+2]}\|_{p+}\|\nabla P_{\leq J_0+2}g\|_{\infty-} \lesssim c_2 2^{j0+}\|P_{\leq J_0+2}g\|_p\|f_{[j-2,j+2]}\|_p.$$

$\square$

Lemma 5.6. *Denote* $f^h = f_{>J_0+2}$, $g^h = g_{>J_0+2}$. *Then for* $j \geq J_0$, $1 \leq p < \infty$, $0 < t \leq 1$, *we have*

$$\|B_j(f^h, g^h)\|_p \lesssim 2^{j\gamma}\|f^h\|_{\dot B_{p,\infty}^{1+\frac{2}{p}-\gamma}}\|g_{[j-2,j+2]}^h\|_p + 2^{j\gamma}\|g^h\|_{\dot B_{p,\infty}^{1+\frac{2}{p}-\gamma}}\|f_{[j-2,j+9]}^h\|_p$$
$$+ 2^j \sum_{k \geq j+8} 2^{k \cdot \frac{2}{p}}\|f_k^h\|_p\|g_{[k-2,k+2]}^h\|_p.$$

Proof. Write

$$B_j(f^h, g^h) = B_j(f_{<j-2}^h, g^h) + B_j(f_{[j-2,j+9]}^h, g^h) + B_j(f_{>j+9}^h, g^h)$$
$$= B_j(f_{<j-2}^h, g_{[j-2,j+2]}^h) + B_j(f_{[j-2,j+9]}^h, g_{<j-4}^h)$$
$$+ B_j(f_{[j-2,j+9]}^h, g_{[j-4,j+12]}^h) + \sum_{k \geq j+10} B_j(f_k^h, g^h)$$
$$=: (1) + (2) + (3) + (4).$$

Estimate of (1).

Note that for given integer $J_1 \geq 2$, the term $B_j(f_{[j-J_1,j-3]}^h, g_{[j-2,j+2]}^h)$ can be included in the estimate of (3).

It suffices for us to estimate $(1A) = T_\sigma(R^\perp f^h_{<j-J_1}, \nabla g^h_{[j-2,j+2]})$ with (here to ensure $|\xi| \ll 2^j$ we need to take J_1 sufficiently large)

$$\sigma(\xi,\eta) = (\phi(\frac{\xi+\eta}{2^j}) - \phi(\frac{\eta}{2^j}))e^{-t(|\xi|)^\gamma}$$
$$+|\eta|^\gamma - |\xi+\eta|^\gamma)\chi_{|\xi|\ll 2^j}\chi_{|\eta|\sim 2^j} - \phi(\frac{\eta}{2^j})(e^{-t(|\xi|^\gamma+|\eta|^\gamma-|\xi+\eta|^\gamma)} - e^{-t|\xi|^\gamma})\chi_{|\xi|\ll 2^j}\chi_{|\eta|\sim 2^j}.$$

By an argument similar to that in Lemma 5.4, we get

$$\|(1A)\|_p$$
$$\lesssim \|\partial R^\perp f^h_{<j-2}\|_\infty- \|g^h_{[j-2,j+2]}\|_{p+} + t\|e^{-tD^\gamma}\partial R^\perp f^h_{<j-2}\|_\infty$$
$$\cdot 2^{j\gamma}\|g^h_j\|_p + 2^{j(\gamma-1)}\cdot t\|e^{-tD^\gamma}\partial^2 R^\perp f^h_{<j-2}\|_\infty- \|g^h_j\|_{p+}$$
$$+ 2^{j(2\gamma-1)}\|D^{2-2\gamma}R^\perp f^h_{<j-2}\|_\infty- \|g^h_j\|_{p+}$$
$$\lesssim 2^{j\gamma}\|f^h\|_{\dot B^{1+\frac{2}{p}-\gamma}_{p,\infty}}\|g^h_{[j-2,j+2]}\|_p.$$

Estimate of (2). Clearly

$$(2) = P_j e^{tD^\gamma}(R^\perp e^{-tD^\gamma}f^h_{[j-2,j+9]}\cdot\nabla e^{-tD^\gamma}g^h_{<j-4}).$$

Thus

$$\|(2)\|_p \lesssim \|f^h_{[j-2,j+9]}\|_{p+}\|\nabla g^h_{<j-4}\|_\infty- \lesssim \|g^h\|_{\dot B^{1-\gamma+\frac{2}{p}}_{p,\infty}}2^{j\gamma}\|f^h_{[j-2,j+9]}\|_p.$$

Estimate of (3). Clearly

$$\|(3)\|_p \lesssim 2^{j\gamma}\|g^h\|_{\dot B^{1-\gamma+\frac{2}{p}}_{p,\infty}}\|f^h_{[j-2,j+9]}\|_p.$$

Estimate of (4). We first note that

$$\sum_{k\geq j+10}\|e^{-tD^\gamma}R^\perp f^h_k\cdot\nabla P_j g\|_p \lesssim 2^{j\gamma}\|f^h\|_{\dot B^{1+\frac{2}{p}-\gamma}_{p,\infty}}\|g^h_{[j-2,j+2]}\|_p. \qquad (5.2)$$

On the other hand by using that $R^\perp f$ is divergence-free, we have

$$\sum_{k\geq j+10}\left\|P_j e^{tD^\gamma}\nabla\cdot\left((e^{-tD^\gamma}R^\perp f^h_k)(e^{-tD^\gamma}g^h_{[k-2,k+2]})\right)\right\|_p$$
$$\lesssim \sum_{k\geq j+10}2^j\|f^h_k\|_{2p}\|g^h_{[k-2,k+2]}\|_{2p}$$
$$\lesssim 2^j\sum_{k\geq j+8}2^{k\cdot\frac{2}{p}}\|f^h_k\|_p\cdot\|g^h_{[k-2,k+2]}\|_p.$$

$\square$

6. Proof of Theorem 1.2

Recall that the initial data $\theta_0 \in B_{p,q}^{1-\gamma+\frac{2}{p}}$, $1 \le p < \infty$, $0 < \gamma < 1$ and $1 \le q < \infty$.

Lemma 6.1. *Let* $\chi \in C_c^\infty(\mathbb{R}^2)$ *and* $\theta_0 \in B_{p,q}^s(\mathbb{R}^2)$ *with* $1 \le p < \infty$, $1 \le q < \infty$, $s > 0$. *Let* $(\lambda_n)_{n=1}^\infty$ *be a sequence of positive numbers such that* $\inf_n \lambda_n > 0$. *Then*

$$\lim_{J_0 \to \infty} \sup_{n \ge 1} \| P_{>J_0}\left(\chi(\lambda_n^{-1}x) P_{\le n+2}\theta_0 \right)\|_{B_{p,q}^s} = 0.$$

Proof. Write $f = \chi(\lambda_n^{-1}x)$, $g = P_{\le n+2}\theta_0$, then

$$fg = \sum_{j \in \mathbb{Z}}(f_j g_{<j-2} + g_j f_{<j-2} + f_j \tilde{g}_j),$$

where $\tilde{g}_j = g_{[j-2, j+2]}$. Clearly

$$(2^{js}\| f_j g_{<j-2}\|_p)_{l_j^q(j \ge J_0)} \lesssim \|g\|_p \|\partial^{2[s]+2} f\|_\infty (2^{-j(2[s]+2-s)})_{l_j^q(j \ge J_0)} \to 0,$$

uniformly in n as $J_0 \to \infty$. A similar estimate also shows that the diagonal piece $f_j \tilde{g}_j$ is OK. On the other hand

$$(2^{js}\| f_{<j-2}g_j\|_p)_{l_j^q(j \ge J_0)} \lesssim \|f\|_\infty (2^{js}\|P_j\theta_0\|_p)_{l_j^q(j \ge J_0)} \to 0,$$

uniformly in n as $J_0 \to \infty$. $\qquad\square$

We now complete the proof of Theorem 1.2. This will be carried out in several steps below.

Step 1. Definition of approximating solutions. Define $\theta^{(0)} \equiv 0$. For $n \ge 0$, define the iterates $\theta^{(n+1)}$ as solutions to the following system

$$\begin{cases} \partial_t \theta^{(n+1)} = -R^\perp \theta^{(n)} \cdot \nabla \theta^{(n+1)} - D^\gamma \theta^{(n+1)}, & (t,x) \in (0,\infty) \times \mathbb{R}^2; \\ \theta^{(n+1)}\big|_{t=0} = \chi(\lambda_n^{-1}x) P_{\le n+2}\theta_0, \end{cases}$$

where $\chi \in C_c^\infty(\mathbb{R}^2)$ satisfies $0 \le \chi \le 1$ for all x, $\chi(x) \equiv 1$ for $|x| \le 1$, and $\chi(x) = 0$ for $|x| \ge 2$. Here we introduce the spatial cut-off χ so that $\theta^{(n+1)}\big|_{t=0} \in H^k$ for all $k \ge 0$ when we only assume θ_0 lies in L^p type spaces. The scaling parameters $\lambda_n \ge 1$ are inductively chosen such that $\lambda_n > \max\{4\lambda_{n-1}, 2^n\}$ and

$$\|\theta_0\|_{L^p(|x| > \frac{1}{100}\lambda_n)} < 2^{-100n}.$$

Easy to check that

$$\|\theta^{(n+1)}(0) - \theta^{(n)}(0)\|_p \lesssim 2^{-n(1-\gamma+\frac{2}{p})},$$

and by interpolation for $0 < \tilde{s} < 1 - \gamma + \frac{2}{p}$, $\tilde{s} = 0+$,

$$\|\theta^{(n+1)}(0) - \theta^{(n)}(0)\|_{B_{p,\infty}^{\tilde{s}}} \lesssim 2^{-n(1-\gamma+\frac{2}{p})+}. \tag{6.1}$$

Also by Lemma 6.1, we have

$$\|\theta^{(n+1)}(0) - \theta_0\|_{B^{1-\gamma+\frac{2}{p}}_{p,q}} \to 0, \qquad \text{as } n \to \infty.$$

These estimates will be needed for the contraction estimate later.

Clearly we have the uniform boundedness of L^p norm:

$$\sup_{n\geq 0}\sup_{0\leq t<\infty} \|\theta^{(n+1)}(t)\|_p \leq \sup_{n\geq 0} \|P_{\leq n+2}\theta_0\|_p \lesssim \|\theta_0\|_p.$$

This will often be used without explicit mentioning below.

Step 2. Denote $A = \frac{1}{2}D^\gamma$, $f^{(n+1)}(t) = e^{tA}\theta^{(n+1)}(t)$. Then

$$\partial_t f^{(n+1)} = -Af^{(n+1)} - e^{tA}(R^\perp e^{-tA} f^{(n)} \cdot \nabla e^{-tA} f^{(n+1)}).$$

One can view $f^{(n+1)}$ as the unique limit of the sequence of solutions $(f^{(n+1)}_m)^\infty_{m=1}$ solving the regularized system

$$\begin{cases} \partial_t f^{(n+1)}_m = -Af^{(n+1)}_m - e^{tA}P_{\leq m}(R^\perp e^{-tA} f^{(n)} \cdot \nabla e^{-tA} P_{\leq m} f^{(n+1)}_m), \\ f^{(n+1)}_m\big|_{t=0} = P_{\leq m}\Big(\chi(\lambda_n^{-1}x)P_{\leq n+2}\theta_0\Big). \end{cases}$$

By using the estimates in Sect. 2 (and the inductive assumption that $f^{(n)} \in C^0_t H^k$ for all $k \geq 0$), we can then obtain $f^{(n+1)} \in C^0_t([0, T], H^k)$ for all $T > 0, k \geq 0$. Write

$$f^{(n+1)}(t) = e^{-tA} f^{(n+1)}(0) - \int_0^t e^{-(t-s)A}e^{sA}(R^\perp e^{-sA} f^{(n)} \cdot \nabla e^{-sA} f^{(n+1)})ds.$$

By using the fact that $f^{(n)}, f^{(n+1)} \in C^0_t H^k$, it is not difficult to check that

$$\sup_{0\leq t\leq T} \|\partial^k f^{(n+1)}(t)\|_p < \infty, \quad \forall T > 0, k \geq 0.$$

It follows that for any $T > 0$

$$\sup_{0\leq t\leq T} \|\partial_t f^{(n+1)}\|_p \leq \sup_{0\leq t\leq T} \|D^\gamma f^{(n+1)}\|_p + \|e^{tA}(R^\perp e^{-tA} f^{(n)} \cdot \nabla e^{-tA} f^{(n+1)})\|_p < \infty.$$

This together with interpolation implies $f^{(n+1)} \in C^0_t([0, T], W^{k,p})$ for any $T > 0$, $k \geq 0$. These estimates establish the (a priori) finiteness of the various Besov norms and associated time continuity needed in the following steps.

Step 3. Besov norm estimates. Denote $f^{(n+1)}_j = P_j f^{(n+1)}$. For any $\epsilon_0 > 0$, we show that there exists J_1 sufficiently large, and $T_1 > 0$ sufficiently small, such that

$$\sup_{n\geq 0}(2^{j(1-\gamma+\frac{2}{p})}\|f^{(n)}_j\|_p)_{l^q_j L^\infty_t(t\in[0,T_1], j\geq J_1)} < \epsilon_0. \tag{6.2}$$

Clearly for each $j \in \mathbb{Z}$,

$$\partial_t f^{(n+1)}_j = -\frac{1}{2}D^\gamma f^{(n+1)}_j - P_j e^{tA}(R^\perp e^{-tA} f^{(n)} \cdot \nabla e^{-tA} f^{(n+1)}).$$

Then by using Lemma 2.1, we get for some constants $\tilde{C}_1 > 0$, $\tilde{C}_2 > 0$,

$$\partial_t(\|f_j^{(n+1)}\|_p) + \tilde{C}_1 2^{j\gamma}\|f_j^{(n+1)}\|_p \leq \tilde{C}_2\|[P_j e^{tA}, e^{-tA}R^\perp f^{(n)}] \cdot \nabla e^{-tA} f^{(n+1)}\|_p.$$

Take an integer $J_0 \geq 10$ which will be made sufficiently large later. By using the nonlinear estimates derived before (see Lemma 5.3–5.6), we then obtain

$$\|f_j^{(n+1)}(t)\|_p \leq e^{-\tilde{C}_1 2^{j\gamma}t}\|f_j^{(n+1)}(0)\|_p + \int_0^t e^{-\tilde{C}_1 2^{j\gamma}(t-s)} N_j ds,$$

where for some constants $\tilde{C}_3 > 0$, $\tilde{C}_4 > 0$, $\tilde{C}_5 > 0$,

$$\begin{aligned}
N_j &= 1_{j \leq J_0+6} \cdot \tilde{C}_3 \cdot (\|P_{\leq J_0+10} f^{(n)}\|_p^2 \\
&\quad + \|P_{\leq J_0+10} f^{(n+1)}\|_p^2 + \|\theta_0\|_p^2) + 1_{j > J_0+6} \cdot \tilde{C}_4 \cdot 2^{j0+}(\|P_{\leq J_0+2} f^{(n)}\|_p \|f_{[j-2,j+2]}^{(n+1)}\|_p) \\
&\quad + 1_{j > J_0+6} \cdot \tilde{C}_4 \cdot (2^{j0+}\|P_{\leq J_0+2} f^{(n+1)}\|_p \|f_{[j-2,j+2]}^{(n)}\|_p \\
&\quad + s\|P_{\leq J_0+2} f^{(n)}\|_p \cdot 2^{j\gamma}\|f_{[j-2,j+2]}^{(n+1)}\|_p) \\
&\quad + \tilde{C}_5 2^{j\gamma}(\|P_{>J_0+2} f^{(n)}\|_{\dot{B}_{p,\infty}^{1+\frac{2}{p}-\gamma}}\|P_{>J_0+2} f_{[j-2,j+2]}^{(n+1)}\|_p \\
&\quad + \|P_{>J_0+2} f^{(n+1)}\|_{\dot{B}_{p,\infty}^{1+\frac{2}{p}-\gamma}}\|P_{>J_0+2} f_{[j-2,j+9]}^{(n)}\|_p) \\
&\quad + \tilde{C}_5 2^j \sum_{k \geq j+8} 2^{k\frac{2}{p}}\|P_{>J_0+2} f_k^{(n)}\|_p \|P_{>J_0+2} f_{[k-2,k+2]}^{(n+1)}\|_p.
\end{aligned}$$

Denote

$$\|f^{(n+1)}\|_{T,J_0} = (2^{j(1-\gamma+\frac{2}{p})}\|f_j^{(n+1)}\|_p)_{l_j^q L_t^\infty(t \in [0,T], j \geq J_0)}.$$

One should note that by the estimates derived in Step 2, the above norm of $f^{(n+1)}$ is finite. Then for $0 < T \leq 1$,

$$\begin{aligned}
\|f^{(n+1)}\|_{T,J_0} &\leq (2^{j(1-\gamma+\frac{2}{p})}\|f_j^{(n+1)}(0)\|_p)_{l_j^q(j \geq J_0)} + C_{J_0}^{(1)} T\|\theta_0\|_p^2 \\
&\quad + C_1 \cdot 2^{-\frac{1}{2}J_0\gamma} \cdot e^{C_3 \cdot 2^{J_0\gamma}T}\|\theta_0\|_p \cdot (\|f^{(n)}\|_{T,J_0} + \|f^{(n+1)}\|_{T,J_0}) \\
&\quad + C_{J_0}^{(2)} \cdot T\|\theta_0\|_p \|f^{(n+1)}\|_{T,J_0} + C_2\|f^{(n)}\|_{T,J_0}\|f^{(n+1)}\|_{T,J_0},
\end{aligned}$$

where $C_{J_0}^{(1)}, C_{J_0}^{(2)} > 0$ are constants depending on (J_0, γ, p, q), $C_1, C_2 > 0$ are constants depending only on (γ, p, q), and $C_3 > 0$ depends only on γ.

By Lemma 6.1, one can find J_0 sufficiently large such that

$$\sup_{n \geq 0}(2^{j(1-\gamma+\frac{2}{p})}\|f_j^{(n+1)}(0)\|_p)_{l_j^q(j \geq J_0)} < \frac{1}{100C_2},$$

$$C_1 \cdot 2^{-\frac{1}{2}J_0\gamma} \cdot 10(1 + \|\theta_0\|_p) < \frac{1}{20}.$$

Fix such J_0 and then choose $T = T_0 \leq 1$ such that

$$C_{J_0}^{(1)} T_0\|\theta_0\|_p^2 < \frac{1}{100C_2}, \quad C_3 \cdot 2^{J_0\gamma} \cdot T_0 < \frac{1}{100}, \quad C_{J_0}^{(2)} T_0\|\theta_0\|_p < \frac{1}{20}.$$

The inductive assumption is $\|f^{(n)}\|_{T_0, J_0} < \frac{1}{4C_2}$. Then clearly

$$\|f^{(n+1)}\|_{T_0, J_0} \leq \frac{1}{100C_2}$$
$$+ \frac{1}{100C_2} + \frac{1}{20}\|f^{(n+1)}\|_{T_0, J_0} + \frac{1}{20} \cdot \frac{1}{4C_2} + \frac{1}{20}\|f^{(n+1)}\|_{T_0, J_0} + \frac{1}{4}\|f^{(n+1)}\|_{T_0, J_0}.$$

This easily implies $\|f^{(n+1)}\|_{T_0, J_0} < \frac{1}{4C_2}$ which completes the argument.

The statement (6.2) clearly follows by a slight modification of the above argument.

Step 4. Contraction in $B_{p,\infty}^{s_0}$ where $s_0 > 0$ is a sufficiently small number.

Remark. We chose the space $C_t^0 B_{p,\infty}^{0+}$ since it contains L^p and its norm coincides with the usual Chemin-Lerner space $\tilde{L}_t^\infty B_{p,\infty}^{0+}$ (see (6.10)). This way one can make full use of the smoothing effect of the linear semigroup on each dyadic frequency block which is needed for this critical problem.

Set $\eta^{(n+1)} = f^{(n+1)} - f^{(n)}$. Then

$$\partial_t \eta^{(n+1)} = -A\eta^{(n+1)} - e^{tA}(R^\perp e^{-tA}\eta^{(n)} \cdot \nabla e^{-tA} f^{(n+1)}) - e^{tA}(R^\perp e^{-tA} f^{(n-1)} \cdot \nabla e^{-tA}\eta^{(n+1)}).$$

It is easy to check for $0 \leq t \leq T_0$, $J_1 \in \mathbb{Z}$ (below we work with $p+$ to avoid the end-point situation $p = 1$)

$$\partial_t \|P_{\leq 2J_1}\eta^{(n+1)}\|_p$$
$$\lesssim_{J_1, T_0, p, \gamma} \|e^{-tA} R^\perp \eta^{(n)}\|_{p+}\|f^{(n+1)}\|_{\infty-} + \|R^\perp f^{(n-1)}\|_{\infty-}\|e^{-tA}\eta^{(n+1)}\|_{p+}$$
$$\lesssim_{J_1, T_0, p, \gamma} \|\eta^{(n)}\|_p\|f^{(n+1)}\|_{\infty-} + \|f^{(n-1)}\|_{\infty-}\|\eta^{(n+1)}\|_p$$
$$\leq C_{T_0, J_1} \cdot (\|\eta^{(n)}\|_p + \|\eta^{(n+1)}\|_p),$$

where C_{T_0, J_1} is a constant depending only on $(\theta_0, J_1, T_0, \gamma, p, q)$. Here in the last inequality we used the estimates obtained in Step 3.

On the other hand for $j \geq J_1$, denoting $\eta_j^{(n+1)} = P_j\eta^{(n+1)}$, we have

$$\partial_t \|\eta_j^{(n+1)}\|_p + \tilde{C}_1 2^{j\gamma}\|\eta_j^{(n+1)}\|_p$$
$$\lesssim \|P_j e^{tA}(R^\perp e^{-tA}\eta^{(n)} \cdot \nabla e^{-tA} f^{(n+1)})\|_p + \|[P_j e^{tA}, R^\perp e^{-tA} f^{(n-1)}] \cdot \nabla e^{-tA}\eta^{(n+1)}\|_p.$$

We now need a simple lemma.

Lemma 6.2. *Let $0 < t \le 1$, $1 \le p < \infty$, $J_1 \ge 10$. We have for any $j \ge J_1$,*

$$\|P_j e^{tA}(R^\perp e^{-tA}\eta \cdot \nabla e^{-tA}f)\|_p \lesssim 2^{j(1+\frac{2}{p}-s_0)}\|f_{[j-2,j+2]}\|_p \|\eta\|_{B^{s_0}_{p,\infty}}$$

$$+ 2^{j\gamma}\|\eta_{[j-2,j+3]}\|_p \cdot (2^{j_1(1-\gamma+\frac{2}{p})}\|f_{j_1}\|_p)_{l^\infty_{j_1}(j_1 \ge 2J_1)}$$

$$+ 2^{3J_1(1+\frac{2}{p})}\|P_{\le 3J_1}f\|_p \cdot 2^{j0+}\|\eta_{[j-2,j+3]}\|_p + 2^j\|\eta\|_{B^{s_0}_{p,\infty}} \sum_{k\ge j+4} 2^{k(\frac{2}{p}-s_0)}\|f_{[k-2,k+2]}\|_p;$$

$$\|P_j e^{tA}(R^\perp e^{-tA}f \cdot \nabla e^{-tA}\eta) - R^\perp e^{-tA}f \cdot \nabla P_j \eta\|_p$$

$$\lesssim 2^{j\gamma}\|f_{>J_1}\|_{\dot{B}^{1+\frac{2}{p}-\gamma}_{p,\infty}}\|\eta_{[j-2,j+3]}\|_p + C_{J_1} \cdot \|f_{\le J_1+10}\|_p \cdot 2^{j0+}\|\eta_{[j-2,j+3]}\|_p$$

$$+ C_{J_1} \cdot t\|f_{\le J_1+10}\|_p \cdot 2^{j\gamma}\|\eta_{[j-2,j+3]}\|_p$$

$$+ \|\eta\|_{B^{s_0}_{p,\infty}} \cdot 2^{j(1+\frac{2}{p}-s_0)}\|f_{[j-5,j+5]}\|_p + 2^{j\gamma}\|P_{\ge j+6}f\|_{\dot{B}^{1+\frac{2}{p}-\gamma}_{p,\infty}}\|\eta_j\|_p,$$

where C_{J_1} is a constant depending on J_1.

Proof of Lemma 6.2. For the first inequality we denote $\widetilde{N}_j(g,h) = P_j e^{tA}(R^\perp e^{-tA}g \cdot \nabla e^{-tA}h)$. By frequency localization, we write

$$\widetilde{N}_j(\eta, f) = \widetilde{N}_j(\eta_{<j-2}, f_{[j-2,j+2]}) + \widetilde{N}_j(\eta_{[j-2,j+3]}, f_{\le j+5}) + \sum_{k\ge j+4} \widetilde{N}_j(\eta_k, f_{[k-2,k+2]}).$$

Clearly

$$\|\widetilde{N}_j(\eta_{<j-2}, f_{[j-2,j+2]})\|_p$$

$$\lesssim 2^j\|f_{[j-2,j+2]}\|_p \|\eta_{<j-2}\|_\infty - \lesssim 2^{j(1+\frac{2}{p}-s_0)}\|f_{[j-2,j+2]}\|_p \|\eta\|_{B^{s_0}_{p,\infty}}.$$

For the second term we split f as $f = f_{>3J_1} + f_{\le 3J_1}$. Then (below we work again with $p+$ for the η-term which give rises to 2^{j0+}; the reason for $p+$ is to avoid the end-point case $p = 1$)

$$\|\widetilde{N}_j(\eta_{[j-2,j+3]}, f_{\le j+5})\|_p$$

$$\lesssim 2^{j\gamma}\|\eta_{[j-2,j+3]}\|_p \cdot (2^{j_1(1-\gamma+\frac{2}{p})}\|f_{j_1}\|_p)_{l^\infty_{j_1}(j_1 \ge 2J_1)} + 2^{3J_1(1+\frac{2}{p})}\|P_{\le 3J_1}f\|_p \cdot 2^{j0+}\|\eta_{[j-2,j+3]}\|_p.$$

For the diagonal piece, we have

$$\sum_{k\ge j+4} \|\widetilde{N}_j(\eta_k, f_{[k-2,k+2]})\|_p$$

$$\lesssim 2^j \sum_{k\ge j+4} 2^{k\frac{2}{p}}\|\eta_k\|_p \|f_{[k-2,k+2]}\|_p \lesssim 2^j\|\eta\|_{B^{s_0}_{p,\infty}} \sum_{k\ge j+4} 2^{k(\frac{2}{p}-s_0)}\|f_{[k-2,k+2]}\|_p.$$

For the second inequality, we denote

$$N_j(f, \eta) = P_j e^{tA}(R^\perp e^{-tA}f \cdot \nabla e^{-tA}\eta) - R^\perp e^{-tA}f \cdot \nabla P_j \eta. \tag{6.3}$$

Observe that

$$N_j(f, \eta_{<j-2})$$
$$= P_j e^{tA}(R^\perp e^{-tA} f \cdot \nabla e^{-tA} \eta_{<j-2}) = P_j e^{tA}(R^\perp e^{-tA} f_{[j-2,j+2]} \cdot \nabla e^{-tA} \eta_{<j-2}).$$

Thus

$$\|N_j(f, \eta_{<j-2})\|_p \lesssim \|f_{[j-2,j+2]}\|_p \cdot 2^{j(1+\frac{2}{p}-s_0)} \|\eta\|_{B^{s_0}_{p,\infty}}. \tag{6.4}$$

On the other hand,

$$N_j(f, \eta_{>j+3}) = P_j e^{tA}(R^\perp e^{-tA} f \cdot \nabla e^{-tA} \eta_{>j+3})$$
$$= \sum_{k \geq j+4} P_j e^{tA}(R^\perp e^{-tA} f_{[k-2,k+2]} \cdot \nabla e^{-tA} \eta_k). \tag{6.5}$$

Thus

$$\|N_j(f, \eta_{>j+3})\|_p \lesssim 2^j \|\eta\|_{B^{s_0}_{p,\infty}} \sum_{k \geq j+4} 2^{k(\frac{2}{p}-s_0)} \|f_{[k-2,k+2]}\|_p. \tag{6.6}$$

It remains to estimate $N_j(f, \eta_{[j-2,j+3]})$. We first note that

$$\|N_j(f_{\geq j+6}, \eta_{[j-2,j+3]})\|_p = \|R^\perp e^{-tA} f_{\geq j+6} \cdot \nabla P_j \eta\|_p$$
$$\lesssim 2^{j\gamma} \|P_{\geq j+6} f\|_{\dot{B}^{1+\frac{2}{p}-\gamma}_{p,\infty}} \|\eta_j\|_p. \tag{6.7}$$

On the other hand,

$$\|N_j(f_{[j-5,j+5]}, \eta_{[j-2,j+3]})\|_p \lesssim 2^{j(1+\frac{2}{p})} \|f_{[j-5,j+5]}\|_p \|\eta_{[j-2,j+3]}\|_p. \tag{6.8}$$

Finally to deal with the piece $N_j(f_{\leq j-6}, \eta_{[j-2,j+3]})$, we appeal to similar estimates in Lemmas 5.4 and 5.6. We obtain

$$\|N_j(f_{[j-5,j+5]}, \eta_{[j-2,j+3]})\|_p \lesssim C_{J_1} \cdot (2^{j0+} + t \cdot 2^{j\gamma}) \|P_{\leq J_1+10} f\|_p \|\eta_{[j-2,j+3]}\|_p$$
$$+ 2^{j\gamma} \|P_{>J_1} f_{<j-6}\|_{\dot{B}^{1+\frac{2}{p}-\gamma}_{p,\infty}} \|\eta_{[j-2,j+3]}\|_p. \tag{6.9}$$

The desired result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is clear that for any $T > 0$,

$$\|\eta^{(n+1)}\|_{C^0_t B^{s_0}_{p,\infty}([0,T])} \sim \|P_{\leq 1}\eta^{(n+1)}\|_{L^\infty_t L^p_x([0,T])} + (2^{js_0} \|\eta_j^{(n+1)}\|_p)_{l^\infty_t l^\infty_j (t \in [0,T], \, j \geq 2)}$$
$$= \|P_{\leq 1}\eta^{(n+1)}\|_{L^\infty_t L^p_x([0,T])} + (2^{js_0} \|\eta_j^{(n+1)}\|_p)_{l^\infty_j l^\infty_t (t \in [0,T], \, j \geq 2)}, \tag{6.10}$$

where the implied constant (in the notation "$\sim$") depends only on (s_0, p).

By this simple observation, using Lemma 6.2, (6.2), (6.1), and choosing first J_1 sufficiently large and then T_1 sufficiently small, we obtain

$$\|\eta^{(n+1)}\|_{C^0_t B^{s_0}_{p,\infty}([0,T_1])} \leq \frac{1}{2} \|\eta^{(n)}\|_{C^0_t B^{s_0}_{p,\infty}([0,T_1])} + C_{\theta_0} \cdot 2^{-n\sigma_0},$$

where C_{θ_0} is a constant depending on $(\theta_0, s_0, p, \gamma, q)$ and $\sigma_0 > 0$ depends on (s_0, γ, p). This clearly yields the desired contraction in the Banach space $C_t^0([0, T_1], B_{p,\infty}^{s_0})$.

Step 5. Time continuity in $B_{p,q}^{1-\gamma+\frac{2}{p}}$. By the previous step and interpolation, we get $f^{(n)}$ converges strongly to the limit f also in $C_t^0 B_{p,1}^{s'}([0, T_1])$ for any $0 < s' < 1-\gamma+\frac{2}{p}$. We still have to show $f \in C_t^0 B_{p,q}^{1-\gamma+\frac{2}{p}}([0, T_1])$. Since $f^{(n)} \to f$ in $C_t^0 L_x^p$ we only need to consider the high frequency part. Denote $s = 1 - \gamma + \frac{2}{p}$. By using the estimates in Step 3 and strong convergence in each dyadic frequency block, we have for any $M \geq 10$,

$$\sum_{1 \leq j \leq M} 2^{jsq} \|f_j\|_{L_t^\infty L_x^p([0,T_1])}^q = \lim_{n \to \infty} \sum_{1 \leq j \leq M} 2^{jsq} \|f_j^{(n)}\|_{L_t^\infty L_x^p([0,T_1])}^q < A_1 < \infty,$$

where $A_1 > 0$ is a constant independent of M. Thus $\|(f_j)\|_{l_j^q L_t^\infty L_x^p (j \geq 1, t \in [0,T_1])} < \infty$. Since $P_{\leq M} f \in C_t^0 B_{p,q}^s$ for any M, and

$$\|P_{\leq M} f - P_{\leq M'} f\|_{C_t^0 B_{p,q}^s} \lesssim \left(\sum_{M-2 \leq j \leq M'+2} 2^{jsq} \|f_j\|_{L_t^\infty L_x^p}^q \right)^{\frac{1}{q}} \to 0, \quad \text{as } M' > M \to \infty,$$

we obtain $f \in C_t^0 B_{p,q}^s$.

Remark. An alternative argument to show time continuity is to use directly (6.2) to get time continuity at $t = 0$. For $t > 0$ one can proceed similarly as the last part of Sect. 3 and show $e^{\epsilon_0 At} f \in L_t^\infty B_{p,q}^{1-\gamma+\frac{2}{p}}$ for some $\epsilon_0 > 0$ small and use it to "damp" the high frequencies.

Step 6. Set $\theta(t) = e^{-tA} f(t, \cdot)$. Clearly $\theta \in C_t^0 B_{p,q}^{1-\gamma+\frac{2}{p}}$. Recall $\theta_n = e^{-tA} f_n(t, \cdot)$. In view of strong convergence of f_n to f, we have $\theta_n \to \theta$ strongly in $C_t^0 B_{p,1}^{s'}$ for any $0 < s' < 1 - \gamma + \frac{2}{p}$. Since for any $0 \leq t_0 < t \leq T_0$ we have

$$\theta^{(n+1)}(t) = e^{-(t-t_0)D^\gamma} \theta^{(n+1)}(t_0) - \int_{t_0}^t \nabla \cdot e^{-(t-s)D^\gamma} (R^\perp \theta^{(n)} \theta^{(n+1)})(s) ds.$$

Taking the limit $n \to \infty$ yields

$$\theta(t) = e^{-(t-t_0)D^\gamma} \theta(t_0) - \int_{t_0}^t \nabla \cdot e^{-(t-s)D^\gamma} (R^\perp \theta(s) \theta(s)) ds. \qquad (6.11)$$

It should be mentioned that the above equality holds in the sense of L_x^p and even stronger topology. It is easy to check the absolute convergence of the integral on the RHS since

$$\|\nabla \cdot e^{-(t-s)D^\gamma} (R^\perp \theta(s) \theta(s))\|_p$$
$$\lesssim (t-s)^{-1+} \|D^{(1-\gamma)+}(R^\perp \theta(s) \theta(s))\|_p \lesssim (t-s)^{-1+} \|\theta(s)\|_{B_{p,\infty}^{1-\gamma+\frac{2}{p}}}^2.$$

Thus θ is the desired local solution. One can regard (6.11) (together with some regularity assumptions) as a variant of the usual mild solution. Note that $\theta_j = P_j\theta$ is smooth, and one can easy deduce from the integral formulation (6.11) the point-wise identity:

$$\partial_t\theta_j = -D^\gamma\theta_j - \nabla \cdot P_j(\theta R^\perp\theta).$$

From this one can proceed with the localized energy estimates and easily check the uniqueness of solution in $C_t^0 B_{p,\infty}^{1-\gamma+\frac{2}{p}}$. We omit the details.

Remark. Much better uniqueness results can be obtained by exploiting the specific form of $R^\perp\theta$ in connection with the $H^{-1/2}$ conservation law for non-dissipative SQG. Since this is not the focus of this work, we will not dwell on this issue here.

Declarations

Conflict of interest None.

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